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Results in Mathematics

A Diophantine Equation With Powers of Three Consecutive k−Fibonacci Numbers

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Abstract. The k–generalized Fibonacci sequence $\{F_n^{(k)}\}_{n\geq 2-k}$ is the linear recurrent sequence of order k whose first k terms are $0,\ldots,0,1$ and each term afterwards is the sum of the preceding k terms. The case $k=2$ corresponds to the well known Fibonacci sequence $\{F_n\}_{n\geq 0}$. In this paper we extend the study of the exponential Diophantine equation $(F_{n+1})^x+(F_n)^x-(F_{n-1})^x=F_m$ with terms $F_r^{(k)}$ instead of F_r, where $r\in\{n+1,n,n-1,m\}$.

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1. Introduction

Let $k\geq 2$ be a fixed integer. The $k-$generalized Fibonacci sequence, denoted by $F^{(k)}=\{F_n^{(k)}\}_{n\geq -(2-k)}$ is given by the linear recurrence

$$F_n^{(k)}=F_{n-1}^{(k)}+\cdots+F_{n-k}^{(k)}\quad \text{for all} \quad n\geq 2,$$

with the initial values

$$F_i^{(k)}=0 \quad \text{for} \quad i=2-k,\ldots,0 \quad \text{and} \quad F_1^{(k)}=1.$$

Here, we refer to $F_n^{(k)}$ as the nth $k-$Fibonacci number. In the particular case $k=2$, we recover the classical Fibonacci sequence denoted by $F^{(2)}=\{F_n\}$.

Due to the vast amount of identities that Fibonacci numbers satisfy, several authors looked at such identities but used $k-$Fibonacci numbers instead.

This led to several interesting Diophantine equations. This is, for example, the case of Bednařík et al [1], who showed that the equation

$$\left(F_{n+1}^{(k)}\right)^2 + \left(F_n^{(k)}\right)^2 = F_m^{(l)},$$

has no positive integral solutions (k, l, n, m), with $2 \leq k < l$ and $n \geq 2$, in an attempt to generalize the identity $F_{n+1}^2 + F_n^2 = F_{2n+1}$. An analogous identity with Fibonacci numbers namely $F_{n+1}^2 - F_{n-1}^2 = F_{2n}$, was also generalized for the $k-$Fibonacci numbers leading to considering the Diophantine equation

$$(F_{n+1}^{(k)})^x - (F_{n-1}^{(k)})^x = F_m^{(k)}$$

with solutions in positive integral quadruples (k, n, m, x). The study of the above equation was initiated by Bensella, Patel and Behloul [2] and completed by us in [10], where we found, apart from some trivial cases, two parametric families of solutions.

Another well known identity discovered by the French mathematician Francois Edouard Anatole Lucas in 1876 is

$$F_{n+1}^3 + F_n^3 - F_{n-1}^3 = F_{3n} \quad \text{valid for all} \quad n \geq 1,$$

which was initially generalized by Patel and Teh [18] who considered the equation

$$F_{n+1}^x + F_n^x - F_{n-1}^x = F_m$$

and showed that only the triplets (n, m, x) with $m = 3n$ and $x = 3$ are solutions. Our aim is to look for the positive integral quadruples (k, m, n, x), with $k \geq 2$, that solve the more general Diophantine equation

$$\left(F_{n+1}^{(k)}\right)^x + \left(F_n^{(k)}\right)^x - \left(F_{n-1}^{(k)}\right)^x = F_m^{(k)}. \tag{1}$$

Our main result is the following.

Main Theorem. *The Diophantine Eq. (1) does not have non-trivial solutions* (k, n, m, x) *with* $k \geq 2$, $n \geq 1$, $m \geq 2$ *and* $x \geq 1$.

For the trivial ones, we invite the reader to consult Sect. 3.2.

2. Tools

2.1. Linear Forms in Logarithms

Here we present the necessary results related to non–zero linear forms in logarithms of algebraic numbers; i.e., some results of Baker's theory, such as the following result of Matveev [15].

Theorem 1 *(Matveev's theorem). Let* $\mathbb{K}$ *be a number field of degree* D *over* $\mathbb{Q}$, $\gamma_1, \ldots, \gamma_t$ *be positive real numbers in* $\mathbb{K}$, *and* $b_1, \ldots, b_t$ *rational integers. Put*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1 \quad \textit{and} \quad B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and let $A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$ *for* $i = 1, \ldots, t$. *If* $\Lambda \neq 0$, *then*

$$|\Lambda| > \exp\left(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t\right),$$

whete $h(\gamma_i)$ *is the logarithmic height of* γ_i.

For more details and properties of the logarithmic height see [19].

2.2. Analytical Arguments

For real algebraic numbers $\gamma_1, \ldots, \gamma_t$ if we set

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1 \quad \text{and} \quad \Gamma := b_1 \log \eta_1 + \cdots + b_t \log \eta_t,$$

then we have $\Lambda = e^{\Gamma} - 1$. So, it is a straight–forward exercise to show that $|\Gamma| < (1 - c)^{-1}|\Lambda|$, when $|\Lambda| < c$, for all c in $(0, 1)$. We will use this argument without further mention.

Moreover, the following analytic result will be used in some specific parts of our work (see Lemma 7 from [13]).

Lemma 1. *If* $r \geq 1$ *and* $T > (4r^2)^r$, *then*

$$y < T(\log y)^r, \qquad \text{implies} \qquad y < 2^r T(\log T)^r.$$

2.3. Reduction by Continued Fractions

Since in a first stage of our process we get some large bounds on the integer variables appearing in the Diophantine Eq. (1), we require some results from the theory of continued fractions to reduce them.

When we treat with homogeneous linear forms in two integer variables we use a classical theorem of Legendre.

Lemma 2. *Let* M *be a positive integer and* $P_1/Q_1, P_2/Q_2, \ldots$ *be the convergents of the continued fraction* $[a_0, a_1, \ldots]$ *for the real number* τ. *Let* N *be a positive integer such that* $M < Q_{N+1}$. *If* $a_M := \max\{a_\ell : 0 \leq \ell \leq N + 1\}$, *then the inequality*

$$\left|\tau - \frac{v}{u}\right| > \frac{1}{(a_M + 2)u^2},$$

holds for all pairs (u, v) *of integers with* $0 < u < M$.

Besides, to treat non-homogeneous linear forms in two integer variables, we need the following slight variation of a Dujella and Pethő result (Lemma 5a in [8]). For $X \in \mathbb{R}$, we use $||X|| := \min\{|X - n| : n \in \mathbb{Z}\}$ to denote the distance from X to its nearest integer.

