Results Math (2024) 79:201

1422-6383/24/050001-15
published online July 15, 2024
https://doi.org/10.1007/s00025-024-02226-z

Results in Mathematics

Almost Invertible Operators

Zakariae Aznay, Abdelmalek Ouahab, and Hassan Zariouh

Abstract. This paper aims to provide a thorough characterization of the family of all Cantor-Bendixson derivatives of the spectrum, Browder spectrum, and the Drazin spectrum of bounded linear operators using projections and invariant subspaces. Furthermore, our findings demonstrate that if two commuting operators, R and T, satisfy the conditions that R is Riesz and T is a direct sum of an invertible operator and an operator with an at most countable spectrum, then $T+R$ can also be represented as a direct sum of an invertible operator and an operator with an at most countable spectrum.

Mathematics Subject Classification. Primary 03Exx, 47A10, 47Axx.

Keywords. Cantor-Bendixson derivative, $g^{\alpha}_{\sigma_*}$-invertible, almost invertible, at most countable spectrum, accumulation points.

1. Introduction

Let X be a complex Banach space and $L(X)$ be the set of all bounded linear operators on X. The spectrum, Browder spectrum, and Drazin spectrum of an operator $T \in L(X)$ are denoted by $\sigma(T)$, $\sigma_b(T)$, and $\sigma_d(T)$, respectively. A subset $\tilde{\sigma} \subseteq \sigma(T)$ is called a spectral set or an isolated part of T if it is both closed and open in $\sigma(T)$. Moreover, a subspace $M \subseteq X$ is said to be T-invariant if $T(M) \subseteq M$, and in this case, the restriction of T on M is denoted by T_M. We use $(M, N) \in \text{Red}(T)$ to denote that M and N are closed, T-invariant subspaces such that $X = M \oplus N$. We also use the notation:

$\mathrm{comm}(T)$	:	The set of all operators that commute with T
$\mathrm{comm}^2(T)$	:	The set of all operators that commute with every operator that commutes with T
$\overline{A}$	:	The closure of a complex subset A
A^C	:	The complementary of a complex subset A
$C(\lambda, \epsilon)$	:	The circle ball of radius ϵ centered at λ
$B(\lambda, \epsilon)$	:	The open ball of radius ϵ centered at λ
$D(\lambda, \epsilon)$	:	The closed ball of radius ϵ centered at λ

Descriptive set theory employs the Cantor-Bendixson derivative to gauge the complexity of a subset of the complex plane $\mathbb{C}$. Given an ordinal number α, the α-th Cantor-Bendixson derivative of a set $A \subseteq \mathbb{C}$ is defined by a transfinite recursion process, as follows:

$$\begin{cases} \mathrm{acc}^0 A = A \\ \mathrm{acc}^\alpha A = \mathrm{acc}\,(\mathrm{acc}^{\alpha-1} A) \text{ if } \alpha \text{ is a successor ordinal} \\ \mathrm{acc}^\alpha A = \bigcap\limits_{\beta<\alpha} \mathrm{acc}^\beta A \text{ if } \alpha \text{ is a limit ordinal} \end{cases}$$

Here, $\mathrm{acc}\, A$ denotes the set of all accumulation points of A, and $\alpha - 1$ denotes the predecessor of α if it is a successor. The Cantor-Bendixson rank of A, denoted $\mathrm{CBR}(A)$, is the smallest ordinal α such that $\mathrm{acc}^\alpha A = \mathrm{acc}^{\alpha+1} A$. In other words, $\mathrm{CBR}(A)$ measures the smallest number of times we need to apply the derived set operation before the set of accumulation points stops changing. The Cantor-Bendixson rank finds significant application in the study of Polish spaces, which are separable topological spaces that can be completely metrized. In fact, for every non-empty perfect Polish space, its Cantor-Bendixson rank is a countable ordinal (see, for instance, [13, Theorem 4.9]). Thus, Cantor-Bendixson rank offers a useful tool for measuring the complexity of subsets of $\mathbb{C}$.

For every $\sigma_* \in \{\sigma, \sigma_b, \sigma_d\}$, it can be shown that $\sigma_*(T)$ is a Polish space for every $T \in L(X)$, and therefore, $\mathrm{CBR}(\sigma_*(T)) < \omega_1$, where ω_1 is the first uncountable ordinal. Let $\lambda := \mathrm{CBR}(\sigma_*(T))$. Note that the family $(\mathrm{acc}^\alpha \sigma_*(T))_{\alpha \leq \lambda}$ is strictly decreasing and consists of compact subsets. Furthermore, for every ordinals α and β such that $\alpha > \beta$, we have:

$$\mathrm{acc}^\beta \sigma_*(T) \setminus \mathrm{acc}^\alpha \sigma_*(T) = \bigsqcup_{\beta \leq \gamma < \alpha} \mathrm{iso}\ (\mathrm{acc}^\gamma \sigma_*(T)),$$

where $\bigsqcup$ denotes the mutually disjoint union. Therefore

$$\mathrm{acc}^\beta \sigma_*(T) = \mathrm{acc}^\alpha \sigma_*(T) \bigsqcup \left[\bigsqcup_{\beta \leq \gamma < \alpha} \mathrm{iso}\ (\mathrm{acc}^\gamma \sigma_*(T)) \right].$$

Thus, we can write:

$$\sigma_*(T) = \operatorname{acc}^\alpha \sigma_*(T) \bigsqcup \left[\bigsqcup_{\beta<\alpha} \operatorname{iso}\left(\operatorname{acc}^\beta \sigma_*(T)\right)\right] \text{ for every ordinals } \alpha.$$

Since we have $\operatorname{acc}^{\alpha+1}\sigma(T) \subset \operatorname{acc}^\alpha \sigma_d(T) \subset \operatorname{acc}^\alpha \sigma_b(T) \subset \operatorname{acc}^\alpha \sigma(T)$ for every ordinals α, it follows that $\operatorname{acc}^\alpha \sigma(T) = \operatorname{acc}^\alpha \sigma_b(T) = \operatorname{acc}^\alpha \sigma_d(T)$ for every ordinals $\alpha \geq \omega$, where ω denotes the first infinite ordinal. Therefore, we have $\operatorname{CBR}(\sigma(T)) = \operatorname{CBR}(\sigma_b(T)) = \operatorname{CBR}(\sigma_d(T))$ if $\operatorname{CBR}(\sigma_d(T)) \geq \omega$. If $\operatorname{CBR}(\sigma_d(T)) < \omega$, then we have:

$$\operatorname{CBR}(\sigma(T)) - 1 \leq \operatorname{CBR}(\sigma_d(T)) \leq \operatorname{CBR}(\sigma_b(T)) \leq \operatorname{CBR}(\sigma(T)) \leq \operatorname{CBR}(\sigma_b(T)) + 1.$$

Finally, we can conclude that:

$$\sigma_*(T) = \operatorname{acc}^\alpha \sigma(T) \bigsqcup \left[\bigsqcup_{\beta<\alpha} \operatorname{iso}\left(\operatorname{acc}^\beta \sigma_*(T)\right)\right] \text{ if } \alpha \geq \omega.$$

