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Results in Mathematics

Formulas for Bernoulli Numbers and Polynomials

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Abstract. We present several formulas involving the classical Bernoulli numbers and polynomials. Among others, we extend an identity for Bernoulli polynomials published by Wu et al. (Fibonacci Quart 42:295-299, 2004).

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1. Introduction and Statement of the Results

In 2001, Momiyama [7] used methods from p-adic analysis to prove the following remarkable identity for the classical Bernoulli numbers B_ν, defined by

$$\frac{z}{e^z-1}=\sum_{\nu=0}^{\infty} B_\nu \frac{z^\nu}{\nu!}.$$

Proposition 1. *For all nonnegative integers m and n with $m+n>0$, we have*

$$(-1)^m \sum_{\nu=0}^{m} (\nu+n+1)\binom{m+1}{\nu} B_{\nu+n} = (-1)^{n+1} \sum_{\nu=0}^{n} (\nu+m+1)\binom{n+1}{\nu} B_{\nu+m}. \tag{1.1}$$

From (1.1) with $m=n$ we obtain

$$\sum_{\nu=0}^{m} (\nu+m+1)\binom{m+1}{\nu} B_{\nu+m}=0,$$

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which is attributed to von Ettingshausen [8, pp. 284-285]; see also Kaneko [5]. In 2004, Wu et al. [9] used properties of power series to prove an interesting generalization of (1.1) involving the Bernoulli polynomials

$$B_n(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} B_{n-\nu} z^{\nu}.$$

Proposition 2. *For all nonnegative integers m and n and complex numbers z, we have*

$$\begin{aligned}(-1)^m \sum_{\nu=0}^{m} &(\nu+n+1)\binom{m+1}{\nu} B_{\nu+n}(z) + (-1)^n \sum_{\nu=0}^{n} (\nu+m+1) \\ &\times \binom{n+1}{\nu} B_{\nu+m}(-z) \\ &= (-1)^m (m+n+1)(m+n+2) z^{m+n}. \end{aligned} \tag{1.2}$$

If $m+n>0$ and $z=0$, then (1.2) reduces to (1.1). Chen and Sun [2] applied Zeilberger's algorithm to prove (1.2). We show that (1.2) can be written in a slightly shorter and more elegant form as follows.

Theorem 1. *Let m and n be nonnegative integers. Then, for $z \in \mathbb{C}$,*

$$\begin{aligned}(-1)^m &\sum_{\nu=0}^{m+1} (\nu+n+1)\binom{m+1}{\nu} B_{\nu+n}(z) \\ &= (-1)^{n+1} \sum_{\nu=0}^{n+1} (\nu+m+1)\binom{n+1}{\nu} B_{\nu+m}(-z). \end{aligned} \tag{1.3}$$

We apply some basic properties of Bernoulli polynomials to settle (1.3). In particular, we provide a new proof of (1.2).

The main aim of this paper is to present a new generalization of (1.2). We define

$$S(m,n;k;z) = (-1)^m \sum_{\nu=0}^{m} (\nu+n+1)\binom{m+1}{\nu}\binom{\nu+n}{k} B_{\nu+n-k}(z).$$

The following reciprocity formula holds. (As usual, if $p<0$, then $B_p(z)=0$.)

Theorem 2. *Let m, n and k be nonnegative integers. Then, for $z \in \mathbb{C}$,*

$$\begin{aligned} S(m,n;k;z) + (-1)^k S(n,m;k;-z) \ &= (-1)^m (m+n+1)(m+n+2) \\ &\quad \times \binom{m+n}{k} z^{m+n-k}. \end{aligned} \tag{1.4}$$

Remark 1. The special case $k=0$ gives (1.2).

If we set $z=0$, then (1.4) yields an extension of Momiyama's identity (1.1).

Corollary 1. *Let m, n, k be nonnegative integers. Then*

$$S^*(m,n;k) = (-1)^{k+1} S^*(n,m;k), \tag{1.5}$$

where

$$S^*(m,n;k) = (-1)^m \sum_{\nu=0}^{m} (\nu+n+1) \binom{m+1}{\nu} \binom{\nu+n}{k} B_{\nu+n-k}.$$

An application of Theorem 2 leads to an identity for the alternating sum

$$A(m,n;k) = \sum_{\nu=0}^{m} (-1)^\nu (\nu+n+1) \binom{m+1}{\nu} \binom{\nu+n}{k} B_{\nu+n-k}.$$

We obtain the following companion to (1.5).

