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Results in Mathematics

Ahlfors–David Regular Sets, Point Spectrum and Dirichlet Spaces

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Abstract. Let E be a closed subset of the unit circle $\mathbb{T}$, and let $\alpha \in (0,1)$. Nikolski's result states that if the Hausdorff dimension of E is strictly greater than α, then for any operator T on a separable Hilbert space such that the point spectrum $\sigma_p(T)$ of T contains E, the series $\sum_n n^{\alpha-1}\|T^n\|^{-2}$ converges. A partial converse of this result has been obtained by El-Fallah and Ransford. Namely they constructed, for any α strictly greater than the upper box dimension of E, an operator T on a separable Hilbert space such that $\sigma_p(T)$ contains E and $\frac{1}{n}\sum_{k=0}^{n-1}\left\|T^k\right\|^2 \lesssim n^{\alpha}$. In this paper, we improve on this latter result for regular sets. Indeed, for any Ahlfors–David regular set E and for any α strictly greater than the Hausdorff dimension of E there exists an operator T on a separable Hilbert space such that $\sigma_p(T)$ contains E and $\|T^n\|^2 \asymp n^{\alpha}$.

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1. Introduction

Let T be a bounded linear operator on a complex Banach space X. The unit circle is denoted by $\mathbb{T}$, while $\sigma_p(T) := \{\lambda \in \mathbb{C} : \ker(T - \lambda I) \neq \emptyset\}$ represents the point spectrum of T. Jamison [10] showed that if X is separable and T is power-bounded, then $\sigma_p(T) \cap \mathbb{T}$ is at most countable. Later on, several authors (see, e.g., [1,2,4,15,16] and the references therein) have shown interest in the study of the relationship between the size of the set $\sigma_p(T) \cap \mathbb{T}$ and the growth of $\|T^n\|$ as $n \to \infty$.

In the case of separable Hilbert spaces, Nikolski [13] proved that if $\sigma_p(T)\cap\mathbb{T}$ has a positive γ-capacity, where $\gamma:\mathbb{T}\to(0,\infty)$ is integrable with positive Fourier coefficients, then there exists $N\in\mathbb{N}$ such that $\sum_n \widehat{\gamma}(n+N)\|T^n\|^{-2}$ converges (see [11, Chapter 3, p. 31] for the definition of γ-capacity). As a by-product of this, if $\sigma_p(T)\cap\mathbb{T}$ contains a subset E of $\mathbb{T}$ with $\dim_H(E)>\alpha>0$, where $\dim_H(E)$ is the Hausdorff dimension of E, then the series $\sum_n n^{\alpha-1}\|T^n\|^{-2}$ converges. El-Fallah-Ransford [8] proved that, as a partial converse of the preceding result, if α is strictly greater than the upper box dimension of E, then there exists an operator T on a separable Hilbert space such that $\sigma_p(T)\cap\mathbb{T}$ contains E and the series $\sum_n n^{\alpha-1}\|T^n\|^{-2}$ diverges. (See [9, p. 41] for more details about the box dimension). Precisely, they constructed an operator T such that $\sigma_p(T)$ contains E and

$$\frac{1}{n}\sum_{k=0}^{n-1}\left\|T^k\right\|^2 \lesssim \omega(n)^2\left|E_{1/n}\right|, \quad n\geqslant 1,$$

where $\omega:\mathbb{Z}\to(0,\infty)$ is a regular weight function satisfying $\sum_n \frac{1}{\omega(n)^2}<\infty$, $E_{1/n}:=\{\zeta\in\mathbb{T}:\ \mathrm{dist}(E,\zeta)<\frac{1}{n}\}$ with $\mathrm{dist}(.,.)$ being the arc-length distance, and $\left|E_{1/n}\right|$ is its Lebesgue measure. In particular, one can obtain

$$\inf_{n\le k\le 2n}\|T^k\|^2\lesssim\omega(n)^2|E_{1/n}|, \quad n\ge 1.$$

The main result of this paper is the following theorem.

Theorem 1.1. *Assume that $E\subset\mathbb{T}$ is a closed Ahlfors–David regular set. If $\alpha>\dim_H(E)$, then there exists an operator T on a separable Hilbert space such that $\sigma_p(T)\cap\mathbb{T}$ contains E and*

$$\|T^n\|^2\asymp n^\alpha.$$

The definitions of Ahlfors–David regular sets and Hausdorff dimension are recalled in Sect. 4. Theorem 1.1 is a corollary from the following more general result.

