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Results in Mathematics

Characterizations of Composition Operators on Bloch and Hardy Type Spaces

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Abstract. The main purpose of this paper is to investigate characterizations of composition operators on Bloch and Hardy type spaces. Initially, we use general doubling weights to study the composition operators from harmonic Bloch type spaces on the unit disc $\mathbb{D}$ to pluriharmonic Hardy spaces on the Euclidean unit ball $\mathbb{B}^n$. Furthermore, we develop some new methods to study the composition operators from harmonic Bloch type spaces on $\mathbb{D}$ to pluriharmonic Bloch type spaces on $\mathbb{D}$. Additionally, some application to new characterizations of the composition operators between pluriharmonic Lipschitz type spaces to be bounded or compact will be presented. The obtained results of this paper provide the improvements and extensions of the corresponding known results.

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1. Introduction

The study of composition operators on various Banach spaces of holomorphic functions and planar harmonic functions is currently a very active field of complex and functional analysis (see [1,6,7,13,15,19,20,28,32,37]). This paper continues the study of previous work of authors [6,7] and is mainly motivated by the articles of Kwon [19], Pavlović [25], Wulan et al. [35] and Zhao [36]. First, we use general doubling weights to study the composition operators from harmonic Bloch type spaces on the unit disc $\mathbb{D}$ to pluriharmonic Hardy spaces on the Euclidean unit ball $\mathbb{B}^n$. In addition, we develop some new methods to study the composition operators from harmonic Bloch type spaces on $\mathbb{D}$ to pluriharmonic Bloch type spaces on $\mathbb{D}$. At last, some application to new characterizations of the composition operators between pluriharmonic Lipschitz type spaces to be bounded or compact will be given. The obtained results of this paper provide the improvements and extensions of the corresponding known results. In particular, we improve and extend the main results of Chen et al. [7] and Kwon [19], and we also establish a completely different characterization from Pavlović [25]. In order to state our main results, we need to recall some basic definitions and introduce some necessary terminologies.

Let $\mathbb{C}^n$ be the complex space of dimension n, and let $\mathbb{C} := \mathbb{C}^1$ be the complex plane, where n is a positive integer. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, we write $\langle z, w\rangle := \sum_{k=1}^{n} z_k \overline{w}_k$ and $|z| := \langle z, z\rangle^{1/2}$. For $a \in \mathbb{C}^n$, we set $\mathbb{B}^n(a, r) = \{z \in \mathbb{C}^n : |z - a| < r\}$. In particular, let $\mathbb{B}^n := \mathbb{B}^n(0, 1)$ and $\mathbb{D} := \mathbb{B}^1$.

A twice continuously differentiable complex-valued function f defined on a domain $\Omega \subset \mathbb{C}^n$ is called a pluriharmonic function if for each fixed $z \in \Omega$ and $\theta \in \partial\mathbb{B}^n$, the function $f(z + \zeta\theta)$ is harmonic in $\{\zeta : |\zeta| < d_\Omega(z)\}$, where $d_\Omega(z)$ is the distance from z to the boundary $\partial\Omega$ of Ω. Recently, pluriharmonic functions have been widely studied (cf. [4,8,10,16,29–31,34]). If $\Omega \subset \mathbb{C}^n$ is a simply connected domain, then a function $f : \Omega \to \mathbb{C}$ is pluriharmonic if and only if f has a decomposition $f = h + \overline{g}$, where h and g are holomorphic in Ω (see [34]). This decomposition is unique up to an additive constant. From this decomposition, it is easy to know that the class of pluriharmonic functions is broader than that of holomorphic functions. Furthermore, a twice continuously differentiable real-valued function in a simply connected domain Ω is pluriharmonic if and only if it is the real part of some holomorphic function on Ω. Obviously, all pluriharmonic functions are harmonic. In particular, if $n = 1$, then the converse holds (cf. [9]). Throughout this paper, we use $\mathscr{H}(\Omega)$ and $\mathscr{P}\mathscr{H}(\Omega)$ to denote the set of all holomorphic functions of a domain $\Omega \subset \mathbb{C}^n$ into $\mathbb{C}$ and that of all pluriharmonic functions of Ω into $\mathbb{C}$, respectively.

1.1. Hardy Spaces

For $p \in (0, \infty]$, the pluriharmonic Hardy space $\mathscr{P}\mathscr{H}^p(\mathbb{B}^n)$ consists of all those functions $f \in \mathscr{P}\mathscr{H}(\mathbb{B}^n)$ such that, for $p \in (0, \infty)$,

$$\|f\|_p := \sup_{r\in[0,1)} M_p(r, f) < \infty,$$

and, for $p = \infty$,

$$\|f\|_\infty := \sup_{r\in[0,1)} M_\infty(r, f) < \infty,$$

where

$$M_p(r, f) = \left(\int_{\partial\mathbb{B}^n} |f(r\zeta)|^p d\sigma(\zeta)\right)^{\frac{1}{p}},\ M_\infty(r, f) = \sup_{\zeta\in\partial\mathbb{B}^n} |f(r\zeta)|$$

and $d\sigma$ denotes the normalized Lebesgue surface measure on $\partial\mathbb{B}^n$. In particular, we use $\mathscr{H}^p(\mathbb{B}^n) := \mathscr{H}(\mathbb{B}^n) \cap \mathscr{PH}^p(\mathbb{B}^n)$ to denote the holomorphic Hardy space. If $f \in \mathscr{PH}^p(\mathbb{B}^n)$ for some $p \in (1, \infty)$, then the radial limits

$$f(\zeta) = \lim_{r \to 1^-} f(r\zeta)$$

exist for almost every $\zeta \in \partial\mathbb{B}^n$ (see [3, Theorems 6.7, 6.13 and 6.39]). Moreover, if $p \in [1, \infty)$, then $\mathscr{PH}^p(\mathbb{B}^n)$ is a normed space with respect to the norm $\|\cdot\|_p$ and if $p \in (0, 1)$, then $\mathscr{PH}^p(\mathbb{B}^n)$ is a metric space with respect to the metric $d(f, g) = \|f - g\|_p^p$.

1.2. Bloch Type Spaces

For a pluriharmonic function f of $\mathbb{B}^n$ into $\mathbb{C}$, let

$$\nabla f := \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right), \ \overline{\nabla} f := \left(\frac{\partial f}{\partial \overline{z}_1}, \ldots, \frac{\partial f}{\partial \overline{z}_n}\right)$$

and

$$\Lambda_f(z) := \max_{\theta \in \partial\mathbb{B}^n} |\langle \nabla f(z), \overline{\theta} \rangle + \langle \overline{\nabla} f(z), \theta \rangle|$$

for $z = (z_1, \ldots, z_n) \in \mathbb{B}^n$.

A continuous non-decreasing function $\omega : [0, 1) \to (0, \infty)$ is called a weight if ω is unbounded (see [1]). Moreover, a weight ω is called doubling if there is a constant $C > 1$ such that

$$\omega(1 - s/2) < C\omega(1 - s)$$

for $s \in (0, 1]$.

For a weight ω, we use $\mathscr{B}_\omega(\mathbb{B}^n)$ to denote the pluriharmonic Bloch type space consisting of all complex-valued pluriharmonic functions defined in $\mathbb{B}^n$ with the norm

$$\|f\|_{\mathscr{B}_\omega(\mathbb{B}^n)} := |f(0)| + \sup_{z \in \mathbb{B}^n} \mathscr{B}_\omega^f(z) < \infty,$$

where $\mathscr{B}_\omega^f(z) = \Lambda_f(z)/\omega(|z|)$. It is easy to know that $\mathscr{B}_\omega(\mathbb{B}^n)$ is a complex Banach space. Furthermore, let

$$\|f\|_{\mathscr{B}_\omega(\mathbb{B}^n),s} := \sup_{z \in \mathbb{B}^n} \mathscr{B}_\omega^f(z) < \infty$$

be the semi-norm. If $f \in \mathscr{B}_\omega(\mathbb{B}^n)$, then we call f a pluriharmonic Bloch function.

1.3. Composition Operators

Given a holomorphic function ϕ of $\mathbb{B}^n$ into $\mathbb{D}$, the composition operator $C_\phi : \mathscr{PH}(\mathbb{D}) \to \mathscr{PH}(\mathbb{B}^n)$ is defined by

$$C_\phi(f) = f \circ \phi,$$

where $f \in \mathscr{PH}(\mathbb{D})$.

Shapiro [32] gave a complete characterization of compact composition operators on $\mathscr{H}^2(\mathbb{D})$, with a number of interesting consequences for peak sets, essential norm of composition operators, and so on. Recently, the studies of composition operators on holomorphic function spaces have been attracted much attention of many mathematicians (see [1,6,7,13,15,19,20,28,37]). In particular, Kwon [19] investigated some characterizations of composition operators from the holomorphic Bloch spaces to the holomorphic Hardy spaces to be bounded or compact. Let us recall the main result in [19] as follows.

For $\zeta \in \partial\mathbb{B}^n$ and $\alpha \in (1,\infty)$, we use $D^n_\alpha(\zeta)$ to denote the Koranyi approach domain defined by

$$D^n_\alpha(\zeta) = \left\{ z \in \mathbb{B}^n : |1 - \langle z, \zeta\rangle| < \frac{\alpha}{2}(1-|z|^2) \right\}.$$

For $\phi : \mathbb{B}^n \to \mathbb{D}$ holomorphic, let

$$M_\alpha\phi(\zeta) = \sup\left\{ \log\frac{1}{1-|\phi(z)|^2} : z \in D^n_\alpha(\zeta) \right\}$$

be the maximal function (see [17]). Let $\mathrm{Aut}(\mathbb{B}^n)$ denote the group of holomorphic automorphisms of $\mathbb{B}^n$.

