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Results in Mathematics

Conformal Kaehler Submanifolds

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Abstract. This paper presents two results in the realm of conformal Kaehler submanifolds. These are conformal immersions of Kaehler manifolds into the standard flat Euclidean space. The proofs are obtained by making a rather strong use of several facts and techniques developed in Chion and Dajczer (Proc Edinb Math Soc 66:810–833, 2023) for the study of isometric immersions of Kaehler manifolds into the standard hyperbolic space.

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Let $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ denote a *conformal Kaehler submanifold*. Thus (M^{2n}, J) is a Kaehler manifold of complex dimension $n \geq 2$ and f a conformal immersion into Euclidean space that lies in codimension p. Thus, there is a positive function $\lambda \in C^\infty(M)$ such that the Kaehler metric and the one induced by f relate by $\langle , \rangle_f = \lambda^2 \langle , \rangle_{M^{2n}}$.

Conformal Kaehler submanifolds laying in the low codimensions $p = 1$ and $p = 2$ have already been considered in [1]. In this paper, we are interested in higher codimensions although not too large in comparison to the dimension of the manifold.

Our first result, provides a necessary condition for the existence of a conformal immersion in codimension at most $n - 3$ in terms of the sectional curvature of the Kaehler manifold.

Theorem 1. *Let $f\colon M^{2n} \to \mathbb{R}^{2n+p}$, $p \leq n-3$, be a conformal Kaehler submanifold. Then at any $x \in M^{2n}$ there is a complex vector subspace $V^{2m} \subset T_xM$ with $m \geq n-p$ such that the sectional curvature of M^{2n} satisfies $K_M(S, JS)(x) \leq 0$ for any $S \in V^{2m}$.*

Notice that the conclusion of Theorem 1 remains valid if the Euclidean ambient space $\mathbb{R}^{2n+p}$ is replaced by any locally conformally flat manifold of the same dimension.

Our second result characterizes a submanifold, in terms of the degree of positiveness of the sectional curvature, as being locally the Example 2 presented below.

The light-cone $\mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$ of the standard flat Lorentzian space is any one of the two connected components of the set of all light-like vectors, namely,

$$\{v \in \mathbb{L}^{m+2} : \langle v, v\rangle = 0,\ v \neq 0\}$$

endowed with the (degenerate) induced metric. The Euclidean space $\mathbb{R}^m$ can be realized as an umbilic hypersurface of $\mathbb{V}^{m+1}$ as follows: Take vectors $v, w \in \mathbb{V}^{m+1}$ such that $\langle v, w\rangle = 1$ and a linear isometry $C\colon \mathbb{R}^m \to \{v, w\}^{\perp}$. Let $\psi\colon \mathbb{R}^m \to \mathbb{V}^{m+1} \subset \mathbb{L}^{m+2}$ be defined by

$$\psi(x) = v + Cx - \frac{1}{2}\|x\|^2 w. \tag{1}$$

Then ψ is an isometric embedding of $\mathbb{R}^m$ as an umbilical hypersurface in the light cone which is the intersection of $\mathbb{V}^{m+1}$ with an affine hyperplane. Namely, we have that

$$\psi(\mathbb{R}^m) = \{y \in \mathbb{V}^{m+1} \colon \langle y, w\rangle = 1\}.$$

The normal bundle of ψ is $N_\psi\mathbb{R}^m = \operatorname{span}\{\psi, w\}$ and its second fundamental form is

$$\alpha^\psi(X, Y) = -\langle X, Y\rangle_{\mathbb{R}^m} w.$$

Proposition 9.9 in [3] gives an elementary correspondence between the conformal immersions in Euclidean space and the isometric immersions into the light cone which goes as follows: Associated to a given conformal immersion $f\colon M^m \to \mathbb{R}^{m+p}$ with conformal factor $\lambda \in C^\infty(M)$ there is the associated isometric immersion defined by

$$F = \frac{1}{\lambda}\psi \circ f\colon M^m \to \mathbb{V}^{m+p+1} \subset \mathbb{L}^{m+p+2}.$$

Conversely, any isometric immersion $F\colon M^m \to \mathbb{V}^{m+p+1}\backslash\mathbb{R}w \subset \mathbb{L}^{m+p+2}$ gives rise to an associated conformal immersion $f\colon M^m \to \mathbb{R}^{m+p}$ given by $\psi \circ f = \pi \circ F$ with conformal factor $1/\langle F, w\rangle$. Here $\pi\colon \mathbb{V}^{m+p+1} \setminus \mathbb{R}w \to \mathbb{R}^{m+p}$ is the projection $\pi(x) = x/\langle x, w\rangle$.

