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Results in Mathematics

Uniformity and Loewner Condtions of Metric Spaces

Xiantao Wang and Zhiqiang Yang

Abstract. In this paper, we establish four equivalent conditions in the metric space setting. These conditions concern the (inner) uniformity of metric spaces, and (locally) Loewner conditions, weak slice conditions and k-cap conditions of metric measure spaces. This investigation completes the related study started by Bonk, Heinonen, and Koskela in 2001. Also, two examples are constructed to demonstrate that the assumption of the regularity of the underlying spaces can not be removed.

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1. Introduction

A metric space (X, d) is called *uniform* if every pair of points in X can be joined by a so called uniform curve (see Sect. 2.1 below for the precise definition). If a domain, i.e., an open and connected subset, in X satisfies this condition, then it is called a *uniform* domain. The concept of uniformity in the metric space setting was introduced by Bonk et al. [2]. Uniform domains in $\mathbb{R}^n$ with $n \geq 2$ were introduced by Martio and Sarvas [19]. Since its introduction, this concept has played a significant role in the study of geometric function theory, see [16] and references therein.

In order to study the theory of quasiconformal mappings in metric spaces, Heinonen and Koskela introduced a class of metric spaces which is called Loewner spaces [12]. A metric measure space is called *Loewner* if it satisfies a Loewner condition (see Definition 2.2 below for the details). All Euclidean spaces $\mathbb{R}^n$ with $n \geq 2$ are Loewner. See [4,12,18] for more examples of Loewner spaces.

In 2001, Bonk, Heinonen, and Koskela considered the relation between uniformity and Loewner condition, and established the following sufficient condition for a metric measure space to be Loewner.

Theorem A [2, Theorem 6.4]. *Suppose that (X, d, μ) is a locally compact and noncomplete metric measure space. If (X, d, μ) is a uniform and locally Q-Loewner space with $Q > 1$, then it is Q-Loewner. This implication is quantitative.*

See Chapters 6 and 7 in [2] for the significance and the importance of Theorem A. Throughout the paper, for a metric measure space (X, d, μ), we assume that the measure μ is locally finite and Borel regular with dense support. The terminology in Theorem A and in the rest of this section will be introduced in the second section unless otherwise stated.

Here and hereafter, if a condition ϖ with data χ implies a condition ϖ' with data χ' so that χ' depends only on χ and other given quantities, then we say that ϖ implies ϖ' quantitatively, or the implication is quantitative. If also ϖ' implies ϖ quantitatively, then we say that ϖ and ϖ' are quantitatively equivalent, or the equivalence is quantitative.

In [22], Yang constructed an L^1_p domain D in $\mathbb{R}^n$ for all $p \geq 1$, which is not uniform; see [22, Theorem 1.1]. Moreover, it is known that every L^1_n domain is QED (the abbreviated form of the phrase "quasiextremal distance") (cf. [17]), and every QED domain is Loewner (cf. [3]). These facts guarantee that the domain D constructed in [22] is Loewner. Since D is not uniform, we conclude that the converse of Theorem A is invalid. Naturally, one will ask if there is any condition which, by combining with the Loewner condition, constitutes a necessary and sufficient condition for a metric measure space to be uniform and locally Loewner. The purpose of this paper is to study this question, and the following are our answers.

Theorem 1.1. *Suppose that (X, d, μ) is Q-regular with $Q > 1$, and $G \subseteq X$ is a locally compact and c-quasiconvex domain with $c \geq 1$, where the case $G = X$ is included when (X, d) is noncomplete. Then the following are quantitatively equivalent:*

($\mathfrak{a}$) *G is uniform and locally Q-Loewner;*
($\mathfrak{b}$) *G is Q-Loewner and inner uniform;*
($\mathfrak{c}$) *G is Q-Loewner and satisfies a weak slice condition;*
($\mathfrak{d}$) *G is Q-Loewner and satisfies a k-cap condition.*

We shall construct two examples, i.e., Examples 5.1 and 5.2 below, to demonstrate that the assumption on the regularity of the underlying spaces in Theorem 1.1 cannot be weakened to the one of being doubling, and thus, this implies that such an assumption cannot be removed from Theorem 1.1.

The rest of this paper is arranged as follows. In Sect. 2, necessary preliminaries will be introduced and two lemmas will be proved. Several auxiliary lemmas will be shown in Sect. 3. Section 4 will be devoted to the proof of Theorem 1.1, and in Sect. 5, the aforementioned examples will be exhibited.

2. Preliminaries

Let (X, d) denote a metric space, $\overline{X}$ the metric completion of X, and $\partial X = \overline{X} \backslash X$, the metric boundary of X. For a metric space (X, d), we write $|x-y| = d(x, y)$ for the *distance* between two pints x and y in X. The symbol $\mathbb{B}(x, r)$ denotes the *open ball* in X with center x and radius $r > 0$, i.e., $\mathbb{B}(x, r) = \{y \in X : \ |y-x| < r\}$. For $\lambda > 0$, define $\lambda\mathbb{B}(x, r) = \mathbb{B}(x, \lambda r)$. When working in a given domain $G \subsetneq X$, for any $x \in G$, let $\lambda\mathbb{B}(x) = \lambda\mathbb{B}(x, d_G(x))$, where $d_G(x)$ denotes the distance of x to the boundary ∂G of G. Observe that every ball $\lambda\mathbb{B}(x)$ in $\mathbb{R}^n$ is contained in G whenever $x \in G$ and $\lambda \in (0, 1)$. But in the metric space setting, this property is no longer valid. To overcome this deficiency, we introduce a new notation $\lambda\mathbb{B}_G(x) = \lambda\mathbb{B}(x) \cap G$, which is called the *Whitney ball* at x with parameter $\lambda \in (0, 1)$. Notice that not every Whitney ball is connected. Also, we use $\lambda\overline{\mathbb{B}}_G(x)$ to denote the closure of $\lambda\mathbb{B}_G(x)$.

2.1. Uniform Metric Spaces and Quasi-hyperbolic Metrics

A *curve* in (X, d) is a continuous function $\gamma : \ I \to X$ from an interval $I \subset \mathbb{R}$ to X. We use γ to denote both the function and its image set. The *length* $\ell(\gamma)$ of γ with respect to the metric d is defined in the usual way. The parameter interval I is allowed to be closed, open or half-open. If $\ell(\gamma) < \infty$, then γ is said to be *rectifiable*. Obviously, when I is compact, γ is a continuum (i.e., a connected and compact set) in X since both the connectedness and the compactness are invariant under continuous functions.

A metric space (X, d) is called *geodesic* if every pair of points x and $y \in X$ can be joined by a curve γ such that

$$\ell(\gamma) = |x-y|,$$

and the curve γ is also called a *geodesic* in X joining x and y.

A metric space (X, d) is called *a-uniform* if there is a constant $a \ge 1$ such that each pair of points x and y in X can be joined by a rectifiable curve γ in X satisfying

(1) $\min\{\ell(\gamma[x, z]), \ell(\gamma[z, y])\} \le a\, d_X(z)$ for all $z \in \gamma$, and
(2) $\ell(\gamma) \le a\,|x-y|$.

If the metric d is replaced by its inner metric σ_X, then uniformity is called *inner uniformity*, where the inner metric σ_X is defined by

$$\sigma_X(x, y) = \inf_{\gamma \in \Gamma(x, y : X)} \big\{\ell(\gamma)\big\}$$

for any x and y in X. Here and in what follows, $\Gamma(x, y : X)$ denotes the family of all rectifiable curves in X connecting x and y.

When our uniform space (resp. our inner uniform space) is a domain of a metric space, we call it a *uniform domain* (resp. an *inner uniform domain*).

Assume that (X, d) is a rectifiably connected, non-complete and locally compact metric space. The *quasi-hyperbolic metric* k_X in X (with respect to d) is defined by

$$k_X(x,y) = \inf\Big\{ \int_\gamma \frac{1}{d_X(z)}\,|dz| \Big\},$$

where the infimum is taken over all curves in $\Gamma(x, y : X)$, $d_X(z)$ denotes the distance from z to ∂X, and $|dz|$ denotes the arc-length element with respect to d. The quantity $\int_\gamma \frac{1}{d_X(z)}\,|dz| = \ell_{k,X}(\gamma)$ is called the *quasi-hyperbolic length* of γ. A curve in X connecting x and y is called a *quasihyperbolic geodesic* if $\ell_{k,X}(\gamma) = k_X(x, y)$.

For x and $y \in X$, their *relative distance* is defined by the number:

$$r_X(x,y) = \frac{|x-y|}{\min\{d_X(x), d_X(y)\}}.$$

The following lemma is useful.

Lemma 2.1. *Suppose that (X, d) is a noncomplete and c-quasiconvex metric space, and let $\tau > 0$ be a constant. For any pair of points x, y in X,*

1 *if $|x - y| \le \tau \max\{d_X(x), d_X(y)\}/c(\tau + 1)$, then*

$$k_X(x,y) \le c(\tau+1)\frac{|x-y|}{\max\{d_X(x), d_X(y)\}} \le c(\tau+1) r_X(x,y);$$

2 *if $k_X(x, y) \le \tau$, then*

$$k_X(x,y) \le c(\tau+1) r_X(x,y);$$

3 *if $k_X(x, y) \ge \tau$, then*

$$|x-y| \ge \frac{\tau}{c(\tau+1)} \max\{d_X(x), d_X(y)\};$$

Proof. (1) Assume that $|x - y| \le \tau \max\{d_X(x), d_X(y)\}/c(\tau + 1)$. Without loss of generality, assume that $\max\{d_X(x), d_X(y)\} = d_X(x)$. Since X is c-quasiconvex, we know that there is a curve γ_{xy} in X connecting x and y such that

$$\ell(\gamma_{xy}) \le c|x-y|,$$

and so,

$$\ell(\gamma_{xy}) \le \frac{\tau}{\tau+1} d_X(x).$$

Since for every $z \in \gamma_{xy}$,

$$d_X(z) \ge d_X(x) - \ell(\gamma_{xy}) \ge \frac{1}{\tau+1} d_X(x),$$

and since

$$k_X(x,y) \le \int_{\gamma_{xy}} \frac{|dz|}{d_X(z)},$$

we get

$$k_X(x,y) \le c(\tau+1)\frac{|x-y|}{d_X(x)} \le c(\tau+1)r_X(x,y).$$

(2) Assume that $k_X(x,y) \le \tau$. By the statement (1) of the lemma, we may assume that $|x-y| \ge \tau \max\{d_X(x), d_X(y)\}/c(\tau+1)$. Then we deduce that

$$k_X(x,y) \le \tau \le c(\tau+1)\frac{|x-y|}{\max\{d_X(x), d_X(y)\}} \le c(\tau+1)r_X(x,y).$$

(3) Assume that $k_X(x,y) \ge \tau$. Without loss of generality, we assume that $\max\{d_X(x), d_X(y)\} = d_X(x)$. Suppose on the contrary that

$$|x-y| < \tau d_X(x)/c(\tau+1).$$

Then by the statement (1) of the lemma, we get

$$k_X(x,y) \le c(\tau+1)\frac{|x-y|}{d_X(x)} < \tau.$$

This obvious contradiction proves this statement. □

For two nondegenerate and bounded sets E and F in X, let

$$\Delta(E,F) = \frac{\text{dist}(E,F)}{\min\{\text{diam}E, \text{diam}F\}},$$

which is called the *relative distance* between E and F.