Theorem A [19, Theorems 5.1 and 5.10]. *Let $p \in (0,\infty)$, $1 < \alpha, \beta < \infty$ and $\omega(t) = 1/(1-t^2)$ for $t \in [0,1)$. If $\phi : \mathbb{B}^n \to \mathbb{D}$ is holomorphic, then the followings are equivalent:*

(1) $\sup_{r\in(0,1)} \int_{\partial\mathbb{B}^n} \left(\frac{1}{2}\log\frac{1+|\phi(r\zeta)|}{1-|\phi(r\zeta)|}\right)^{\frac{p}{2}} d\sigma(\zeta) < \infty$;

(2) $\int_{\partial\mathbb{B}^n} (M_\beta\phi(\zeta))^{\frac{p}{2}} d\sigma(\zeta) < \infty$;

(3) $\int_{\partial\mathbb{B}^n} \left(\int_0^1 \frac{|\nabla(\phi\circ\varphi_{r\zeta})(0)|^2}{(1-|\phi(r\zeta)|^2)^2}\frac{dr}{1-r}\right)^{\frac{p}{2}} d\sigma(\zeta) < \infty$, *where $\varphi_{r\zeta} \in Aut(\mathbb{B}^n)$ with $\varphi_{r\zeta}(0) = r\zeta$;*

(4) $\int_{\partial\mathbb{B}^n} \left(\int_{D^n_\alpha(\zeta)} \frac{|\nabla(\phi\circ\varphi_z)(0)|^2}{(1-|\phi(z)|^2)^2}\frac{dV(z)}{(1-|z|^2)^{n+1}}\right)^{\frac{p}{2}} d\sigma(\zeta) < \infty$, *where dV denotes the Lebesgue volume measure of $\mathbb{C}^n$, and $\varphi_z \in Aut(\mathbb{B}^n)$ with $\varphi_z(0) = z$;*

(5) $C_\phi : \mathscr{B}_\omega(\mathbb{D}) \cap \mathscr{H}(\mathbb{D}) \to \mathscr{H}^p(\mathbb{B}^n)$ *is a bounded operator;*

(6) $C_\phi : \mathscr{B}_\omega(\mathbb{D}) \cap \mathscr{H}(\mathbb{D}) \to \mathscr{H}^p(\mathbb{B}^n)$ *is compact.*

Also, the study of composition operators and linear operators on holomorphic function spaces by using weight has aroused great interest of many mathematicians (cf. [7,13,22,27]). However, there are few literatures on the theory of composition operators of harmonic functions. In the following, by using general doubling weights, we will establish the characterizations of composition operators from harmonic Bloch type spaces on $\mathbb{D}$ to pluriharmonic Hardy spaces on $\mathbb{B}^n$ to be bounded or compact.

Theorem 1.1. *Let $p \in (0,\infty)$, $\alpha \in (1,\infty)$ and ω be a doubling function. If $\phi : \mathbb{B}^n \to \mathbb{D}$ is holomorphic, then the followings are equivalent:*

(1) $C_\phi : \mathscr{B}_\omega(\mathbb{D}) \to \mathscr{P}\mathscr{H}^p(\mathbb{B}^n)$ *is a bounded operator;*

(2) $\int_{\partial\mathbb{B}^n}\left(\int_0^1 |\nabla\phi(r\zeta)|^2\omega^2(|\phi(r\zeta)|)(1-r)dr\right)^{\frac{p}{2}} d\sigma(\zeta)<\infty$;
(3) $\int_{\partial\mathbb{B}^n}\left(\int_{D_\alpha^n(\zeta)} |\nabla\phi(z)|^2\omega^2(|\phi(z)|)(1-|z|)^{1-n}dV(z)\right)^{\frac{p}{2}} d\sigma(\zeta)<\infty$;
(4) $C_\phi:\mathscr{B}_\omega(\mathbb{D})\to\mathscr{P}\mathscr{H}^p(\mathbb{B}^n)$ *is compact.*

If we take $\omega(t)=1/(1-t^2)$ for $t\in[0,1)$ in Theorem 1.1, then we extend Theorem A into the following form.

Corollary 1.2. *Let $p\in(0,\infty)$ and $1<\alpha,\beta<\infty$. If $\phi:\mathbb{B}^n\to\mathbb{D}$ is holomorphic, then the followings are equivalent:*

(1) $\sup_{r\in(0,1)}\int_{\partial\mathbb{B}^n}\left(\frac{1}{2}\log\frac{1+|\phi(r\zeta)|}{1-|\phi(r\zeta)|}\right)^{\frac{p}{2}} d\sigma(\zeta)<\infty$*;*
(2) $\int_{\partial\mathbb{B}^n}(M_\beta\phi(\zeta))^{\frac{p}{2}} d\sigma(\zeta)<\infty$*;*
(3) $\int_{\partial\mathbb{B}^n}\left(\int_0^1\frac{|\nabla(\phi\circ\varphi_{r\zeta})(0)|^2}{(1-|\phi(r\zeta)|^2)^2}\frac{dr}{1-r}\right)^{\frac{p}{2}} d\sigma(\zeta)<\infty$*, where* $\varphi_{r\zeta}\in Aut(\mathbb{B}^n)$ *with* $\varphi_{r\zeta}(0)=r\zeta$*;*
(4) $\int_{\partial\mathbb{B}^n}\left(\int_{D_\alpha^n(\zeta)}\frac{|\nabla(\phi\circ\varphi_z)(0)|^2}{(1-|\phi(z)|^2)^2}\frac{dV(z)}{(1-|z|^2)^{n+1}}\right)^{\frac{p}{2}} d\sigma(\zeta)<\infty$*, where* $\varphi_z\in Aut(\mathbb{B}^n)$ *with* $\varphi_z(0)=z$*;*
(5) $\int_{\partial\mathbb{B}^n}\left(\int_0^1 |\nabla\phi(r\zeta)|^2\frac{(1-r)}{(1-|\phi(r\zeta)|^2)^2}dr\right)^{\frac{p}{2}} d\sigma(\zeta)<\infty$;
(6) $\int_{\partial\mathbb{B}^n}\left(\int_{D_\alpha^n(\zeta)}\frac{|\nabla\phi(z)|^2}{(1-|\phi(z)|^2)^2}(1-|z|)^{1-n}dV(z)\right)^{\frac{p}{2}} d\sigma(\zeta)<\infty$;
(7) $C_\phi:\mathscr{B}_\omega(\mathbb{D})\to\mathscr{P}\mathscr{H}^p(\mathbb{B}^n)$ *is a bounded operator;*
(8) $C_\phi:\mathscr{B}_\omega(\mathbb{D})\to\mathscr{P}\mathscr{H}^p(\mathbb{B}^n)$ *is compact.*

In the case ϕ maps $\mathbb{D}$ into $\mathbb{D}$, Chen et al. [6] proved the following result.

Theorem B [6, Theorem 6]. *Let* $\omega(t)=1/\left((1-t^2)^\alpha\left(\log\frac{e}{1-t^2}\right)^\beta\right)$ *and* $\phi:\mathbb{D}\to\mathbb{D}$ *be an analytic function, where* $\alpha\in(0,\infty)$ *and* $\beta\le\alpha$*. Then the followings are equivalent:*

(1) $C_\phi:\mathscr{B}_\omega(\mathbb{D})\cap\mathscr{H}(\mathbb{D})\to\mathscr{P}\mathscr{H}^2(\mathbb{D})$ *is a bounded operator;*
(2) $\frac{1}{2\pi}\int_0^{2\pi}\int_0^1\frac{|\phi'(re^{i\theta})|^2}{(1-|\phi(re^{i\theta})|)^{2\alpha}\left(\log\frac{e}{1-|\phi(re^{i\theta})|}\right)^{2\beta}}(1-r)\,dr\,d\theta<\infty.$

A continuous increasing function $\psi:[0,\infty)\to[0,\infty)$ with $\psi(0)=0$ is called a majorant if $\psi(t)/t$ is non-increasing for $t>0$ (see [11,24]). By using a special doubling function $\omega(t)=1/\psi\left((1-t^2)^\alpha\left(\log\frac{e}{1-t^2}\right)^\beta\right)$ for $t\in[0,1)$, the characterization of composition operators from $\mathscr{B}_\omega(\mathbb{D})$ to $\mathscr{P}\mathscr{H}^p(\mathbb{D})$ to be bounded or compact was established in [7] as follows, which is the improvement of Theorem B, where $p\in(0,\infty)$, $\alpha\in(0,\infty)$ and $\beta\in(-\infty,\alpha]$ are constants.

Theorem C [7, Theorem 2.4]. *Let* $p\in(0,\infty)$*,* $\alpha\in(0,\infty)$*,* $\beta\in(-\infty,\alpha]$ *and* $\omega(t)=1/\psi\left((1-t^2)^\alpha\left(\log\frac{e}{1-t^2}\right)^\beta\right)$ *for* $t\in[0,1)$*, where* ψ *is a majorant. If* $\phi:\mathbb{D}\to\mathbb{D}$ *is a holomorphic function, then the followings are equivalent:*

(1) $C_\phi : \mathscr{B}_\omega(\mathbb{D}) \to \mathscr{P}\mathscr{H}^p(\mathbb{D})$ *is a bounded operator;*
(2) $\int_0^{2\pi} \left(\int_0^1 |\phi'(re^{i\theta})|^2 \omega^2(|\phi(re^{i\theta})|)(1-r)dr\right)^{\frac{p}{2}} \frac{d\theta}{2\pi} < \infty$;
(3) $C_\phi : \mathscr{B}_\omega(\mathbb{D}) \to \mathscr{P}\mathscr{H}^p(\mathbb{D})$ *is compact.*

In the following, by using a general doubling weight, we give the characterizations of composition operators from harmonic Bloch type spaces on $\mathbb{D}$ to harmonic Hardy spaces on $\mathbb{D}$ to be bounded or compact, which is an improvement of Theorem C.

Also, the characterizations (3), (4) and (5) are new.