Lemma 3. *Let* M *and* Q *be positive integers such that* $Q > 6M$, *and* A, B, τ, μ *be real numbers with* $A > 0$ *and* $B > 1$. *Let* $\varepsilon := ||\mu Q|| - M||\tau Q||$. *If* $\varepsilon > 0$, *then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < A \cdot B^{-w},$$

in positive integers u, v *and* w *with* $u \leq M$ *and* $w \geq \log(AQ/\varepsilon)/\log B$.

For practical applications, Q is taken to be the denominator of a continued fraction convergent for the real number τ.

3. Some Preliminaries

3.1. On k–Fibonacci Numbers

In this section we present some $k-$Fibonacci numbers properties necessary for the development of this paper. For related results, we invite the reader to consult [2–5,7,9,11,12,14,16,17,20].

For the $k-$Fibonacci sequence, it is well known that its characteristic polynomial is given by

$$\Psi_k(z) = z^k - z^{k-1} - \cdots - z - 1.$$

This is irreducible over $\mathbb{Q}[z]$ and has only one real root outside the unit circle. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be all its roots with $\alpha := \alpha_1$ denoting the real positive one.

Lemma 4. *The sequence $F^{(k)}$ satisfies the following:*

(*i*) *For all $k \geq 2$,*

$$2(1 - 2^{-k}) < \alpha < 2.$$

(*ii*) *The first $k+1$ non–zero terms in $F^{(k)}$ are powers of two, namely*

$$F_1^{(k)} = 1 \quad \textit{and} \quad F_n^{(k)} = 2^{n-2} \quad \textit{for all} \quad 2 \leq n \leq k+1.$$

Also, $F_{k+2}^{(k)} = 2^k - 1$ and, moreover,

$$F_n^{(k)} < 2^{n-2} \quad \textit{for all} \quad n \geq k+2.$$

(*iii*) *Let $f_k(z) := (z-1)/(2+(k+1)(z-2))$. For all $n \geq 1$ and $k \geq 2$,*

$$F_n^{(k)} = \sum_{i=1}^{k} f_k(\alpha_i){\alpha_i}^{n-1} = f_k(\alpha)\alpha^{n-1} + e_k(n)$$

with $|e_k(n)| < 1/2$.

(*iv*) *For all $k \geq 2$ and $i = 2, \ldots, k$*

$$f_k(\alpha) \in [1/2, 3/4] \qquad \text{and} \qquad |f_k(\alpha_i)| < 1.$$

Thus, $f_k(\alpha)$ is not an algebraic integer, for any $k \geq 2$.

(*v*) *For all $n \geq 1$ and $k \geq 2$,*

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1}.$$

(*vi*) *For all $n \geq 3$ and $k \geq 3$, we have*

$$F_{n-1}^{(k)}/F_{n+1}^{(k)} \leq 3/7 \quad \textit{and} \quad F_n^{(k)}/F_{n+1}^{(k)} \leq 4/7.$$

The following lemma is a consequence of an identity of Cooper and Howard with $k-$Fibonacci numbers that shows these numbers can be written as linear combinations of powers of two with coefficients involving binomial coefficients (see [6]).

Lemma 5. *If $k+2 \leq r < 2^{ck}$ for some $c \in (0,1)$, then the following estimates hold:*

(*i*) $F_r^{(k)} = 2^{r-2}(1+\zeta'),$ *with* $|\zeta'| < \dfrac{2r}{2^k} < \dfrac{2}{2^{k(1-c)}}.$

(*ii*) $F_r^{(k)} = 2^{r-2}\left(1 - \dfrac{r-k}{2^{k+1}} + \zeta''\right),$ *with* $|\zeta''| < \dfrac{4r^2}{2^{2k+2}} < \dfrac{1}{2^{2k(1-c)}}.$

Since our equation involves the expression $(F_r^{(k)})^x$ for $r \in \{n-1, n, n+1\}$, we need the following lemma.

Lemma 6. *Let* $k \geq 2$, x, r *positive integers and* $i \in \{-1, 0, 1\}$. *Then*

(*i*) *For all* $n \geq 2$, *the estimate*

$$(F_{n+i}^{(k)})^x = f_k(\alpha)^x \alpha^{(n+i-1)x}(1+\eta_i),$$

holds with

$$|\eta_i| < \frac{xe^{x/\alpha^{n+i-1}}}{\alpha^{n+i-1}}.$$

(*ii*) *If* $n + i \geq k + 2$ *and* $\max\{n+i, x\} < 2^{ck}$ *for some* $c \in (0, 1/2)$, *then*

$$\left(F_{n+i}^{(k)}\right)^x = 2^{(n+i-2)x}\left(1 - \frac{x(n+i-k)}{2^{k+1}} + \xi_i\right),$$

holds with

$$|\xi_i| < \frac{24(nx)^2}{2^{2k+2}} < \frac{6}{2^{2k(1-2c)}}.$$

The proof of the previous lemma can be found in [10]. However, we pointed out that item (*i*) is a direct consequence of Lemma 4 parts (*iii*) and (*iv*). To establish item (*ii*) it is sufficient to use the item (*ii*) of Lemma 5.

Finally, note that, since $\Psi_k(z)$ is the minimal primitive polynomial of α, we have $\mathbb{Q}(\alpha) = \mathbb{Q}(f_k(\alpha))$. Thus, by Lemma 4 part (*iv*), we have that

$$h(\alpha) = (\log\alpha)/k \quad \text{and} \quad h(f_k(\alpha)) < 2\log k, \quad \text{for all} \quad k \geq 2. \tag{2}$$

See [5] for further details.

3.2. The Trivial Cases

Recall that we work with integral quadruples (k, n, m, x) with $k \geq 2$, $n \geq 1$, $m \geq 2$ and $x \geq 1$. Here we study some cases that provide *trivial solutions* of our Diophantine equation.

Theorem 2. *The trivial solutions* (k, n, m, x) *of Diophantine Eq. (1) are*

$$(2, n, 3n, 3) \quad \textit{for} \quad n \geq 2.$$

Also, for every $k \geq 2$,

$$(k, 1, 3, x) \quad \textit{for} \quad x \geq 1,$$

and

$$(k, 2, m, m-2) \quad \textit{whenever} \quad 2 \leq m \leq k+1.$$

Proof. By the work of Patel and Teh [18], when $k=2$ and $n \geq 2$ we have the parametric solutions $(2, n, 3n, 3)$.

Now, let us check some particular values of n:

- If $n=1$, then Eq. (1) corresponds to $2(1)^x = F_m^{(k)}$, which has the solutions $(k, 1, 3, x)$ for every $k \geq 2$ and $x \geq 1$.
- If $n=2$, then Eq. (1) corresponds to $2^x = F_m^{(k)}$. Thus, by Lemma 4 part (ii), we get $2 \leq m \leq k+1$ for all $k \geq 2$ and $x = m-2$. Note here we that we also have the quadruple $(2, 2, 3, 1)$ that does not belong to the parametric family given by Patel and Teh.