Example 2.. Let the Kaehler manifold M^{2n} be the Riemannian product of one hyperbolic plane and a set of two-dimensional round spheres such that

$$M^{2n} = \mathbb{H}^2_c \times \mathbb{S}^2_{c_2} \times \cdots \times \mathbb{S}^2_{c_n} \ \text{ with } \ 1/c_2 + \cdots + 1/c_n = -1/c.$$

If f_1 is the inclusion $\mathbb{H}^2_c \subset \mathbb{L}^3$ and $f_2\colon \mathbb{S}^2_{c_2} \times \cdots \times \mathbb{S}^2_{c_n} \to \mathbb{S}^{3n-4}_c \subset \mathbb{R}^{3n-3}$ is the product of umbilical spheres then the map $\psi^{-1} \circ (f_1 \times f_2)\colon M^{2n} \to \mathbb{R}^{3n-2}$ is a conformal Kaehler submanifold.

Theorem 3.. *Let $f\colon M^{2n} \to \mathbb{R}^{2n+p}$, $2 \le p \le n-2$, be a connected conformal Kaehler submanifold. Assume that at a point $x_0 \in M^{2n}$ there is a complex tangent vector subspace $V^{2m} \subset T_{x_0}M$ with $m \ge p+1$ such that the sectional curvature of M^{2n} satisfies $K_M(S,JS)(x) > 0$ for any $0 \ne S \in V^{2m}$. Then $p = n-2$ and $f(M)$ is an open subset of the submanifold given by Example 2.*

1. The Proofs

Let V^{2n} and $\mathbb{L}^p$, $p \ge 2$, be real vector spaces such that there is $J \in Aut(V)$ which satisfies $J^2 = -I$ and $\mathbb{L}^p$ is endowed with a Lorentzian inner product $\langle , \rangle$. Then let $W^{p,p} = \mathbb{L}^p \oplus \mathbb{L}^p$ be endowed with the inner product of signature (p,p) defined by

$$\langle\langle (\xi,\bar{\xi}),(\eta,\bar{\eta})\rangle\rangle = \langle \xi,\eta\rangle - \langle \bar{\xi},\bar{\eta}\rangle.$$

A vector subspace $L \subset W^{p,p}$ is called *degenerate* if $L \cap L^\perp \ne 0$.

Let $\alpha\colon V^{2n}\times V^{2n} \to \mathbb{L}^p$ be a symmetric bilinear form and $\beta\colon V^{2n}\times V^{2n} \to W^{p,p}$ the associated bilinear form given by

$$\beta(X,Y) = (\alpha(X,Y) + \alpha(JX,JY), \alpha(X,JY) - \alpha(JX,Y)).$$

We have that if $\beta(X,Y) = (\xi,\eta)$ then

$$\beta(X,JY) = (\eta,-\xi) \ \text{and} \ \beta(Y,X) = (\xi,-\eta). \tag{2}$$

We denote the vector subspace of $W^{p,p}$ generated by β by

$$\mathcal{S}(\beta) = \operatorname{span}\{\beta(X,Y)\colon X,Y \in V^{2n}\}$$

and say that β is *surjective* if $\mathcal{S}(\beta) = W^{p,p}$. The (right) kernel β is defined by

$$\mathcal{N}(\beta) = \{Y \in V^{2n}\colon \beta(X,Y) = 0 \ \text{for all} \ X \in V^{2n}\}.$$

A vector $X \in V^{2n}$ is called a (left) *regular element* of β if $\dim B_X(V) = r$ where $r = \max\{\dim B_X(V)\colon X \in V\}$ and $B_X\colon V \to W^{p,p}$ is the linear transformation defined by $B_XY = \beta(X,Y)$. The set $RE(\beta)$ of regular elements of β is easily seen to be an open dense subset of V^{2n}, for instance see Proposition 4.4 in [3].