Theorem 1.3. *Let $p \in (0, \infty)$, ω be a doubling function and $\phi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. Then the followings are equivalent:*

(1) $C_\phi : \mathscr{B}_\omega(\mathbb{D}) \to \mathscr{P}\mathscr{H}^p(\mathbb{D})$ *is a bounded operator;*
(2) $\int_0^{2\pi} \left(\int_0^1 |\phi'(re^{i\theta})|^2 \omega^2(|\phi(re^{i\theta})|)(1-r)dr\right)^{\frac{p}{2}} \frac{d\theta}{2\pi} < \infty$;
(3) $\int_0^{2\pi} \left(\sum_{k=0}^{\infty} 2^{-2k} |\phi'(r_k e^{i\theta})|^2 \omega^2(|\phi(r_k e^{i\theta})|)\right)^{\frac{p}{2}} \frac{d\theta}{2\pi} < \infty$, *where* $r_k = 1 - 2^{-k}$;
(4) $\int_0^{2\pi} \left(\int_0^1 (1-r) \sup_{0<\rho<r} \left(|\phi'(\rho\, e^{i\theta})|^2 \omega^2(|\phi(\rho\, e^{i\theta})|)\right) dr\right)^{\frac{p}{2}} \frac{d\theta}{2\pi} < \infty$;
(5) $\int_0^{2\pi} \left(\int_{D_\alpha^1(\zeta)} |\nabla \phi(z)|^2 \omega^2(|\phi(z)|) dA(z)\right)^{\frac{p}{2}} \frac{d\theta}{2\pi} < \infty$, *where dA denotes the Lebesgue area measure of $\mathbb{C}$;*
(6) $C_\phi : \mathscr{B}_\omega(\mathbb{D}) \to \mathscr{P}\mathscr{H}^p(\mathbb{D})$ *is compact.*

Recently, the studies of composition operators between the classical analytic Bloch spaces have attracted much attention of many mathematicians (cf. [20–22,35,36]). In particular, Zhao [36] gave characterizations of composition operators from the analytic α-Bloch space $\mathscr{B}_{\omega_1}(\mathbb{D}) \cap \mathscr{H}(\mathbb{D})$ to the analytic β-Bloch space $\mathscr{B}_{\omega_2}(\mathbb{D}) \cap \mathscr{H}(\mathbb{D})$ to be bounded or compact, where $\alpha, \beta \in (0, \infty)$, $\omega_1(t) = 1/(1-t^2)^\alpha$ and $\omega_2(t) = 1/(1-t^2)^\beta$ for $t \in [0, 1)$. Pavlović [25] gave some derivative-free characterizations of bounded composition operators between analytic Lipschitz spaces. In the following, we will develop some new methods to give new characterizations of the composition operators between the Bloch type spaces with weights to be bounded or compact, and give some application to new characterizations of the composition operators between the Lipschitz spaces to be bounded or compact.

For $k \in \{1, 2, \ldots\}$ and a weight ω, let

$$\mathscr{E}_\omega(k) = \left\{\varrho_k : \mu_{\omega,k}(\varrho_k) = \max_{x \in [0,1)} \mu_{\omega,k}(x)\right\},$$

where $\mu_{\omega,k}(x) = x^{k-1}/\omega(x), x \in [0, 1)$.

Proposition 1.4. *Let $r_1 = 0 \in \mathscr{E}_\omega(1)$ and let $r_k \in \mathscr{E}_\omega(k)$, $k \in \{2, 3, \ldots\}$, be arbitrarily chosen. Then $\{r_k\}$ is a non-decreasing sequence.*

Theorem 1.5. *For $k \in \{1, 2, \ldots\}$, suppose that ω_1 is a weight so that a point r_k can be selected from each set $\mathscr{E}_{\omega_1}(k)$ with $r_1 = 0$ and*

$$\lim_{k\to\infty} r_k^{k-1} = \gamma > 0.$$

Let ω_2 be a weight, and let ϕ be a holomorphic function of $\mathbb{B}^n$ into $\mathbb{D}$. Then

(1) *$C_\phi : \mathscr{B}_{\omega_1}(\mathbb{D}) \to \mathscr{B}_{\omega_2}(\mathbb{B}^n)$ is bounded if and only if*

$$\sup_{k\geq 1} \frac{\omega_1(r_k)}{k} \|\phi^k\|_{\mathscr{B}_{\omega_2}(\mathbb{B}^n)} < \infty. \tag{1.1}$$

(2) *$C_\phi : \mathscr{B}_{\omega_1}(\mathbb{D}) \to \mathscr{B}_{\omega_2}(\mathbb{B}^n)$ is compact if and only if*

$$\lim_{k\to\infty} \frac{\omega_1(r_k)}{k} \|\phi^k\|_{\mathscr{B}_{\omega_2}(\mathbb{B}^n)} = 0. \tag{1.2}$$

For $\alpha \in (0, 1]$, the Lipschitz space $\mathscr{L}_\alpha(\mathbb{B}^n)$ consists of all those functions $f \in \mathcal{C}(\mathbb{B}^n)$ satisfying

$$\|f\|_{\mathscr{L}_\alpha(\mathbb{B}^n),s} := \sup_{z,w\in\mathbb{B}^n, z\neq w} \frac{|f(z) - f(w)|}{|z - w|^\alpha} < \infty,$$

where $\mathcal{C}(\mathbb{B}^n)$ is a set of all continuous functions defined in $\mathbb{B}^n$.

By using Theorem 1.5, we give a characterization of the composition operators between pluriharmonic Lipschitz type spaces to be bounded or compact which is completely different from Pavlović [25] as follows.

Theorem 1.6. *Suppose that $k \in \{1, 2, \ldots\}$, $\alpha, \beta \in (0, 1)$ and $\omega(t) = 1/(1-t)^{1-\beta}$ for $t \in [0, 1)$. Let ϕ be a holomorphic function of $\mathbb{B}^n$ into $\mathbb{D}$. Then,*

(1) *$C_\phi : \mathscr{L}_\alpha(\mathbb{D}) \cap \mathscr{PH}(\mathbb{D}) \to \mathscr{L}_\beta(\mathbb{B}^n) \cap \mathscr{PH}(\mathbb{B}^n)$ is bounded if and only if*

$$\sup_{k\geq 1} \left\{ k^{-\alpha} \|\phi^k\|_{\mathscr{B}_\omega(\mathbb{B}^n)} \right\} < \infty;$$

(2) *$C_\phi : \mathscr{L}_\alpha(\mathbb{D}) \cap \mathscr{PH}(\mathbb{D}) \to \mathscr{L}_\beta(\mathbb{B}^n) \cap \mathscr{PH}(\mathbb{B}^n)$ is compact if and only if*

$$\lim_{k\to\infty} \left\{ k^{-\alpha} \|\phi^k\|_{\mathscr{B}_\omega(\mathbb{B}^n)} \right\} = 0.$$

The proofs of Theorems 1.1–1.6 and Proposition 1.4 will be presented in Sect. 2.

2. The Proofs of the Main Results

Denote by $L^p(\partial\mathbb{B}^n)$ ($p \in (0, \infty)$) the set of all measurable functions F of $\partial\mathbb{B}^n$ into $\mathbb{C}$ with

$$\|F\|_{L^p} = \left(\int_{\partial\mathbb{B}^n} |F(\zeta)|^p d\sigma(\zeta) \right)^{\frac{1}{p}} < \infty.$$

Given $f \in \mathscr{H}^p(\mathbb{B}^n)$, the Littlewood–Paley type $\mathscr{G}$-function is defined as follows

$$\mathscr{G}(f)(\zeta) = \left(\int_0^1 |\nabla f(r\zeta)|^2 (1-r) dr \right)^{\frac{1}{2}}, \quad \zeta \in \partial\mathbb{B}^n.$$

Then

$$f \in \mathscr{H}^p(\mathbb{B}^n) \quad \text{if and only if} \quad \mathscr{G}(f) \in L^p(\partial\mathbb{B}^n) \tag{2.1}$$

for $p \in (0, \infty)$ (see [2, 18, 33]). The conclusion of (2.1) also can be rewritten in the following form. There exists a positive constant C, depending only on p, such that

$$\frac{1}{C}\|f\|_p^p \leq |f(0)|^p + \int_{\partial\mathbb{B}^n} \big(\mathscr{G}(f)(\zeta)\big)^p d\sigma(\zeta) \leq C\|f\|_p^p \tag{2.2}$$

for $p \in (0, \infty)$ (see [2, 18, 33]). For $\alpha \in (1, \infty)$ and $p \in (0, \infty)$, it follows from [2, Theorem 3.1] (or [18, Theorem 1.1]) that there is a positive constant C such that

$$\begin{aligned} \frac{1}{C}\int_{\partial\mathbb{B}^n} \big(\mathscr{G}(f)(\zeta)\big)^p d\sigma(\zeta) &\leq \int_{\partial\mathbb{B}^n} \left(\mathscr{A}_\alpha f(\zeta)\right)^p d\sigma(\zeta) \\ &\leq C\int_{\partial\mathbb{B}^n} \big(\mathscr{G}(f)(\zeta)\big)^p d\sigma(\zeta), \end{aligned} \tag{2.3}$$

where $f \in \mathscr{H}^p(\mathbb{B}^n)$ and

$$\mathscr{A}_\alpha f(\zeta) = \left(\int_{D_\alpha^n(\zeta)} |\nabla f(z)|^2 (1-|z|)^{1-n} dV(z) \right)^{1/2}.$$

It follows from (2.2) and (2.3) that there is a positive constant C such that

$$\frac{1}{C}\|f\|_p^p \leq |f(0)|^p + \int_{\partial\mathbb{B}^n} \left(\mathscr{A}_\alpha f(\zeta)\right)^p d\sigma(\zeta) \leq C\|f\|_p^p. \tag{2.4}$$

The following result easily follows from [1, Lemma 1] and [1, Theorem 2].

Lemma 2.1. *Let ω be a doubling function. Then there exist functions $f_j \in \mathscr{B}_\omega(\mathbb{D}) \cap \mathscr{H}(\mathbb{D})$ $(j \in \{1, 2\})$ such that, for $z \in \mathbb{D}$,*

$$\sum_{j=1}^{2} |f_j'(z)| \geq \omega(|z|).$$

The following result is well-known.