The previous analysis allows us to work with $k \geq 3$ and $n \geq 3$. Since $x \geq 1$, we have

$$(F_{n+1}^{(k)})^x + (F_n^{(k)})^x > (F_{n+1}^{(k)})^x + F_3^{(k)} > (F_{n-1}^{(k)})^x + F_3^{(k)},$$

which implies that $m \geq 4$.

Now, note that by Lemma 4 part (v), we have that

$$\alpha^{m-2} \leq F_m^{(k)} = (F_{n+1}^{(k)})^x + (F_n^{(k)})^x - (F_{n-1}^{(k)})^x < 2(F_{n+1}^{(k)})^x \leq \alpha^{nx+2},$$

and

$$\alpha^{(n-1)x-1} < \alpha^{(n-1)x} - \alpha^{(n-2)x} \leq (F_{n+1}^{(k)})^x + (F_n^{(k)})^x - (F_{n-1}^{(k)})^x = F_m^{(k)} \leq \alpha^{m-1}.$$

Thus, we can conclude that

$$(n-1)x < m < nx + 4. \tag{3}$$

So, if we take $x=1$, then the previous inequality implies that

$$m \in \{n, n+1, n+2, n+3\}.$$

It is a straightforward verification argument to show that none of these options provide solutions for the Diophantine Eq. (1). □

Therefore, now our problem is reduced to finding the quadruples (k, n, m, x) that solve the Diophantine Eq. (1) with $k \geq 3$, $n \geq 3$, $m \geq 4$ and $x \geq 2$.

4. The Case $n \leq k$

By (ii) in Lemma 4, the Diophantine Eq. (1) is equivalent to

$$F_m^{(k)} = 2^{(n-1)x} + 2^{(n-2)x} - 2^{(n-3)x}, \tag{4}$$

which implies that

$$F_m^{(k)} = 2^{(n-3)x}\left(2^{2x} + 2^x - 1\right),$$

but $n \geq 3$ and $x \geq 2$, so $F_m^{(k)}$ cannot be a power of 2. Thus, by item (ii) from Lemma 4, we have that $m \geq k+2$.

Now, using Lemma 4 parts (ii) and (v) together with Eq. (4), we get

$$2^{(n-1)x-0.2} < 2^{(n-1)x} - 2^{(n-3)x} < F_m^{(k)} < 2^{m-2}$$

and

$$2^{(m-2)/2} < \alpha^{m-2} < F_m^{(k)} < 2^{(n-1)x+1},$$

which give us

$$(n-1)x + 1.8 \leq m \leq 2(n-1)x + 4. \tag{5}$$

Next, we look for some additional relations between the variables in the Diophantine Eq. (4).

Lemma 7. *Let* $3 \leq n \leq k$. *If* (k, n, m, x) *is a solution of (4) with* $x \geq 2$, *then*

$$x < 6.4 \times 10^{11} k^4 (\log k)^2 \log m. \tag{6}$$

and

$$m < 1.8 \times 10^{30} k^9 (\log k)^6. \tag{7}$$

Proof. The Binet formula in Lemma 4 part (*iii*) allow us to rewrite Eq. (4) like

$$f_k(\alpha)\alpha^{m-1}2^{-(n-1)x} - 1 = \frac{1}{2^x} - \frac{1}{2^{2x}} - \frac{e_k(n)}{2^{(n-1)x}},$$

which implies that

$$|\Lambda_1| := \left| f_k(\alpha)\alpha^{m-1}2^{-(n-1)x} - 1 \right| < \frac{3}{2^x}. \tag{8}$$

Let us set the conditions to apply Theorem 1 on inequality (8). First, note that $\Lambda_1 = 0$ implies $f_k(\alpha) = \alpha^{1-m}2^{(n-1)x}$, which, due to the fact that α is a unit in $\mathcal{O}_{\mathbb{K}}$, leads to $f_k(\alpha)$ being an algebraic integer, a contradiction with item (*iv*) from Lemma 4. Thus, $\Lambda_1 \neq 0$. So, let us take $t := 3$, $\mathbb{K} := \mathbb{Q}(\alpha)$, $D := k$, $B := m$ and

$$\begin{aligned} &(\gamma_1, b_1) := (f_k(\alpha), 1), \ (\gamma_2, b_2) := (\alpha, m-1), \ (\gamma_3, b_3) := (2, -(n-1)x); \\ &\quad A_1 := 2k \log k, \qquad A_2 := 0.7, \qquad A_3 := 0.7k, \end{aligned}$$

where we have used (2) to calculate upper bounds for the A_i's. Thanks to Matveev's result, we have

$$\exp\left(-4.4 \times 10^{11} \times k^4 (\log k)^2 \log m\right) < |\Lambda_1| < 2^{-(x-2)},$$

which implies that

$$x < 6.4 \times 10^{11} k^4 (\log k)^2 \log m,$$

as we wanted to prove.

Returning to Eq. (4), we have

$$f_k(\alpha)\alpha^{m-1} + e_k(m) - 2^{(n-2)x} = 2^{(n-3)x}(2^{2x} - 1).$$

Thus, we get

$$(f_k(\alpha))^{-1}\alpha^{-(m-1)}2^{(n-3)x}(2^{2x} - 1) - 1 = \frac{e_k(m)}{f_k(\alpha)\alpha^{m-1}} - \frac{2^{(n-2)x}}{f_k(\alpha)\alpha^{m-1}}.$$

By Lemma 4 items (iii) and (iv) and inequality (3), we have that

$$|\Lambda_2| := \left|(f_k(\alpha))^{-1}\alpha^{-(m-1)}2^{(n-3)x}(2^{2x}-1)-1\right| < \frac{2}{\alpha^m}. \tag{9}$$