Towards the end of this paper, we will present the following result:

Theorem 1.1. *Let $T \in L(X)$ be a bounded linear operator. The following statements are equivalent:*

- *(i) T is a direct sum of an invertible operator and an operator with an at most countable spectrum;*
- *(ii) $0 \notin acc^\alpha \sigma(T)$ for some ordinal α;*
- *(iii) $0 \notin acc^{\omega_1} \sigma(T)$, where ω_1 is the first uncountable ordinal;*
- *(iv) There exists a bounded projection $P \in comm(T)$ such that $T + P$ is invertible and $\sigma(TP)$ is at most countable;*
- *(v) There exists a bounded projection $P \in comm^2(T)$ such that $T + P$ is invertible and $\sigma(TP)$ is at most countable;*
- *(vi) There exists a pair of T-invariant subspaces (M, N) such that N is closed, $X = M \oplus N$, T_M is invertible and $\sigma(T_N)$ is at most countable;*
- *(vii) There exists $(M, N) \in \bigcap_{S \in comm(T)} Red(S)$ such that T_M is invertible and $\sigma(T_N)$ is at most countable;*
- *(viii) There exists $S \in comm(T)$ such that $S^2T = S$ and $\sigma(T - T^2S)$ is at most countable;*
- *(ix) There exists $S \in comm^2(T)$ such that $S^2T = S$ and $\sigma(T - T^2S)$ is at most countable;*
- *(x) There exists an at most countable spectral set σ of T such that $0 \notin \sigma(T) \setminus \sigma$;*
- *(xi) There exists a spectral set $\tilde{\sigma}$ of T such that $0 \notin \sigma(T) \setminus \tilde{\sigma}$ and $\tilde{\sigma} \setminus \{0\} \subset \bigsqcup_{\beta<\omega_1} iso\,(acc^\beta \sigma(T))$.*

To establish the validity of the previous result, we will first prove the following result, which generalizes previous findings, such as those presented in [1,2,6,14].

Theorem 1.2. *Let $\alpha \geq 1$ be a successor ordinal, and let $\sigma_* \in \{\sigma, \sigma_b, \sigma_d\}$. The following statements are equivalent for $T \in L(X)$:*

(i) $T = D \oplus Q$, *D is an invertible operator and Q is an operator with* $\mathrm{acc}^{\alpha-1}\,\sigma_*(Q) \subset \{0\}$;
(ii) $0 \notin \mathrm{acc}^{\alpha}\,\sigma_*(T)$;
(iii) *There exists a bounded projection* $P \in \mathrm{comm}(T)$ *such that* $T+P$ *is invertible and* $\mathrm{acc}^{\alpha-1}\,\sigma_*(TP) \subset \{0\}$;
(iv) *There exists a bounded projection* $P \in \mathrm{comm}^2(T)$ *such that* $T+P$ *is invertible and* $\mathrm{acc}^{\alpha-1}\,\sigma_*(TP) \subset \{0\}$;
(v) *There exists a pair of T-invariant subspaces* (M,N) *such that N is closed,* $X = M \oplus N$, T_M *is invertible and* $\mathrm{acc}^{\alpha-1}\,\sigma_*(T_N) \subset \{0\}$;
(vi) *There exists* $(M,N) \in \bigcap_{S \in \mathrm{comm}(T)} \mathrm{Red}(S)$ *such that* T_M *is invertible and* $\mathrm{acc}^{\alpha-1}\,\sigma_*(T_N) \subset \{0\}$;
(vii) *There exists* $S \in \mathrm{comm}(T)$ *such that* $S^2T = S$ *and* $\mathrm{acc}^{\alpha-1}\,\sigma_*(T - T^2S) \subset \{0\}$;
(viii) *There exists* $S \in \mathrm{comm}^2(T)$ *such that* $S^2T = S$ *and* $\mathrm{acc}^{\alpha-1}\,\sigma_*(T - T^2S) \subset \{0\}$.
(ix) *There exists a spectral set* $\tilde{\sigma}$ *of T such that* $0 \notin \sigma(T) \setminus \tilde{\sigma}$ *and* $\tilde{\sigma} \setminus \{0\} \subset \mathrm{iso}\,(\mathrm{acc}^{\alpha-1}\,\sigma_*(T))$.

Furthermore, we establish that if R is a Riesz operator, that is $\sigma_b(T) \subset \{0\}$, which commutes with T, where $T \in L(X)$ is a direct sum of an invertible operator and an operator with an at most countable spectrum, then $T+R$ is also a direct sum of an invertible operator and an operator with an at most countable spectrum. Moreover, we demonstrate that if T has a topological uniform descent [10], then T is Drazin invertible if and only if $0 \notin \mathrm{acc}^{\alpha}\,\sigma(T)$ for some ordinal α.

2. $g^{\alpha}_{\sigma_*}$-Invertible Operators

For the sake of completeness, we begin this section by including the two following definitions which will be useful for proving Proposition 2.3.

Definition 2.1. [16, Definition I.1.23]. Let M be a subset of a Banach algebra $\mathcal{A}$. The commutant of $\mathcal{A}$ is defined by $M^{'} := \{a \in \mathcal{A} \,|\, am = ma, \text{ for every } m \in M\}$. We write $M^{''}$ instead of $(M^{'})^{'}$ for the second commutant of M.

Definition 2.2. [16, Definition I.2.2]. Let $\mathcal{A}$ be a commutative Banach algebra. A linear functional $\phi : \mathcal{A} \longrightarrow \mathbb{C}$ is called multiplicative if it is an homomorphism, that is $\phi(1_A) = 1$ and $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \mathcal{A}$, where 1_A is the unit element of $\mathcal{A}$.

According to the terminology introduced in [2], an operator $T \in L(X)$ is said to be zeroloid if zero is the only possible accumulation point of its spectrum. The following proposition establishes that the product of commuting zeroloid operators is also zeroloid.

Proposition 2.3. *Consider $T, S \in L(X)$ such that $TS = ST$.*

(i) If both T and S are zeroloid, then TS is also zeroloid.
(ii) If $0 \in acc\,\sigma(TS)$, then $0 \in acc\,\sigma(T) \cup acc\,\sigma(S)$.

Proof. Let $\mathcal{A} = \{T, S\}^{''}$ be the second commutant of $\{T, S\}$. It is widely known that $\mathcal{A}$ is a unital commutative Banach algebra, and if $\sigma_{\mathcal{A}}(T)$ denotes the spectrum of T in $\mathcal{A}$, then $\sigma_{\mathcal{A}}(T) = \sigma(T)$.