It is said that β is *flat* if it satisfies that

$$\langle\langle \beta(X,Y),\beta(Z,T)\rangle\rangle - \langle\langle \beta(X,T),\beta(Z,Y)\rangle\rangle = 0 \ \text{for all} \ X,Z,Y,T \in V^{2n}.$$

If β is flat and $X \in RE(\beta)$ we have from Proposition 4.6 in [3] that

$$\mathcal{S}(\beta|_{V\times \ker \beta_X}) \subset B_X(V) \cap (B_X(V))^\perp. \tag{3}$$

Proposition 4. *Let $\beta\colon V^{2n}\times V^{2n} \to W^{p,p}$, $p \le n$, be flat and surjective. Then*

$$\dim \mathcal{N}(\beta) \ge 2n - 2p.$$

Proof. This is condition (9) in Proposition 11 of [2]. □

Let V^{2n} be endowed with a positive definite inner product $(\,,\,)$ with respect to which $J \in Aut(V)$ is an isometry. Assume that there is a light-like vector $w \in \mathbb{L}^p$ such that

$$\langle \alpha(X,Y), w\rangle = -(X,Y) \text{ for any } X,Y \in V^{2n}. \tag{4}$$

Let $U_0^s \subset \mathbb{L}^p$ be the s-dimensional vector subspace given by

$$U_0^s = \pi_1(\mathcal{S}(\beta)) = \operatorname{span}\{\alpha(X,Y) + \alpha(X,JY)\colon X,Y \in V^{2n}\},$$

where $\pi_1\colon W^{p,p} \to \mathbb{L}^p$ denotes the projection onto the first component of $W^{p,p}$. From Proposition 9 in [2] we know that

$$\mathcal{S}(\beta) = U_0^s \oplus U_0^s. \tag{5}$$

In addition, if $\mathcal{S}(\beta)$ is a degenerate vector subspace then $1 \le s \le p-1$ and there is a light-like vector $v \in U_0^s$ such that

$$\mathcal{S}(\beta) \cap \mathcal{S}(\beta)^\perp = \operatorname{span}\{v\} \oplus \operatorname{span}\{v\}. \tag{6}$$

Proposition 5. *Let the bilinear form $\beta\colon V^{2n} \times V^{2n} \to W^{p,p}$ be flat and the vector subspace $\mathcal{S}(\beta)$ degenerate. Then $L = span\{v,w\} \subset \mathbb{L}^p$ is a Lorentzian plane. Moreover, choosing v such that $\langle v,w\rangle = -1$ and setting $\beta_1 = \pi_{L^\perp \times L^\perp} \circ \beta$, we have*

$$\beta(X,Y) = \beta_1(X,Y) + 2((X,Y)v,(X,JY)v) \text{ for any } X,Y \in V^{2n}. \tag{7}$$

Furthermore, if $s \le n$ then

$$\dim \mathcal{N}(\beta_1) \ge 2n - 2s + 2. \tag{8}$$

Proof. We obtain from (6) that

$$0 = \langle\langle \beta(X,Y),(v,0)\rangle\rangle = \langle \alpha(X,Y) + \alpha(JX,JY), v\rangle \text{ for any } X,Y \in V^{2n}. \tag{9}$$

From (4) and the fact that J is an isometry with respect to $(\,,\,)$, we also have that

$$\begin{aligned}\langle\langle \beta(X,Y),(w,0)\rangle\rangle &= \langle \alpha(X,Y) + \alpha(JX,JY), w\rangle \\ &= -2(X,Y) \text{ for any } X,Y \in V^{2n}. \end{aligned}\tag{10}$$

In particular, we have $\langle\langle \beta(X,X),(w,0)\rangle\rangle < 0$ for any $0 \ne X \in V^{2n}$, which jointly with (9) implies that v and w are linearly independent and thus span a Lorentzian plane.