Lemma D cf. [6, Lemma 5]. *Suppose that $a, b \in [0, \infty)$ and $q \in (0, \infty)$. Then*

$$(a+b)^q \leq 2^{\max\{q-1,0\}}(a^q + b^q).$$

2.1. The Proof of Theorem 1.1

We first prove (1) ⇒ (2). By Lemma 2.1, there exist functions $f_j \in \mathscr{B}_\omega(\mathbb{D}) \cap \mathscr{H}(\mathbb{D})$ $(j \in \{1,2\})$ such that, for $z \in \mathbb{D}$,

$$\sum_{j=1}^{2} |f_j'(z)| \geq \omega(|z|). \tag{2.5}$$

Since $C_\phi : \mathscr{B}_\omega(\mathbb{D}) \to \mathscr{P}\mathscr{H}^p(\mathbb{B}^n)$ is a bounded operator, by (2.1), we see that

$$\begin{aligned}\infty &> \int_{\partial\mathbb{B}^n} \big(\mathscr{G}(C_\phi(f_j))(\zeta)\big)^p d\sigma(\zeta)\\ &= \int_{\partial\mathbb{B}^n} \left(\int_0^1 |f_j'(\phi(r\zeta))|^2 |\nabla\,\phi(r\zeta)|^2 (1-r)dr\right)^{\frac{p}{2}} d\sigma(\zeta)\end{aligned} \tag{2.6}$$

for $j \in \{1,2\}$. It follows from (2.5), (2.6) and Lemma D that

$$\begin{aligned}\infty &> \sum_{j=1}^{2} \int_{\partial\mathbb{B}^n} \big(\mathscr{G}(C_\phi(f_j))(\zeta)\big)^p d\sigma(\zeta)\\ &\geq \mathscr{M}_p \int_{\partial\mathbb{B}^n} \left(\int_0^1 \left(\sum_{j=1}^{2} |f_j'(\phi(r\zeta))|\right)^2 |\nabla\,\phi(r\zeta)|^2 (1-r)dr\right)^{\frac{p}{2}} d\sigma(\zeta)\\ &\geq \mathscr{M}_p \int_{\partial\mathbb{B}^n} \left(\int_0^1 \omega^2(|\phi(r\zeta)|) |\nabla\,\phi(r\zeta)|^2 (1-r)dr\right)^{\frac{p}{2}} d\sigma(\zeta),\end{aligned}$$

where $\mathscr{M}_p = 2^{-p/2-\max\{p/2-1,0\}}$.

Next, we prove (2) ⇒ (1). We split the proof of this case into two steps.

Step 1. We first prove $C_\phi(f) \in \mathscr{P}\mathscr{H}^p(\mathbb{B}^n)$. Since $\mathbb{D}$ is a simply connected domain, we see that f admits the canonical decomposition $f = f_1 + \overline{f_2}$, where f_1 and f_2 are analytic in $\mathbb{D}$ with $f_2(0) = 0$. For $j \in \{1,2\}$, elementary calculations lead to

$$\begin{aligned}|\nabla C_\phi(f_j)(z)| &= |f_j'(\phi(z))||\nabla\,\phi(z)| \leq \|f_j\|_{\mathscr{B}_\omega(\mathbb{D})} |\nabla\,\phi(z)| \omega(|\phi(z)|)\\ &\leq \|f\|_{\mathscr{B}_\omega(\mathbb{D})} |\nabla\,\phi(z)| \omega(|\phi(z)|).\end{aligned} \tag{2.7}$$

For $j \in \{1,2\}$, let

$$\mathscr{Y}_j = \int_{\partial\mathbb{B}^n} \big(\mathscr{G}(C_\phi(f_j))(\zeta)\big)^p d\sigma(\zeta).$$

Consequently, there is a constant $C=\|f\|^p_{\mathscr{B}_\omega(\mathbb{D})}$ such that

$$\begin{aligned}\mathscr{Y}_j&=\int_{\partial\mathbb{B}^n}\left(\int_0^1|f_j'(\phi(r\zeta))|^2|\nabla\,\phi(r\zeta)|^2(1-r)dr\right)^{\frac{p}{2}}d\sigma(\zeta)\\&\leq C\int_{\partial\mathbb{B}^n}\left(\int_0^1\omega^2(|\phi(r\zeta)|)|\nabla\,\phi(r\zeta)|^2(1-r)dr\right)^{\frac{p}{2}}d\sigma(\zeta)\\&<\infty,\end{aligned}\tag{2.8}$$

which, together with (2.1), implies that $C_\phi(f_j)\in\mathscr{H}^p(\mathbb{B}^n)$ for $j\in\{1,2\}$. Hence $C_\phi(f)\in\mathscr{P}\mathscr{H}^p(\mathbb{B}^n)$.

Step 2. In this step, we will show that C_ϕ is a bounded operator. Without loss of generality, we assume that $\|f\|_{\mathscr{B}_\omega(\mathbb{D})}\neq 0$, and f_j are not constant functions for $j\in\{1,2\}$. Since, for $j\in\{1,2\}$,

$$\begin{aligned}|C_\phi(f_j)(0)|&\leq|f_j(0)|+|f_j(\phi(0))-f_j(0)|\\&\leq|f_j(0)|+|\phi(0)|\int_0^1|f_j'(\phi(0)t)|dt\\&\leq|f_j(0)|+\|f_j\|_{\mathscr{B}_\omega(\mathbb{D})}|\phi(0)|\int_0^1\omega(|\phi(0)|t)dt\\&\leq\|f_j\|_{\mathscr{B}_\omega(\mathbb{D})}\left(1+|\phi(0)|\omega(|\phi(0)|)\right),\end{aligned}\tag{2.9}$$

by (2.2) and (2.8), we see that there is a positive constant C such that

$$\begin{aligned}\|C_\phi(f_j)\|_p^p&\leq C\left(|C_\phi(f_j)(0)|^p+\int_{\partial\mathbb{B}^n}\big(\mathscr{G}(C_\phi(f_j))(\zeta)\big)^pd\sigma(\zeta)\right)\\&\leq C\|f_j\|^p_{\mathscr{B}_\omega(\mathbb{D})}C(\phi),\end{aligned}\tag{2.10}$$

where

$$\begin{aligned}C(\phi)&=\big(1+|\phi(0)|\omega(|\phi(0)|)\big)^p\\&\quad+\int_{\partial\mathbb{B}^n}\left(\int_0^1\omega^2(|\phi(r\zeta)|)|\nabla\,\phi(r\zeta)|^2(1-r)dr\right)^{\frac{p}{2}}d\sigma(\zeta).\end{aligned}$$

Combining (2.10) and Lemma D gives

$$\begin{aligned}\|C_\phi(f)\|_p^p&\leq 2^{\max\{p-1,0\}}\left(\|C_\phi(f_1)\|_p^p+\|C_\phi(f_2)\|_p^p\right)\\&\leq 2^{\max\{p-1,0\}}CC(\phi)\big(\|f_1\|^p_{\mathscr{B}_\omega(\mathbb{D})}+\|f_2\|^p_{\mathscr{B}_\omega(\mathbb{D})}\big)\\&\leq 2^{1+\max\{p-1,0\}}CC(\phi)\|f\|^p_{\mathscr{B}_\omega(\mathbb{D})}.\end{aligned}$$

Therefore, $C_\phi:\mathscr{B}_\omega(\mathbb{D})\to\mathscr{P}\mathscr{H}^p(\mathbb{B}^n)$ is a bounded operator.

Now, we prove (1) $\Rightarrow$ (3). By Lemma 2.1, there exist functions $f_j\in\mathscr{B}_\omega(\mathbb{D})\cap\mathscr{H}(\mathbb{D})$ ($j\in\{1,2\}$) such that, for $z\in\mathbb{D}$,

$$\sum_{j=1}^2|f_j'(z)|\geq\omega(|z|).\tag{2.11}$$

Since $C_\phi : \mathscr{B}_\omega(\mathbb{D}) \to \mathscr{P}\mathscr{H}^p(\mathbb{B}^n)$ is a bounded operator, by (2.3), we see that

$$\begin{aligned}\infty &> \int_{\partial\mathbb{B}^n} \left(\mathscr{A}_\alpha C_\phi(f_j)(\zeta)\right)^p d\sigma(\zeta)\\ &= \int_{\partial\mathbb{B}^n}\left(\int_{D_\alpha^n(\zeta)} |f_j'(\phi(z))|^2 |\nabla\,\phi(z)|^2(1-|z|)^{1-n} dV(z)\right)^{\frac{p}{2}} d\sigma(\zeta)\end{aligned} \tag{2.12}$$

for $j \in \{1,2\}$. It follows from (2.11), (2.12) and Lemma D that

$$\begin{aligned}\infty &> \sum_{j=1}^{2}\int_{\partial\mathbb{B}^n} \left(\mathscr{A}_\alpha C_\phi(f_j)(\zeta)\right)^p d\sigma(\zeta)\\ &\geq \mathscr{M}_p \int_{\partial\mathbb{B}^n}\left(\int_{D_\alpha^n(\zeta)}\left(\sum_{j=1}^{2}|f_j'(\phi(z)))|\right)^2 \mathscr{V}(z)dV(z)\right)^{\frac{p}{2}} d\sigma(\zeta)\\ &\geq \mathscr{M}_p \int_{\partial\mathbb{B}^n}\left(\int_{D_\alpha^n(\zeta)} \omega^2(|\phi(z)|)\mathscr{V}(z)dV(z)\right)^{\frac{p}{2}} d\sigma(\zeta),\end{aligned}$$

where $\mathscr{V}(z) = |\nabla\,\phi(z)|^2(1-|z|)^{1-n}$.