Based on Lemma 2.1, we have the following sufficient condition for two Whitney balls to be disjoint.

Lemma 2.2. *Suppose that X is a metric space and $G \subset X$ is a c-quasiconvex domain. Let $\alpha > 1/2$ and $0 < \lambda \le 1/4(\alpha+2)c$. For x, $y \in G$, if $k_G(x,y) \ge \alpha$, then*

$$\Delta(\lambda\overline{\mathbb{B}}_G(x), \lambda\overline{\mathbb{B}}_G(y)) \ge 2\alpha - 1 \quad \textit{and} \quad \lambda\overline{\mathbb{B}}_G(x) \cap \lambda\overline{\mathbb{B}}_G(y) = \emptyset.$$

Proof. Let x and y be two points in G with $k_G(x,y) \ge \alpha$. Then it follows from Lemma 2.1(3) that

$$|x-y| > \frac{\alpha}{c(\alpha+2)} \max\{d_G(x), d_G(y)\}.$$

Obviously, $\lambda\overline{\mathbb{B}}_G(x) \subset \lambda\overline{\mathbb{B}}(x)$ and $\lambda\overline{\mathbb{B}}_G(y) \subset \lambda\overline{\mathbb{B}}(y)$, and thus, we have

$$\Delta(\lambda\overline{\mathbb{B}}_G(x), \lambda\overline{\mathbb{B}}_G(y)) \ge \Delta(\lambda\overline{\mathbb{B}}(x), \lambda\overline{\mathbb{B}}(y)).$$

Since

$$\text{dist}(\lambda\overline{\mathbb{B}}(x), \lambda\overline{\mathbb{B}}(y)) \ge |x-y| - \lambda d_G(x) - \lambda d_G(y)$$

and

$$\mathrm{diam}(\lambda\overline{\mathbb{B}}(x)) \leq 2\lambda d_G(x),$$

we know that

$$\Delta(\lambda\overline{\mathbb{B}}_G(x), \lambda\overline{\mathbb{B}}_G(y)) \geq 2\alpha - 1 > 0,$$

since $\alpha > 1/2$. Also, this implies that

$$\lambda\overline{\mathbb{B}}_G(x) \cap \lambda\overline{\mathbb{B}}_G(y) = \emptyset.$$

The proof of the lemma is complete. □

Remark 2.1. For any pair of points x, $y \in G \subsetneq \mathbb{R}^n$, if $k_G(x,y) \geq \alpha > 2\log 2$, then

$$\mathbb{B}_{k_G}\left(x, \frac{\alpha}{2}\right) \cap \mathbb{B}_{k_G}\left(y, \frac{\alpha}{2}\right) = \emptyset,$$

where $\mathbb{B}_{k_G}\left(x, \frac{\alpha}{2}\right) = \{z \in G : \ k_G(z,x) < \frac{\alpha}{2}\}$, the quasihyperbolic ball with center x and radius $\frac{\alpha}{2}$.

Since for any $z \in G$, $\left(1 - e^{-\frac{\alpha}{2}}\right)\mathbb{B}(z) \subset \mathbb{B}_{k_G}\left(z, \frac{\alpha}{2}\right)$ (cf. [10, (5.15)]), we get

$$\left(1 - e^{-\frac{\alpha}{2}}\right)\mathbb{B}(x) \cap \left(1 - e^{-\frac{\alpha}{2}}\right)\mathbb{B}(y) = \emptyset.$$

Obviously,

$$\frac{1}{2}\overline{\mathbb{B}}(x) \cap \frac{1}{2}\overline{\mathbb{B}}(y) = \emptyset,$$

since $1 - e^{-\frac{\alpha}{2}} > \frac{1}{2}$.

A minimally nice (i.e., rectifiably connected, non-complete and locally complete) metric space (X,d) is called *quasihyperbolic φ-uniform* (briefly, ***QH*** *φ-uniform*) if there is a self-homeomorphism φ of $[0,+\infty)$ such that for any pair of points x and $y \in X$,

$$k_X(x,y) \leq \varphi(r_X(x,y)).$$

When the QH φ-uniform space is a domain of a metric space, it is called a ***QH*** *φ-uniform domain*.

We remark that local compactness implies local completeness. Also, it is known that uniformity is quantitatively equivalent to QH φ-uniformity with φ being slow, i.e., $\limsup_{t\to\infty} \varphi(t)/t < 1$, in Banach spaces; see [21, Theorem 6.16]. For general metric spaces, Buckley and Herron established the following generalization.

Theorem 2 [7, Theorem 3.1]. *A minimally nice and locally (c,λ)-quasiconvex space (X,d) is uniform if and only if it is* QH *φ-uniform with a slow φ. The uniformity constant depends only on c and φ, and conversely in an a-uniform space, one can always take $\varphi(t) = 4a^2\log(1+t)$.*

2.2. Modulus and Capacity

Let (X, d, μ) denote a metric measure space, and let Γ (sometimes with a subscript) be a family of curves in X. For a number $1 \leq p < \infty$, the p-modulus of Γ is defined as

$$\text{mod}_p\Gamma = \inf \Big\{ \int_X \varrho^p d\mu \Big\},$$

where the infimum is taken over all nonnegative Borel functions $\varrho : X \to [0, \infty]$ satisfying

$$\int_\gamma \varrho ds \geq 1 \tag{2.1}$$

for all locally rectifiable curves $\gamma \in \Gamma$. Functions ϱ satisfying (2.1) are called *admissible densities* for Γ.

The moduli of families of curves possess the following elementary and useful properties.

Theorem 3 [12, §2.3]. *For $1 \leq p < \infty$, the following statements are true:*

1 *If $\Gamma = \emptyset$, the empty set, then $\text{mod}_p(\Gamma) = 0$;*
2 *For two families of curves Γ_1 and Γ_2 in X, if $\Gamma_1 \subset \Gamma_2$, then*

$$\text{mod}_p\Gamma_1 \leq \text{mod}_p\Gamma_2;$$

3 *For two families of curves Γ_3 and Γ_4 in X, if each curve $\gamma_4 \in \Gamma_4$ has a subcurve γ_3 which belongs to Γ_3, then*

$$\text{mod}_p\Gamma_4 \leq \text{mod}_p\Gamma_3;$$

4 *For a sequence of families of curves $\{\Gamma_i\}_{i=1}^{+\infty}$ in X,*

$$\text{mod}_p \left(\bigcup_{i=1}^{+\infty} \Gamma_i \right) \leq \sum_{i=1}^{+\infty} \text{mod}_p\Gamma_i.$$

Let G be an open set in X, and let u be an arbitrary real-valued function in G. We say that a Borel function $\rho : G \to [0, \infty]$ is a *very weak gradient* of u in G if for every pair of points x and $y \in G$,

$$|u(x) - u(y)| \leq \int_\gamma \rho ds$$

whenever γ belongs to $\Gamma(x, y : G)$, the set of all rectifiable curves in G connecting x and y.

For an open subset G of X, and its two disjoint closed subsets E and F, let the triple $(E, F; G)$, which is called a *condenser*, denote the family of all curves in G joining E and F. With the aid of very weak gradients, Heinonen and Koskela gave the following definition of capacity in the metric space setting.

Definition 2.1. For a condenser $(E, F; G)$ of X, its *p-capacity* for $1 \leq p < \infty$ is defined as

$$\mathrm{cap}_p(E, F; G) = \inf \Big\{ \int_G \rho^p d\mu \Big\},$$

where the infimum is taken over all very weak gradients ρ of each admissible function u for the condenser $(E, F; G)$. Here, a function u in G is called *admissible* for a condenser $(E, F; G)$ if the restriction $u|_E \geq 1$ and the restriction $u|_F \leq 0$.

Heinonen and Koskela showed that for all $1 \leq p < \infty$,

$$\mathrm{cap}_p(E, F; G) = \mathrm{mod}_p(E, F; G)$$

(see [12, Proposition 2.17]). Thus, in the following, we use the notation $\mathrm{cap}_p(E, F; G)$ instead of $\mathrm{mod}_p(E, F; G)$.

2.3. Capacity Conditions and Regularity

In order to generalize the theory of quasiconformal mappings to the metric space setting, in [12], Heinonen and Koskela introduced the concept of a Loewner space. The definition is as follows.

Definition 2.2. Suppose that (X, d, μ) is a rectifiably connected metric measure space. It is called

(1) a *Q-Loewner space* if there is a function $\phi : (0, +\infty) \to (0, +\infty)$ so that

$$\mathrm{cap}_Q(E, F; X) \geq \phi(t) \tag{2.2}$$

whenever E and F are two disjoint and nondegenerate continua in X, and

$$t \geq \Delta(E, F),$$

where ϕ is called a *Loenwer control function* of X. Moreover, if a domain $G \subset X$ satisfies the Q-Loewner condition (2.2), then we say that G is a *Q-Loewner domain.*

(2) a *locally Q-Loewner space* if X is locally compact and there exist numbers $\kappa \geq 1$, $\varepsilon_0 \in (0, \kappa^{-1}]$, and a function $\phi : (0, +\infty) \to (0, +\infty)$ so that

$$\mathrm{cap}_Q(E, F; \mathbb{B}(x, \varepsilon\kappa d_G(x))) \geq \phi(t) \tag{2.3}$$

whenever $x \in X$, $0 < \varepsilon < \varepsilon_0$, E and $F \subset \mathbb{B}(x, \varepsilon d_G(x))$ are two disjoint and nondegenerate continua, and $t \geq \Delta(E, F)$.

By [12, Theorem 3.13], every Q-regular Loewner space (see Definition 2.3 for the definition of Q-regularity) satisfies the following properties.

Quasiconvexity A metric space (X, d) is said to be *c-quasiconvex* if there is a constant $c \geq 1$ such that for each pair of points x and $y \in X$, there exists a curve γ joining x and y satisfying

$$\ell(\gamma) \leq c|x - y|.$$

Such a curve is also called a *c-quasiconvex* curve.

Local quasiconvexity A metric space (X, d) is said to be *locally* (c, α)-*quasiconvex* if there are constants $c \geq 1$ and $0 < \alpha < 1$ such that for each point $w \in X$, every pair of points in $\alpha B(w)$ can be joined by a c-quasiconvex curve.

Linearly local connectedness A metric space (X, d) is said to be *b-linearly locally connected* (briefly, b-LLC) if there is a constant $b \geq 1$ so that for each $x \in X$ and $r > 0$, the following conditions hold:

LLC_1 : any pair of points in $\mathbb{B}(x, r)$ can be joined by a rectifiable curve in $\mathbb{B}(x, br)$;

LLC_2 : any pair of points in $X \setminus \overline{\mathbb{B}}(x, r)$ can be joined by a rectifiable curve in $X \setminus \overline{\mathbb{B}}(x, r/b)$.