Corollary 2. *Let k, m, n be nonnegative integers with $k+2 \le \min(m,n)$. Then*

$$A(m,n;k) = (-1)^{k+1} A(n,m;k). \tag{1.6}$$

A second application of Theorem 2 gives a reciprocity formula for the polynomial

$$P(m,n;k;z) = \sum_{\nu=0}^{m} \binom{m}{\nu} \binom{\nu+n}{k} z^{\nu+n-k}.$$

Corollary 3. *Let k, m, n be nonnegative integers. Then, for $z \in \mathbb{C}$,*

$$P(m,n;k;z) = (-1)^{m+n+k} P(n,m;k;-z-1). \tag{1.7}$$

Remark 2. From (1.7) with $z = -1/2$ we get

$$(-1)^m \sum_{\nu=0}^{m} (-2)^\nu \binom{m+n-\nu}{k} \binom{m}{\nu} = (-1)^{k+n} \sum_{\nu=0}^{n} (-2)^\nu \binom{m+n-\nu}{k} \binom{n}{\nu}.$$

By using differentiation or integration certain summation formulas involving Bernoulli polynomials lead to interesting new identities. Hereby, the recurrence relation $B_n'(x) = nB_{n-1}(x)$ plays an important role. We show that by applying an integral formula and (1.3) we obtain the following counterpart of

$$(-1)^m \sum_{\nu=0}^{m} \binom{m}{\nu} B_{\nu+n} = (-1)^n \sum_{\nu=0}^{n} \binom{n}{\nu} B_{\nu+m} \tag{1.8}$$

which is due to Gessel [3].

Corollary 4. *Let m and n be nonnegative integers. Then*

$$(-1)^n \sum_{\nu=0}^{m} \binom{m}{\nu} (2^{-\nu-n}-1) B_{\nu+n} - (-1)^m \sum_{\nu=0}^{n} \binom{n}{\nu} (2^{-\nu-m}-1) B_{\nu+m}$$
$$= \frac{m-n}{2^{m+n}}. \tag{1.9}$$

Remark 3. Combining (1.8) and (1.9) gives

$$(-1)^n \sum_{\nu=0}^{m} \binom{m}{\nu} 2^{m-\nu} B_{\nu+n} - (-1)^m \sum_{\nu=0}^{n} \binom{n}{\nu} 2^{n-\nu} B_{\nu+m} = m - n.$$

2. Proofs

I. We need the following formulas:

$$B_n(x+y) = \sum_{\nu=0}^{n} \binom{n}{\nu} B_\nu(x) y^{n-\nu} \quad (n = 0, 1, \ldots), \tag{2.1}$$

$$B_n(x+1) = (-1)^n B_n(-x) = B_n(x) + n x^{n-1} \quad (n = 0, 1, \ldots). \tag{2.2}$$

These formulas can be found in Abramowitz and Stegun [1, (23.1.7), (23.1.8), (23.1.9)]. The integral formula

$$\int_0^{1/2} B_n(x) dx = \frac{1 - 2^{n+1}}{(n+1)2^n} B_{n+1} \quad (n = 0, 1, \ldots) \tag{2.3}$$

is given in Moll and Vignat [6]. Applying (2.2) and (2.3) leads to

$$\int_0^{1/2} B_n(-x) dx = \left(\frac{-1}{2}\right)^n \left(\frac{1 - 2^{n+1}}{n+1} B_{n+1} + 1\right) \quad (n = 1, 2, \ldots). \tag{2.4}$$

The next two identities can be found in Gould [4, (1.13), (3.49)]:

$$\sum_{\nu=0}^{m+1} (-1)^\nu \binom{m+1}{\nu} \nu^p = 0 \quad (0 \le p \le m), \tag{2.5}$$

$$\sum_{\nu=0}^{k} (-1)^\nu \binom{k}{\nu} \binom{x-\nu}{r} = \binom{x-k}{r-k} \quad (k = 0, 1, \ldots). \tag{2.6}$$