Now, if $\Lambda_2 = 0$, then $f_k(\alpha) = \alpha^{1-m}2^{(n-3)x}(2^{2x}-1)$, which, analogous to the reasoning invoked when studying $\Lambda_1 = 0$, implies that $f_k(\alpha)$ is an algebraic integer, which is not the case. Thus, we have that $\Lambda_2 \neq 0$. Let us take $t := 4$ and $\mathbb{K}$, D and B as before and apply Theorem 1 with

$$(\gamma_1, b_1, A_1) := (f_k(\alpha), -1, 2k\log k),\ \ (\gamma_2, b_2, A_2) := (\alpha, -(m-1), 0.7),$$
$$(\gamma_3, b_3, A_3) := (2, (n-3)x, 0.7k),\ \ \ (\gamma_4, b_4, A_4) := (2^{2x}-1, 1, 1.4kx),$$

where we have used the properties of the logarithmic height and (2) to calculate the A_i's. Thus, Theorem 1 together with inequality (9) yield

$$\exp(-1.6\times 10^{13}k^2(1+\log k)(1+\log m)(2k\log k)(0.7)(0.7k)(1.4kx)) < \frac{2}{\alpha^m},$$

which implies

$$\begin{aligned} m &< 1.5\times 10^{14}k^4(kx)(\log k)^2(\log m)\\ &< 9.6\times 10^{25}k^9(\log k)^4(\log m)^2, \end{aligned}$$

where we have used inequality (6). Hence, by Lemma 1 with $(y, r) := (m, 2)$ and $T := 9.6\times 10^{25}k^9(\log k)^4$, we have

$$m < 1.8\times 10^{30}k^9(\log k)^6,$$

which corresponds to inequality (7). □

Let us assume that $k > 600$. Therefore, we have that

$$m < 1.8\times 10^{30}k^9(\log k)^6 < 2^{k/2},$$

by equality (4) and item (i) from Lemma 5, with $c = 1/2$ and $r = m$,

$$|2^{(n-1)x} + 2^{(n-2)x} - 2^{(n-3)x} - 2^{m-2}| < \frac{2^{m-1}}{2^{k/2}}.$$

Let us take $M := \max\{(n-1)x, m-2\}$ and $N := \min\{(n-1)x, m-2\}$. Dividing by 2^M both sides of the previous inequality, we get

$$\min\left\{\frac{1}{4}, \frac{1}{2^{x+1}}\right\} < \frac{1}{2^M}|2^{(n-1)x} + 2^{(n-2)x} - 2^{(n-3)x} - 2^{m-2}| < \frac{2}{2^{k/2}}.$$

If $M > N$, then the left–hand side is at least $1/4$. We get $2^{k/2} < 8$, which is a contradiction since $k > 600$. Thus, we only need to look at the instance $M = N$, or, equivalently, $(n-1)x = m-2$. By item (ii) from Lemma 5, with $c = 1/2$ and $r = m$, we get

$$\frac{m-k}{2^{k+1}} < \left|\frac{1}{2^x} - \frac{1}{2^{2x}} + \frac{m-k}{2^{k+1}}\right| < \frac{4m^2}{2^{2k+2}}.$$

Since, we have $m \geq k+2$, the left–hand side of the previous inequality is at least $1/2^k$, thus we get that

$$2^{k/2} < 2m < 3.6 \times 10^{30} k^9 (\log k)^6.$$

However, this implies $k \leq 388$, a contradiction with our assumption.

From now on we work with $k \leq 600$. Let us start by looking for an upper bound on x. So, we take

$$\Gamma_1 := (m-1)\log\alpha - (n-1)x\log 2 + \log(f_k(\alpha)).$$

Since $x \geq 2$, by inequality (8) we have that

$$0 < |(m-1)\log\alpha - (n-1)x\log 2 + \log(f_k(\alpha))| < \frac{12}{2^x}.$$

Dividing both sides of the previous inequality by $\log 2$ and taking

$$\tau_k := (\log\alpha)/\log 2, \quad \mu_k := \log(f_k(\alpha))/\log 2, \quad A := 18, \quad B := 2,$$

we then get

$$0 < |(m-1)\tau_k - (n-1)x + \mu_k| < AB^{-x}. \tag{10}$$

For each $k \in [4, 600]$, we consider $M := 1.8 \times 10^{30} k^9 (\log k)^6$, which is an upper bound to $m-1$, according to inequality (7). A computer search shows that

$$\max_{k\in[4,600]} \left\{ \lfloor \log \left(AQ^{(k)}/\epsilon_k \right) / \log B \rfloor \right\} \leq 1189.$$

Hence, by Lemma 3, we can conclude that $x \leq 1189$.

Now that we have bounded x, let us fix it in $[2, 1189]$ and consider

$$\Gamma_2 := (m-1)\log\alpha - (n-3)x\log 2 - \log\left(\frac{2^{2x}-1}{f_k(\alpha)}\right).$$

Using inequality (9) in its logarithmic form, we obtain a similar inequality to (10), namely

$$0 < |(m-1)\tau_k - (n-3)x + \mu_{k,x}| < AB_k^{-m}, \tag{11}$$

where we have taken

$$\tau_k := \frac{\log\alpha}{\log 2}, \quad \mu_{k,x} := -\frac{\log((2^{2x}-1)/f_k(\alpha))}{\log 2}, \quad A := 6 \quad \text{and} \quad B_k := \alpha.$$

Therefore, for $k \in [4, 600]$ and $x \in [2, 1189]$, we apply Lemma 3 to inequality (11) using $M := 1.8 \times 10^{30} k^9 (\log k)^6$. With computational support, we obtain

$$\max_{k\in[4,600],\ x\in[2,1189]} \left\{ \lfloor \log \left(AQ^{(k,x)}/\epsilon_{k,x} \right) / \log B_k \rfloor \right\} \leq 1511.$$

Thus, by Lemma 3, we have that $m \leq 1511$.

In summary, for $n \leq k$, the integer solutions (k, n, m, x) of (4) must satisfy $k \in [4, 600]$, $x \in [2, 1189]$, $m \in [k+2, 1511]$ and, by (5)

$$n \in [4, N_0] \quad \text{with} \quad N_0 := \min\{k, 1 + \lfloor (m-1)/x \rfloor\}.$$

A computational search in the above range for the solutions of the Diophantine Eq. (4) gave us only those that we have indicated in the statement of the Main Theorem.

5. The Case $n > k$

As before, here we establish some relations between the variables in our Diophantine equation.

Lemma 8. *Let* (k, n, m, x) *be an integral solution of (1) with* $n > k \geq 3$ *and* $x \geq 2$*, then*

$$x < 1.5 \times 10^{16} n k^4 (\log k)^2 \log n. \tag{12}$$

Proof. By item (iii) from Lemma 4, Eq. (1) can be rewritten as

$$f_k(\alpha)\alpha^{m-1} - (F_{n+1}^{(k)})^x = (F_n^{(k)})^x - (F_{n-1}^{(k)})^x - e_k(m).$$

Dividing both sides by $(F_{n+1}^{(k)})^x$ and taking absolute values, we get

$$|\Lambda_3| := \left| f_k(\alpha)\alpha^{m-1}(F_{n+1}^{(k)})^{-x} - 1 \right| < 3\left(\frac{F_n^{(k)}}{F_{n+1}^{(k)}}\right)^x < \frac{3}{1.7^x}, \tag{13}$$

where we have used item (vi) from Lemma 4.