(i) Assuming that T and S are non-zero, we know that $B(0, \epsilon)^C \cap \sigma(T)$ and $B(0, \epsilon)^C \cap \sigma(S)$ are finite sets for all $\epsilon > 0$ since T and S are zeroloid. Using [16, Theorem I.2.9], we can deduce the existence of multiplicative functionals $\phi_1, \ldots, \phi_n, \psi_1, \ldots, \psi_m$ on $\mathcal{A}$ such that $B(0, \mu)^C \cap \sigma(T) = \{\phi_1(T), \ldots, \phi_n(T)\}$ and $B(0, \mu)^C \cap \sigma(S) = \{\psi_1(S), \ldots, \psi_m(S)\}$, where $\mu := \frac{\epsilon}{\max\{\|T\|, \|S\|\}}$. Let ϕ be a multiplicative functional such that $|\phi(TS)| \geq \epsilon$. Then, we can find $1 \leq i \leq n$ and $1 \leq j \leq m$ such that $\phi(TS) = \phi_i(T)\psi_j(S)$, since $\min\{|\phi(T)|, |\phi(S)|\} \geq \mu$. Therefore, $B(0, \epsilon)^C \cap \sigma(TS)$ is a finite set, and consequently, TS is zeroloid.

(ii) If $0 \in \operatorname{acc} \sigma(TS)$, then $B(0, \epsilon) \cap (\sigma(TS) \setminus 0) \neq \emptyset$ for every $\epsilon > 0$. So, there exists a multiplicative functional ϕ_ϵ on $\mathcal{A}$ such that $0 < |\phi_\epsilon(TS)| < \epsilon^2$. Hence, $|\phi_\epsilon(T)| < \epsilon$ or $|\phi_\epsilon(S)| < \epsilon$. This implies that $B(0, \epsilon) \cap ([\sigma(T) \cup \sigma(S)] \setminus 0) \neq \emptyset$, and as a result, $0 \in \operatorname{acc} \sigma(T) \cup \operatorname{acc} \sigma(S)$.

□

Note that the assertion (ii) of Proposition 2.3 is already proved in [14, Theorem 5.5], and the proof which gave is new. Throughout our discussion, we will use the notation σ_* to refer to either the spectrum, Browder spectrum, or Drazin spectrum of an operator. The following definition introduces the classes of $g^{\alpha}_{\sigma_*}$-invertible operators.

Definition 2.4. Let $T \in L(X)$ and let α be an ordinal.

(1) If α is a successor ordinal, we say that T is $g^{\alpha}_{\sigma_*}$-invertible provided that there exists a pair $(M, N) \in \operatorname{Red}(T)$ such that T_M is invertible and $\operatorname{acc}^{\alpha-1} \sigma_*(T_N) \subset \{0\}$, Here, $\alpha - 1$ denotes the predecessor of α.
(2) If α is a limit ordinal, we say that T is $g^{\alpha}_{\sigma_*}$-invertible if there exists a successor ordinal $\beta < \alpha$ such that T is $g^{\beta}_{\sigma_*}$-invertible.
(3) T is called $g^{0}_{\sigma_*}$-invertible if $0 \notin \sigma_*(T)$.

We use the notation $\mathcal{G}^{\alpha}_{\sigma_*}(X)$ to represent the class of $g^{\alpha}_{\sigma_*}$-invertible operators acting on the Banach space X.

Remark 2.5. (1) The inclusions $\mathcal{G}^{\alpha}_{\sigma}(X) \subset \mathcal{G}^{\alpha}_{\sigma_b}(X) \subset \mathcal{G}^{\alpha}_{\sigma_d}(X) \subset \mathcal{G}^{\alpha+1}_{\sigma}(X)$ and $\mathcal{G}^{\alpha}_{\sigma_*}(X) \subset \mathcal{G}^{\beta}_{\sigma_*}(X)$ hold for all ordinals $\alpha \leq \beta$.
(2) If α is a limit ordinal, then T is $g^{\alpha}_{\sigma_*}$-invertible if and only if there exists an ordinal $\beta < \alpha$ such that T is $g^{\beta}_{\sigma_*}$-invertible. Therefore, we have $\mathcal{G}^{\alpha}_{\sigma_*}(X) = \bigcup_{\beta<\alpha} \mathcal{G}^{\beta}_{\sigma_*}(X)$.

(3) Notice that T is g^1_σ-invertible (resp., T is $g^1_{\sigma_b}$-invertible, T is $g^1_{\sigma_d}$-invertible) if and only if T is generalized Drazin invertible [14] (resp., T is generalized Drazin-Riesz invertible [18], T is generalized Drazin-meromorphic invertible [19]).

The following theorem gives a characterization of $g^\alpha_{\sigma_*}$-invertible operators in terms of $\operatorname{acc}^\alpha \sigma_*(T)$.

Theorem 2.6. *Let $T \in L(X)$ and let α be an ordinal. Then $T - \mu I$ is $g^\alpha_{\sigma_*}$-invertible if and only if $\mu \notin \operatorname{acc}^\alpha \sigma_*(T)$.*

Proof. Let us firstly prove that the result is true for all successor ordinals α. Let $\mu \notin \operatorname{acc}^\alpha \sigma_*(T)$. Since $\operatorname{acc}^\alpha \sigma_*(T - zI) = \operatorname{acc}^\alpha \sigma_*(T) - z$ for every complex number z, we may assume that $\mu = 0$. The case of $0 \notin \operatorname{acc} \sigma(T)$ is clear. Suppose now that $0 \in \operatorname{acc} \sigma(T)$. Then $0 \in \operatorname{acc}(\Pi^\alpha_{\sigma_*}(T))$, where $\Pi^\alpha_{\sigma_*}(T) = \sigma(T) \setminus \operatorname{acc}^{\alpha-1} \sigma_*(T)$. As $0 \notin \operatorname{acc}^\alpha \sigma_*(T)$ then $D(0, \epsilon) \cap [\operatorname{acc}^{\alpha-1} \sigma_*(T) \setminus \{0\}] = \emptyset$ for some $1 > \epsilon > 0$. Let $0 < \mu < v \leq \epsilon$. The set $\Pi^\alpha_{\sigma_*}(T)$ is countable, since $\lambda = \operatorname{CBR}(\sigma(T)) < \omega_1$ and $\Pi^\alpha_{\sigma_*}(T) \subset \sigma(T) \setminus \operatorname{acc}^\lambda \sigma(T) = \bigsqcup_{\beta<\lambda} \operatorname{iso}(\operatorname{acc}^\beta \sigma(T))$. Thus there exists $\mu \leq \epsilon_{(\mu,v)} \leq v$ such that $C(0, \epsilon_{(\mu,v)}) \cap \sigma(T) = \emptyset$. Hence $\sigma_{(\mu,v)} := D(0, \epsilon_{(\mu,v)}) \cap \sigma(T) \subset \Pi^\alpha_{\sigma_*}(T) \cup \{0\}$ is a countable spectral set of $\sigma(T)$. So there exists $(M_{(\mu,v)}, N_{(\mu,v)}) \in \operatorname{Red}(T)$ such that $\sigma(T_{M_{(\mu,v)}}) = \sigma(T) \setminus \sigma_{(\mu,v)}$ and $\sigma(T_{N_{(\mu,v)}}) = \sigma_{(\mu,v)}$. Further, we have $-1 \notin \sigma(T_{N_{(\mu,v)}})$ and $\sigma_{(\mu,v)} \cap (\operatorname{acc}^{\alpha-1} \sigma_*(T) \setminus \{0\}) = \emptyset$, and thus $\operatorname{acc}^{\alpha-1} \sigma_*(T_{N_{(\mu,v)}}) \subset \{0\}$. This proves that T is $g^\alpha_{\sigma_*}$-invertible. Assume now that α is a limit ordinal. If $0 \notin \operatorname{acc}^\alpha \sigma_*(T)$, then there exists a successor odinal $\beta < \alpha$ such that $0 \notin \operatorname{acc}^\beta \sigma_*(T)$, which implies by the first case that T is $g^\beta_{\sigma_*}$ -invertible, and thus T is $g^\alpha_{\sigma_*}$-invertible. Conversely, assume that T is $g^\alpha_{\sigma_*}$-invertible for some ordinal α. Assume that α is a successor ordinal, then there exists $(M, N) \in \operatorname{Red}(T)$ such that T_M is invertible and $\operatorname{acc}^{\alpha-1} \sigma_*(T_N) \subset \{0\}$. So there exists $\epsilon > 0$ such that $B(0, \epsilon) \setminus \{0\} \subset (\sigma(T_M))^C \cap (\operatorname{acc}^{\alpha-1} \sigma_*(T_N))^C \subset (\operatorname{acc}^{\alpha-1} \sigma_*(T_M))^C \cap (\operatorname{acc}^{\alpha-1} \sigma_*(T_N))^C = (\operatorname{acc}^{\alpha-1} \sigma_*(T))^C$. So $0 \notin \operatorname{acc}^\alpha \sigma_*(T)$. The case of α is a limit ordinal isbreak clear.