Locally external connectedness If a metric space (X, d) is locally compact and the LLC_2 condition holds for all $x \in X$ and all $r \in (0, d_X(x)/b)$ with $b > 1$, then we say that X is *b-locally externally connected* (briefly, b-LEC).

Next is the definition of (Ahlfors-David) regularity.

Definition 2.3. Suppose that (X, d, μ) is a metric measure space. If there is a constant $C_r \geq 1$ such that

$$C_r^{-1} R^Q \leq \mu(\mathbb{B}(x, R)) \leq C_r R^Q$$

for all balls $\mathbb{B}(x, R)$ in X of radius $R < \mathrm{diam} X$, then X is said to be *Q-regular*. In particular, when only the upper bound above holds, X is called *upper Q-regular*, and when only the lower bound above holds, X is called *lower Q-regular*.

We make a notational convention: In the following, C_r always denotes the constant from the definition of regularity, and also, it is called a *regularity constant* of X. Its value may be different at different occasions.

It is known that every Q-regular metric measure space with $Q > 1$ has Hausdorff dimension precisely Q (cf. [11, §8.7]). The converse is invalid. This can be seen from the first statement of Example 5.1 or Example 5.2 below.

In [12], Heinonen and Koskela discussed the relationship between regularity and capacity; see [12, Lemma 3.14]. In [12, Lemma 3.14], the underlying spaces are assumed to be Loewner. But this assumption is not used in its proof. Thus we formulate it in the following form which we need.

Theorem 4 *[12, Lemma 3.14] Suppose that* (X, d, μ) *is Q-regular with* $Q > 1$*. Then there is a constant* $C_1 > 0$ *such that for any* r *with* $0 < 2r < R < \mathrm{diam} X$ *and* $y \in X$*,*

$$\mathrm{cap}_Q(\overline{\mathbb{B}}(y, r), X \setminus \mathbb{B}(y, R); X) \leq C_1 \left(\log \frac{R}{r}\right)^{1-Q}.$$

The following result is from [11].

Theorem 5 [11, Proposition 8.19 and Theorem 8.23]. *Let (X, d, μ) be a Q-Loewner space. Then there is a constant $C_2 \geq 1$ such that*

$$C_2^{-1} R^Q \leq \mu(\mathbb{B}(x, R)) \tag{2.4}$$

for all balls $\mathbb{B}(x, R)$ in X of radius $R < \mathrm{diam} X$. If X is upper Q-regular, then (X, d, μ) is Q-regular and there is a decreasing self-homeomorphism ψ of $(0, \infty)$ such that

$$\mathrm{cap}_Q(E, F; X) \geq \psi(\Delta(E, F)) \tag{2.5}$$

for all disjoint and nondegenerate continua E and F in X. Moreover, we can select ψ so as to satisfy

$$\psi(t) \approx \log \frac{1}{t}$$

for all sufficiently small t, and

$$\psi(t) \approx (\log t)^{1-Q} \tag{2.6}$$

for all sufficiently large t. The statement is quantitative.

It is known that all Euclidean spaces $\mathbb{R}^n$ are 1-quasiconvex. In combination with Lemma 2.2, Remark 2.1 and the definitions of the k-cap conditions in $\mathbb{R}^n$ (cf. [5,6]), we introduce the following definition in the quasiconvex metric space setting.

Definition 2.4. Suppose that (X, d, μ) is a metric measure space, and $G \subset X$ is a locally compact and c-quasiconvex domain. Let $\tau > 0$ and $0 < \lambda \leq 1/(16c)$ be constants. Then G is said to satisfy a (τ, λ)-*k-cap condition* if for every pair of points x and $y \in G$ with $k_G(x, y) \geq 2$,

$$\mathrm{cap}_Q\big(\lambda \overline{\mathbb{B}}_G(x), \lambda \overline{\mathbb{B}}_G(y); G\big) \leq \frac{\tau}{k_G(x, y)^{Q-1}}. \tag{2.7}$$

Also, G is called a (τ, λ)-*k-cap domain*. When the values of the parameters are not important, briefly, we say that G is a *k-cap domain*.

Remark 2.2. By [12], Heinonen and Koskela proved that the combination of Loewner property and regularity guarantees quasiconvexity (see [12, Theorem 3.13] or Lemma 3.5 below). Based on this, it follows from the proof of Theorem 1.1 that the assumption of G being quasiconvex can be removed from Theorem 1.1. As we see from Definition 2.4 that the concept of k-cap condition is introduced in quasiconvex metric measure spaces or their domains, such an assumption is kept in Theorem 1.1.

2.4. Weak Slice Conditions

In order to study the geometrical characterizations of domains which support Sobolev–Poincaré-type imbeddings, Buckley and Koskela introduced a concept which is the so-called slice condition (cf. [8]). Later, in order to establish sharp inequalities of Trudinger-type on general domains, Buckley and O'Shea generalized the slice condition into a weaker one which is called the weak slice condition (cf. [9]). See [7] for an application of this condition. The following is the definition of weak slice conditions.

Definition 2.5. Suppose that (X, d) is a b-LEC metric space, x and $y \in X$ are two points.

(1) A non-empty bounded open subset $S \subset X$ is an A-*slice* separating x and y if there is a constant $A \geq 2$ such that
$$A^{-1}\mathbb{B}(x) \cap S = \emptyset = S \cap A^{-1}\mathbb{B}(y),$$
and for all $\gamma \in \Gamma(x, y : X)$,
$$\ell(\gamma \cap S) \geq A^{-1}\mathrm{diam}(S);$$

(2) For x and y in X, let $\mathcal{S}(x, y)$ denote a collection of pairwise disjoint A-slices separating x and y. Define
$$d_{ws}(x, y) = d_{ws}(x, y; A) = 1 + \sup\left\{\mathrm{card}\mathcal{S}(x, y)\right\},$$
where the supremum is taken over all $\mathcal{S}(x, y)$ and card means cardinality. We say that X satisfies a *weak A-slice condition* if for each pair of points x and y in X,
$$k_X(x, y) \leq A d_{ws}(x, y). \tag{2.8}$$

In [7], Buckley and Herron considered the characterization of uniform spaces in terms of the weak slice condition. The obtained result is as follows.

Theorem 6 [7, Theorem 4.2]. *A minimally nice metric space (X, d) is uniform and LEC if and only if it is quasiconvex, LLC and satisfies a weak slice condition. These implications are quantitative.*

3. Auxiliary Results

The aim of this section is to establish several auxiliary lemmas, which are useful for the proof of Theorem 1.1. The first lemma compares the relative distance between the closures of two Whitney balls with the one between their centers.

Lemma 3.1. *Suppose that $G \subset X$ is a c-quasiconvex domain, where the case $G = X$ is included when (X, d) is noncomplete. For any pair of points x, y in G and $0 < \lambda \leq 1/(8c)$, if $\lambda\overline{\mathbb{B}}_G(x) \cap \lambda\overline{\mathbb{B}}_G(y) = \emptyset$ and $k_G(x, y) > 1$, then*
$$\frac{1}{4\lambda} r_G(x, y) \leq \Delta(\lambda\overline{\mathbb{B}}_G(x), \lambda\overline{\mathbb{B}}_G(y)) \leq \frac{1}{\lambda} r_G(x, y). \tag{3.1}$$

Proof. We only need to prove the case $G \subsetneq X$ since the proof of the case $G = X$ is similar. Without loss of generality, we may assume that $d_G(x) \leq d_G(y)$. Since G is c-quasiconvex and open, we infer from [15, Lemma 3.1] that
$$\partial\lambda\mathbb{B}_G(x) \neq \emptyset,$$

and from [15, Lemma 3.5] that for any $u \in \tau\overline{\mathbb{B}}_G(z)$,

$$|u-z| = \tau d_G(z) \text{ if and only if } u \in \partial\tau\mathbb{B}_G(z).$$

These imply that there is a point $z \in \partial\lambda\mathbb{B}_G(x)$ such that

$$\text{diam}\lambda\overline{\mathbb{B}}_G(x) \geq |x-z| = \lambda d_G(x).$$

Since $|x-y| \geq \text{dist}(\lambda\overline{\mathbb{B}}_G(x), \lambda\overline{\mathbb{B}}_G(y))$, we obtain that

$$\Delta(\lambda\overline{\mathbb{B}}_G(x), \lambda\overline{\mathbb{B}}_G(y)) \leq \frac{|x-y|}{\lambda d_G(x)} = \frac{1}{\lambda} r_G(x,y).$$

This proves the right side of (3.1).

Since $k_G(x,y) > 1$, it follows from Lemma 2.1(3) that

$$|x-y| \geq \frac{1}{2c} d_G(y).$$

Then the assumption $0 < \lambda \leq 1/(8c)$ implies that

$$\text{dist}(\lambda\overline{\mathbb{B}}_G(x), \lambda\overline{\mathbb{B}}_G(y)) \geq |x-y| - \lambda d_G(x) - \lambda d_G(y) \geq \frac{1}{2}|x-y|,$$

and thus, we get

$$\Delta(\lambda\overline{\mathbb{B}}_G(x), \lambda\overline{\mathbb{B}}_G(y)) \geq \frac{|x-y|}{4\lambda d_G(x)} = \frac{1}{4\lambda} r_G(x,y),$$

since $\min\{\text{diam}\lambda\overline{\mathbb{B}}_G(x), \text{diam}\lambda\overline{\mathbb{B}}_G(x)\} \leq 2\lambda d_G(x)$. This proves the left side of (3.1), and hence the lemma is proved. □

It is known that for any domain $G \subset \mathbb{R}^n$, and for any $\lambda \in (0,1)$ and $x \in G$, the closure $\lambda\overline{\mathbb{B}}_G(x)$ of the Whitney ball $\lambda\mathbb{B}_G(x) = \lambda\mathbb{B}(x)$ is a continuum. But the fact is invalid in the metric space setting. The next lemma establishes the existence of continua in such closures of Whitney balls, which also satisfy certain comparability conditions.