If $\Lambda_3 = 0$, then we get $f_k(\alpha) = \alpha^{1-m}(F_{n+1}^{(k)})^x$, which implies that $f_k(\alpha)$ is an algebraic integer, again, a contradiction with item (iv) from Lemma 4. Thus, we have $\Lambda_3 \neq 0$ and we can apply Theorem 1 with $t := 3$,

$$(\gamma_1, b_1, A_1) := (f_k(\alpha), 1, 2k\log k), \qquad (\gamma_2, b_2, A_2) := (\alpha, m-1, 0.7),$$
$$(\gamma_3, b_3, A_3) := (F_{n+1}^{(k)}, -x, 0.7nk)$$

and $\mathbb{K}, D, B$ as for Λ_1.

Now, Theorem 1 combined with inequality (13) yields

$$\begin{aligned} x &< 1.82 \times 10^{14} n k^4 (\log k)^2 \, \log m \\ &< 1.9 \times 10^{14} n k^4 (\log k)^2 \, \log(nx), \end{aligned} \tag{14}$$

where we used the fact that $m < nx + 4$, which follows from (3).

We next extract from (14) an upper bound for x depending on n and k. Multiplying by n both sides of the inequality (14) we obtain

$$nx < 1.9 \times 10^{14} n^2 k^4 (\log k)^2 \log(nx).$$

Taking $y := nx$ and $T := 1.9 \times 10^{14} n^2 k^4 (\log k)^2$, by Lemma 1 and the fact that $k < n$,

$$nx < 1.5 \times 10^{16} n^2 k^4 (\log k)^2 \log n.$$

It remains to divide by n both sides of the previous inequality. □

We now work under the assumption that $n > 700$ in order to find an upper bound for n, m and x in terms of k only.

Lemma 9. *Let (k, n, m, x) be an integral solution of (1) with $n > \max\{k, 700\}$. Then the following inequalities*

$$n < 2.2 \times 10^{26} k^6 (\log k)^6, \qquad x < 4.9 \times 10^{30} k^7 (\log k)^6,$$
$$m < 4 \times 10^{43} k^{10} (\log k)^9 \tag{15}$$

hold.

Proof. Given that $n > k$, from (12), we have that

$$x < 1.5 \times 10^{16} n^5 (\log n)^3. \tag{16}$$

Thus, for $i \in \{-1, 0, 1\}$,

$$\frac{x}{\alpha^{n+i-1}} < \frac{1.5 \times 10^{16} n^5 (\log n)^3}{\alpha^{n-2}} < \frac{1}{\alpha^{0.8n}}, \qquad \text{since } n > 700.$$

Then, by Lemma 6, we can write

$$\left(F_{n+i}^{(k)}\right)^x = f_k(\alpha)^x \alpha^{(n+i-1)x}(1 + \eta_n), \quad \text{with} \quad |\eta_n| < \frac{2}{\alpha^{0.8n}}. \tag{17}$$

We now use (17) to rewrite the Eq. (1) as

$$\left|f_k(\alpha)\alpha^{m-1} - f_k(\alpha)^x \alpha^{nx} \beta_x\right| < |\eta_n| f_k(\alpha)^x \alpha^{nx} \beta_x + \frac{1}{2}, \tag{18}$$

where $\beta_x := 1 + \alpha^{-x} - \alpha^{-2x}$. Dividing both sides of the previous inequality by $f_k(\alpha)^x \alpha^{nx}$, we conclude that

$$\begin{aligned} \left|f_k(\alpha)^{1-x} \alpha^{m-1-nx} - \beta_x\right| &< |\eta_n| \beta_x + \frac{1}{2 f_k(\alpha)^x \alpha^{nx}} \\ &< \frac{3|\eta_n|}{2} + \frac{1}{2}\left(\frac{1}{\alpha^{n-2}}\right)^x \\ &< \frac{4}{\alpha^{0.8n}}, \end{aligned}$$

where we have used the fact that $\beta_x < 1 + \alpha^{-x} < 3/2$, $f_k(\alpha)\alpha^n > \alpha^{n-2}$ and $(n-2)x + 1 \geq 0.8n$ for all $n > 700$, $k \geq 3$ and $x \geq 2$. Hence,

$$|\Lambda_4| := \left|f_k(\alpha)^{1-x} \alpha^{m-1-nx} - 1\right| < \frac{4}{\alpha^{0.8n}} + \frac{1}{\alpha^x} + \frac{1}{\alpha^{2x}} < \frac{6}{\alpha^\kappa}, \tag{19}$$

with $\kappa := \min\{0.8n, x\}$.

Now, if $\Lambda_4 = 0$, then we have $f_k(\alpha)^{x-1} = \alpha^{(m-1)-(n-2)x}$, which implies that $f_k(\alpha)$ is an algebraic integer, again, a contradiction. Thus, we have $\Lambda_4 \neq 0$ and we can apply Theorem 1 with the parameters $t := 2$,

$$(\gamma_1, b_1, A_1) := (f_k(\alpha), 1 - x, 2k \log k), \quad (\gamma_2, b_2, A_2) := (\alpha, m - 1 - nx, 0.7),$$

and $\mathbb{K}$ and D as before. Moreover, we can take $B := x$, since $|m - 1 - nx| \leq x$ by inequality (3).

The conclusion of Theorem 1 and the inequality (19) yield, after taking logarithms, the following upper bound for κ:

$$\kappa < 9.3 \times 10^9\, k^3 (\log k)^2 \log x. \tag{20}$$

Now, let us take $\kappa = 0.8n$, then from (20),

$$n < 1.2 \times 10^{10} k^3 (\log k)^2 \log x,$$

and using the inequality (16), we obtain that

$$\begin{aligned} n &< 1.2 \times 10^{10} k^3 (\log k)^2 \left(\log(1.5 \times 10^{16}) + 5 \log n + 3 \log\log n\right) \\ &< 1.2 \times 10^{10} k^3 (\log k)^2 (14 \log n) \\ &< 1.7 \times 10^{11} k^3 (\log k)^2 \log n, \end{aligned}$$

since $n > 700$. Hence, we apply Lemma 1 with $T := 1.7 \times 10^{11} k^3 (\log k)^2$ and $(y, r) := (n, 1)$ to obtain

$$n < 9.8 \times 10^{12} k^3 (\log k)^3,$$

an upper bound on n depending only on k. Further, inserting it in inequality (12) and using the inequality (3), we have that

$$\begin{aligned} n &< 8.1 \times 10^{12} k^3 (\log k)^3, \qquad x < 4.9 \times 10^{30} k^7 (\log k)^6, \\ m &< 4 \times 10^{43} k^{10} (\log k)^9. \end{aligned} \tag{21}$$