□

The following corollary is a consequence of the previous theorem:

Corollary 2.7. *Let $T \in L(X)$ and let α be an ordinal. The following statements hold:*

(i) T is $g^\alpha_{\sigma_*}$-invertible if and only if T^* is $g^\alpha_{\sigma_*}$-invertible.
(ii) If Y is a complex Banach space and $S \in L(Y)$, then $S \oplus T$ is $g^\alpha_{\sigma_*}$-invertible if and only if both S and T are $g^\alpha_{\sigma_*}$-invertible.
(iii) T is $g^\alpha_{\sigma_*}$-invertible if and only if T^m is $g^\alpha_{\sigma_*}$-invertible for some (equivalently for every) integer $m \geq 1$.

Since $\operatorname{CBR}(\sigma(T)) < \omega_1$ and $\operatorname{acc}^\omega \sigma(T) = \operatorname{acc}^\omega \sigma_b(T) = \operatorname{acc}^\omega \sigma_d(T)$ for every $T \in L(X)$, we can derive the following corollary from Theorem 2.6.

This corollary indicates that $g^{\alpha}_{\sigma_*}$-invertibility is only significant for countable ordinals.

Corollary 2.8. *Let $T \in L(X)$. The following assertions hold:*

(i) If $\alpha \geq \omega_1$ is an ordinal, then T is $g^{\alpha}_{\sigma_*}$-invertible if and only if T is $g^{\beta}_{\sigma_*}$-invertible for some $\beta < \omega_1$.

(ii) If $\alpha \geq \omega$, then T is $g^{\alpha}_{\sigma_*}$-invertible if and only if T is $g^{\alpha}_{\sigma_{**}}$-invertible, where $\sigma_{**} \in \{\sigma, \sigma_b, \sigma_d\}$.

Remark 2.9. Consider $T \in L(X)$ and define the set $\mathcal{C}(T)$ as:

$$\mathcal{C}(T) := \{(T - zI)_M : z \in \mathbb{C} \text{ and } M \text{ is a closed } T\text{-invariant subspace}\}.$$

Let $\mathcal{P}(\mathbb{C})$ denote the set of all subsets of $\mathbb{C}$. From the proof of Theorem 2.6, it follows that if there exists a mapping $\tilde{\sigma} : \mathcal{C}(T) \longrightarrow \mathcal{P}(\mathbb{C})$ that satisfies the following conditions:

(i) $\tilde{\sigma}(T_M)$ is closed and $\tilde{\sigma}(T_M) \subset \sigma(T_M)$ for all closed T-invariant subspaces M.

(ii) $\tilde{\sigma}(T) = \tilde{\sigma}(T_M) \cup \tilde{\sigma}(T_N)$ for all $(M, N) \in \text{Red}(T)$.

(iii) $\tilde{\sigma}(T - zI) = \tilde{\sigma}(T) - z$ for all complex numbers z.

(iv) $\text{acc}^{\beta}\, \sigma(T) \subset \text{acc}^{\gamma}\, \tilde{\sigma}(T)$ for some ordinals β, γ.

Then, for all successor ordinal α, we have $z \notin \text{acc}^{\alpha}\, \tilde{\sigma}(T)$ if and only if there exists $(M, N) \in \text{Red}(T)$ such that $(T - zI)_M$ is invertible and $\text{acc}^{\alpha-1}\, \tilde{\sigma}((T - zI)_N) \subset 0$.

Definition 2.10. Let α be an ordinal and let $T \in L(X)$ be $g^{\alpha}_{\sigma_*}$-invertible. We define the degree of g_{σ_*}-invertibility of T, denoted $\beta = \text{d}_{\sigma*}(T)$, to be the smallest ordinal β such that $0 \notin \text{acc}^{\beta}\, \sigma_*(T)$.

Remark 2.11. (1) Let $T \in L(X)$ be a $g^{\alpha}_{\sigma_*}$-invertible such that $0 \in \sigma_*(T)$. The degree of g_{σ_*}-invertibility of T is known by the Cantor-Bendixson rank of the point 0 of $\sigma_*(T)$.

(2) If $T \in L(X)$ is $g^{\alpha}_{\sigma_*}$-invertible and $0 \in \sigma_*(T)$, then the degree of g_{σ_*}-invertibility of T is an ordinal β such that $0 \in \text{iso}\,\text{acc}^{\beta}\, \sigma_*(T)$. Moreover, it cannot be a limit ordinal.

(3) Clearly, if $\alpha > 0$ is an ordinal, then

$$\text{d}_{\sigma}(T) \leq \text{d}_{\sigma_b}(T) \leq \text{d}_{\sigma_d}(T) \leq \text{CBR}(\sigma_d(T))$$

for every $g^{\alpha}_{\sigma_*}$-invertible operator T, and if $\omega \leq \text{d}_{\sigma}(T)$, then $\text{d}_{\sigma}(T) = \text{d}_{\sigma_b}(T) = \text{d}_{\sigma_d}(T)$

Recall from [14] that an operator T is called generalized Drazin invertible if there exists a pair $(M, N) \in \text{Red}(T)$ such that T_M is invertible and T_N is quasi-nilpotent. It has been shown that T is generalized Drazin invertible if and only if there exists a bounded projection $P \in \text{comm}^2(T)$ such that $T + P$ is invertible and TP is quasi-nilpotent. This is also equivalent to 0 not being an accumulation point of $\sigma(T)$.