Lemma 3.2. *Suppose that $G \subset X$ is a c-quasiconvex domain, where the case $G = X$ is included when (X,d) is noncomplete. For any pair of points x, y in G and $0 < \lambda \leq 1/(8c)$, if $\lambda\overline{\mathbb{B}}_G(x) \cap \lambda\overline{\mathbb{B}}_G(y) = \emptyset$ and $k_G(x,y) > 1$, then there are curves $\gamma_x \subset \lambda\overline{\mathbb{B}}_G(x)$ and $\gamma_y \subset \lambda\overline{\mathbb{B}}_G(y)$ such that both γ_x and γ_y are continua, and*

$$\frac{1}{4\lambda} r_G(x,y) \leq \Delta(\gamma_x, \gamma_y) \leq \frac{3c}{2\lambda} r_G(x,y). \tag{3.2}$$

Proof. We only need to prove the case $G \subsetneq X$ since the proof of the case $G = X$ is similar. Since G is c-quasiconvex and open, we know from [15, Lemmas 3.1 and 3.5] that there are points $x_1 \in \partial(\lambda/c)\mathbb{B}_G(x)$ and $y_1 \in \partial(\lambda/c)\mathbb{B}_G(y)$ such that

$$|x_1 - x| = \frac{1}{c}\lambda d_G(x) \text{ and } |y_1 - y| = \frac{1}{c}\lambda d_G(y).$$

Also, we know that there are a c-quasiconvex curve γ_x in G connecting x_1 and x, and another c-quasiconvex curve γ_y in G connecting y_1 and y. Then $\gamma_x \subset \lambda\overline{\mathbb{B}}_G(x)$ and $\gamma_y \subset \lambda\overline{\mathbb{B}}_G(y)$, and both of them are continua. Since

$$\operatorname{dist}(\gamma_x,\gamma_y) \geq \operatorname{dist}(\lambda\overline{\mathbb{B}}_G(x),\lambda\overline{\mathbb{B}}_G(y)) \text{ and } \frac{\lambda}{c}d_G(x) \leq \operatorname{diam}(\gamma_x) \leq 2\lambda d_G(x),$$

it follows from Lemma 3.1 that

$$\Delta(\gamma_x,\gamma_y) \geq \Delta(\lambda\overline{\mathbb{B}}_G(x),\lambda\overline{\mathbb{B}}_G(y)) \geq \frac{1}{4\lambda}r_G(x,y).$$

Next, we check the upper bound in (3.2). Since $k_G(x,y) > 1$, by Lemma 2.1(3), we have that

$$|x-y| \geq \frac{1}{2c}d_G(y).$$

This implies that

$$\operatorname{dist}(\gamma_x,\gamma_y) \leq |x-y| + 2\lambda d_G(y) \leq (1+4c\lambda)|x-y| \leq \frac{3}{2}|x-y|.$$

Hence we obtain that

$$\Delta(\gamma_x,\gamma_y) \leq \frac{c}{\lambda}\frac{\operatorname{dist}(\gamma_x,\gamma_y)}{\min\{d_G(x),d_G(y)\}} \leq \frac{3c}{2\lambda}r_G(x,y),$$

which is what we need. □

The following lemma is an analog of [6, Lemma 2.2] in the Q-regular metric measure space setting.

Lemma 3.3. *Suppose that (X,d,μ) is a Q-regular metric measure space with $Q > 1$. Let E, F be disjoint bounded closed sets in X. Then the following two statements are true:*

1 *$\operatorname{cap}_Q(E,F;X) \leq C_r(1+1/\Delta(E,F))^Q$; (Recall that the constant C_r denotes a regularity constant of X.)*

2 *If $\Delta(E,F) \geq 2$, then*

$$\operatorname{cap}_Q(E,F;X) \leq C_1\left(\log\Delta(E,F)\right)^{1-Q},$$

where the constant C_1 is from Theorem 4.

Proof. Let $a = \operatorname{diam}(E)$ and $b = \operatorname{dist}(E,F)$. Without loss of generality, we assume that $\operatorname{diam}(E) \leq \operatorname{diam}(F)$.

(1) For $x \in E$, let

$$\varrho(y) = \begin{cases} 1/b, & \text{if} \quad y \in \mathbb{B}(x,a+b), \\ 0, & \text{otherwise}. \end{cases}$$

Then ϱ is admissible for the curve family $\Gamma(E,F : X)$. Since (X,d,μ) is Q-regular, we deduce that

$$\operatorname{cap}_Q(E,F;X) \leq \int_X \varrho^Q d\mu \leq \mu(\mathbb{B}(x,a+b))/b^Q \leq C_r(1+1/\Delta(F,F))^Q,$$

where $C_r \geq 1$ is a regularity constant of X.

(2). The assumption $\Delta(E,F) \geq 2$ implies that $b \geq 2a$. Then for $x \in E$, we see that $E \subset \mathbb{B}(x,a)$ and $F \subset X \setminus \mathbb{B}(x,b)$. By Theorems 3(3) and 4, we have that

$$\mathrm{cap}_Q(E,F;X) \leq \mathrm{cap}_Q(\overline{\mathbb{B}}(y,a), X \setminus \mathbb{B}(y,b); X) \leq C_1 \left(\log \Delta(E,F)\right)^{1-Q},$$

where the constant C_1 is from Theorem 4. □

Lemma 3.4. *Suppose that (X,d,μ) is a Q-regular metric measure space and $G \subset X$ is a domain. If G is Q-Loewner, then G is Q-regular.*

Proof. Let $x \in G$ and $0 < r < \mathrm{diam} G$. Then the inequality (2.4) in Theorem 5 ensures that there is a constant $C_2 \geq 1$ such that

$$\mu(\mathbb{B}_G(x,r)) \geq C_2^{-1} r^Q.$$

Moreover, since X is Q-regular, we see that

$$\mu(\mathbb{B}_G(x,r)) \leq \mu(\mathbb{B}(x,r)) \leq C_r r^Q.$$

Let $C_3 = \max\{C_r, C_2\}$. Then G is Q-regular with a regularity constant C_3. □

Lemma 3.5. *Suppose that (X,d,μ) is a Q-regular metric measure space and $G \subset X$ is a domain. If G is Q-Loewner, then G is c-quasiconvex and c-LLC.*

Proof. It follows from Lemma 3.4 that G is Q-regular, and thus, the lemma follows from [12, Theorem 3.13]. □

4. Proof of Theorem 1.1

The aim of this section is to prove Theorem 1.1. We shall prove the theorem by checking the implications: $(\mathfrak{a}) \Rightarrow (\mathfrak{b}) \Rightarrow (\mathfrak{c}) \Rightarrow (\mathfrak{d}) \Rightarrow (\mathfrak{a})$. We only need to prove the case when G is a proper domain in X since the proof of the case when $G = X$ is similar.

$(\mathfrak{a}) \Rightarrow (\mathfrak{b})$. Assume that G is a-uniform and locally Q-Loewner. This implication follows from Theorem A since uniformity implies inner uniformity.

$(\mathfrak{b}) \Rightarrow (\mathfrak{c})$. Assume that G is Q-Loewner and a-inner uniform. Since G is c-quasiconvex, it follows that for any pair of points x, $y \in G$, there is a curve γ_{xy} in G connecting x and y such that

$$\sigma_G(x,y) \leq l(\gamma_{xy}) \leq c|x-y|.$$

Moreover, the assumption of G being a-inner uniform and Theorem 2 guarantee that

$$k_G(x,y) \leq 4a^2 \log\left(1 + \frac{\sigma_G(x,y)}{\min\{d_G(x), d_G(y)\}}\right).$$

Then it follows that

$$k_G(x,y) \leq 4a^2 \log\left(1 + c r_G(x,y)\right).$$

By letting $\varphi(t) = 4a^2 \log(1+ct)$, we see that G is QH φ-uniform with φ being slow. Again, we know from Theorem 2 that G is a_1-uniform. Since Lemma 3.5 indicates G is c-LLC, and thus, it is c-LEC. Hence we know from Theorem 6 that G satisfies a weak A-slice condition, which completes the proof of this implication.

($\mathfrak{c}$) $\Rightarrow$ ($\mathfrak{d}$). Assume that G is Q-Loewner and satisfies a weak A-slice condition. Let x, y be two points in G with $k_G(x,y) \geq 2$. To prove this implication, it suffices to show that there are constants $\tau > 0$ and $0 < \lambda \leq 1/(16c)$, independent of x and y, such that

$$\mathrm{cap}_Q\Big(\lambda\overline{\mathbb{B}}_G(x), \lambda\overline{\mathbb{B}}_G(y); G\Big) \leq \frac{\tau}{k_G(x,y)^{Q-1}}. \tag{4.1}$$

Since G is Loewner, Lemma 3.5 ensures that G is c-LLC, and then, we know from the assumptions of G being c-quasiconvex and satisfying a weak slice condition, together with Theorem 6, that it is a-uniform. Without loss of generality, we assume that $a = c$ and $d_G(x) \leq d_G(y)$. Let

$$\lambda = \min\Big\{\frac{1}{16c}, \frac{1}{4}(1 - e^{-1/(2c^2)})\Big\}. \tag{4.2}$$

Then the assumption $k_G(x,y) \geq 2$ and Lemma 2.2 imply that

$$\Delta(\lambda\overline{\mathbb{B}}_G(x), \lambda\overline{\mathbb{B}}_G(y)) \geq 3 \quad \text{and} \quad \lambda\overline{\mathbb{B}}_G(x) \cap \lambda\overline{\mathbb{B}}_G(y) = \emptyset. \tag{4.3}$$

By the assumptions that X is Q-regular and G is Loewner, we know from Lemma 3.4 that G is Q-regular. Then Lemma 3.3(2) ensures that there is a constant $C_1 \geq 1$ such that

$$\mathrm{cap}_Q(\lambda\overline{\mathbb{B}}_G(x), \lambda\overline{\mathbb{B}}_G(y); G) \leq C_1 \left(\log \Delta(\lambda\overline{\mathbb{B}}_G(x), \lambda\overline{\mathbb{B}}_G(y))\right)^{1-Q}. \tag{4.4}$$

Moreover, by the assumption $k_G(x,y) \geq 2$ and Lemma 2.1(3), we have that $r_G(x,y) \geq 2/(3c)$. Then we deduce from (4.2) that

$$\frac{1}{4\lambda} r_G(x,y) \geq \frac{8}{3}. \tag{4.5}$$

Since the assumption $k_G(x,y) \geq 2$, (4.3) and Lemma 3.1 guarantee that

$$\Delta(\lambda\overline{\mathbb{B}}_G(x), \lambda\overline{\mathbb{B}}_G(y)) \geq \frac{1}{4\lambda} r_G(x,y),$$

we infer from (4.4) and (4.5) that

$$\mathrm{cap}_Q(\lambda\overline{\mathbb{B}}_G(x), \lambda\overline{\mathbb{B}}_G(y); G) \leq C_1 \left(\log \frac{1}{4\lambda} r_G(x,y)\right)^{1-Q}. \tag{4.6}$$

We still need the following relation between $r_G(x,y)$ and $k_G(x,y)$:

$$r_G(x,y) \geq e^{k_G(x,y)/4c^2} - 1. \tag{4.7}$$

This estimate directly follows from Theorem 2 since G is a-uniform.

Since it follows from (4.2), together with the assumption $k_G(x,y) \geq 2$, that

$$\frac{1}{4\lambda}\left(e^{k_G(x,y)/4c^2} - 1\right) > 1,$$

by substituting (4.7) into (4.6), we get

$$\mathrm{cap}_Q(\lambda\overline{\mathbb{B}}_G(x), \lambda\overline{\mathbb{B}}_G(x); G) \leq C_1 \left(\log \frac{1}{4\lambda}\left(e^{k_G(x,y)/(2c^2)}(1 - 1/e^{1/(2c^2)})\right)\right)^{1-Q}.$$

Again, by (4.2), we have

$$\mathrm{cap}_Q(\lambda\overline{\mathbb{B}}_G(x), \lambda\overline{\mathbb{B}}_G(x); G) \leq \tau k_G(x,y)^{1-Q},$$

where $\tau = (2c^2)^{Q-1}C_1$. This proves (4.1), which indicates that G satisfies the (τ, λ)-k-cap condition.