Since w is light-like and v satisfies $\langle v,w\rangle = -1$, we have

$$\begin{aligned}\alpha(X,Y) + \alpha(JX,JY) = \alpha_{L^\perp}(X,Y) + \alpha_{L^\perp}(JX,JY) - \langle \alpha(X,Y) \\ + \alpha(JX,JY), v\rangle w - \langle \alpha(X,Y) + \alpha(JX,JY), w\rangle v,\end{aligned}$$

where $\alpha_{L^\perp}$ denotes the $L^\perp$-component of α. Then (9) and (10) yield

$$\alpha(X,Y) + \alpha(JX,JY) = \alpha_{L^\perp}(X,Y) + \alpha_{L^\perp}(JX,JY) + 2(X,Y)v$$

and

$$\alpha(X,JY)-\alpha(JX,Y)=\alpha_{L^\perp}(X,JY)-\alpha_{L^\perp}(JX,Y)+2(X,JY)v,$$

from which we obtain (7).

We have from (5) and (6) that $w\notin U_0^s+L^\perp$. Hence $\dim(U_0^s+L^\perp)=p-1$. It then follows from

$$p-1=\dim(U_0^s+L^\perp)=\dim U_0^s+\dim L^\perp-\dim U_0^s\cap L^\perp$$

that $U_1=U_0^s\cap L^\perp$ satisfies

$$\dim U_1=s-1 \tag{11}$$

and we have from (5), (6) and (7) that $\mathcal{S}(\beta_1)=U_1^{s-1}\oplus U_1^{s-1}$.

From (7) we obtain that

$$\langle\langle\beta(X,Y),\beta(Z,T)\rangle\rangle=\langle\langle\beta_1(X,Y),\beta_1(Z,T)\rangle\rangle \text{ for any } X,Y,Z,T\in V^{2n},$$

and hence also the bilinear form $\beta_1\colon V^{2n}\times V^{2n}\to L^\perp\oplus L^\perp$ is flat. Let $X\in RE(\beta_1)$ and set $N_1(X)=\ker B_{1X}$ where $B_{1X}Y=\beta_1(X,Y)$. To obtain (8) it suffices to show that $N_1(X)=\mathcal{N}(\beta_1)$ since then $\dim\mathcal{N}(\beta_1)=\dim N_1(X)\geq 2n-2\dim U_1=2n-2s+2$.

If $\beta_1(Y,Z)=(\xi,\eta)$ then by (2) and (7) we have $\beta_1(Z,Y)=(\xi,-\eta)$. If $Y,Z\in N_1(X)$ it follows from (3) that

$$0=\langle\langle\beta_1(Y,Z),\beta_1(Z,Y)\rangle\rangle=\langle\langle(\xi,\eta),(\xi,-\eta)\rangle\rangle=\|\xi\|^2+\|\eta\|^2.$$

Hence $\beta_1|_{N_1(X)\times N_1(X)}=0$ since the inner product induced on U_1^{s-1} is positive definite. Now let $\beta_1(Y,Z)=(\delta,\zeta)$ where $Y\in V^{2n}$ and $Z\in N_1(X)$. Then the flatness of β_1 yields

$$0=\langle\langle\beta_1(Y,Z),\beta_1(Z,Y)\rangle\rangle=\langle\langle(\delta,\zeta),(\delta,-\zeta)\rangle\rangle=\|\delta\|^2+\|\zeta\|^2$$

and therefore $\beta_1|_{V\times N_1(X)}=0$. □

Proposition 6. *Let the bilinear form $\beta\colon V^{2n}\times V^{2n}\to W^{p,p}$ be flat and satisfy*

$$\langle\langle\beta(X,Y),\gamma(Z,T)\rangle\rangle=\langle\langle\beta(X,T),\gamma(Z,Y)\rangle\rangle \text{ for any } X,Y,Z,T\in V^{2n} \tag{12}$$