Applying the binomial theorem gives

$$\sum_{\nu=0}^{r} (\nu + s) \binom{r}{\nu} t^\nu = (t+1)^{r-1} \big((r+s)t + s\big) \quad (r = 0, 1, \ldots). \tag{2.7}$$

II. We show that the identities (1.2) and (1.3) are equivalent. Let

$$T(m, n; z) = \sum_{\nu=0}^{m+1} a_\nu(m, n; z) \quad \text{and} \quad T^*(m, n; z) = \sum_{\nu=0}^{m} a_\nu(m, n; z) \tag{2.8}$$

with

$$a_\nu(m, n; z) = (-1)^m (\nu + n + 1) \binom{m+1}{\nu} B_{\nu+n}(z).$$

Using (2.2) gives

$$T(m, n; z) + T(n, m; -z) = T^*(m, n; z) + a_{m+1}(m, n; z) + T^*(n, m; -z) + a_{n+1}(n, m; -z)$$

$$= T^*(m,n;z) + T^*(n,m;-z)$$
$$-(-1)^m(m+n+1)(m+n+2)z^{m+n}.$$

This implies that (1.2) and (1.3) are equivalent.

III. Now, we prove the theorems and corollaries stated in Sect. 1.

Proof of Theorem 1. Using (2.8), (2.2), (2.1) with $x = -z$, $y = 1$ and the elementary formula

$$(r+1)\binom{r}{j} = (j+1)\binom{r+1}{j+1}$$

with $r = \nu + n$ gives

$$\begin{aligned} T(m,n;z) &= (-1)^{m+n}\sum_{\nu=0}^{m+1}(-1)^\nu(\nu+n+1)\binom{m+1}{\nu}B_{\nu+n}(1-z) \\ &= (-1)^{m+n}\sum_{\nu=0}^{m+1}(-1)^\nu(\nu+n+1)\binom{m+1}{\nu}\sum_{j=0}^{\nu+n}\binom{\nu+n}{j}B_j(-z) \\ &= (-1)^{m+n}\sum_{\nu=0}^{m+1}(-1)^\nu\binom{m+1}{\nu}\sum_{j=0}^{\nu+n}(j+1)\binom{\nu+n+1}{j+1}B_j(-z) \\ &= (-1)^{m+n}\sum_{j=0}^{m+n+1}(j+1)B_j(-z)\sum_{\nu=0}^{m+1}(-1)^\nu\binom{m+1}{\nu}\binom{\nu+n+1}{j+1}. \end{aligned}$$

Applying (2.5) we conclude that the inner sum is equal to zero, if $0 \le j \le m-1$. Thus

$$\begin{aligned} T(m,n;z) &= (-1)^{m+n}\sum_{j=m}^{m+n+1}(j+1)B_j(-z)\sum_{\nu=0}^{m+1}(-1)^\nu\binom{m+1}{\nu}\binom{\nu+n+1}{j+1} \\ &= (-1)^{m+n}\sum_{j=0}^{n+1}(j+m+1)B_{j+m}(-z)\sum_{\nu=0}^{m+1}(-1)^\nu\binom{m+1}{\nu}\binom{\nu+n+1}{j+m+1}. \end{aligned} \tag{2.9}$$

Using (2.6) with $k = m+1$, $r = j+m+1$, $x = m+n+2$ we get

$$\begin{aligned} \sum_{\nu=0}^{m+1}(-1)^\nu\binom{m+1}{\nu}\binom{\nu+n+1}{j+m+1} &= (-1)^{m+1}\sum_{\nu=0}^{m+1}(-1)^\nu\binom{m+1}{\nu}\binom{m+n+2-\nu}{j+m+1} \\ &= (-1)^{m+1}\binom{n+1}{j}. \end{aligned} \tag{2.10}$$

From (2.9) and (2.10) we obtain

$$T(m,n;z) = (-1)^{n+1}\sum_{j=0}^{n+1}(j+m+1)\binom{n+1}{j}B_{j+m}(-z) = -T(n,m;-z).$$