Theorem 1.2. *Let E be closed subset of $\mathbb{T}$, if there exist an increasing function $\Lambda:[0,1]\to[0,+\infty)$ such that $\frac{\Lambda(t)}{t^c}$ is decreasing for some $c>0$, and a positive finite Borel measure μ on $\mathbb{T}$ satisfying*

$$\int_0\frac{dt}{\Lambda(t)\mu(\overline{\zeta},t)}<\infty, \quad \zeta\in E, \tag{1}$$

where $\mu(\overline{\zeta},t)=\mu(\overline{\zeta}e^{-it},\overline{\zeta}e^{it})$, then there exists an operator T on a separable Hilbert space such that $E\subset\sigma_p(T)$ and

$$\|T^n\|^2\asymp n\Lambda\left(\frac{1}{n}\right), \quad n\ge 1.$$

In Sect. 2, we study the point spectrum of adjoint of the shift operator acting on some weighted Dirichlet spaces. In Sect. 3, we determine the growth of power of the adjoint of this shift operator. In Sect. 4, we give the proofs of the main theorems.

2. Point Spectrum

Let $\Lambda : [0, 1] \to [0, +\infty)$ be a positive function, let $\mathbb{D}$ be the unit disc, and let μ be a positive finite Borel measure on the unit circle $\mathbb{T}$. The weighted Dirichlet integral of $f \in \mathrm{Hol}(\mathbb{D})$ associated with Λ and μ is defined as follows:

$$\mathcal{D}_{\Lambda,\mu}(f) = \int_{\mathbb{D}} |f'(z)|^2 \Lambda(1 - |z|^2) P_\mu(z) dA(z),$$

where dA denotes the normalized area measure on $\mathbb{D}$, and P_μ is the Poisson integral of μ on $\mathbb{T}$ given by

$$P_\mu(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} d\mu(\zeta), \quad z \in \mathbb{D}.$$

The associated weighted Dirichlet spaces $\mathcal{D}_{\Lambda,\mu}$ consist of all analytic functions on $\mathbb{D}$ with finite weighted Dirichlet integral, i.e.,

$$\mathcal{D}_{\Lambda,\mu} := \{f \in \mathrm{Hol}(\mathbb{D}) : \mathcal{D}_{\Lambda,\mu}(f) < \infty\}.$$

We associate to $\mathcal{D}_{\Lambda,\mu}$ the following inner product

$$\langle f, g\rangle_{\Lambda,\mu} := f(0)\overline{g(0)} + \mathcal{D}_{\Lambda,\mu}(f, g), \quad f, g \in \mathrm{Hol}(\mathbb{D}),$$

where

$$\mathcal{D}_{\Lambda,\mu}(f, g) = \int_{\mathbb{D}} f'(z))\overline{g'(z)}\Lambda(1 - |z|^2) P_\mu(z) dA(z).$$

$\mathcal{D}_{\Lambda,\mu}$ is a reproducing kernel Hilbert space, and denote K (or $K_{\Lambda,\mu}$ if necessary) its reproducing kernel. The standard weighted Dirichlet spaces on $\mathbb{D}$, denoted $\mathcal{D}_\alpha$, correspond to $\Lambda(t) = t^\alpha$ and $\mu = m$ the normalized arc measure on $\mathbb{T}$. If $\Lambda = 1$, then $\mathcal{D}_{\Lambda,\mu}$ is the harmonically weighted Dirichlet spaces. (See, e.g., [6,7]). Note that, in general, $\mathcal{D}_{\Lambda,\mu}$ is not contained in the Hardy space H^2. In the following proposition, using a reasoning similar to that in [3], we establish the density of polynomials in $\mathcal{D}_{\Lambda,\mu}$, for some regular weights Λ.