where $\gamma\colon V^{2n}\times V^{2n}\to W^{p,p}$ is the bilinear form defined by

$$\gamma(X,Y)=(\alpha(X,Y),\alpha(X,JY)) \text{ for any } X,Y\in V^{2n}.$$

If the vector subspace $\mathcal{S}(\beta)$ is degenerate and $s\leq n-1$ then there is a J-invariant vector subspace $P^{2m}\subset V^{2n}$, $m\geq n-s+1$, such that

$$\langle\alpha(S,S),\alpha(JS,JS)\rangle-\|\alpha(S,JS)\|^2\leq 0 \text{ for any } S\in P^{2m}.$$

Proof. Let $v\in U_0^s$ be given by (6). We claim that

$$\langle\alpha(X,Y),v\rangle=0 \text{ for any } X,Y\in V^{2n}. \tag{13}$$

Since $s\leq n-1$ then (8) gives $\dim\mathcal{N}(\beta_1)\geq 4$. Hence (7) yields $\beta(S,S)=2((S,S)v,0)$ for any $S\in\mathcal{N}(\beta_1)$. Thus

$$\langle\langle\gamma(X,Y),\beta(S,S)\rangle\rangle=2\langle\alpha(X,Y),v\rangle \tag{14}$$

for any $S \in \mathcal{N}(\beta_1)$ of unit length. On the other hand, we obtain from (2) and (7) that $\beta(S,Y) = \beta(Y,S) = 0$ for any $S \in \mathcal{N}(\beta_1)$ and $Y \in \{S, JS\}^\perp$. Then (12) and (14) give $\langle \alpha(X,Y), v\rangle = 0$ for any $X \in V^{2n}$ and $Y \in \{S, JS\}^\perp$ where $S \in \mathcal{N}(\beta_1)$. Now that $\dim \mathcal{N}(\beta_1) \geq 4$ yields the claim.

Choosing $v \in U_0^s$ as in Proposition 5 it follows from (4) and (13) that

$$\alpha(X,Y) = \alpha_{L^\perp}(X,Y) + (X,Y)v \text{ for any } X, Y \in V^{2n}. \tag{15}$$

Then we obtain from (15) that

$$\gamma(X,Y) = (\alpha_{L^\perp}(X,Y) + (X,Y)v, \alpha_{L^\perp}(X,JY) + (X,JY)v) \text{ for any } X,Y \in V^{2n}.$$

Set $P^{2m} = \mathcal{N}(\beta_1)$ where $2m = \dim \mathcal{N}(\beta_1) \geq 2n - 2s + 2$ by (8). From (7) we have $\beta(Z,S) = 2((Z,S)v, (Z,JS)v)$ for any $S \in P^{2m}$ and $Z \in V^{2n}$. Then (12) gives

$$\langle\langle \gamma(X,S), \beta(Z,Y)\rangle\rangle = \langle\langle \gamma(X,Y), \beta(Z,S)\rangle\rangle = 0$$

for any $S \in P^{2m}$ and $X,Y,Z \in V^{2n}$. Hence $\mathcal{S}(\gamma|_{V\times P})$ and $\mathcal{S}(\beta)$ are orthogonal vector subspaces. From (11) we have

$$U_0^s = U_1^{s-1} \oplus \operatorname{span}\{v\} \text{ where } U_1^{s-1} = U_0^s \cap L^\perp. \tag{16}$$

Then by (5) the vector subspaces $\mathcal{S}(\gamma|_{V\times P})$ and $U_1^{s-1} \oplus U_1^{s-1}$ are orthogonal, and thus

$$\langle \alpha(X,S), \xi\rangle = \langle\langle \gamma(X,S), (\xi, 0)\rangle\rangle = 0$$

for any $X \in V^{2n}$, $S \in P^{2m}$ and $\xi \in U_1^{s-1}$. Since $U_1^{s-1} \subset L^\perp$ then

$$\alpha_{U_1}(X,S) = 0 \text{ for any } X \in V^{2n} \text{ and } S \in P^{2m}. \tag{17}$$