This settles (1.3). □

Proof of Theorem 2. Using (2.1) gives

$$\begin{aligned}(-1)^m \sum_{\nu=0}^{m} (\nu+n+1)\binom{m+1}{\nu} B_{\nu+n}(x+z) \\ = (-1)^m \sum_{\nu=0}^{m} (\nu+n+1)\binom{m+1}{\nu} \sum_{k=0}^{\infty} \binom{\nu+n}{k} B_{\nu+n-k}(z)x^k \\ = (-1)^m \sum_{k=0}^{\infty} \sum_{\nu=0}^{m} (\nu+n+1)\binom{m+1}{\nu}\binom{\nu+n}{k} B_{\nu+n-k}(z)x^k.\end{aligned} \tag{2.11}$$

Next, we exchange m and n and we replace x and z by $-x$ and $-z$, respectively. Then we obtain from (2.11):

$$\begin{aligned}(-1)^n \sum_{\nu=0}^{n} (\nu+m+1)\binom{n+1}{\nu} B_{\nu+m}(-(x+z)) \\ = (-1)^n \sum_{k=0}^{\infty} \sum_{\nu=0}^{n} (\nu+m+1)\binom{n+1}{\nu}\binom{\nu+m}{k} B_{\nu+m-k}(-z)(-1)^k x^k.\end{aligned} \tag{2.12}$$

Moreover, we have

$$\begin{aligned}(-1)^m(m+n+1)(m+n+2)(x+z)^{m+n} \\ = (-1)^m(m+n+1)(m+m+2) \sum_{k=0}^{\infty} \binom{m+n}{k} z^{m+n-k}x^k.\end{aligned} \tag{2.13}$$

We apply (1.2) with $x+z$ instead of z and (2.11), (2.12), (2.13). A comparison of the coefficients yields

$$\begin{aligned}(-1)^m \sum_{\nu=0}^{m} (\nu+n+1)\binom{m+1}{\nu}\binom{\nu+n}{k} B_{\nu+n-k}(z) \\ +(-1)^{k+n} \sum_{\nu=0}^{n} (\nu+m+1)\binom{n+1}{\nu}\binom{\nu+m}{k} B_{\nu+m-k}(-z) \\ = (-1)^m(m+n+1)(m+n+2)\binom{m+n}{k} z^{m+n-k},\end{aligned}$$

as claimed. □

Proof of Corollary 2. We define

$$C(m,n;k) = \sum_{\nu=0}^{m} (-1)^\nu (\nu+n+1)(\nu+n-k)\binom{m+1}{\nu}\binom{\nu+n}{k}$$

and

$$D(m,n;k)=\sum_{\nu=0}^{m}(-1)^{\nu}\binom{m+1}{\nu}\binom{\nu+n+1}{k}.$$

Then

$$C(m,n;k)=(k+1)(k+2)D(m,n;k+2).$$

Using

$$B_n(1)=(-1)^nB_n \quad \text{and} \quad B_n(-1)=B_n+(-1)^n n \quad (n=0,1,\dots)$$

gives

$$\begin{aligned} S(m,n;k;1) &= (-1)^m\sum_{\nu=0}^{m}(\nu+n+1)\binom{m+1}{\nu}\binom{\nu+n}{k}B_{\nu+n-k}(1) \\ &= (-1)^{m+n+k}A(m,n;k) \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} S(n,m;k;-1) &= (-1)^n\sum_{\nu=0}^{n}(\nu+m+1)\binom{n+1}{\nu}\binom{\nu+m}{k}B_{\nu+m-k}(-1) \\ &= S^*(n,m;k)+(-1)^{m+n+k}C(n,m;k). \end{aligned} \tag{2.15}$$

We apply Theorem 2 with $z=1$ and (2.14), (2.15). This yields

$$\begin{aligned} &(-1)^{m+k}(m+n+1)(m+n+2)\binom{m+n}{k} \\ &\quad=(-1)^kS(m,n;k;1)+S(n,m;k;-1) \\ &\quad=(-1)^{m+n}A(m,n;k)+S^*(n,m;k)+(-1)^{m+n+k}C(n,m;k). \end{aligned} \tag{2.16}$$