$(\mathfrak{d}) \Rightarrow (\mathfrak{a})$. Assume that G is a Q-Loewner space and satisfies a (τ, λ)-k-cap condition, where $\tau > 0$ and $0 < \lambda < 1/(16c)$. Since G is a Q-Loewner space, we know from Lemma 3.4 that G is Q-regular. Then it follows from [2, Proposition 6.48] that there are functions ψ: $(0, +\infty) \to (0, +\infty)$ and κ: $(0, +\infty) \to [1, +\infty)$ with the following property: If E and F are two disjoint nondegenerate continua in a ball $\mathbb{B}(x, r)$ in G, then

$$\mathrm{cap}_Q(E, F; \mathbb{B}(x, \kappa(t)r)) \geq \psi(t) \tag{4.8}$$

whenever $\Delta(E, F) \leq t$.

Let $t = 16$ and $\varepsilon_0 \in (0, \kappa(16)^{-1}]$. If E, F are disjoint nondegenerate continua in a ball $\mathbb{B}(x, \varepsilon d_G(x))$ in G with $0 < \varepsilon \leq \varepsilon_0$ and $\Delta(E, F) \leq 16$, then we know from (4.8) that

$$\mathrm{cap}_Q(E, F; \mathbb{B}(x, \kappa(16)\varepsilon d_G(x))) \geq \psi(16).$$

Since G is c-quasiconvex, it follows from [2, Propositions 6.49] that G is locally Q-Loewner. This shows that, to prove this implication, we only need to check the uniformity of G. We are going to reach this goal by applying Theorem 2, that is, it suffices to show that there is a slow η such that for any x, $y \in G$,

$$k_G(x,y) \leq \eta(r_G(x,y)). \tag{4.9}$$

To establish the existence of η, we consider two cases: $k_G(x,y) < 2$ and $k_G(x,y) \geq 2$. For the former, it follows from Lemma 2.1 (2) that

$$k_G(x,y) \leq 3cr_G(x,y). \tag{4.10}$$

For the latter, that is, $k_G(x,y) \geq 2$, since G is Q-regular and Q-Loewner, (2.5) in Theorem 5 guarantees that there is a decreasing self-homeomorphism ψ of $(0, \infty)$ such that

$$\mathrm{cap}_Q(E, F; G) \geq \psi(\Delta(E, F))$$

for all disjoint and nondegenerate continua E and F in G. Also, (2.6) in Theorem 5 ensures that there is a sufficiently large β and a constant $C' > 0$ satisfying that for all $t > \beta$,

$$\psi(t) \geq C' (\log t)^{1-Q}.$$

These show that for any pair of disjoint and nondegenerate continua E and F in G, if $\Delta(E, F) > \beta$, then

$$\mathrm{cap}_Q(E, F; G) \geq C' (\log \Delta(E, F))^{1-Q}. \tag{4.11}$$

Without loss of generality, we may assume that $\beta \geq 3$. The following assertion is useful for our arguments.

Claim 4.1. *Suppose that λ_1 is a constant with $0 < \lambda_1 \leq \frac{1}{3(\beta+1)c}$. Then for any continua $E_1 \subset \lambda_1 \overline{\mathbb{B}}_G(x)$ and $F_1 \subset \lambda_1 \overline{\mathbb{B}}_G(y)$, we have*

$$\Delta(E_1, F_1) \geq \beta.$$

Let $E_1 \subset \lambda_1 \overline{\mathbb{B}}_G(x)$ and $F_1 \subset \lambda_1 \overline{\mathbb{B}}_G(y)$ be continua. Obviously,

$$\Delta(E_1, F_1) \geq \Delta(\lambda_1 \overline{\mathbb{B}}_G(x), \lambda_1 \overline{\mathbb{B}}_G(y)).$$

Since

$$\begin{aligned} |x - y| &\leq \mathrm{dist}(\lambda_1 \overline{\mathbb{B}}_G(x), \lambda_1 \overline{\mathbb{B}}_G(y)) + \lambda_1 d_G(x) + \lambda_1 d_G(y) \\ &\leq \mathrm{dist}(\lambda_1 \overline{\mathbb{B}}_G(x), \lambda_1 \overline{\mathbb{B}}_G(y)) + 2\lambda_1 \max\{d_G(x), d_G(y)\}, \\ \max\{\mathrm{diam}&(\lambda_1 \overline{\mathbb{B}}_G(x)), \mathrm{diam}(\lambda_1 \overline{\mathbb{B}}_G(y))\} \leq 2\lambda_1 \max\{d_G(x), d_G(y)\} \end{aligned}$$

and because Lemma 2.1(3) gives

$$|x - y| \geq \frac{2}{3c} \max\{d_G(x), d_G(y)\},$$

we get

$$\Delta(E_1, F_1) \geq \frac{1}{3c\lambda_1} - 1 \geq \beta,$$

which is what we need.

Let

$$\lambda_2 = \min\Big\{\lambda, \frac{1}{4(\beta+1)c}\Big\}.$$

Then the assumptions $k_G(x, y) \geq 2$ and $\beta \geq 3$, together with Lemma 2.2, ensure that

$$\lambda_2 \overline{\mathbb{B}}_G(x) \cap \lambda_2 \overline{\mathbb{B}}_G(y) = \emptyset,$$

and so, Lemma 3.2 guarantees that there are two curves $\gamma_x \subset \lambda_2 \overline{\mathbb{B}}_G(x)$ and $\gamma_y \subset \lambda_2 \overline{\mathbb{B}}_G(y)$ such that both γ_x and γ_y are continua, and

$$\frac{1}{4\lambda_2} r_G(x, y) \leq \Delta(\gamma_x, \gamma_y) \leq \frac{3c}{2\lambda_2} r_G(x, y). \tag{4.12}$$

Moreover, Claim 4.1 implies that

$$\Delta(\gamma_x, \gamma_y) \geq \beta \geq 3. \tag{4.13}$$

Obviously, the fact $\lambda_2 \leq \lambda$ implies that $\gamma_x \subset \lambda\overline{\mathbb{B}}_G(x)$ and $\gamma_y \subset \lambda\overline{\mathbb{B}}_G(y)$. By Theorem 3(3), we have

$$\operatorname{cap}_Q(\gamma_x, \gamma_y; G) \leq \operatorname{cap}_Q(\lambda\overline{\mathbb{B}}_G(x), \lambda\overline{\mathbb{B}}_G(y); G).$$

Since $k_G(x, y) \geq 2$ and $\lambda\overline{\mathbb{B}}_G(x) \cap \lambda\overline{\mathbb{B}}_G(y) = \emptyset$ by Lemma 2.2, the assumption that G satisfies a (τ, λ)-k-cap condition indicates that

$$\operatorname{cap}_Q(\lambda\overline{\mathbb{B}}_G(x), \lambda\overline{\mathbb{B}}_G(y); G) \leq \frac{\tau}{k_G(x, y)^{Q-1}},$$

where $\tau > 0$ and $0 < \lambda \leq 1/(16c)$. Thus we get

$$\operatorname{cap}_Q(\gamma_x, \gamma_y; G) \leq \frac{\tau}{k_G(x, y)^{Q-1}}.$$

Because it follows from (4.11) and (4.13) that

$$\operatorname{cap}_Q(\gamma_x, \gamma_y; G) \geq C' \big(\log \Delta(\gamma_x, \gamma_y) \big)^{1-Q},$$

we have

$$k_G(x, y) \leq C_0 \log \Delta(\gamma_x, \gamma_y),$$

where $C_0 = \big(\tau/C'\big)^{1/(Q-1)}$, and then, (4.12) leads to

$$k_G(x, y) \leq C_0 \log \left(\frac{3c}{2\lambda_2} r_G(x, y) \right). \tag{4.14}$$

To construct the needed control function η in (4.9) based on (4.10) and (4.14), let

$$\eta_0(t) = \max\{\eta_1(t), \eta_2(t)\}$$

for $t \in (0, \infty)$, where

$$\eta_1(t) = 3ct \ \text{ and } \ \eta_2(t) = C_0 \log(3ct/2\lambda_2).$$

Since both η_1 and η_2 are strictly increasing homeomorphisms, by elementary computations, we see that η_0 is also a strictly increasing homeomorphism on $(0, +\infty)$ as well.

Since G is c-quasiconvex, [14, Theorem 2.7(1)] implies that for any x, $y \in G$,

$$k_G(x, y) \geq \log \big(1 + r_G(x, y)\big).$$

This shows that if $r_G(x, y) \geq e^2 - 1$, then $k_G(x, y) \geq 2$. Let

$$t_0 = e^2 - 1.$$

Then we infer from the fact $\eta_2(t) \to +\infty$ as $t \to +\infty$ that there is $t_1 \in (t_0, +\infty)$ such that $\eta_2(t_1) = 2\eta_0(t_0)$. Let

$$\eta(t) = \begin{cases} \eta_0(t) & \text{if} \quad t \in [0, t_0], \\ \max\{\eta_2(t), \eta_3(t)\} & \text{if} \quad t \in (t_0, t_1], \\ \eta_2(t) & \text{if} \quad t \in (t_1, +\infty), \end{cases}$$

where

$$\eta_3(t) = \frac{\eta_0(t_0)}{t_1 - t_0} t + \frac{(t_1 - 2t_0)\eta_0(t_0)}{t_1 - t_0}.$$

Then η is a strictly increasing homeomorphism. Moreover,

$$\lim_{t \to +\infty} \frac{\eta(t)}{t} = \lim_{t \to +\infty} \frac{\eta_2(t)}{t} = 0,$$

which shows that the function η is slow. Then we conclude from (4.10) and (4.14) that η satisfies (4.9), and hence, it follows from Theorem 2 that the implication is true. $\square$

5. An Example

In this section, we shall provide two examples which show that the assumption of regularity in Theorem 1.1 can not be removed. Also, these examples demonstrate that for a metric measure space with Hausdorff dimension Q, it may not be Q-regular. Before the statement of these examples, let us introduce the following definitions.

Definition 5.1. A metric measure space (X, d, μ) is said to be *doubling* if there is a constant $C_d \geq 1$ such that

$$\mu(\mathbb{B}(x, 2R)) \leq C_d \mu(\mathbb{B}(x, R))$$

for all balls $\mathbb{B}(x, R)$ in X of radius $0 < R < \operatorname{diam} X$. Also, we call C_d a *doubling constant* of X.

Definition 5.2. Suppose that (X, d, μ) is a metric measure space and $1 \leq p < \infty$. We say that (X, d, μ) admits a *weak* $(1, p)$-*Poincaré inequality* if there are constants $\beta \geq 1$ and $C_p \geq 1$ such that for all balls $\mathbb{B}(x, R)$ in X with $R > 0$,

$$⨍_{\mathbb{B}(x,R)} \left| u - u_{\mathbb{B}(x,R)} \right| d\mu \leq C_p \left(\operatorname{diam} \mathbb{B}(x, R) \right) \left(⨍_{\beta \mathbb{B}(x,R)} \rho^p d\mu \right)^{\frac{1}{p}}$$

whenever u is a bounded continuous function on $\beta \mathbb{B}(x, R)$ and ρ is its very weak gradient. Also, β is called a *dilation constant* of X. If $\beta = 1$, then we say that (X, d, μ) admits a $(1, p)$-*Poincaré inequality*. We use the standard notation

$$u_A = ⨍_A u d\mu = \frac{1}{\mu(A)} \int_A u d\mu$$

for the mean value of a function u in a measurable set A of positive measure.