Let $\mathbb{L}^p = U_1^{s-1} \oplus U_2^{p-s-1} \oplus L$ be an orthogonal decomposition. Then (5) and (16) give

$$\langle \alpha(X,Y) + \alpha(JX,JY), \xi_2\rangle = \langle\langle \beta(X,Y), (\xi_2, 0)\rangle\rangle = 0$$

for any $X, Y \in V^{2n}$ and $\xi_2 \in U_2^{p-s-1}$. Thus

$$\alpha_{U_2}(X,Y) = -\alpha_{U_2}(JX,JY) \text{ for any } X,Y \in V^{2n}. \tag{18}$$

Having U_2^{p-s-1} a positive definite induced inner product, we obtain from (15), (17) and (18) that

$$\langle \alpha(S,S), \alpha(JS,JS)\rangle - \|\alpha(S,JS)\|^2 = -\|\alpha_{U_2}(S,S)\|^2 - \|\alpha_{U_2}(S,JS)\|^2 \leq 0$$

for any $S \in P^{2m}$. □

Given a conformal immersion $f\colon M^{2n} \to \mathbb{R}^{2n+p}$ with conformal factor $\lambda \in C^\infty(M)$ we have the associated isometric immersion $F = \frac{1}{\lambda}\psi \circ f\colon M^{2n} \to \mathbb{V}^{2n+p+1} \subset \mathbb{L}^{2n+p+2}$ where ψ is given by (1). Differentiating $\langle F, F\rangle = 0$ once gives $F \in \Gamma(N_F M)$ and twice yields that the second fundamental form $\alpha^F\colon TM \times TM \to N_F M$ of F satisfies

$$\langle \alpha^F(X,Y), F\rangle = -\langle X, Y\rangle \text{ for any } X, Y \in \mathfrak{X}(M). \tag{19}$$

Since $\psi_* N_f M \subset N_F M$ the normal bundle of F decomposes as $N_F M = \psi_* N_f M \oplus L^2 \equiv \mathbb{L}^{p+2}$ where L^2 is the Lorentzian plane subbundle orthogonal to $\psi_* N_f M$ such that $F \in \Gamma(L^2)$.

Let the bilinear forms $\gamma, \beta\colon T_xM \times T_xM \to N_F M(x) \oplus N_F M(x)$ be defined by

$$\gamma(X,Y) = (\alpha^F(X,Y), \alpha^F(X,JY))$$

and

$$\beta(X,Y) = (\alpha^F(X,Y) + \alpha^F(JX,JY), \alpha^F(X,JY) - \alpha^F(JX,Y)).$$

Proposition 7. *Let $N_F M(x) \oplus N_F M(x)$ be endowed with the inner product defined by*

$$\langle\langle(\xi,\bar{\xi}),(\eta,\bar{\eta})\rangle\rangle = \langle\xi,\eta\rangle - \langle\bar{\xi},\bar{\eta}\rangle.$$

Then the bilinear form β is flat and

$$\langle\langle\beta(X,Y),\gamma(Z,T)\rangle\rangle = \langle\langle\beta(X,T),\gamma(Z,Y)\rangle\rangle \ \textit{for any}\ X,Y,Z,T \in T_xM.$$

Proof. The proof is straightforward using that $\beta(X,JY) = -\beta(JX,Y)$, that the curvature tensor satisfies $R(X,Y)JZ = JR(X,Y)Z$ for any $X,Y,Z \in T_xM$ and the Gauss equation for f; for details see the proof of Proposition 16 in [2] □

Proof of Theorem 1. It suffices to show that the vector subspace $\mathcal{S}(\beta)$ is degenerate since then the proof follows from the Gauss equation jointly with Propositions 6 and 7. If $\mathcal{S}(\beta)$ is not degenerate, and since we have the result given by Proposition 7, then Proposition 4 yields $\dim \mathcal{N}(\beta) \geq 2n - 2p - 4 > 0$. But this is a contradiction since from (19) we have that $\mathcal{N}(\beta) = 0$. □