Next, we exchange m and n, multiply by $(-1)^{k+1}$, and use (1.5). Then

$$\begin{aligned} &(-1)^{n+1}(m+n+1)(m+n+2)\binom{m+n}{k} \\ &\quad=(-1)^{m+n+k+1}A(n,m;k)+(-1)^{k+1}S^*(m,n;k) \\ &\qquad+(-1)^{m+n+1}C(m,n;k) \\ &\quad=(-1)^{m+n+k+1}A(n,m;k)+S^*(n,m;k)+(-1)^{m+n+1}C(m,n;k). \end{aligned} \tag{2.17}$$

From (2.16) and (2.17) we obtain

$$A(m,n;k)+(-1)^kA(n,m;k)=F(m,n;k) \tag{2.18}$$

with

$$\begin{aligned} F(m,n;k) &= \big((-1)^{n+k}+(-1)^m\big)(m+n+1)(m+n+2)\binom{m+n}{k} \\ &\quad -(-1)^kC(n,m;k)-C(m,n;k) \end{aligned}$$

$$= (-1)^k (k+1)(k+2)\left[(-1)^n \binom{m+n+2}{k+2} - D(n,m;k+2)\right]$$
$$+(k+1)(k+2)\left[(-1)^m \binom{m+n+2}{k+2} - D(m,n;k+2)\right]. \tag{2.19}$$

If $0 \le p \le m$, then we conclude from (2.5) that

$$D(m,n;p) = \sum_{\nu=0}^{m+1} (-1)^\nu \binom{m+1}{\nu}\binom{\nu+n+1}{p} + (-1)^m \binom{m+n+2}{p}$$
$$= (-1)^m \binom{m+n+2}{p}.$$

Since $k+2 \le \min(m,n)$, we obtain from (2.19) that $F(m,n;k) = 0$, so that (2.18) implies (1.6). □

Proof of Corollary 3. We present two proofs.
First proof. A short calculation gives that (1.7) holds for $k=0$ and $k=1$. Moreover, since

$$(-1)^{n+k} P(n,0;k;-z-1) = (-1)^{n+k} \sum_{\nu=0}^{n} \binom{\nu}{k}\binom{n}{\nu}(-z-1)^{\nu-k}$$
$$= (-1)^{n+k}\binom{n}{k} \sum_{\nu=0}^{n} \binom{n-k}{\nu-k}(-z-1)^{\nu-k}$$
$$= \binom{n}{k} z^{n-k} = P(0,n;k;z),$$

we conclude that (1.7) is valid for $m=0$ and $n=0$. Let $m \ge 1$, $n \ge 1$, $k \ge 2$. Using (2.2) gives

$$S(m-1,n-1;k-2;z+1) - S(m-1,n-1;k-2;z)$$
$$= (-1)^{m-1} \sum_{\nu=0}^{m-1} (\nu+n)\binom{m}{\nu}\binom{\nu+n-1}{k-2}\big(B_{\nu+n+1-k}(z+1) - B_{\nu+n+1-k}(z)\big)$$
$$= (-1)^{m-1} \sum_{\nu=0}^{m-1} (\nu+n)\binom{m}{\nu}\binom{\nu+n-1}{k-2}(\nu+n+1-k) z^{\nu+n-k}$$
$$= (k-1)k(-1)^{m-1} \sum_{\nu=0}^{m-1} \binom{m}{\nu}\binom{\nu+n}{k} z^{\nu+n-k}$$
$$= (k-1)k\left[(-1)^{m-1} P(m,n;k;z) - (-1)^{m-1}\binom{m+n}{k} z^{m+n-k}\right] \tag{2.20}$$

and

$$\begin{aligned}(-1)^k\big(&S(n-1,m-1;k-2;-z-1)-S(n-1,m-1;k-2;-z)\big)\\&=(-1)^{k+n}\sum_{\nu=0}^{n-1}(\nu+m)\binom{n}{\nu}\binom{\nu+m-1}{k-2}\big(B_{\nu+m+1-k}(-z)-B_{\nu+m+1-k}(-z-1)\big)\\&=(-1)^{m+n}\sum_{\nu=0}^{n-1}(-1)^{\nu}(\nu+m)(\nu+m+1-k)\binom{\nu+m-1}{k-2}\binom{n}{\nu}(z+1)^{\nu+m-k}\\&=(k-1)k(-1)^{m+n}\sum_{\nu=0}^{n-1}(-1)^{\nu}\binom{\nu+m}{k}\binom{n}{\nu}(z+1)^{\nu+m-k}\\&=(k-1)k\left[(-1)^{n+k}P(n,m;k;-z-1)+(-1)^{m-1}\binom{m+n}{k}(z+1)^{m+n-k}\right].\end{aligned} \tag{2.21}$$