If $\kappa = x$, then by (20) and Lemma 1 with $T := 9.3 \times 10^9 k^3 (\log k)^2$ and $(y, r) := (x, 1)$, we get

$$x < 4.9 \times 10^{11} k^3 (\log k)^3. \tag{22}$$

Furthermore, since $x \leq 0.8n$, by Lemma 6, that for $i \in \{-1, 0, 1\}$,

$$x/\alpha^{n+i-1} < 0.8n/\alpha^{n-2} < 1/\alpha^{0.98n},$$

where we have used the fact that $n > 700$. Thus,

$$\left(F_{n+i}^{(k)}\right)^x = f_k(\alpha)^x \alpha^{(n+i-1)x} (1 + \eta_n), \quad \text{with} \quad |\eta_n| < \frac{1}{\alpha^{0.98n}}.$$

We return to the inequality (18) and dividing both sides by $f_k(\alpha)\alpha^{m-1}$, we obtain

$$\begin{aligned} |f_k(\alpha)^{x-1} \alpha^{nx-(m-1)} \beta_x - 1| &< |\eta_n| f_k(\alpha)^{x-1} \alpha^{nx-(m-1)} \beta_x + \frac{1}{2 f_k(\alpha) \alpha^{m-1}} \\ &< \frac{\alpha (f_k(\alpha)\alpha)^{x-1}}{\alpha^{0.98n}} \beta_x + \frac{1}{\alpha^{m-1}} \\ &< 3 \left(\frac{(3/2)^{0.8n}}{\alpha^{0.98n}} \right) + \frac{1}{\alpha^{0.38n}} \\ &< \frac{2}{\alpha^{0.38n}}, \end{aligned}$$

where we have used the inequalities:

$$x \leq 0.8n, \quad nx - (m-1) \leq x, \quad m - 1 > 0.38n,$$

$$\alpha < 2, \quad f_k(\alpha)\alpha < 3/2, \quad 3(3/2)^{0.8n}/\alpha^{0.98n} < 1/\alpha^{0.38n} \text{ and } \beta_x < 3/2,$$

valid for $n > 700$, $x \geq 2$ and $k \geq 3$. In conclusion, we have shown that

$$|\Lambda_5| := |f_k(\alpha)^{x-1}\alpha^{nx-(m-1)}(1 + \alpha^{-x} - \alpha^{-2x}) - 1| < \frac{2}{\alpha^{0.38n}}. \tag{23}$$

So, if $\Lambda_5 = 0$, we get $f_k(\alpha)^{x-1} = \alpha^{nx-(m-1)}(1 + \alpha^{-x} - \alpha^{-2x})$, which implies that $f_k(\alpha)$ is an algebraic integer or $x = 1$, a contradiction. Thus, we have $\Lambda_5 \neq 0$ and again we can apply Theorem 1 with the parameters $t := 3$,

$$(\gamma_1, b_1, A_1) := (f_k(\alpha), x - 1, 2k \log k), \quad (\gamma_2, b_2, A_2) := (\alpha, nx - (m - 1), 0.7),$$
$$(\gamma_3, b_3, A_3) := (1 + \alpha^{-x} - \alpha^{-2x}, 1, 3x)$$

and $\mathbb{K}$ and D as before. Moreover, again we can take $B := x$. Combining the conclusion of Theorem 1 with inequality (23), we get

$$n < 1.4 \times 10^{13} \times k^3(\log k)^2 x \log x. \tag{24}$$

By (22), we have $x < 4.9 \times 10^{11} k^3(\log k)^3$, therefore

$$\log x < \log(4.9 \times 10^{11}) + 3 \log k + 3 \log\log k < 31 \log k$$

since $k \geq 3$. Hence, returning to inequality (24) and taking into account that $m < nx + 2$, we have in summary

$$\begin{aligned} &n < 2.2 \times 10^{26} k^6(\log k)^6, \qquad x < 4.9 \times 10^{11} k^3(\log k)^3, \\ &m < 1.1 \times 10^{38} k^9(\log k)^9. \end{aligned} \tag{25}$$

Comparing inequalities (21) and (25), we get that

$$\begin{aligned} &n < 2.2 \times 10^{26} k^6(\log k)^6, \qquad x < 4.9 \times 10^{30} k^7(\log k)^6, \\ &m < 4 \times 10^{43} k^{10}(\log k)^9, \end{aligned}$$

as we wanted to show. □

The inequalities in Lemma 9 were obtained under the assumptions that $n > 700$. However, when $n \leq 700$ the inequalities (3) and (16) yield smaller upper bounds for x and m in terms of k.

From now on let us assume that $k > 700$. Thus, from (15), we have that

$$n + i < 2.2 \times 10^{26} k^6(\log k)^6 < 2^{0.24k}, \quad m < 4 \times 10^{43} k^{10}(\log k)^9 < 2^{0.48k},$$

for $i \in \{-1, 0, 1\}$. We recall that $n \geq k + 1$, so $m > k + 1$ according to (3). By item (ii) of Lemma 5 (for m with $c := 0.48$) and item (ii) Lemma 6 (for $n + i$ with $i \in \{-1, 0, 1\}$ and $c := 0.24$), we conclude that

$$F_m^{(k)} = 2^{m-2}\left(1 - \frac{m-k}{2^{k+1}} + \zeta''\right), \qquad |\zeta''| < \frac{1}{2^{1.04k}};$$
$$(F_{n+i}^{(k)})^x = 2^{(n+i-2)x}\left(1 - \delta_i \frac{x(n+i-k)}{2^{k+1}} + \delta_i \xi_i\right), \qquad |\xi_i| < \frac{6}{2^{1.04k}},$$

where $\delta_1 = 1$ for all $n \geq k+1$,

$$\delta_0 = \begin{cases} 1, & \text{for } n \geq k+2; \\ 0, & \text{for } n = k+1, \end{cases} \quad \text{and} \quad \delta_{-1} = \begin{cases} 1, & \text{for } n \geq k+3; \\ 0, & \text{for } n \in \{k+1, k+2\}. \end{cases}$$

Now, let us take again

$$M := \max\{(n-1)x, m-2\} \text{ and } N := \min\{(n-1)x, m-2\}.$$

We get

$$\begin{aligned} |2^{(n-1)x} + 2^{(n-2)x} - 2^{(n-3)x} - 2^{m-2}| &\leq 2^M \left(\frac{6(\delta_1 + \delta_0 + \delta_{-1}) + 1}{2^{1.04k}} \right. \\ &\quad \left. + \frac{\delta_1 + \delta_0 + \delta_{-1} + 1}{2^{0.52k+1}} \right) \\ &\leq 2^M \left(\frac{19}{2^{1.04k}} + \frac{2}{2^{0.52k}} \right) \\ &< \frac{2^{M+2}}{2^{0.52k}}. \end{aligned}$$