Remark 2.12. The proof of Theorem 2.6 yields further insights into g_{σ_*}-invertibility. Specifically, if T is $g^{\alpha}_{\sigma_*}$-invertible for some successor ordinal $\alpha > 0$, then there exists $(M, N) \in \text{Red}(T)$ with the following properties: T_M is invertible, $\text{acc}^{\alpha-1}\,\sigma_*(T_N) \subset \{0\}$, $\sigma(T_M) \cap \sigma(T_N) = \emptyset$, and $\sigma(T_N) \setminus \{0\} \subset \Pi^{\alpha}_{\sigma_*}(T)$. Consequently, the set $\tilde{\sigma} := \sigma(T_N)$ is at most countable and serves as a spectral set. We also have $N = \mathcal{R}(P_{\tilde{\sigma}})$ and $M = \mathcal{N}(P_{\tilde{\sigma}})$, where $P_{\tilde{\sigma}}$ is the spectral projection of T corresponding to $\tilde{\sigma}$. In addition, we can define the Drazin inverse of T with respect to $\tilde{\sigma}$ as $T^{D_\alpha}_{\tilde{\sigma}} := (T_M)^{-1} \oplus 0_N \in \text{comm}^2(T)$. It follows that $T^{D_\alpha}_{\tilde{\sigma}}$ is Drazin invertible, $T^{D_\alpha}_{\tilde{\sigma}} T T^{D_\alpha}_{\tilde{\sigma}} = T^{D_\alpha}_{\tilde{\sigma}}$, and $\text{acc}^{\alpha-1}\,\sigma_*(T - T^2 T^{D_\alpha}_{\tilde{\sigma}}) \subset \{0\}$.

Moreover, we observe that $\text{acc}^{\alpha-1}\,\sigma_*(TP_\sigma) \subset \{0\}$ and $T + P_{\tilde{\sigma}} = T_M \oplus (T + I)_N$ is invertible.

For a $g^{\alpha}_{\sigma_*}$-invertible operator $T \in L(X)$ with an at most countable successor ordinal α, we can define the set $g^{\alpha}_{\sigma_*}\mathcal{D}(T)$ as follows:

$$g^{\alpha}_{\sigma_*}\mathcal{D}(T) := \{(M, N) \in \text{Red}(T) \text{ such that } T_M \text{ is invertible and } \text{acc}^{\alpha-1}\,\sigma_*(T_N) \subset \{0\}\}.$$

We can show that $g^{\alpha}_{\sigma_*}\mathcal{D}(T) \subset g^{\alpha+1}_{\sigma_*}\mathcal{D}(T)$. Furthermore, the following proposition demonstrates that if T is not generalized Drazin invertible, then $g^{\alpha}_{\sigma_*}\mathcal{D}(T)$ is at least countable.

Proposition 2.13. *Let $\alpha \geq 1$ be a successor ordinal. If $T \in L(X)$ is $g^{\alpha}_{\sigma_*}$-invertible and not generalized Drazin invertible, then there exists $\{(M_m, N_m)\}_m \subset g^{\alpha}_{\sigma_*}\mathcal{D}(T)$ such that $(\sigma(T_{M_m}))_m$ is a strictly increasing sequence, $(\sigma(T_{N_m}))_m$ is a strictly decreasing sequence and $\sigma(T_{M_m}) \cap \sigma(T_{N_m}) = \emptyset$ for all $m \in \mathbb{N}$.*

Proof. Since T is $g^{\alpha}_{\sigma_*}$-invertible and not generalized Drazin invertible, there exists $\epsilon > 0$ such that $D(0, \epsilon) \cap [\text{acc}^{\alpha-1}\,\sigma_*(T) \setminus \{0\}] = \emptyset$. Moreover, there exists a positive strictly decreasing sequence $(\epsilon_m)_m \subset \mathbb{R}$ such that $\epsilon = \epsilon_0$, $C(0, \epsilon_m) \cap \sigma(T) = \emptyset$ and $D(0, \epsilon_{m+1}) \cap \sigma(T) \subsetneq D(0, \epsilon_m) \cap \sigma(T)$. Denote by $\sigma_m := D(0, \epsilon_m) \cap \sigma(T)$. It follows that $(\sigma_m)_m$ is a strictly decreasing sequence of countable spectral sets of $\sigma(T)$. By taking $(M_m, N_m) = (\mathcal{N}(P_{\sigma_m}), \mathcal{R}(P_{\sigma_m}))$, where P_{σ_m} is the spectral projection of T corresponding to σ_m, we get the desired result. □

Hereafter, denote by $\text{Red}^2(T) := \{(M, N) \in \text{Red}(S) : S \in \text{comm}(T)\}$.

Theorem 2.14. *Let $\alpha \geq 1$ be a successor ordinal. The following statements are equivalent for $T \in L(X)$:*

(i) *T is $g^{\alpha}_{\sigma_*}$-invertible;*
(ii) *$0 \notin \text{acc}^{\alpha}\,\sigma_*(T)$;*
(iii) *There exists $(M, N) \in \text{Red}^2(T) \cap g^{\alpha}_{\sigma_*}\mathcal{D}(T)$;*
(iv) *There exists a spectral set $\tilde{\sigma}$ of T such that $0 \notin \sigma(T) \setminus \tilde{\sigma}$ and $\tilde{\sigma} \setminus \{0\} \subset \Pi^{\alpha}_{\sigma_*}(T)$;*
(v) *There exists a bounded projection $P \in \text{comm}(T)$ such that $T + P$ is $g^{\alpha-1}_{\sigma_*}$-invertible and $\text{acc}^{\alpha-1}\,\sigma_*(TP) \subset \{0\}$;*

(vi) There exists a bounded projection $P \in comm^2(T)$ such that $T+P$ is $g^{\alpha-1}_{\sigma_*}$-invertible and $acc^{\alpha-1}\,\sigma_*(TP) \subset \{0\}$;
(vii) There exists a bounded projection $P \in comm(T)$ such that $T+P$ is invertible and $acc^{\alpha-1}\,\sigma_*(TP) \subset \{0\}$;
(viii) There exists a bounded projection $P \in comm^2(T)$ such that $T+P$ is invertible and $acc^{\alpha-1}\,\sigma_*(TP) \subset \{0\}$;
(ix) There exists $S \in comm(T)$ such that $S^2T = S$ and $acc^{\alpha-1}\,\sigma_*(T-T^2S) \subset \{0\}$;
(x) There exists $S \in comm^2(T)$ such that $S^2T = S$ and $acc^{\alpha-1}\,\sigma_*(T-T^2S) \subset \{0\}$.