From Theorem 2 we obtain

$$\begin{aligned}S(m-1,&n-1;k-2;z+1)-S(m-1,n-1;k-2;z)\\&+(-1)^kS(n-1,m-1;k-2;-z-1)-(-1)^kS(n-1,m-1;k-2;-z)\\&=(-1)^{m+1}(m+n-1)(m+n)\binom{m+n-2}{k-2}\big((z+1)^{m+n-k}-z^{m+n-k}\big)\\&=(k-1)k(-1)^{m-1}\binom{m+n}{k}\big((z+1)^{m+n-k}-z^{m+n-k}\big).\end{aligned} \tag{2.22}$$

Combining (2.20), (2.21) and (2.22) leads to (1.7). □

Second proof. We have

$$\begin{aligned}P(m,n;k;z)&=\sum_{\nu=0}^{m}\binom{m}{\nu}\binom{\nu+n}{k}z^{\nu+n-k}=\frac{1}{k!}\sum_{\nu=0}^{m}\binom{m}{\nu}(z^{\nu+n})^{(k)}\\&=\frac{1}{k!}\big(z^n(z+1)^m\big)^{(k)}\end{aligned}$$

and

$$\begin{aligned}(-1)^{m+n+k}P(n,m;k;-z-1)&=\sum_{\nu=0}^{n}(-1)^{n-\nu}\binom{n}{\nu}\binom{\nu+m}{k}(z+1)^{\nu+m-k}\\&=\frac{1}{k!}\sum_{\nu=0}^{n}(-1)^{n-\nu}\binom{n}{\nu}\big((z+1)^{\nu+m}\big)^{(k)}\\&=\frac{1}{k!}\big(z^n(z+1)^m\big)^{(k)}.\end{aligned}$$

□

Proof of Corollary 4. We assume that $m>n\geq 0$. Since, for $m\geq 1$,

$$\sum_{\nu=0}^{m}\binom{m}{\nu}(2^{-\nu}-1)B_\nu=2^{-m}B_m(2)-B_m(1)=(-1)^m(2^{-m}-1)B_m+m2^{-m},$$

we conclude that (1.9) holds for $n = 0$. Next, let $m > n \geq 1$. Using (2.3) gives

$$\frac{1}{2}\int_0^{1/2}(-1)^{m-1}\sum_{\nu=0}^{m}(\nu+n)\binom{m}{\nu}B_{\nu+n-1}(z)dz = (-1)^{m-1}\sum_{\nu=0}^{m}\binom{m}{\nu}(2^{-\nu-n}-1)B_{\nu+n} \tag{2.23}$$

and from (2.4) and (2.7) with $r = n$, $s = m$, $t = -1/2$ we get

$$\begin{aligned}\frac{1}{2}\int_0^{1/2}&(-1)^{n}\sum_{\nu=0}^{n}(\nu+m)\binom{n}{\nu}B_{\nu+m-1}(-z)dz\\ &= \frac{(-1)^n}{2}\sum_{\nu=0}^{n}\binom{n}{\nu}\Big(-\frac{1}{2}\Big)^{\nu+m-1}\big[(1-2^{\nu+m})B_{\nu+m}+\nu+m\big]\\ &= (-1)^{m+n-1}\left[\sum_{\nu=0}^{n}(-1)^{\nu}\binom{n}{\nu}\big(2^{-\nu-m}-1\big)B_{\nu+m}+\frac{m-n}{2^{m+n}}\right].\end{aligned} \tag{2.24}$$

Applying (1.3), (2.23), (2.24) and

$$\sum_{\nu=0}^{n}(-1)^{\nu}\binom{n}{\nu}(2^{-\nu-m}-1)B_{\nu+m} = (-1)^{m}\sum_{\nu=0}^{n}\binom{n}{\nu}(2^{-\nu-m}-1)B_{\nu+m}$$

we obtain (1.9). □

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