In the above, we used that $x(n+i-k) < x(n-1) < m < 2^{0.48k}$ for $i \in \{-1, 0, 1\}$, where the last inequality is due to the fact that $k > 700$. After dividing by 2^M and by a previous argument, we get

$$\min\left\{\frac{1}{4}, \frac{1}{2^{x+1}}\right\} < \frac{1}{2^M} \left| 2^{(n-1)x} + 2^{(n-2)x} - 2^{(n-3)x} - 2^{m-2} \right| < \frac{4}{2^{0.52k}}.$$

If $M > N$, then the left–hand side is at least $1/4$, so $2^{0.52k} < 16$, which is a contradiction since $k > 700$. Thus, we only need to consider the case $M = N$, or, equivalently, $(n-1)x = m-2$. We get

$$\left| \frac{x(n-k+1)}{2^{k+1}} - \frac{m-k}{2^{k+1}} + \frac{1}{2^{2x}} - \frac{1}{2^x} \right| < \frac{20}{2^{1.04k}},$$

or

$$\left| \frac{x(n-k+1) - (m-k)}{2^{k+1}} + \frac{1}{2^{2x}} - \frac{1}{2^x} \right| < \frac{20}{2^{1.04k}}.$$

But

$$\begin{aligned} x(n-k+1) - (m-k) &= x(n-k+1) - (m-2) + (k-2) \\ &= x(n-k+1-(n-1)) + k - 2 \\ &= (k-2)(1-x) < 0. \end{aligned}$$

Thus,

$$\left| \frac{x(n-k+1) - (m-k)}{2^{k+1}} - \left(\frac{1}{2^x} - \frac{1}{2^{2x}} \right) \right| > \frac{(k-2)(x-1)}{2^{k+1}} \geq \frac{1}{2^{k+1}},$$

and we get

$$\frac{1}{2^{k+1}} < \frac{20}{2^{1.04k}},$$

or $2^{0.04k} < 40$, again a contradiction with our assumption that $k > 700$. Therefore, we can conclude that $k \leq 700$.

Next we prove the following result which narrows down the computational ranges to look for possible solutions of Diophantine Eq. (1).

Lemma 10. *Let (k, n, m, x) be an integral solution of Diophantine Eq. (1) with $n > k \geq 3$, $k \leq 700$ and $x \geq 2$. Then $m \in [M_0, M_1]$ with*

$$M_0 := \lceil (n-1)x + 1.8 \rceil \quad \textit{and} \quad M_1 := 2(n-1)x + 4. \tag{26}$$

Furthermore, if $n > 700$, then $n \leq 1459$ and $x \leq 2733$, otherwise $x \leq 1822$.

Proof. The range for m is given by inequality (5). Now, let us assume that $n > 700$, so, we can use Lemma 9 to obtain upper bounds on n, x and m.

We assume that $x \leq 7$. Using inequality (23), we take

$$\Gamma_5 := (x-1)\log(f_k(\alpha)) + (nx - (m-1))\log\alpha + \log\left(1 - \alpha^{-2x}\right).$$

Since $|\Gamma_5| < 4/\alpha^{0.38n}$, dividing by $\log\alpha$, we obtain

$$\left|(x-1)\frac{\log(f_k(\alpha))}{\log\alpha} + \frac{\log\left(1-\alpha^{-2x}\right)}{\log\alpha} - (m-1-nx)\right| < \frac{7}{\alpha^{0.38n}}.$$

Now, let us take

$$\tau_{k,x} := (x-1)\frac{\log(f_k(\alpha))}{\log\alpha} + \frac{\log\left(1-\alpha^{-2x}\right)}{\log\alpha}.$$

Thus, we have that

$$\min_{k\in[3,700],\ x\in[2,7]} \|\tau_{k,x}\| < |\tau_{k,x} - (m-1-nx)| < \frac{7}{\alpha^{0.38n}}.$$

Computationally, we found that the minimum on the left–hand of the previous inequality is at least 0.1×10^{-4}, which implies that $n \leq 73$, a contradiction. Thus, we can work from now on with $x \geq 8$.

Due to inequality (19), we take

$$\Gamma_4 := (x-1)\log(f_k(\alpha)^{-1}) + (m-1-nx)\log\alpha.$$

By the analytical argument given in Sect. 2.2, we get

$$|\Gamma_4| < \frac{12}{\alpha^{\kappa}}, \quad \text{with} \quad \kappa := \min\{0.8n, x\},$$

where we have used the fact that $\kappa \geq 8$. Dividing both sides of the above inequality by $(x-1)\log\alpha$, we obtain

$$\left|\frac{\log(f_k(\alpha)^{-1})}{\log\alpha} - \frac{nx+1-m}{x-1}\right| < \frac{12}{\alpha^{\kappa}(x-1)\log\alpha} < \frac{25}{\alpha^{\kappa}(x-1)}, \tag{27}$$

and we need to proceed by cases:

- **Case** $m = 1 + nx$. Here, the inequality (27) corresponds to

$$\left|\frac{\log(f_k(\alpha)^{-1})}{\log \alpha}\right| < \frac{25}{\alpha^{\kappa}(x-1)}.$$

A quick computational search shows that the left–hand side of the previous inequality is greater than 0.7 for all $k \in [3, 700]$. Thus, since $x \geq 8$, we get

$$0.7 < \frac{4}{\alpha^{\kappa}}. \tag{28}$$

Now, if we take $\kappa = 0.8n$ or $\kappa = x$, then inequality (28) implies $n \leq 4$ or $x \leq 3$, respectively, which contradicts our assumptions that $n > 700$ and $x \geq 8$.

- **Case** $m \neq 1 + nx$. Here, we apply Lemma 2 to inequality (27) using $k \in [3, 700]$. In order to do it, let us take $\tau_k := \log(f_k(\alpha)^{-1})/\log \alpha$. By inequality (15), we look for the integer t_k such that

$$Q_{t_k}^{(k)} > 4.9 \times 10^{30} k^7 (\log k)^6 > x - 1,$$

and take $a_M := \max\{a_i^{(k)} : 0 \leq i \leq t_k,\ 3 \leq k \leq 700\}$. Then, by Lemma 2, we have that

$$\left|\tau_k - \frac{nx - (m-1)}{x-1}\right| > \frac{1}{(a_M + 2)(x-1)^2}. \tag{29}$$