The proof of (3) ⇒ (1) is similar to the proof of (2) ⇒ (1). We only need to replace "(2.2)" and "$\int_{\partial\mathbb{B}^n}\left(\mathscr{G}(C_\phi(f_j))(\zeta)\right)^p d\sigma(\zeta)$" by "(2.4)" and

$$\text{“}\int_{\partial\mathbb{B}^n} \left(\mathscr{A}_\alpha C_\phi(f_j)(\zeta)\right)^p d\sigma(\zeta)\text{”},$$

respectively, in the proof of (2) ⇒ (1), where $j \in \{1,2\}$.

At last, we prove (1) ⇔ (4). Since (4) ⇒ (1) is obvious, we only need to prove (1) ⇒ (4). Let $C_\phi : \mathscr{B}_\omega(\mathbb{D}) \to \mathscr{P}\mathscr{H}^p(\mathbb{B}^n)$ be a bounded operator. Then $f\circ\phi \in \mathscr{P}\mathscr{H}^p(\mathbb{B}^n)$ for all $f \in \mathscr{B}_\omega(\mathbb{D})$, and

$$|\phi(\zeta)| := |\lim_{r\to 1^-}\phi(r\zeta)| \leq 1$$

exists for almost every $\zeta \in \partial\mathbb{B}^n$. Suppose that $\{f_k = h_k + \overline{g_k}\}$ is a sequence of $\mathscr{B}_\omega(\mathbb{D})$ such that $\|f_k\|_{\mathscr{B}_\omega(\mathbb{D})} \leq 1$, where h_k and g_k are analytic in $\mathbb{D}$ with $g_k(0) = 0$. Next, we will show that $\{C_\phi(f_k)\}$ has a convergent subsequence in $\mathscr{P}\mathscr{H}^p(\mathbb{B}^n)$. Since

$$\begin{aligned}\max\{|h_k(z)|, |g_k(z)|\} &\leq |f_k(0)| + \|f_k\|_{\mathscr{B}_\omega(\mathbb{D})}\int_0^1 \omega(t|z|)|z|dt\\ &\leq 1 + \int_0^1 \omega(t|z|)|z|dt\\ &\leq 1 + \omega(|z|) < \infty,\end{aligned}$$

$\{h_k\}$ and $\{g_k\}$ are normal families. Then there are subsequences of $\{h_k\}$ and $\{g_k\}$ that converge uniformly on compact subsets of $\mathbb{D}$ to holomorphic functions h ang g, respectively. Without loss of generality, we assume that the sequences $\{h_k\}$ and $\{g_k\}$ themselves converge to h and g, respectively. Then the sequence $f_k = h_k + \overline{g_k}$ itself converges uniformly on compact subsets of $\mathbb{D}$ to the harmonic function $f = h + \overline{g}$ with $g(0) = 0$. Consequently,

$$1 \geq \lim_{k\to\infty} \left\{|f_k(0)| + \mathscr{B}_\omega^{f_k}(z)\right\} = |f(0)| + \mathscr{B}_\omega^{f}(z),$$

which gives that $f \in \mathscr{B}_\omega(\mathbb{D})$ with $\|f\|_{\mathscr{B}_\omega(\mathbb{D})} \leq 1$. It follows from (1) $\Rightarrow$ (2) that

$$\int_0^1 \omega^2(|\phi(r\zeta)|)|\nabla\phi(r\zeta)|^2(1-r)dr < \infty, \quad \text{a.e. } \zeta \in \partial\mathbb{B}^n \tag{2.13}$$

and

$$\int_{\partial\mathbb{B}^n} \left(\int_0^1 |\nabla\phi(r\zeta)|^2\omega^2(|\phi(r\zeta)|)(1-r)dr\right)^{\frac{p}{2}} d\sigma(\zeta) < \infty. \tag{2.14}$$

Also, we have

$$\begin{aligned} &|(h_k - h)' \circ \phi(r\zeta)|^2 \, |\nabla\phi(r\zeta)|^2(1-r) \\ &\quad \leq \|h_k - h\|^2_{\mathscr{B}_\omega(\mathbb{D})}\omega^2(|\phi(r\zeta)|)|\nabla\phi(r\zeta)|^2(1-r) \\ &\quad \leq 4\omega^2(|\phi(r\zeta)|)|\nabla\phi(r\zeta)|^2(1-r) \end{aligned} \tag{2.15}$$

for $\zeta \in \partial\mathbb{B}^n$.

Applying (2.13), (2.14), (2.15) and the control convergence theorem twice, we have

$$\begin{aligned} &\lim_{k\to\infty} \int_{\partial\mathbb{B}^n} \left(\int_0^1 |(h_k - h)' \circ \phi(r\zeta)|^2 \, |\nabla\,\phi(r\zeta)|^2(1-r)dr\right)^{\frac{p}{2}} d\sigma(\zeta) \\ &\quad = \int_{\partial\mathbb{B}^n} \left(\int_0^1 \lim_{k\to\infty} |(h_k - h)' \circ \phi(r\zeta)|^2 \, |\nabla\,\phi(r\zeta)|^2(1-r)dr\right)^{\frac{p}{2}} d\sigma(\zeta) = 0, \end{aligned}$$

which, together with (2.2), implies that

$$\lim_{k\to\infty} \|h_k \circ \phi - h \circ \phi\|_p = 0.$$

Consequently,

$$C_\phi(h_k) \to C_\phi(h) \quad \text{in } \mathscr{PH}^p(\mathbb{B}^n) \tag{2.16}$$

as $k \to \infty$. By using similar reasoning as in the proof of (2.16), we have

$$C_\phi(g_k) \to C_\phi(g) \quad \text{in } \mathscr{PH}^p(\mathbb{B}^n) \tag{2.17}$$

as $k \to \infty$. Therefore, by (2.16) and (2.17), we conclude that $C_\phi(f) = C_\phi(h) + \overline{C_\phi(g)} \in \mathscr{PH}^p(\mathbb{B}^n)$, and $C_\phi(f_k) \to C_\phi(f)$ in $\mathscr{PH}^p(\mathbb{B}^n)$ as $k \to \infty$. The proof of this theorem is finished. □

Theorem E [26, Theorem 1.1]. *Let $p \in (0,\infty)$ and f be a holomorphic function in $\mathbb{D}$. Then the followings are equivalent:*

(1) $f \in \mathscr{H}^p(\mathbb{D})$;
(2) $\mathcal{G}[f] \in L^p(\mathbb{T})$,*where*

$$\mathcal{G}[f](\zeta) = \left(\int_0^1 (1-r)|f'(r\zeta)|^2 dr\right)^{\frac{1}{2}}, \quad \zeta \in \mathbb{T};$$

(3) $\mathcal{G}_*[f] \in L^p(\mathbb{T})$, *where*

$$\mathcal{G}_*[f](\zeta) = \left(\int_0^1 (1-r) \sup_{\rho\in(0,r)} |f'(\rho\zeta)|^2 dr\right)^{\frac{1}{2}}, \quad \zeta \in \mathbb{T};$$

(4) $\mathcal{G}_d[f] \in L^p(\mathbb{T})$, *where*

$$\mathcal{G}_d[f](\zeta) = \left(\sum_{k=0}^{\infty} 2^{-2k}|f'(r_k\zeta)|^2\right)^{\frac{1}{2}}, \quad \zeta \in \mathbb{T}$$

and $r_k = 1 - 2^{-k}$.

Furthermore, there are constants C_1, C_2, C_3 and C_4 independent of f such that

$$\|f - f(0)\|_p \le C_1\|\mathcal{G}[f]\|_p \le C_2\|\mathcal{G}_*[f]\|_p \le C_3\|\mathcal{G}_d[f]\|_p \le C_4\|f - f(0)\|_p.$$

2.2. The Proof of Theorem 1.3

Since (1) ⇔ (2) ⇔ (5) ⇔ (6) easily follows from Theorem 1.1, we only need to prove (1) ⇔ (3) ⇔ (4). We first prove (1) ⇒ (3). By Lemma 2.1, there are two analytic functions $f_j \in \mathscr{B}_\omega(\mathbb{D}) \cap \mathscr{H}(\mathbb{D})$ $(j \in \{1,2\})$ such that, for $z \in \mathbb{D}$,

$$\sum_{j=1}^{2} |f_j'(z)| \ge \omega(|z|). \tag{2.18}$$

Since $C_\phi : \mathscr{B}_\omega(\mathbb{D}) \to \mathscr{P}\mathscr{H}^p(\mathbb{D})$ is a bounded operator, by Theorem E, we see that

$$\begin{aligned}\infty &> \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{k=0}^{\infty} 2^{-2k} \left|\left(f_j(\phi(r_ke^{i\theta}))\right)'\right|^2\right)^{\frac{p}{2}} d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{k=0}^{\infty} 2^{-2k} \left|f_j'(\phi(r_ke^{i\theta}))\phi'(r_ke^{i\theta})\right|^2\right)^{\frac{p}{2}} d\theta \qquad (2.19)\end{aligned}$$

for $j \in \{1,2\}$. It follows from (2.18), (2.19) and Lemma D that

$$\begin{aligned}
\infty &> \sum_{j=1}^{2} \int_0^{2\pi} \left(\sum_{k=0}^{\infty} 2^{-2k} \left| \left(f_j(\phi(r_k e^{i\theta}) \right)' \right|^2 \right)^{\frac{p}{2}} \frac{d\theta}{2\pi} \\
&\geq \mathscr{M}_p \int_0^{2\pi} \left(\sum_{k=0}^{\infty} 2^{-2k} |\phi'(r_k e^{i\theta})|^2 \left(\sum_{j=1}^{2} |f_j'(\phi(r_k e^{i\theta})| \right)^2 \right)^{\frac{p}{2}} \frac{d\theta}{2\pi} \\
&\geq \mathscr{M}_p \int_0^{2\pi} \left(\sum_{k=0}^{\infty} 2^{-2k} |\phi'(r_k e^{i\theta})|^2 \omega^2(|\phi(r_k e^{i\theta})|) \right)^{\frac{p}{2}} \frac{d\theta}{2\pi},
\end{aligned}$$

where $\mathscr{M}_p$ is defined in the proof of Theorem 1.1.