Proof. The equivalence (i) $\Longleftrightarrow$ (ii) is proved in Theorem 2.6. The implications (i) $\Longrightarrow$ (iii), (i) $\Longrightarrow$ (iv) and (i) $\Longrightarrow$ (viii) are proved in Remark 2.12. (iv) $\Longrightarrow$ (i) Let $\tilde{\sigma}$ be a spectral set of T such that $0 \notin \sigma(T) \setminus \tilde{\sigma}$ and $\tilde{\sigma} \setminus \{0\} \subset \Pi^{\alpha}_{\sigma_*}(T)$. There exists $(M,N) \in \mathrm{Red}(T)$ such that $\sigma(T_M) = \sigma(T) \setminus \tilde{\sigma}$ and $\sigma(T_N) = \tilde{\sigma}$. It is easily seen that $(M,N) \in g^{\alpha}_{\sigma_*}\mathcal{D}(T)$, and then T is $g^{\alpha}_{\sigma_*}$-invertible. (iii) $\Longrightarrow$ (viii) Let $(M,N) \in \mathrm{Red}^2(T) \cap g^{\alpha}_{\sigma_*}\mathcal{D}(T)$. The operator $S = (T_M)^{-1} \oplus 0_N$ is Drazin invertible. Moreover, $S \in \mathrm{comm}^2(T)$, $TS = I_M \oplus 0_N$, $S^2T = S$ and $T - T^2S = 0_M \oplus T_N$. Thus $\mathrm{acc}^{\alpha-1}\,\sigma_*(T-T^2S) \subset \{0\}$. (ix) $\Longrightarrow$ (i) Suppose that there exists $S \in \mathrm{comm}(T)$ such that $S^2T = S$ and $\mathrm{acc}^{\alpha-1}\,\sigma_*(T-T^2S) \subset \{0\}$. Then TS is a projection and $(M,N) := (\mathcal{R}(TS), \mathcal{N}(TS)) \in \mathrm{Red}(T) \cap \mathrm{Red}(S)$. It is not difficult to see that T_M is invertible and $S = (T_M)^{-1} \oplus 0_N$. Thus $T - T^2S = 0 \oplus T_N$. Consequently, T is $g^{\alpha}_{\sigma_*}$-invertible. (v) $\Longrightarrow$ (i) Suppose that there exists a bounded projection $P \in \mathrm{comm}(T)$ such that $T+P$ is $g^{\alpha-1}_{\sigma_*}$-invertible and $\mathrm{acc}^{\alpha-1}\,\sigma_*(TP) \subset \{0\}$. Then $(A,B) := (\mathcal{N}(P), \mathcal{R}(P)) \in \mathrm{Red}(T)$. Therefore T_A is $g^{\alpha-1}_{\sigma_*}$-invertible and $\mathrm{acc}^{\alpha-1}\,\sigma_*(T_B) \subset \mathrm{acc}^{\alpha-1}\,\sigma_*(TP) \subset \{0\}$. Hence there exists $(C,D) \in Red(T_A)$ such that T_C is invertible and $\mathrm{acc}^{\alpha-1}\,\sigma_*(T_D) \subset \{0\}$. Thus $\mathrm{acc}^{\alpha-1}\,\sigma_*(T_{D\oplus B}) \subset \{0\}$, and then T is $g^{\alpha}_{\sigma_*}$-invertible. This completes the proof of the theorem. □

For $T \in L(X)$ and for a subspace M of X, $T_{X/M}$ means the linear map induced by T on the quotient space X/M. The next proposition is a consequence of Theorem 2.6 and the fact that $\sigma_*(T) \subset \sigma_*(T_M) \cup \sigma_*(T_{X/M})$.

Proposition 2.15. *Let α be an ordinal. Let $T \in L(X)$ and let M be a closed T-invariant subspace. If both T_M and $T_{X/M}$ are $g^{\alpha}_{\sigma_*}$-invertible, then T is $g^{\alpha}_{\sigma_*}$-invertible.*

We know that a linear idempotent P (i.e. $P^2 = P$) acting on X is bounded (i.e. a projection) if and only if its nullity $\mathcal{N}(P)$ and range $\mathcal{R}(P)$ are both closed. The following theorem shows that if we replace the bounded projection P in Theorem 2.14 with a linear idempotent P whose range $\mathcal{R}(P)$ is the only possible closed subspace, the conclusion still holds.

Theorem 2.16. *Let $\alpha \geq 1$ be a successor ordinal. The following statements are equivalent for $T \in L(X)$:*

(i) *T is $g^{\alpha}_{\sigma_*}$-invertible;*
(ii) *There exists a pair of T-invariant subspaces (M, N) such that N is closed, $X = M \oplus N$, T_M is invertible and $acc^{\alpha-1}\,\sigma_*(T_N) \subset \{0\}$;*
(iii) *There exists a linear idompotent P acting on X such that $TP = PT$, $\mathcal{R}(P)$ is closed, $P \in comm(T)$, $T+P$ is invertible and $acc^{\alpha-1}\,\sigma_*(T_{\mathcal{R}(P)}) \subset \{0\}$.*

Proof. Assuming (ii), consider a pair (M, N) of subspaces such that N is closed, $X = M \oplus N$, T_M is invertible, and $\mathrm{acc}^{\alpha-1}\,\sigma_*(T_N) \subset 0$. Let $x \in X$ be such that $T_{X/N}(x + N) = N$. Then, we can deduce that $Tx \in N$ because $T_{X/N}(x + N) = N$ implies that $Tx \in T^{-1}_{X/N}(N) = N$. Moreover, since T_M is one-to-one, we also have that $x \in N$. If $x \in X$, then the surjectivity of T_M implies that there exists some $y \in M$ such that $T(y) = P_M(x)$. Thus, $T_{X/N}(y + N) = x + N$ for all $x \in X$. This implies that $T_{X/N}$ is invertible. By Theorem 2.6, there exists $\epsilon > 0$ such that $B(0, \epsilon) \setminus \{0\} \subset (\sigma(T_{X/N}))^C \cap (\mathrm{acc}^{\alpha-1}\,\sigma_*(T_N))^C \subset \mathrm{acc}^{\alpha-1}\,\sigma_*(T)$. Hence, we can conclude that T is $g^{\alpha}_{\sigma_*}$-invertible. The other implications follow directly from the definitions and properties of the various invertibility concepts. □

Let $\mathcal{R}(T^\infty)$ denote the intersection of the ranges of T^n for $n \geq 1$, and let $\mathcal{N}(T^\infty)$ denote the union of the nullspaces of T^n for $n \geq 1$. In the context of Drazin invertibility, we can state a stronger version of Theorem 2.16 as follows:

Theorem 2.17. *Let $T \in L(X)$. The following assertions are equivalent:*
(i) *T is Drazin invertible;*
(ii) *$T_{\mathcal{R}(T^\infty)}$ is one-to-one and $T_{X/\mathcal{N}(T^\infty)}$ is onto. Moreover, in this case $\mathcal{R}(T^\infty)$ and $\mathcal{N}(T^\infty)$ are closed;*
(iii) *$X = \mathcal{R}(T^\infty) \oplus \mathcal{N}(T^\infty)$, where $\oplus$ denotes the algebraic sum;*
(iv) *There exists a pair of T-invariant subspaces (M, N) such that $X = M \oplus N$, T_M is bijective and T_N is nilpotent. Moreover, in this case $M = \mathcal{R}(T^\infty)$ and $N = \mathcal{N}(T^\infty)$ are closed;*
(v) *There exists a linear idompotent P acting on X such that $TP = PT$, $T + P$ is bijective and TP is nilpotent. Moreover, in this case P is a projection.*
(vi) *There exists a linear map S acting on X which commutes with T, $S^2T = S$ and $T^{n+1}S = T^n$ for some positive integer n. Moreover, in this case S is bounded and so $S \in L(X)$.*