The following known results are useful for the arguments in this section.

Theorem 7 [11, Theorem 9.10]. *Suppose that (X, d, μ) is a Q-regular, geodesic and proper metric measure space. Then (X, d, μ) admits a $(1, Q)$-Poincaré inequality if and only if it is a Q-Loewner space.*

Theorem 8 [1, Theorem 4.4]. *Assume that μ supports a weak $(1, p)$-Poincaré inequality on X, and let $G \subset X$ be an A-uniform domain. Then there exists $\alpha_0 > 0$ such that for each $\alpha \geq -\alpha_0$, the measure $d\nu(y) = d_G(y)^\alpha d\mu(y)$ is doubling on G and supports a weak $(1, p)$-Poincaré inequality on G with a dilation constant $\beta = 3A$.*

The following result is from [13, Proposition 2.2] and the proof of [13, Theorem 6.1].

Theorem 9. *Fix $N \geq 3$. Consider points x_0, $x_1 \in G \subset \mathbb{R}^n$ with $|x_0 - x_1| \geq d_G(x_0)/N$, and let γ be a quasihyperbolic geodesic in G joining x_0 and x_1. Then there exist a positive integer k with*

$$\frac{N-1}{2} k_G(x_0, x_1) \leq k \leq 2N k_G(x_0, x_1), \tag{5.1}$$

and successive points $y_0 = x_0, y_1, \ldots, y_k, y_{k+1} = x_1$ in γ along the direction from x_0 to x_1 such that the closed balls $\{B_i = \overline{\mathbb{B}}(y_i, r_i)\}$, where $r_i = d_G(y_i)/N$, satisfy that $\gamma \subset B_0 \cup B_1 \cup \cdots \cup B_{k+1}$, and for each $i \in \{0, 1, \ldots, k\}$,

$$\frac{N-1}{N} r_i \leq r_{i+1} \leq \frac{N+1}{N} r_i \ \text{ and } \ B_{i+1} \subset \left(2 + \frac{1}{N}\right) B_i \subset G. \tag{5.2}$$

Now, we are ready to state and prove our first example.

Example 5.1. Suppose that $X = (\mathbb{R}^2, |\cdot|, \mu)$, where $|\cdot|$ denotes the Euclidean metric, the measure μ is defined by $d\mu(x) = (1 + |x|)dx$ and dx stands for the usual Lebesgue measure on $\mathbb{R}^2$. Let $G = \mathbb{R}^2 \setminus \{0\}$. Then the following statements are true.

1 X is doubling and lower 2-regular, but not 2-regular;
2 G is $\pi/\log 3$-uniform and $\pi/\log 3$-inner uniform;
3 G is a locally 2-Loewner and 2-Loewner space;
4 G satisfies a weak slice condition;
5 G does not satisfy any (τ, λ)-*k-cap* condition with $\tau > 0$ and $0 < \lambda \leq 1/16$.

Proof. (1). The first statement of the example directly follows from [12, 5.3].

(2). By [10, Corollary 6.16], it follows that for each pair of points x, $y \in G$,

$$k_G(x, y) \leq (\pi/\log 3) j_G(x, y),$$

which implies that G is $\pi/\log 3$-uniform by [21, Theorem 6.16]. Obviously, G is $\pi/\log 3$-inner uniform as well.

(3). It is obvious that G is locally compact. Since $(G, |\cdot|, dx)$ is locally 2-Loewner by [2, Theorem 6.47], and since $d\mu \geq dx$, we see that $(G, |\cdot|, d\mu)$ is locally 2-Loewner as well. Similarly, we also know that $(G, |\cdot|, d\mu)$ is 2-Loewner.

(4). We already know that G is $\pi/\log 3$-uniform (see the statement (2) of the example), and also, it follows from [7, Page 202] that G is c-LEC for all $c > 1$. Then Theorem 6 ensures that G satisfies a weak slice condition.

(5). Let $d\mu_1 = dx$ and $d\mu_2 = |x|dx$. Obviously, $d\mu = d\mu_1 + d\mu_2$. Suppose on the contrary that G satisfies a (τ, λ)-k-cap condition, i.e., there are constants $0 < \lambda < 1/16$ and $\tau > 0$ such that for each pair of points x, $y \in G$ with $k_G(x, y) \geq 2$,

$$\mathrm{cap}_2(\lambda\overline{\mathbb{B}}(x), \lambda\overline{\mathbb{B}}(y); G) \leq \frac{\tau}{k_G(x, y)}. \tag{5.3}$$

□

To reach a contradiction, we establish the following proposition.

Proposition 5.1. *For each $t \in (1 - e^{-2}, 1)$, there is a pair of points x_t, $y_t \in G$ with $k_G(x_t, y_t) \geq 2$ such that*

$$\mathrm{cap}_2(\lambda\overline{\mathbb{B}}(x_t), \lambda\overline{\mathbb{B}}(y_t); G) \geq C$$

and

$$k_G(x_t, y_t) \to +\infty \quad as \quad t \to 1^-,$$

where $C > 0$ is a constant and "$t \to 1^-$" means that $0 < t < 1$ and $t \to 1$.

For the proof, let N be an integer such that

$$N > \max\left\{65, \frac{1}{\lambda}\right\}.$$

For each $t \in (1 - e^{-2}, 1)$, let $x_t \in G$ with

$$|x_t| = \frac{2N^2 t}{(N-1)(1-t)}, \tag{5.4}$$

and $y_t = (1 - t)x_t$. Then $y_t \in t\overline{\mathbb{B}}(x_t)$, and by [20, Lemma 2.2(1) and (2)],

$$\log \frac{1}{1-t} \leq k_G(x_t, y_t) \leq \frac{t}{1-t}. \tag{5.5}$$

This implies that

$$k_G(x_t, y_t) \geq 2 \text{ and } k_G(x_t, y_t) \to +\infty \text{ as } t \to 1^-. \tag{5.6}$$

Let

$$E_t = \frac{1}{N}\overline{\mathbb{B}}(x_t) \text{ and } F_t = \frac{1}{N}\overline{\mathbb{B}}(y_t).$$

Since

$$\mathrm{cap}_2(E_t, F_t; G) \leq \mathrm{cap}_2(\lambda\overline{\mathbb{B}}(x_t), \lambda\overline{\mathbb{B}}(y_t); G),$$

it follows from (5.6) that to prove Proposition 5.1, we only need to show the following assertion.

Assertion 5.1. There is a constant $C > 0$ such that

$$\mathrm{cap}_2(E_t, F_t; G) \geq C. \tag{5.7}$$

It follows from [10, Lemma 5.1] that there is a quasihyperbolic geodesic γ in G joining x_t and y_t. Since $t \geq 1 - e^{-2}$ and $N > 65$, we know that

$$|x_t - y_t| = t|x_t| \geq \frac{d_G(x_t)}{N}. \tag{5.8}$$

To get the lower bound in (5.7), we need some preparation which consists of three steps.

Step 5.1. *There is a finite sequence of closed balls $\{B_i\}$ in G such that their centers are in γ and each B_i satisfies (5.12) below.*

It follows from (5.8) and Theorem 9 that there are an integer k and successive points $x_0 = x_t$, x_1, ..., x_k, $x_{k+1} = y_t$ in γ from x_t to y_t such that

$$\frac{N-1}{2} k_G(x_t, y_t) \leq k \leq 2N k_G(x_t, y_t), \tag{5.9}$$

and for each $0 \leq i \leq k$,

$$\frac{N-1}{N} r_i \leq r_{i+1} \leq \frac{N+1}{N} r_i \ \text{ and } \ B_{i+1} \subset \frac{2N+1}{N} B_i, \tag{5.10}$$

where $B_i = \overline{\mathbb{B}}(x_i, r_i)$ and $r_i = d_G(x_i)/N$.

Obviously, $B_0 = E_t$, $B_{k+1} = F_t$, and it follows from (5.6) and (5.9) that

$$k \geq 63. \tag{5.11}$$

Since X is doubling, we get

$$\mu(B_i) \leq \mu((2 + 1/N)B_i) \leq C_d \mu((1 + 1/(2N))B_i) \leq C_d^2 \mu((1/2 + 1/(4N))B_i),$$

where C_d denotes a doubling constant of X. Then it follows from the assumption $N \geq 65$ that

$$\mu(B_i) \leq \mu((2 + 1/N)B_i) \leq C_d^2 \mu(B_i). \tag{5.12}$$

Step 5.2. *For each $i \in \{0, 1, \ldots, k\}$, the quantities $\mu(B_i)$ and $\mu(B_{i+1})$ are equivalent. More precisely, we have*

$$C_0' \mu(B_i) \leq \mu(B_{i+1}) \leq C_0 \mu(B_i), \tag{5.13}$$

where

$$C_0 = \max\left\{ \left(\frac{N+1}{N-1}\right)^2, \frac{(N+1)^4}{N(N-1)^3} \right\}$$

and

$$C_0' = \min\left\{ \left(\frac{N-1}{N+1}\right)^2, \frac{(N-1)^2(N^2 - 2N - 1)}{N(N+1)^3} \right\}.$$

Let $i \in \{0, 1, \ldots, k\}$. Since $\mu_1(B_i) = \pi r_i^2$, it follows from (5.10) that

$$\left(\frac{N-1}{N+1}\right)^2 \mu_1(B_i) \le \mu_1(B_{i+1}) \le \left(\frac{N+1}{N-1}\right)^2 \mu_1(B_i). \tag{5.14}$$

Next, we establish an equivalence between $\mu_2(B_i)$ and $\mu_2(B_{i+1})$ as stated in (5.17) below. For this, let $\xi \in (2 + 1/N)B_i$. Then

$$d_G(x_i) - \frac{2N+1}{N} r_i \le |x_i| - |x_i - \xi| \le |\xi| \le |x_i| + |x_i - \xi| \le d_G(x_i) + \frac{2N+1}{N} r_i,$$

which implies that

$$\frac{N^2 - 2N - 1}{N^2} d_G(x_i) \le |\xi| \le \left(\frac{N+1}{N}\right)^2 d_G(x_i).$$

Moreover, $B_{i+1} \subset (2+1/N)B_i$ by (5.10) and $\mu_2(B_{i+1}) = \int_{B_{i+1}} |\xi| d\xi$. It follows that

$$\frac{N^2 - 2N - 1}{N^2} d_G(x_i)\mu_1(B_{i+1}) \le \mu_2(B_{i+1}) \le \left(\frac{N+1}{N}\right)^2 d_G(x_i)\mu_1(B_{i+1}),$$

and thus, (5.14) gives

$$\mu_2(B_{i+1}) \le \frac{(N+1)^4}{N^2(N-1)^2} d_G(x_i)\mu_1(B_i) \tag{5.15}$$

and

$$\mu_2(B_{i+1}) \ge \frac{(N-1)^2(N^2 - 2N - 1)}{N^2(N+1)^2} d_G(x_i)\mu_1(B_i). \tag{5.16}$$

Moreover, for $\zeta \in B_i$, we have

$$d_G(x_i) - r_i = |x_i| - |\zeta - x_i| \le |\zeta| \le |\zeta - x_i| + |x_i| = r_i + d_G(x_i),$$

which implies that

$$\frac{N-1}{N} d_G(x_i) \le |\zeta| \le \frac{N+1}{N} d_G(x_i).$$

Thus we get

$$\frac{N}{(N+1)d_G(x_i)} \mu_2(B_i) \le \mu_1(B_i) \le \frac{N}{(N-1)d_G(x_i)} \mu_2(B_i).$$

Now, we infer from (5.15) and (5.16) that

$$\frac{(N-1)^2(N^2 - 2N - 1)}{N(N+1)^3} \mu_2(B_i) \le \mu_2(B_{i+1}) \le \frac{(N+1)^4}{N(N-1)^3} \mu_2(B_i). \tag{5.17}$$

Since $\mu = \mu_1 + \mu_2$, we know from (5.14) and (5.17) that (5.13) is true.