Next, we prove (3) $\Rightarrow$ (1). We first show $C_\phi(f) \in \mathscr{PH}^p(\mathbb{D})$. Let $f \in \mathscr{B}_\omega(\mathbb{D})$. Since $\mathbb{D}$ is a simply connected domain, we see that f admits the canonical decomposition $f = f_1 + \overline{f_2}$, where f_1 and f_2 are analytic in $\mathbb{D}$ with $f_2(0) = 0$. For $j \in \{1,2\}$, let

$$\mathscr{L}_j = \int_0^{2\pi} \left(\sum_{k=0}^{\infty} \frac{\left| \left(f_j(\phi(r_k e^{i\theta}) \right)' \right|^2}{2^{2k}} \right)^{\frac{p}{2}} \frac{d\theta}{2\pi}.$$

Then, by (2.7), we see that there is a constant $C = \|f\|^p_{\mathscr{B}_\omega(\mathbb{D})}$ such that

$$\begin{aligned}
\mathscr{L}_j &= \int_0^{2\pi} \left(\sum_{k=0}^{\infty} \frac{\left| f_j'(\phi(r_k e^{i\theta}))\phi'(r_k e^{i\theta}) \right|^2}{2^{2k}} \right)^{\frac{p}{2}} \frac{d\theta}{2\pi} \\
&\leq C \int_0^{2\pi} \left(\sum_{k=0}^{\infty} \frac{\left| \omega(|\phi(r_k e^{i\theta})|)\phi'(r_k e^{i\theta}) \right|^2}{2^{2k}} \right)^{\frac{p}{2}} \frac{d\theta}{2\pi} \\
&< \infty,
\end{aligned}$$

which, together with Theorem E, implies that $C_\phi(f_j) \in \mathscr{PH}^p(\mathbb{D})$. Hence

$$C_\phi(f) = C_\phi(f_1) + \overline{C_\phi(f_2)} \in \mathscr{PH}^p(\mathbb{D}).$$

Now we come to prove C_ϕ is a bounded operator. By (2.9) and Theorem E, we see that there is a positive constant C, depending only on p, such that, for $j \in \{1,2\}$,

$$\begin{aligned}
\|C_\phi(f_j)\|_p^p &\leq C \left(|f_j(\phi(0))|^p + \int_0^{2\pi} \left(\sum_{k=0}^{\infty} \frac{\left| \left(f_j(\phi(r_k e^{i\theta}) \right)' \right|^2}{2^{2k}} \right)^{\frac{p}{2}} \frac{d\theta}{2\pi} \right) \\
&\leq C \|f_j\|^p_{\mathscr{B}_\omega(\mathbb{D})} C^*(\phi),
\end{aligned} \tag{2.20}$$

where

$$C^*(\phi)=(1+|\phi(0)|\omega(|\phi(0)|))^p+\int_0^{2\pi}\left(\sum_{k=0}^{\infty}\frac{\left|\omega(|\phi(r_ke^{i\theta})|)\phi'(r_ke^{i\theta})\right|^2}{2^{2k}}\right)^{\frac{p}{2}}\frac{d\theta}{2\pi}.$$

It follows from (2.20) and Lemma D that

$$\begin{aligned}\|C_\phi(f)\|_p^p &\le 2^{\max\{p-1,0\}}\left(\|C_\phi(f_1)\|_p^p+\|C_\phi(f_2)\|_p^p\right)\\ &\le 2^{\max\{p-1,0\}}CC^*(\phi)\left(\|f_1\|^p_{\mathscr{B}_\omega(\mathbb{D})}+\|f_2\|^p_{\mathscr{B}_\omega(\mathbb{D})}\right)\\ &\le 2^{1+\max\{p-1,0\}}CC^*(\phi)\|f\|^p_{\mathscr{B}_\omega(\mathbb{D})},\end{aligned}$$

which implies that C_ϕ is a bounded operator.

Now we prove (1) $\Rightarrow$ (4). By Lemma 2.1, there are two analytic functions $f_j\in\mathscr{B}_\omega(\mathbb{D})\cap\mathscr{H}(\mathbb{D})$ $(j\in\{1,2\})$ such that, for $z\in\mathbb{D}$,

$$\sum_{j=1}^{2}|f_j'(z)|\ge\omega(|z|). \tag{2.21}$$

Since C_ϕ is a bounded operator, by (2.21), Theorem E and Lemma D, we see that

$$\begin{aligned}\infty &> \frac{1}{\mathscr{M}_p}\sum_{j=1}^{2}\int_0^{2\pi}\left(\int_0^1\delta(r)\sup_{\rho\in(0,r)}\left|\left(f_j(\phi(\rho\, e^{i\theta})\right)'\right|^2dr\right)^{\frac{p}{2}}\frac{d\theta}{2\pi}\\ &\ge\int_0^{2\pi}\left(\int_0^1\delta(r)\sup_{\rho\in(0,r)}\left(|\phi'(\rho\, e^{i\theta})|^2\left(\sum_{j=1}^{2}|f_j'(\phi(\rho e^{i\theta})|\right)^2\right)dr\right)^{\frac{p}{2}}\frac{d\theta}{2\pi}\\ &\ge\int_0^{2\pi}\left(\int_0^1\delta(r)\sup_{\rho\in(0,r)}\left(|\phi'(\rho\, e^{i\theta})|^2\omega^2(|\phi(\rho\, e^{i\theta})|)\right)dr\right)^{\frac{p}{2}}\frac{d\theta}{2\pi},\end{aligned}$$

where $\delta(r)=1-r$ and $\mathscr{M}_p$ is defined in the proof of Theorem 1.1.

At last, we prove (4) $\Rightarrow$ (1). Let $f\in\mathscr{B}_\omega(\mathbb{D})$. It is not difficult to know that, for $z\in\mathbb{D}$, f has the canonical decomposition $f(z)=f_1(z)+\overline{f_2(z)}$, where f_1 and f_2 are analytic in $\mathbb{D}$ with $f_2(0)=0$. Then, by (2.7), we see that

$$\begin{aligned}\infty &> M_j\int_0^{2\pi}\left(\int_0^1\delta(r)\sup_{\rho\in(0,r)}\left(|\phi'(\rho\, e^{i\theta})|^2\omega^2(|\phi(\rho\, e^{i\theta})|)\right)dr\right)^{\frac{p}{2}}\frac{d\theta}{2\pi}\\ &\ge\int_0^{2\pi}\left(\int_0^1\delta(r)\sup_{\rho\in(0,r)}\left(|\phi'(\rho\, e^{i\theta})|^2|f_j'(\phi(\rho e^{i\theta})|^2\right)dr\right)^{\frac{p}{2}}\frac{d\theta}{2\pi}\\ &=\int_0^{2\pi}\left(\int_0^1\delta(r)\sup_{\rho\in(0,r)}\left|\left(f_j(\phi(\rho\, e^{i\theta})\right)'\right|^2dr\right)^{\frac{p}{2}}\frac{d\theta}{2\pi},\end{aligned} \tag{2.22}$$

where $j \in \{1,2\}$ and $M_j = \|f_j\|^p_{\mathscr{B}_\omega(\mathbb{D})}$. It follows from (2.22) and Theorem E that $C_\phi(f_j) \in \mathscr{P}\mathscr{H}^p(\mathbb{D})$ $(j = 1,2)$, which implies that

$$C_\phi(f) = C_\phi(f_1) + \overline{C_\phi(f_2)} \in \mathscr{P}\mathscr{H}^p(\mathbb{D}).$$

Next, we prove C_ϕ is also a bounded operator. By (2.9) and Theorem E, we see that there is a positive constant C, depending only on p, such that, for $j \in \{1,2\}$,

$$\begin{aligned}\|C_\phi(f_j)\|_p^p &\leq C\left(|f_j(\phi(0))|^p + \int_0^{2\pi}\left(\int_0^1 \delta(r) \sup_{\rho\in(0,r)}\left|\left(f_j(\phi(\rho\, e^{i\theta}))\right)'\right|^2 dr\right)^{\frac{p}{2}} \frac{d\theta}{2\pi}\right)\\ &\leq CMM_j, \end{aligned} \tag{2.23}$$

where

$$\begin{aligned}M &= (1+|\phi(0)|\omega(|\phi(0)|))^p\\ &\quad + \int_0^{2\pi}\left(\int_0^1 (1-r) \sup_{\rho\in(0,r)} \left(|\phi'(\rho\, e^{i\theta})|\omega(|\phi(\rho\, e^{i\theta})|)\right)^2 dr\right)^{\frac{p}{2}} \frac{d\theta}{2\pi}.\end{aligned}$$

It follows from (2.23) and Lemma D that

$$\begin{aligned}\|C_\phi(f)\|_p^p &\leq 2^{\max\{p-1,0\}}\left(\|C_\phi(f_1)\|_p^p + \|C_\phi(f_2)\|_p^p\right)\\ &\leq 2^{\max\{p-1,0\}} CM\left(M_1 + M_2\right)\\ &\leq 2^{1+\max\{p-1,0\}}\|f\|^p_{\mathscr{B}_\omega(\mathbb{D})} CM,\end{aligned}$$

which implies that $C_\phi : \mathscr{B}_\omega(\mathbb{D}) \to \mathscr{P}\mathscr{H}^p(\mathbb{D})$ is a bounded operator. The proof of this theorem is finished. □

2.3. The Proof of Proposition 1.4

If $k = 1$, then we can take $r_1 = 0$. Since, for $k \in \{2,3,\ldots\}$,

$$\lim_{x\to 0^+} \mu_{\omega,k}(x) = \lim_{x\to 1^-} \mu_{\omega,k}(x) = 0,$$

we see that $\mathscr{E}_\omega(k)$ is not an empty set. Consequently, for $k \in \{2,3,\ldots\}$, we can choose points $r_k > 0$ and $r_{k+1} > 0$ from the sets $\mathscr{E}_\omega(k)$ and $\mathscr{E}_\omega(k+1)$, respectively, such that $\mu_{\omega,k}(r_k) \geq \mu_{\omega,k}(r_{k+1})$ which is equivalent to

$$\frac{\omega(r_{k+1})}{\omega(r_k)} \geq \frac{r_{k+1}^{k-1}}{r_k^{k-1}}. \tag{2.24}$$