Proof. (i) $\Longleftrightarrow$ (ii) Assume that $T_{\mathcal{R}(T^\infty)}$ is one-to-one and $T_{X/\mathcal{N}(T^\infty)}$ is onto, that is, $\mathcal{N}(T) \cap \mathcal{R}(T^\infty) = \{0\}$ and $X = \mathcal{R}(T) + \mathcal{N}(T^\infty)$. Since $\mathcal{R}(T) + \mathcal{N}(T^n)$ is a paracomplete subspace of $X = \bigcup_{n=0}^{\infty}[\mathcal{R}(T) + \mathcal{N}(T^n)]$, we deduce, from [15, Proposition 2.2.4] and [12, Lemma 3.2], that there exists an integer n such that $T_{\mathcal{R}(T^n)}$ is onto. Hence $\mathcal{N}(T) \cap \mathcal{R}(T^\infty) = \mathcal{N}(T) \cap \mathcal{R}(T^n) = \{0\}$, and thus $T_{\mathcal{R}(T^n)}$ is invertible. So $\mathcal{R}(T^n)$ is closed and T is Drazin invertible. The converse is clear. The implications (iii) $\Longrightarrow$ (ii) and (i) $\Longrightarrow$ (iii) are obvious. (iii) $\Longleftrightarrow$ (iv) Let (M, N) be a pair of T-invariant subspaces such that $X = M \oplus N$, where T_M is invertible and T_N is nilpotent. Then $\mathcal{R}(T^\infty) = \mathcal{R}((T_M)^\infty) = M$ and $\mathcal{N}(T^\infty) = \mathcal{N}((T_N)^\infty) = N$. The converse is obvious.

The equivalences (iv) $\Longleftrightarrow$ (v) and (v) $\Longleftrightarrow$ (vi) go similarly. □

The statement (vi) of Theorem 2.17 gives a generalization of the Banach isomorphism theorem. The next result is a direct consequence of the previous theorem.

Theorem 2.18. *Let $T \in L(X)$. The following assertions are equivalent:*

(i) T is a Browder operator;
(ii) $T_{\mathcal{R}(T^\infty)}$ is one-to-one, $\mathcal{N}(T^\infty)$ has finite dimension and $T_{X/\mathcal{N}(T^\infty)}$ is onto.;
(iii) $\mathcal{N}(T^\infty)$ has finite dimension and $X = \mathcal{R}(T^\infty) \oplus \mathcal{N}(T^\infty)$, where $\oplus$ denotes the algebraic sum;
(iv) There exists a pair of T-invariant subspaces (M, N) such that N has finite dimension, $X = M \oplus N$, T_M is bijective and T_N is nilpotent. Moreover, in this case $M = \mathcal{R}(T^\infty)$ is closed and $N = \mathcal{N}(T^\infty)$;
(v) There exists a linear idompotent P acting on X such that $TP = PT$, $\mathcal{R}(P)$ has finite dimension, $T + P$ is bijective and TP is nilpotent. Moreover, in this case P is a finite rank projection.

3. Almost Invertible Operators

Definition 3.1. Let $T \in L(X)$. We say that T is almost invertible if there exists $(M, N) \in \mathrm{Red}(T)$ such that T_M is invertible and $\sigma(T_N)$ is at most countable. The set of almost invertible operators is denoted by $\mathcal{G}(X)$.

Remark 3.2. (i) Let $T \in L(X)$. Then $\sigma(T)$ is at most countable if and only if $\sigma_*(T)$ is at most countable.
(ii) It is easily seen that if $T, S \in L(X)$ are commuting operators with $\sigma(T)$ and $\sigma(S)$ at most countable, then $\sigma(T+S)$ and $\sigma(TS)$ at most countable.

Denote in the sequel by $\sigma_{al}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not almost invertible}\}$. Based on the previous remark and the fact that $\mathrm{acc}\,\sigma(T) \subset \sigma_*(T)$, we derive the following result, which provides a characterization of almost invertible operators in terms of $\mathrm{acc}^{\omega_1} \sigma_*(T)$.

Theorem 3.3. *Let $T \in L(X)$ and let $r_* := CBR(\sigma_*(T))$. Then T is almost invertible if and only if T is $g^{\omega_1}_{\sigma_*}$-invertible which is equivalent to say that T is $g^{r_*}_{\sigma_*}$-invertible. Consequently, $\mathcal{G}(X) = \mathcal{G}^{\omega_1}_{\sigma_*}(X) = \mathcal{G}^{r_*}_{\sigma_*}(X)$ and $\sigma_{al}(T) = acc^{\omega_1} \sigma_*(T) = acc^{r_*} \sigma_*(T)$.*

Proof. To prove the result, it suffices to invoke Theorem 2.6 and the fact that a compact countable complex subset A is at most countable if and only if $\mathrm{acc}^r A = \emptyset$, where $r = \mathrm{CBR}(A)$ is a successor ordinal when A is nonempty.

□

By Theorem 3.3, we can conclude that $\sigma_{al}(T)$ is a compact set. Furthermore, we have the decomposition:

$$\sigma_*(T) \setminus \sigma_{al}(T) = \bigsqcup_{\beta<\omega_1} \text{iso}\,(\text{acc}^{\beta}\,\sigma_*(T)).$$

For an almost invertible operator $T \in L(X)$, we define:

$\mathcal{D}_{al}(T) := \{(M,N) \in \text{Red}(T)$ such that T_M is invertible and $\sigma(T_N)$ is at most countable$\}$.

Using Theorems 2.14, 2.16, 3.3, we obtain the following theorem:

Theorem 3.4. *The following statements hold for every $T \in L(X)$:*

(i) *T is almost invertible;*
(ii) *T is $g^{\omega_1}_{\sigma_*}$-invertible;*
(iii) $0 \notin acc^{\omega_1}\,\sigma_*(T)$;
(iv) *There exists $(M,N) \in Red^2(T) \cap \mathcal{D}_{al}(T)$;*
(v) *There exists a pair of T-invariant subspaces (M,N) such that N is closed, $X = M \oplus N$, T_M is invertible and $\sigma(T_N)$ is at most countable;*
(vi) *There exists an at most countable spectral set σ of T such that $0 \notin \sigma(T) \setminus \sigma$;*
(vii) *There exists a spectral set σ of T such that $0 \notin \sigma(T) \setminus \sigma$ and $\sigma \setminus \{0\} \subset \bigsqcup_{\beta<\omega_1} iso\,(acc^{\beta}\,\sigma(T))$;*
(viii) *There exists a bounded projection $P \in comm(T)$ such that $T + P$ is invertible and $\sigma(TP)$ is at most countable;*
(ix) *There exists a bounded projection $P \in comm^2(T)$ such that $T + P$ is invertible and $\sigma(TP)$ is at most countable;*
(x) *There exists $S \in comm(T)$ such that $S^2T = S$ and $\sigma(T - T^2S)$ is at most countable;*
(xi) *There exists $S \in comm^2(T)$ such that $S^2T = S$ and $\sigma(T - T^2S)$ is at most countable.*