Step 5.3. *We establish an estimate on the quantity* $\left|u_{B_{i+1}} - u_{B_i}\right|$ *as follows: There is a constant* $C_3 > 0$ *such that for any admissible function* u *of the condenser* $(E_t, F_t; G)$ *and for each very weak gradient* ρ *of* u*,*

$$\left|u_{B_{i+1}} - u_{B_i}\right| \leq C_3 \left(\int_{\omega B_i} \rho^2 d\mu\right)^{\frac{1}{2}} \tag{5.18}$$

hold for all $i \in \{0, 1, \ldots, k\}$*,where*

$$\omega = 3\pi(2N+1)/(N \log 3).$$

Obviously,

$$\omega B_i \subset G, \tag{5.19}$$

since $N \geq 65$*.*

For the proof, let $i \in \{0, 1, \ldots, k\}$, and let u be an admissible function for the condenser $(E_t, F_t; G)$. This means that

$$u|_{E_t} \geq 1 \text{ and } u|_{F_t} \leq 0.$$

Since

$$\left|u_{B_{i+1}} - u_{B_i}\right| \leq I_{i,1} + I_{i,2}, \tag{5.20}$$

where $I_{i,1} = \left|u_{B_{i+1}} - u_{\frac{2N+1}{N} B_i}\right|$ and $I_{i,2} = \left|u_{B_i} - u_{\frac{2N+1}{N} B_i}\right|$, to estimate the quantity $\left|u_{B_{i+1}} - u_{B_i}\right|$, it suffices to estimate $I_{i,1}$ and $I_{i,2}$, respectively. We first work on the quantity $I_{i,1}$. Since

$$I_{i,1} \leq \mu\left(B_{i+1}\right)^{-1} \int_{B_{i+1}} \left|u - u_{\frac{2N+1}{N} B_i}\right| d\mu,$$

we have

$$\begin{aligned} I_{i,1} &\leq \mu\left(B_{i+1}\right)^{-1} \int_{\frac{2N+1}{N} B_i} \left|u - u_{\frac{2N+1}{N} B_i}\right| d\mu && \text{(by (5.10))} \\ &\leq C_0'^{-1} \mu\left(B_i\right)^{-1} \int_{\frac{2N+1}{N} B_i} \left|u - u_{\frac{2N+1}{N} B_i}\right| d\mu && \text{(by (5.13))} \\ &\leq \frac{C_d^2}{C_0'} \fint_{\frac{2N+1}{N} B_i} \left|u - u_{\frac{2N+1}{N} B_i}\right| d\mu. && \text{(by (5.12))} \end{aligned} \tag{5.21}$$

It is known that $(\mathbb{R}^2, |\cdot|, \mu_1)$ is 2-regular, geodesic, proper and 2-Loewner. Then Theorem 7 implies that $(\mathbb{R}^2, |\cdot|, \mu_1)$ admits a $(1, 2)$-Poincaré inequality. This ensures that there is a constant $C_{p,1} \geq 1$ such that for each very weak gradient ρ of u,

$$\fint_{\frac{2N+1}{N} B_i} \left|u - u_{\frac{2N+1}{N} B_i}\right| d\mu_1 \leq C_{p,1} r_i \left(\fint_{\frac{2N+1}{N} B_i} \rho^2 d\mu_1\right)^{\frac{1}{2}}. \tag{5.22}$$

As aforementioned, $(\mathbb{R}^2, |\cdot|, \mu_1)$ admits a $(1, 2)$-Poincaré inequality. By the statement (2) of the example, G is $\pi/\log 3$-uniform. Then it follows from

Theorem 8 that $(G, |\cdot|, \mu_2)$ admits a weak $(1, 2)$-Poincaré inequality with a dilation constant $\beta = 3\pi/\log 3$. This means that there is a constant $C_{p,2} \geq 1$ such that for each very weak gradient ρ of u,

$$\fint_{\frac{2N+1}{N}B_i} \left|u - u_{\frac{2N+1}{N}B_i}\right| d\mu_2 \leq C_{p,2} r_i \left(\fint_{\omega B_i} \rho^2 d\mu_2\right)^{\frac{1}{2}}. \tag{5.23}$$

To continue the estimate on $I_{i,1}$, we consider three cases according to the possible positions of ωB_i in X.

Claim 5.1. $\omega B_i \subset \mathbb{B}(0, 1)$.

Under this assumption, we get

$$d\mu_1 \leq d\mu \leq 2d\mu_1$$

on ωB_i. Then

$$\begin{aligned}\fint_{\frac{2N+1}{N}B_i} \left|u - u_{\frac{2N+1}{N}B_i}\right| d\mu &\leq 2\fint_{\frac{2N+1}{N}B_i} \left|u - u_{\frac{2N+1}{N}B_i}\right| d\mu_1 \\ &\leq 2C_{p,1} r_i \left(\fint_{\frac{2N+1}{N}B_i} \rho^2 d\mu_1\right)^{\frac{1}{2}}, \qquad (\text{by } (5.22))\end{aligned}$$

and so, we get

$$\fint_{\frac{2N+1}{N}B_i} \left|u - u_{\frac{2N+1}{N}B_i}\right| d\mu \leq \frac{6\sqrt{\pi}C_{p,1}}{\omega \log 3} \left(\int_{\omega B_i} \rho^2 d\mu\right)^{\frac{1}{2}}. \tag{5.24}$$

Claim 5.2. $\omega B_i \subset X \setminus \mathbb{B}(0, 1)$.

Under this assumption, we get

$$d\mu_2 \leq d\mu \leq 2d\mu_2$$

on ωB_i. Then

$$\begin{aligned}\fint_{\frac{2N+1}{N}B_i} \left|u - u_{\frac{2N+1}{N}B_i}\right| d\mu &\leq 2\fint_{\frac{2N+1}{N}B_i} \left|u - u_{\frac{2N+1}{N}B_i}\right| d\mu_2 \\ &\leq 2C_{p,2} r_i \left(\fint_{\omega B_i} \rho^2 d\mu_2\right)^{\frac{1}{2}}, \qquad (\text{by } (5.23))\end{aligned}$$

and thus, we have

$$\fint_{\frac{2N+1}{N}B_i} \left|u - u_{\frac{2N+1}{N}B_i}\right| d\mu \leq \frac{2\sqrt{2}C_{p,2}}{\pi^{\frac{1}{2}}\omega} \left(\int_{\omega B_i} \rho^2 d\mu\right)^{\frac{1}{2}}, \tag{5.25}$$

since $\mu_2(\omega B_i) \geq 1/2\mu(\omega B_i) \geq 1/2\mu_1(\omega B_i)$.

Claim 5.3. *$\omega B_i \cap \mathbb{B}(0,1) \neq \emptyset$ and $\omega B_i \cap (X \setminus \mathbb{B}(0,1)) \neq \emptyset$.*

Proof. Let

$$\mathfrak{T}_1 = \fint_{\frac{2N+1}{N}B_i} \left| u - u_{\frac{2N+1}{N}B_i} \right| d\mu_1 \ \text{ and } \ \mathfrak{T}_2 = \fint_{\frac{2N+1}{N}B_i} \left| u - u_{\frac{2N+1}{N}B_i} \right| d\mu_2.$$

Then

$$\fint_{\frac{2N+1}{N}B_i} \left| u - u_{\frac{2N+1}{N}B_i} \right| d\mu \leq \mathfrak{T}_1 + \mathfrak{T}_2, \tag{5.26}$$

since $d\mu_1 \leq d\mu$ and $d\mu_2 \leq d\mu$.

It follows from the fact $\mu_1(((2N+1)/N)B_i) = \pi((2N+1)r_i/N)^2$ and (5.22) that

$$\mathfrak{T}_1 \leq \frac{NC_{p,1}}{(2N+1)\pi^{\frac{1}{2}}} \left(\int_{\frac{2N+1}{N}B_i} \rho^2 d\mu_1 \right)^{\frac{1}{2}}.$$

Thus we get

$$\begin{aligned} \mathfrak{T}_1 &\leq \frac{NC_{p,1}}{(2N+1)\pi^{\frac{1}{2}}} \left(\int_{\omega B_i} \rho^2 d\mu_1 \right)^{\frac{1}{2}} \quad (\text{since} \quad ((2N+1)/N)B_i \subset \omega B_i) \\ &\leq \frac{NC_{p,1}}{(2N+1)\pi^{\frac{1}{2}}} \left(\int_{\omega B_i} \rho^2 d\mu \right)^{\frac{1}{2}}. \quad (\text{since} \quad d\mu_1 \leq d\mu) \end{aligned} \tag{5.27}$$

To estimate the quantity $\mathfrak{T}_2$, we need some preparation. Let $z_1 \in \mathbb{B}(0,1) \cap \omega B_i$. Then we get

$$|x_i| \leq |x_i - z_1| + |z_1| < \frac{\omega}{N}|x_i| + 1,$$

which implies that

$$|x_i| < \frac{N}{N-\omega}.$$

Let $z_2 \in (X \setminus \mathbb{B}(0,1)) \cap \omega B_i$. Then we have

$$|x_i| \geq |z_2| - |x_i - z_2| > 1 - \frac{\omega}{N}|x_i|,$$

which ensures that

$$|x_i| > \frac{N}{N+\omega}.$$

These show that

$$\frac{N}{N+\omega} < |x_i| < \frac{N}{N-\omega},$$

and thus,

$$\frac{1}{N+\omega} < r_i < \frac{1}{N-\omega},$$

since $r_i = |x_i|/N$. Now, we conclude that for any $x \in \omega B_i$,

$$\begin{aligned}\frac{N^2-2N\omega-\omega^2}{N^2-\omega^2} &< |x_i| - |x-x_i| \le |x| \le |x-x_i| + |x_i| \\ &< \frac{N+\omega}{N-\omega}. \end{aligned} \tag{5.28}$$

We are ready to estimate $\mathfrak{T}_2$. Since (5.23) implies

$$\mathfrak{T}_2 \le \frac{C_{p,2} r_i}{\mu_2(\omega B_i)^{\frac{1}{2}}} \left(\int_{\omega B_i} \rho^2 d\mu_2\right)^{\frac{1}{2}},$$

and since (5.28) gives

$$\mu_2(\omega B_i) = \int_{\omega B_i} |z| dz \ge K r_i^2,$$

where $K = \frac{\pi\omega^2(N^2-2N\omega-\omega^2)}{N^2-\omega^2}$, we obtain that

$$\mathfrak{T}_2 \le \frac{C_{p,2}}{K^{\frac{1}{2}}} \left(\int_{\omega B_i} \rho^2 d\mu_2\right)^{\frac{1}{2}} \le \frac{C_{p,2}}{K^{\frac{1}{2}}} \left(\int_{\omega B_i} \rho^2 d\mu\right)^{\frac{1}{2}}, \tag{5.29}$$

where, in the second inequality, the fact $d\mu_2 \le d\mu$ is applied.