Hence, combining the inequalities (27) and (29), and taking into account that $a_M + 2 < 1.1 \times 10^{208}$ (confirmed by computations), we obtain

$$\alpha^{\kappa} < 2.8 \times 10^{209} x.$$

If $\kappa = 0.8n$, since $n > 700 \geq k$, by inequality (16) we have

$$\alpha^{0.8n} < 4.2 \times 10^{225} n^5 (\log n)^3,$$

which implies

$$n \leq 1459. \tag{30}$$

Thus, let us consider

$$\Gamma_3 := \log f_k(\alpha) + (m-1)\log \alpha - x \log F_{n+1}^{(k)}, \tag{31}$$

with $k \in [3, 700]$ and $n \in [701, 1459]$. Note that, by inequality (13), we have

$$|\Gamma_3| < \frac{6}{1.7^x}.$$

Dividing both sides by $\log F_{n+1}^{(k)}$, we get

$$\left|(m-1)\left(\frac{\log \alpha}{\log F_{n+1}^{(k)}}\right) - x + \frac{\log f_k(\alpha)}{\log F_{n+1}^{(k)}}\right| < \frac{4}{1.7^x}, \tag{32}$$

where we used that $\log F_{n+1}^{(k)} \geq \log F_5^{(3)} = \log 7$. In order to apply Lemma 3, we take

$$\gamma_{k,n} := \frac{\log \alpha}{\log F_{n+1}^{(k)}}, \qquad \mu_{k,n} := \frac{\log f_k(\alpha)}{\log F_{n+1}^{(k)}} \qquad A := 4 \quad \text{and} \quad B := 1.7,$$

for $k \in [3, 700]$ and $n \in [701, 1459]$ with $M := 1.5 \times 10^{34} k^7 (\log k)^6$, thanks to inequalities (5) and (15). We obtain

$$\max_{k \in [3,700],\ n \in [701,1459]} \left\{ \left\lfloor \log \left(AQ^{(k,n)} / \epsilon_{k,n} \right) / \log B \right\rfloor \right\} \leq 2733,$$

which, by Lemma 3, implies

$$x \leq 2733. \tag{33}$$

Now, if $\kappa = x$, then $\alpha^x < 2.8 \times 10^{209} x$, which implies

$$x \leq 1016. \tag{34}$$

So, we go back to inequality (23) and take

$$\Gamma_5 := (x-1)\log(f_k(\alpha)) + (nx - (m-1))\log \alpha + \log\left(1 - \alpha^{-2x}\right).$$

Since $|\Gamma_5| < 4/\alpha^{0.38n}$, dividing by $\log \alpha$, we obtain

$$\left| (x-1)\frac{\log(f_k(\alpha))}{\log \alpha} + \frac{\log\left(1-\alpha^{-2x}\right)}{\log \alpha} - (m-1-nx) \right| < \frac{7}{\alpha^{0.38n}}.$$

Here, we take

$$\tau_{k,x} := (x-1)\frac{\log(f_k(\alpha))}{\log \alpha} + \frac{\log\left(1-\alpha^{-2x}\right)}{\log \alpha}.$$

So, we have that

$$\min_{k \in [3,700],\ x \in [8,1016]} \|\tau_{k,x}\| < |\tau_{k,x} - (m-1-nx)| < \frac{7}{\alpha^{0.38n}}.$$

Computationally, we found that the minimum on the left–hand of the previous inequality is at least 0.1×10^{-9}. Therefore, we get $n \leq 136$, a contradiction.

To sum up, by inequalities (30), (33) and (34), we have that the positive integral solutions (k, n, m, x) of Diophantine Eq. (1) with $n > k \geq 3$, $k \leq 700$, $n > 700$ and $x \geq 2$, satisfy $n \leq 1459$ and $x \leq 2733$.

Finally, we consider the case when $n \leq 700$ and, since we are working with $k \leq 700$ and $n > k$, it is clear that $k \leq 699$. Now, we use Γ_3 as we defined it in (31) to proceed as we did with (32). This time we take $k \in [3, 699]$ and

$n \in [k+1, 700]$ with $M := 6.9 \times 10^{33} k^7 (\log k)^6$, which is given by inequalities (5) and (15). We get

$$\max_{k\in[3,699],\ n\in[k+1,700]} \left\{ \lfloor \log \left(AQ^{(k,x)}/\epsilon_{k,x} \right) / \log B_k \rfloor \right\} \le 1822,$$

which, by Lemma 3, implies $x \le 1822$. □

In conclusion, our problem is now reduced to search for integral solutions of the Diophantine Eq. (1) in the ranges indicated by Lemma 10; i.e., $k \in [3, 700]$, $m \in [M_0, M_1]$ (given in (26)),

$$n \in [k+1, 700] \text{ and } x \in [2, 1822] \quad \text{or} \quad n \in [701, 1459] \text{ and } x \in [2, 2733].$$

A computational search allow us to conclude that there are no integral solutions for Eq. (1) in these ranges.

6. On the Final Verifications in (1)

In the verifications of Diophantine Eq. (1), we have the parameters

$n \le k$	$n > k$
$k \in [4, 600]$	$k \in [3, 700]$
$x \in [2, 1189],\ n \in [4, N_0]$	if $n \in [k+1, 700],\ x \in [2, 1822]$;
$N_0 = \min\{k, 1 + \lfloor (m-1)/x \rfloor\}$	if $n \in [701, 1459],\ x \in [2, 2733]$
$m \in [k+2, 1511]$	$m \in [M_0, M_1]$
	$M_0 = \lceil (n-1)x + 1.8 \rceil$;
	$M_1 = 2(n-1)x + 4$

For this task, it is necessary to calculate powers of the form $(F_r^{(k)})^x$, which can lead to cause computational or storage problems. Thus, to speed up our calculations we have considered the following strategy:

(i) Compare the last 30 digits in equality (1), this is, we consider

$$\left(F_{n+1}^{(k)}\right)^x + \left(F_n^{(k)}\right)^x - \left(F_{n-1}^{(k)}\right)^x \equiv F_m^{(k)} \pmod{10^{30}}. \tag{35}$$

(ii) Generate the numbers $F_r^{(k)}$ iterating blocks of k terms, using the identity $F_r^{(k)} = 2F_{r-1}^{(k)} - F_{r-k}^{(k)}$, taking $\pmod{10^{30}}$ at each iteration.

(iii) Create the lists, $L_k := \{(F_{n+1}^{(k)})^x + (F_n^{(k)})^x - (F_{n-1}^{(k)})^x \pmod{10^{30}}\}$ and $R := \{F_m^{(k)} \pmod{10^{30}}\}$. To obtain L_k, it is convenient to compute the list $((F_r^{(k)})^x)$, and suppress a term at the beginning and at the end according to the need for $r = n+1, n$ or $n-1$, and give it a vectorial treatment.

This computation was done with the *Mathematica* software at Computer Center Jurgen Tischer in the Department of Mathematics at the Universidad del Valle on 24 parallel Pc's (Intel Xeon E3-1240 v5, 3.5 GHz, 16 Gb of RAM), using in turn parallelized algorithms. A total calculation time of 6 h revealed that $L_k \cap R_k = \emptyset$ for all k in each case.

This finishes the proof of the main theorem.

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