On the other hand, for $k \in \{2,3,\ldots\}$, we have $\mu_{\omega,k+1}(r_{k+1}) \geq \mu_{\omega,k+1}(r_k)$ which implies that

$$\frac{\omega(r_{k+1})}{\omega(r_k)} \leq \frac{r_{k+1}^{k}}{r_k^{k}}. \tag{2.25}$$

Combining (2.24) and (2.25) gives $r_k \leq r_{k+1}$. □

2.4. The Proof of Theorem 1.5

(1) We first prove the sufficiency. By (1.1), we have

$$\|\phi\|_{\mathscr{B}_{\omega_2}(\mathbb{B}^n)} < \infty. \tag{2.26}$$

For $f \in \mathscr{B}_{\omega_1}(\mathbb{D})$, we have

$$\begin{aligned} |C_\phi(f)(0)| &\leq |f(0)| + |f(\phi(0)) - f(0)| \\ &\leq |f(0)| + |\phi(0)| \int_0^1 \Lambda_f(\phi(0)t)dt \\ &\leq |f(0)| + \|f\|_{\mathscr{B}_{\omega_1}(\mathbb{D})} |\phi(0)| \int_0^1 \omega_1(|\phi(0)|t)dt \\ &\leq \|f\|_{\mathscr{B}_{\omega_1}(\mathbb{D})} \left(1 + |\phi(0)|\omega_1(|\phi(0)|)\right). \end{aligned}$$

So, it suffices to show that there exists a constant independent of f such that

$$\|f \circ \phi\|_{\mathscr{B}_{\omega_2}(\mathbb{B}^n),s} \leq C\|f\|_{\mathscr{B}_{\omega_1}(\mathbb{D}),s}, \quad f \in \mathscr{B}_{\omega_1}(\mathbb{D}).$$

We split the remaining proof into two cases.

Case 1. If

$$\sup_{z\in\mathbb{B}^n} |\phi(z)| < 1,$$

then there is a constant $\rho_0 \in (0, 1)$ such that

$$\sup_{z\in\mathbb{B}^n} |\phi(z)| < \rho_0. \tag{2.27}$$

For $f \in \mathscr{B}_{\omega_1}(\mathbb{D})$, it follows from (2.26) and (2.27) that

$$\begin{aligned} \sup_{z\in\mathbb{B}^n} \left\{ \frac{\Lambda_f(\phi(z))\|D\phi(z)\|}{\omega_2(|z|)} \right\} &= \sup_{z\in\mathbb{B}^n} \left\{ \frac{\Lambda_f(\phi(z))}{\omega_1(|\phi(z)|)} \omega_1(|\phi(z)|) \frac{\|D\phi(z)\|}{\omega_2(|z|)} \right\} \\ &\leq \|f\|_{\mathscr{B}_{\omega_1}(\mathbb{D}),s} \omega_1(\rho_0) \sup_{z\in\mathbb{B}^n} \left\{ \frac{\|D\phi(z)\|}{\omega_2(|z|)} \right\} \\ &< \infty, \end{aligned}$$

which implies that $C_\phi : \mathscr{B}_{\omega_1}(\mathbb{D}) \to \mathscr{B}_{\omega_2}(\mathbb{B}^n)$ is bounded.

Case 2. If

$$\sup_{z\in\mathbb{B}^n} |\phi(z)| = 1, \tag{2.28}$$

then, for $k \in \{1, 2, \ldots\}$, let

$$\Omega_k = \{z \in \mathbb{B}^n : r_k \leq |\phi(z)| \leq r_{k+1}\},$$

where $r_1 = 0$. Since $\{r_k\}$ is a non-decreasing sequence satisfying

$$\lim_{k\to\infty} r_k^{k-1} = \gamma > 0, \tag{2.29}$$

we see that

$$\lim_{k\to\infty} r_k = 1. \tag{2.30}$$

Let m be the smallest positive integer such that Ω_m is not an empty set. By (2.28) and (2.30), we see that Ω_k is not empty for every integer $k \in \{m, m+1, \ldots\}$, and $\mathbb{B}^n = \cup_{k=m}^{\infty} \Omega_k$. Then, for $k \in \{m, m+1, \ldots\}$, we have

$$\min_{z \in \Omega_k} \left\{ \frac{\omega_1(r_k)|\phi(z)|^{k-1}}{\omega_1(|\phi(z)|)} \right\} \geq \left\{ \frac{\omega_1(r_k) r_k^{k-1}}{\omega_1(r_{k+1})} \right\}. \tag{2.31}$$

It follows from $\mu_{\omega_1,k}(r_k) \geq \mu_{\omega_1,k}(r_{k+1})$ and $\mu_{\omega_1,k+1}(r_{k+1}) \geq \mu_{\omega,k+1}(r_k)$ that

$$\frac{r_k^k}{r_{k+1}^k} \leq \frac{\omega_1(r_k)}{\omega_1(r_{k+1})} \leq \frac{r_k^{k-1}}{r_{k+1}^{k-1}}, \tag{2.32}$$

which, together with (2.29) and (2.30), yields that

$$\lim_{k \to \infty} \frac{\omega_1(r_k)}{\omega_1(r_{k+1})} = 1. \tag{2.33}$$

Combining (2.31) and (2.33) gives

$$\lim_{k \to \infty} \min_{z \in \Omega_k} \left\{ \frac{\omega_1(r_k)|\phi(z)|^{k-1}}{\omega_1(|\phi(z)|)} \right\} \geq \lim_{k \to \infty} \left\{ \frac{\omega_1(r_k) r_k^{k-1}}{\omega_1(r_{k+1})} \right\} = \gamma.$$

Hence there is a positive integer $m_0 \geq m$ such that, for all $k \in \{m_0, m_0 + 1, \ldots\}$,

$$\min_{z \in \Omega_k} \left\{ \frac{\omega_1(r_k)|\phi(z)|^{k-1}}{\omega_1(|\phi(z)|)} \right\} \geq \frac{\gamma}{2}, \tag{2.34}$$

which implies that, for $f \in \mathscr{B}_{\omega_1}(\mathbb{D})$,

$$\begin{aligned} \|C_\phi(f)\|_{\mathscr{B}_{\omega_2}(\mathbb{B}^n),s} &\leq \sup_{z \in \mathbb{B}^n} \left\{ \frac{\Lambda_f(\phi(z))\|D\phi(z)\|}{\omega_2(|z|)} \right\} \\ &= \sup_{k \geq m} \sup_{z \in \Omega_k} \left\{ \frac{\Lambda_f(\phi(z))\|D\phi(z)\|}{\omega_2(|z|)} \right\} \\ &= \max\{J_1(f, m_0), J_2(f, m_0)\}, \end{aligned} \tag{2.35}$$

where

$$J_1(f, m_0) = \sup_{k \geq m_0} \sup_{z \in \Omega_k} \left\{ \frac{\Lambda_f(\phi(z))\|D\phi(z)\|}{\omega_2(|z|)} \right\}$$

and

$$J_2(f, m_0) = \sup_{m_0 - 1 \geq k \geq m} \sup_{z \in \Omega_k} \left\{ \frac{\Lambda_f(\phi(z))\|D\phi(z)\|}{\omega_2(|z|)} \right\}.$$

By (2.34), we see that

$$\begin{aligned} J_1(f,m_0) &= \sup_{k\geq m_0}\sup_{z\in\Omega_k} \frac{\frac{\Lambda_f(\phi(z))\|D\phi(z)\|\omega_1(r_k)|\phi(z)|^{k-1}}{\omega_1(|\phi(z)|)}}{\frac{\omega_2(|z|)\omega_1(r_k)|\phi(z)|^{k-1}}{\omega_1(|\phi(z)|)}} \\ &\leq \frac{2}{\gamma}\sup_{k\geq m_0}\sup_{z\in\Omega_k}\frac{\omega_1(r_k)}{k}\frac{\|D(\phi^k(z))\|\Lambda_f(\phi(z))}{\omega_1(|\phi(z)|)\omega_2(|z|)} \\ &\leq \frac{2}{\gamma}\sup_{k\geq 1}\frac{\omega_1(r_k)}{k}\|\phi^k\|_{\mathscr{B}_{\omega_2}(\mathbb{B}^n)}\|f\|_{\mathscr{B}_{\omega_1}(\mathbb{D}),s} \\ &< \infty \end{aligned}$$

and

$$J_2(f,m_0) \leq \omega_1(r_{m_0})\|\phi\|_{\mathscr{B}_{\omega_2}(\mathbb{B}^n)}\|f\|_{\mathscr{B}_{\omega_1}(\mathbb{D}),s},$$

which imply that $C_\phi : \mathscr{B}_{\omega_1}(\mathbb{D}) \to \mathscr{B}_{\omega_2}(\mathbb{B}^n)$ is bounded.