Let

$$K' = \frac{N C_{p,1}}{(2N+1)\pi^{\frac{1}{2}}} + \frac{C_{p,2}}{K^{\frac{1}{2}}}.$$

Then we conclude from (5.26), (5.27) and (5.29) that

$$⨍_{\frac{2N+1}{N} B_i} \left|u - u_{\frac{2N+1}{N} B_i}\right| d\mu \le K' \left(\int_{\omega B_i} \rho^2 d\mu\right)^{\frac{1}{2}}. \tag{5.30}$$

Let

$$M = \max\left\{\frac{6\sqrt{\pi} C_{p,1}}{\omega \log 3}, \frac{2\sqrt{2} C_{p,2}}{\pi^{\frac{1}{2}}\omega}, K'\right\}.$$

Then the combination of (5.24), (5.25) and (5.30) ensures that

$$⨍_{\frac{2N+1}{N} B_i} \left|u - u_{\frac{2N+1}{N} B_i}\right| d\mu \le M \left(\int_{\omega B_i} \rho^2 d\mu\right)^{\frac{1}{2}}, \tag{5.31}$$

and so, we get from (5.21) that

$$I_{i,1} \le H \left(\int_{\omega B_i} \rho^2 d\mu\right)^{\frac{1}{2}}, \tag{5.32}$$

where $H = M C_d^2 / C_0'$. This is an upper bound on $I_{i,1}$ which we need.

Next, we work on the quantity $I_{i,2}$. Since

$$I_{i,2} = \left|\mu\left(B_i\right)^{-1} \int_{B_i} \left(u - u_{\frac{2N+1}{N} B_i}\right) d\mu\right|,$$

we have that

$$\begin{aligned} I_{i,2} &\le \mu(B_i)^{-1} \int_{B_i} \left| u - u_{\frac{2N+1}{N}B_i} \right| d\mu \\ &\le \mu(B_i)^{-1} \int_{\frac{2N+1}{N}B_i} \left| u - u_{\frac{2N+1}{N}B_i} \right| d\mu \ \ (\text{since } B_i \subset (2+1/N)B_i) \\ &\le C_d^2 \fint_{\frac{2N+1}{N}B_i} \left| u - u_{\frac{2N+1}{N}B_i} \right| d\mu \qquad (\text{by (5.12)}) \\ &\le H' \left(\int_{\omega B_i} \rho^2 d\mu \right)^{\frac{1}{2}}, \qquad (\text{by (5.31)}) \end{aligned} \tag{5.33}$$

where $H' = MC_d^2$. This is an upper bound on $I_{i,2}$ which we need.

By letting $C_3 = H + H'$, we know from (5.20), (5.32) and (5.33) that (5.18) is true.

In the following, based on the relation (5.18), we are going to check the estimate (5.7) in Assertion 5.1.

First, we need a lower bound for $|x|$ for any $x \in E_t = (1/N)\overline{\mathbb{B}}(x_t)$. Let $x \in E_t$. Then it follows from (5.4) that

$$|x| \ge |x_t| - |x_t - x| \ge \frac{2N^2 t}{(N-1)(1-t)} - \frac{2Nt}{(N-1)(1-t)} = \frac{2Nt}{1-t}.$$

Since $u|_{E_t} \ge 1$ and $u|_{F_t} \le 0$, we have that

$$u_{E_t} - u_{F_t} \ge \fint_{E_t} (1+|x|)dx \ge \frac{2Nt}{1-t},$$

and so, we obtain from (5.5) and (5.9) that

$$k \le 2Nk_G(x_t, y_t) \le \frac{2Nt}{1-t} \le u_{E_t} - u_{F_t}. \tag{5.34}$$

Moreover, by (5.18), we get

$$u_{E_t} - u_{F_t} \le \sum_{i=0}^{k} |u_{B_{i+1}} - u_{B_i}| \le C_3(k+1)^{\frac{1}{2}} \left(\sum_{i=0}^{k} \int_{\omega B_i} \rho^2 d\mu \right)^{\frac{1}{2}}. \tag{5.35}$$

Then we deduce that

$$\begin{aligned} k &\le C_3(k+1)^{\frac{1}{2}} \left(\sum_{i=0}^{k} \int_{\omega B_i} \rho^2 d\mu \right)^{\frac{1}{2}} \quad (\text{by (5.34) and (5.35)}) \\ &\le C_3(k+1) \left(\int_G \rho^2 d\mu \right)^{\frac{1}{2}}. \qquad (\text{by (5.19)}) \end{aligned} \tag{5.36}$$

It follows from (5.11) that $1 + k \le 2k$, and thus, we infer from (5.36) that

$$\int_G \rho^2 d\mu \ge \left(\frac{1}{2C_3} \right)^2 = C,$$

from which the estimate (5.7) in Assertion 5.1 follows. Hence Proposition 5.1 is proved as well.

Now, we are ready to get a contradiction. Since (5.6) shows that for each $t \in (1-e^{-2}, 1)$, $k_G(x_t, y_t) \geq 2$, we know from (5.3) and Proposition 5.1 that

$$C \leq \frac{\tau}{k_G(x_t, y_t)}$$

hold for all $t \in (1-e^{-2}, 1)$. This is impossible since (5.6) ensures that $k_G(x_t, y_t) \to +\infty$ as $t \to 1^-$. This contradiction shows that G does not satisfy any (τ, λ)-*k-cap* condition. □

Similar arguments as in the proof of Example 5.1 guarantee that the following example is true as well.

Example 5.2. Suppose that $X = (\mathbb{R}^2 \setminus \{0\}, |\cdot|, \mu)$, where $|\cdot|$ denotes the Euclidean metric, the measure μ is defined by $d\mu(x) = (1+|x|)dx$, and dx stands for the usual Lebesgue measure on $\mathbb{R}^2$. Then the following statements are true.

(1) X is doubling and lower 2-regular, but not 2-regular;
(2) X is $\pi/\log 3$-uniform and $\pi/\log 3$-inner uniform;
(3) X is a locally 2-Loewner and 2-Loewner space;
(4) X satisfies a weak slice condition;
(5) X does not satisfy any (τ, λ)-*k-cap* condition with $\tau > 0$ and $0 < \lambda \leq 1/16$.

We end this paper with the following problem: Under the assumptions of Theorem 1.1, are the following statements quantitatively equivalent: (1) G is uniform and locally Q-Loewner; (2) G is Q-Loewner and Gromov hyperbolic?

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Declarations

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References

[1] Björn, J., Shanmugalingam, N.: Poincaré inequalities, uniform domains and extension properties for Newton–Sobolev functions in metric spaces. J. Math. Anal. Appl. **332**, 190–208 (2007)

[2] Bonk, M., Heinonen, J., Koskela, P.: Uniformizing Gromov hyperbolic spaces. Astérisque **270**, viii+99 pp (2001)

[3] Brania, A., Yang, S.: Domains with controlled modulus and quasiconformal mappings. Nonlinear Stud. **9**, 57–73 (2002)

[4] Bourdon, M., Pajot, H.: Poincaré inequalities and quasiconformal structure on the boundary of some hyperbolic buildings. Proc. Am. Math. Soc. **127**, 2315–2324 (1999)

[5] Buckley, S.: Quasiconformal images of Hölder domains. Ann. Acad. Sci. Fenn. Math. **29**, 21–42 (2004)

[6] Buckley, S., Herron, D.: Uniform domain and capacity. Isr. J. Math. **158**, 129–157 (2007)

[7] Buckley, S., Herron, D.: Uniform spaces and weak slice spaces. Conform. Geom. Dyn. **11**, 191–206 (2007)

[8] Buckley, S., Koskela, P.: Criteria for imbeddings of Sobolev–Poincaré type. Int. Math. Res. Not. **18**, 881–901 (1996)

[9] Buckley, S., O'Shea, J.: Weighted Trudinger-type inequalities. Indiana Univ. Math. J. **48**, 85–114 (1999)

[10] Hariri, P., Klén, R., Vuorinen, M.: Conformally Invariant Metrics and Quasiconformal Mappings, Springer Monographs in Mathematics. Springer, Cham, xix+502 pp (2020)

[11] Heinonen, J.: Lectures on Analysis on Metric Spaces, Universitext. Springer, New York, x+144 pp (2001)

[12] Heinonen, J., Koskela, P.: Quasiconformal maps in metric spaces with controlled geometry. Acta Math. **181**, 1–61 (1998)

[13] Herron, D., Koskela, P.: Conformal capacity and the quasihyperbolic metric. Indiana Univ. Math. J. **45**, 333–359 (1996)

[14] Huang, X., Liu, J.: Quasihyperbolic metric and quasisymmetric mappings in metric spaces. Trans. Am. Math. Soc. **367**, 6225–6246 (2015)

[15] Huang, M., Rasila, A., Wang, X., Zhou, Q.: Semisolidity and locally weak quasisymmetry of homeomorphisms in metric spaces. Studia Math. **242**, 267–301 (2018)

[16] Huang, M., Rasila, A., Wang, X., Zhou, Q.: Quasisymmetry and quasihyperbolicity of mappings and John domains. Comput. Methods Funct. Theory (2022). https://doi.org/10.1007/s40315-022-00440-w

[17] Koskela, P.: Capacity Extension Domains. Dissertation, University of Jyväskylä, Jyväskylä, 1990. Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes, No. 73 , 45 pp (1990)

[18] Laakso, T.J.: Ahlfors Q-regular spaces with arbitrary $Q>1$ admitting weak Poincaré inequality. Geom. Funct. Anal. **10**, 111–123 (2000)

[19] Martio, O., Sarvas, J.: Injectivity theorems in plane and space. Ann. Acad. Sci. Fenn. Ser. A I Math. **4**, 383–401 (1979)

[20] Väisälä, J.: Free quasiconformality in Banach spaces. I. Ann. Acad. Sci. Fenn. Ser. A I Math. **15**, 355–379 (1990)

[21] Väisälä, J.: Free quasiconformality in Banach spaces. II. Ann. Acad. Sci. Fenn. Ser. A I Math. **16**, 255–310 (1991)

[22] Yang, S.: A Sobolev extension domain that is not uniform. Manuscr. Math. **120**, 241–251 (2006)

Xiantao Wang and Zhiqiang Yang
MOE-LCSM, School of Mathematics and Statistics
Hunan Normal University
Changsha 410081, Hunan
People's Republic of China
e-mail: xtwang@hunnu.edu.cn;
931490599@qq.com

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