Proposition 8. *Let the bilinear form $\beta\colon V^{2n} \times V^{2n} \to W^{p,p}$, $s \leq n$, be flat. Assume that the vector subspace $\mathcal{S}(\beta)$ is nondegenerate and that (12) holds. For $p \geq 4$ assume further that there is no non-trivial J-invariant vector subspace $V_1 \subset V^{2n}$ such that the subspace $\mathcal{S}(\beta|_{V_1\times V_1})$ is degenerate and $\dim \mathcal{S}(\beta|_{V_1\times V_1}) \leq \dim V_1 - 2$. Then $s = n$ and there is an orthogonal basis $\{X_i, JX_i\}_{1\leq i\leq n}$ of V^{2n} such that:*

(i) $\beta(Y_i, Y_j) = 0$ if $i \neq j$ and $Y_k \in$ span$\{X_k, JX_k\}$ for k=i,j.

(ii) The vectors $\{\beta(X_j,X_j), \beta(X_j,JX_j)\}_{1\leq j\leq n}$ form an orthonormal basis of $\mathcal{S}(\beta)$.

Proof. It follows from Proposition 15 in [2]. □

Proof of Theorem 3. Theorem 1 gives that $p = n-2$. In an open neighborhood U of x_0 in M^{2n} there is a complex vector subbundle $\bar{V} \subset TM$ such that $\bar{V}(x_0) = V^{2m}$ and $K_M(S,JS) > 0$ for any $0 \neq S \in \bar{V}$. At any point of U the vector subspace $\mathcal{S}(\beta)$ is nondegenerate. In fact, if otherwise then by Proposition 6 there is a point $y \in U$ and a complex vector subspace $P^{2\ell} \subset T_yM$

with $\ell \geq 2$ such that the sectional curvature satisfies $K_M(S, JS) \leq 0$ for any $0 \neq S \in P^{2\ell}$, in contradiction with our assumption.

By Proposition 8, there is at any $y \in U$ an orthogonal basis $\{X_j, JX_j\}_{1\leq j\leq n}$ of T_yM such that both parts hold. By part (ii) the vectors $(\xi_j, 0) = \beta(X_j, X_j) \in N_F M(y)$, $1 \leq j \leq n$, are orthonormal. Then the argument used for the proof of Lemma 18 in [2] gives that $F|_U$ has flat normal bundle, that rank $A_{\xi_j}=2$ for $1 \leq j \leq n$ and that the normal vector fields $\xi_1, \ldots, \xi_n$ are smooth on connected components of an open dense subset of U. Moreover, we obtain from the Codazzi equation and the use of the de Rham theorem that M^{2n} is locally a Riemannian product of surfaces $M_1^2 \times \cdots \times M_n^2$.

Having that the codimension is $n = p + 2$ and that $\alpha^F(Y_i, Y_j) = 0$ if $Y_i \in (E_i)$ and $Y_j \in (E_j)$, $i \neq j$, then by Theorem 8.7 in [3] there are isometric immersions $g_1 \colon M_1^2 \to \mathbb{L}^3$ and $g_j \colon M_j^2 \to \mathbb{R}^3$, $2 \leq j \leq n$, such that

$$F(x_1, \ldots, x_n) = (g_1(x_1), g_2(x_2), \ldots, g_n(x_n)).$$

Since $F(M) \subset \mathbb{V}^{3n-1} \subset \mathbb{L}^{3n}$ then $\langle F, F\rangle = 0$. Hence $\langle g_{j*}X_j, g_j\rangle = \langle g_* X_j, g_j\rangle = 0$ and thus $\|g_j\| = r_j$ with $-r_1^2 + \sum_{j=2}^n r_j^2 = 0$. This gives that $F(U) \subset \mathbb{H}^2_{c_1} \times \mathbb{S}^2_{c_2} \times \cdots \times \mathbb{S}^n_{c_n}$ where $1/c_i = r_i^2$ and, by continuity, this also holds for $F(M)$. $\square$

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Data availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest No potential conflict of interest was reported by the authors.

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