Next, we begin to prove the necessity. For $k \in \{2,3,\ldots\}$, let $f(w) = w^k, w \in \mathbb{D}$. Since

$$\|f\|_{\mathscr{B}_{\omega_1}(\mathbb{D})} = \sup_{w\in\mathbb{D}}\left\{\frac{k|w|^{k-1}}{\omega_1(|w|)}\right\} = \frac{kr_k^{k-1}}{\omega_1(r_k)},$$

we see that

$$\lim_{k\to\infty}\frac{\|f\|_{\mathscr{B}_{\omega_1}(\mathbb{D})}\omega_1(r_k)}{k} = \lim_{k\to\infty} r_k^{k-1} = \gamma > 0. \tag{2.36}$$

From (2.36), we see that there is a positive constant, independent of k, such that

$$\|f\|_{\mathscr{B}_{\omega_1}(\mathbb{D})} \leq C\frac{k}{\omega_1(r_k)}$$

which is equivalent to

$$\frac{1}{\|f\|_{\mathscr{B}_{\omega_1}(\mathbb{D})}} \geq \frac{\omega_1(r_k)}{kC}.$$

For $w \in \mathbb{D}$, let $F(w) = w^k/\|w^k\|_{\mathscr{B}_{\omega_1}(\mathbb{D})}$. Then $\|F\|_{\mathscr{B}_{\omega_1}(\mathbb{D})} = 1$. Therefore,

$$\infty > \|C_\phi\| \geq \|C_\phi(F)\|_{\mathscr{B}_{\omega_2}(\mathbb{B}^n)} = \frac{\|\phi^k\|_{\mathscr{B}_{\omega_2}(\mathbb{B}^n)}}{\|f\|_{\mathscr{B}_{\omega_1}(\mathbb{D})}} \geq \frac{\omega_1(r_k)\|\phi^k\|_{\mathscr{B}_{\omega_2}(\mathbb{B}^n)}}{kC},$$

which implies (1.1) is true.

(2) Assume that C_ϕ is compact. For $k \in \{1,2,\ldots\}$ and $w \in \mathbb{D}$, let $F_k(w) = w^k/\|w^k\|_{\mathscr{B}_{\omega_1}(\mathbb{D})}$. Then $\|F_k\|_{\mathscr{B}_{\omega_1}(\mathbb{D})} = 1$. For $k \in \{2,3,\ldots\}$ and $m \in \{1,2,\ldots\}$, it follows from (2.32) that

$$\frac{\omega_1(r_k)}{\omega_1(r_{k+m})} \geq \frac{r_k^k r_{k+1}\cdots r_{k+m-1}}{r_{k+m}^{k+m-1}},$$

which implies that

$$\frac{\omega_1(r_{k+m})r_k^{k+m}}{k+m} \leq \frac{\omega_1(r_k)r_k^m r_{k+m}^{k+m-1}}{(k+m)r_{k+1}\cdots r_{k+m-1}} \leq \frac{\omega_1(r_k)r_k r_{k+m}^{k+m-1}}{k+m}. \quad (2.37)$$

For $k \in \{2, 3, \ldots\}$ and $w \in \mathbb{D}$, elementary calculations lead to

$$F_k(w) = \frac{\omega_1(r_k)w^k}{kr_k^{k-1}},$$

which, together with (2.29), (2.30) and (2.37), yields that $F_k \to 0$ locally uniformly in $\mathbb{D}$ as $k \to \infty$. Since C_ϕ is compact, we deduce that

$$\lim_{k\to\infty} \frac{\omega_1(r_k)}{kr_k^{k-1}} \|\phi^k\|_{\mathscr{B}_{\omega_2}(\mathbb{B}^n)} = \lim_{k\to\infty} \|C_\phi(F_k)\|_{\mathscr{B}_{\omega_2}(\mathbb{B}^n)} = 0.$$

Consequently, (1.2) follows from

$$\lim_{k\to\infty} r_k^{k-1} = \gamma > 0.$$

Conversely, assume that (1.2) holds. Suppose that $\{f_j = h_j + \overline{g_j}\}$ is a sequence of $\mathscr{B}_{\omega_1}(\mathbb{D})$ such that $\|f_j\|_{\mathscr{B}_{\omega_1}(\mathbb{D})} \leq 1$, where h_j and g_j are analytic in $\mathbb{D}$ with $g_j(0) = 0$. Then there are subsequences of $\{h_j\}$ and $\{g_j\}$ that converge uniformly on compact subsets of $\mathbb{D}$ to holomorphic functions h ang g, respectively. Without loss of generality, we assume that the sequences $\{h_j\}$ and $\{g_j\}$ themselves converge to h and g, respectively. Then the sequence $f_j = h_j + \overline{g_j}$ itself converges uniformly on compact subsets of $\mathbb{D}$ to the harmonic function $f = h + \overline{g}$ with $g(0) = 0$. Consequently,

$$1 \geq \lim_{j\to\infty} \left\{|f_j(0)| + \mathscr{B}_{\omega_1}^{f_j}(w)\right\} = |f(0)| + \mathscr{B}_{\omega_1}^{f}(w),$$

which gives that $f \in \mathscr{B}_{\omega_1}(\mathbb{D})$ with $\|f\|_{\mathscr{B}_{\omega_1}(\mathbb{D})} \leq 1$. Let $\varepsilon > 0$ be arbitrarily fixed. Since (1.2) holds, there exists a positive integer $N > m_0$ such that, for $k \in \{N, N+1, \ldots\}$,

$$\frac{4}{\gamma}\frac{\omega_1(r_k)}{k} \|\phi^k\|_{\mathscr{B}_{\omega_2}(\mathbb{B}^n)} < \varepsilon,$$

which implies that, for $j \geq 1$,

$$J_1(f_j - f, N) \leq \varepsilon. \quad (2.38)$$

Since the sequence $\{\Lambda_{f_j - f}\}$ converges to 0 locally uniformly in $\mathbb{D}$, there exists an integer j_0 such that, for $j \geq j_0$,

$$J_2(f_j - f, N) \leq \varepsilon. \quad (2.39)$$

Combining (2.35), (2.38) and (2.39) gives that, for $j \geq j_0$,

$$\|C_\phi(f_j - f)\|_{\mathscr{B}_{\omega_2}(\mathbb{B}^n),s} \leq \varepsilon,$$

which implies that C_ϕ is compact. The proof of this theorem is finished. □

2.5. The Proof of Theorem 1.6

For $\alpha, \beta \in (0,1)$, let $f \in \mathscr{PH}(\mathbb{B}^n) \cap \mathscr{L}_{\varpi}(\mathbb{B}^n)$, where $\varpi = \alpha$ or β. Since $\mathbb{B}^n$ is a simply connected domain, we see that f admits the canonical decomposition $f = f_1 + \overline{f_2}$, where f_1 and f_2 are analytic in $\mathbb{B}^n$ with $f_2(0) = 0$. Next, we prove $f_1, f_2 \in \mathscr{L}_{\varpi}(\mathbb{B}^n)$. Let $f = f_1 + \overline{f_2} = u + iv$, where $f_1 = u_1 + iv_1$ and $f_2 = u_2 + iv_2$. Then $f \in \mathscr{L}_{\varpi}(\mathbb{B}^n)$ if and only if $u,\ v \in \mathscr{L}_{\varpi}(\mathbb{B}^n)$. Let $F = f_1 + f_2$ and $\widetilde{v} = \mathrm{Im}(F)$, where "Im" is the imaginary part of a complex number. Then iF is holomorphic in $\mathbb{B}^n$. Since $\mathrm{Re}(iF) = -\widetilde{v}$, where "Re" is the real part of a complex number, and $\mathrm{Im}(iF) = \mathrm{Im}(if) = u$, u and $\widetilde{v}$ satisfy $\mathrm{div}(\nabla u) = \mathrm{div}(\nabla \widetilde{v}) = 0$ and $|\nabla u| = |\nabla \widetilde{v}|$. Then by [23, Corollary 3.11], we see that

$$\|\mathrm{Re}(iF)\|_{\mathscr{L}_{\varpi}(\mathbb{B}^n),s} = \|\widetilde{v}\|_{\mathscr{L}_{\varpi}(\mathbb{B}^n),s} \le C\|\mathrm{Im}(iF)\|_{\mathscr{L}_{\varpi}(\mathbb{B}^n),s} = C\|u\|_{\mathscr{L}_{\varpi}(\mathbb{B}^n),s}. \tag{2.40}$$

By (2.40), we have

$$v_1 = \frac{v + \widetilde{v}}{2} \in \mathscr{L}_{\varpi}(\mathbb{B}^n) \tag{2.41}$$

and

$$v_2 = \frac{\widetilde{v} - v}{2} \in \mathscr{L}_{\varpi}(\mathbb{B}^n). \tag{2.42}$$

Applying [23, Corollary 3.11] to f_1 and f_2 again, we see that

$$u_1 \in \mathscr{L}_{\varpi}(\mathbb{B}^n) \tag{2.43}$$

and

$$u_2 \in \mathscr{L}_{\varpi}(\mathbb{B}^n). \tag{2.44}$$

Consequently, $f_1 \in \mathscr{L}_{\varpi}(\mathbb{B}^n)$ follows from (2.41) and (2.43), and $f_2 \in \mathscr{L}_{\varpi}(\mathbb{B}^n)$ follows from (2.42) and (2.44). It follows from [31, Section 6.4] that $f_1,\ f_2 \in \mathscr{B}_{\omega}(\mathbb{B}^n)$, where $\omega(t) = 1/(1-t)^{1-\varpi}$ for $t \in [0,1)$. Hence $f \in \mathscr{PH}(\mathbb{B}^n) \cap \mathscr{L}_{\varpi}(\mathbb{B}^n)$ is equivalent to $f \in \mathscr{B}_{\omega}(\mathbb{B}^n)$. Hence we only need to show that $C_\phi : \mathscr{B}_{\omega_1}(\mathbb{D}) \to \mathscr{B}_{\omega_2}(\mathbb{B}^n)$ is bounded if and only if

$$\sup_{k \ge 1} \left\{ k^{-\alpha} \|\phi^k\|_{\mathscr{B}_{\omega_2}(\mathbb{B}^n)} \right\} < \infty,$$

and C_ϕ is compact if and only if

$$\lim_{k \to \infty} \left\{ k^{-\alpha} \|\phi^k\|_{\mathscr{B}_{\omega_2}(\mathbb{B}^n)} \right\} = 0,$$

where $\omega_1(t) = 1/(1-t)^{1-\alpha}$ and $\omega_2(t) = 1/(1-t)^{1-\beta}$ for $t \in [0,1)$. By taking $r_k = (k-1)/(k-\alpha)$ in Theorem 1.5, we can obtain the desired result. The proof of this theorem is finished. $\square$

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