Corollary 3.5. *Let $T \in L(X)$. The following statements hold:*

(i) *T is almost invertible if and only if T^* is almost invertible.*
(ii) *If S is a bounded operator acts on a complex Banach space Y, then $T \oplus S$ is almost invertible if and only if both T and S are almost invertible.*
(iii) *T is almost invertible if and only if T^m is almost invertible for some (equivalently for every) integer $m \geq 1$.*
(iv) *Let M be a closed T-invariant subspace. If both T_M and $T_{X/M}$ are almost invertible, then T is almost invertible.*
(v) *If $R \in comm(T)$ is a Riesz operator, then $\sigma_{al}(T + R) = \sigma_{al}(T)$.*

Proposition 3.6. *Let $T, R \in L(X)$ be almost invertible operators such that $TR = RT = 0$. Then $T + R$ is almost invertible.*

Proof. Let $S \in \mathrm{comm}^2(T)$ and $S^{'} \in \mathrm{comm}^2(R)$ such that $S^2T = S$, $(S^{'})^2R = S^{'}$, $\sigma(T - T^2S)$ and $\sigma(R - R^2S^{'})$ are at most countable. It is easy to see that $(S + S^{'}) \in \mathrm{comm}^2(T)$, $(S + S^{'})^2(T + R) = S^2T + (S^{'})^2T = S + S^{'}$ and $(T + R) - (T + R)^2(S + S^{'}) = T - T^2S + R - R^2S^{'}$. We conclude from Remark 3.2 that $T + R$ is almost countable. □

According to [10], an operator T is said to have a topological uniform descent if $\mathcal{R}(T) + \mathcal{N}(T^d)$ is closed for some integer d and $\mathcal{R}(T) + \mathcal{N}(T^n) = \mathcal{R}(T) + \mathcal{N}(T^n)$ for all $n \geq d$. For the definition of the SVEP, one can see [11].

Proposition 3.7. *Let $T \in L(X)$. The following assertions hold:*

(i) If $0 \notin acc^{\omega_1}\,\sigma(T)$, then T and its dual adjoint T^* have the SVEP at 0.
(ii) If T has a topological uniform descent, then T is Drazin invertible if and only if T is almost invertible.

Proof. The point (i) is a consequence of Theorem 3.3, and the point (ii) is a consequence of [11, Theorems 3.2, 3.4]. □

In the next corollary, we denote, respectively, the essential spectrum and the B-Fredholm spectrum of $T \in L(X)$ by $\sigma_e(T)$ and $\sigma_{bf}(T)$. For the definition and the properties of the B-Fredholm spectrum, one can see [3,5].

Corollary 3.8. $\sigma_b(T) = \sigma_e(T) \cup acc^{\omega_1}\,\sigma(T)$ *and* $\sigma_d(T) = \sigma_{bf}(T) \cup acc^{\omega_1}\,\sigma(T)$, *for all $T \in L(X)$.*

Funding The authors have not disclosed any funding.

Data Availability Not applicable.

Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] Abad, O., Zguitti, H.: A note on the generalized Drazin-Riesz invertible operators. Ann. Funct. Anal. **12**, 55 (2021)

[2] Aznay, Z., Ouahab, A., Zariouh, H.: On the g_z-Kato decomposition and generalization of Koliha Drazin invertibility. Filomat **37**, 2087–2103 (2023)

[3] Aznay, Z., Ouahab, A., Zariouh, H.: On the index of pseudo B-Fredholm operator. Monatsh. Math. **200**, 195–207 (2023)

[4] Aznay, Z., Ouahab, A., Zariouh, H.: Generalization of Kato's decomposition. Linear Algebra Appl. **672**, 108–121 (2023)

[5] Berkani, M.: On a class of quasi-Fredholm operators. Integral Equ. Oper. Theory **34**, 244–249 (1999)

[6] Berkani, M.: On the p-invertibility in rings and Banach algebras. Aequ. Math. **96**, 1125–1140 (2022)

[7] Dajić, A., Koliha, J.J.: The σ_g-Drazin inverse and the generalized Mbekhta decomposition. Integral Equ. Oper. Theory **57**, 309–326 (2007)

[8] Drazin, M.P.: Pseudo-inverses in associative rings and semigroups. Amer. Math. Mon. **65**, 506–514 (1958)

[9] Gohberg, I., Goldberg, S., Kaashoek, M.A.: Classes of Linear Operators, vol. I, Birkhäuser Basel (1990)

[10] Grabiner, S.: Uniform ascent and descent of bounded operators. J. Math. Soc. Jpn. **34**, 317–337 (1982)

[11] Jiang, Q., Zhong, H., Zeng, Q.: Topological uniform descent and localized SVEP. J. Math. Anal. Appl. **390**, 355–361 (2012)

[12] Kaashoek, M.A.: Ascent, descent, nullity and defect, a note on a paper by A. E. Taylor, Math. Ann **172**, 105–115 (1967)

[13] Kechris, A.S.: Classical descriptive set theory. Springer New York, NY (1995)

[14] Koliha, J.J.: A generalized Drazin inverse. Glasg. Math. J. **38**, 367–381 (1996)

[15] Labrousse, J.P.: Les opérateurs quasi Fredholm: une généralisation des opérateurs semi Fredholm. Rend. Circ. Mat. Palermo **29**, 161–258 (1980)

[16] Müller, V.: *Spectral theory of linear operators and spectral systems in Banach algebras*, Oper. Theory Adv. Appl., vol 139, 2*nd* edn., Birkhaüser Basel (2007)

[17] Schmoeger, C.: On isolated points of the spectrum of a bounded linear operator. Proc. Am. Math. Soc. **117**, 715–719 (1993)

[18] Živković-Zlatanović, S.Č, Cvetković, M.D.: Generalized Kato-Riesz decomposition and generalized Drazin-Riesz invertible operators. Linear Multilinear Algebra **65**, 1171–1193 (2017)

[19] Živković-Zlatanović, S.Č, Duggal, B.P.: Generalized Kato-meromorphic decomposition, generalized Drazin-meromorphic invertible operators and single-valued extension property. Banach J. Math. Anal. **14**, 894–914 (2020)

Zakariae Aznay
Laboratory (A.F.N.L.A), Department of Mathematics, Faculty of Science
Abdelmalek Essaadi University
Tetouan
Morocco
e-mail: aznayzakariae@gmail.com

Abdelmalek Ouahab
Laboratory (L.A.N.O), Department of Mathematics, Faculty of Science
Mohammed I University
60000 Oujda
Morocco
e-mail: ouahab05@yahoo.fr

Hassan Zariouh
Department of Mathematics (CRMEFO), and Laboratory (L.A.N.O),
Faculty of Science
Mohammed I University
60000 Oujda
Morocco
e-mail: h.zariouh@yahoo.fr

Received: May 10, 2023.
Accepted: June 20, 2024.

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