Proposition 2.1. *Let μ be a positive finite Borel measure on $\mathbb{T}$, and let $\Lambda : [0, 1] \to [0, +\infty)$ be an increasing function such that $\frac{\Lambda(t)}{t^c}$ is decreasing for some $c > 0$. Then the polynomials are dense in $\mathcal{D}_{\Lambda,\mu}$.*

Proof. The proof uses the fact that the dilations $f_r(z) := f(rz)$ for $r \in [0, 1)$ tend to f in the norm, i.e.,

$$\lim_{r\to 1} \|f_r - f\|_{\Lambda,\mu} = 0, \quad f \in \mathcal{D}_{\Lambda,\mu}.$$

To this end, it is sufficient to show that

$$\|f_r\|_{\Lambda,\mu} \lesssim \|f\|_{\Lambda,\mu}, \quad f \in \mathcal{D}_{\Lambda,\mu}, \quad r \in [0,1).$$

Since $\Lambda(2t) \asymp \Lambda(t)$, we have

$$\Lambda(1-|z|^2) \asymp \Lambda(1-|w|^2), \quad z \in D(w, \rho(w)),$$

where $\rho(w) = (1-|w|)/2$. Hence,

$$|f'(w)|^2\Lambda(1-|w|^2) \lesssim \int_{z\in\mathbb{D}} |f'(z)|^2\Lambda(1-|z|^2)\frac{(1-|z|^2)^2}{|1-\overline{z}w|^4}dA(z), \quad w \in \mathbb{D},$$

Denote

$$J(z,\zeta) := \int_{w\in\mathbb{D}} \frac{(1-|w|^2)}{|1-w\overline{z}|^4|1-w\overline{\zeta}|^2}dA(w), \quad z \in \mathbb{D}.$$

Using the following inequality (cf. [14, Lemma 2.5]):

$$J(rz,\zeta) \lesssim \frac{1}{(1-r^2|z|^2)|1-r\overline{\zeta}z|^2}, \quad z \in \mathbb{D},\ \zeta \in \mathbb{T},\ r \in [0,1),$$

we obtain

$$\begin{aligned}
\|f_r\|^2_{\Lambda,\mu} &= |f_r(0)|^2 + \int_{w\in\mathbb{D}} |rf'(rw)|^2\Lambda(1-|w|^2)P_\mu(w)dA(w)\\
&\le |f(0)|^2 + \int_{\zeta\in\mathbb{T}}\int_{w\in\mathbb{D}} |f'(rw)|^2\Lambda(1-|rw|^2)\frac{(1-|w|^2)}{|1-w\overline{\zeta}|^2}dA(w)d\mu(\zeta)\\
&\lesssim |f(0)|^2 + \int_{\zeta\in\mathbb{T}}\int_{w\in\mathbb{D}}\int_{z\in\mathbb{D}} |f'(z)|^2\Lambda(1-|z|^2)\\
&\qquad \frac{(1-|z|^2)^2(1-|w|^2)}{|1-rw\overline{z}|^4|1-w\overline{\zeta}|^2}dA(z)dA(w)d\mu(\zeta)\\
&= |f(0)|^2 + \int_{\zeta\in\mathbb{T}}\int_{z\in\mathbb{D}} |f'(z)|^2\Lambda(1-|z|^2)(1-|z|^2)^2J(rz,\zeta)dA(z)d\mu(\zeta)\\
&\lesssim |f(0)|^2 + \int_{\zeta\in\mathbb{T}}\int_{z\in\mathbb{D}} |f'(z)|^2\Lambda(1-|z|^2)\frac{(1-|z|^2)}{|1-\overline{\zeta}z|^2}dA(z)d\mu(\zeta)\\
&= |f(0)|^2 + \mathcal{D}_{\Lambda,\mu}(f)\\
&= \|f\|^2_{\Lambda,\mu}.
\end{aligned}$$

□

Let $f \in \mathrm{Hol}(\mathbb{D})$, and let $\zeta \in \mathbb{T}$, we write $f^*(\zeta) := \lim_{r\to1} f(r\zeta)$ its radial limit (if it exists) at ζ. Note that, under conditions of Proposition 2.1, if $\sup_{0\le r<1} K(r\zeta, r\zeta) < \infty$, then $f^*(\zeta)$ exists for every $f \in \mathcal{D}_{\Lambda,\mu}$.

We denote $S_{\Lambda,\mu}$ (or simply S) the shift operator acting on $\mathcal{D}_{\Lambda,\mu}$ defined as follows:

$$S : \mathcal{D}_{\Lambda,\mu} \longrightarrow \mathcal{D}_{\Lambda,\mu},\ f \mapsto Sf(z) = zf(z),$$

and S^* is its adjoint operator.

Theorem 2.1. *Let μ be a positive finite Borel measure on $\mathbb{T}$, and let $\Lambda : [0,1] \to [0,+\infty)$ be an increasing function such that $\frac{\Lambda(t)}{t^c}$ is decreasing for some $c > 0$, then*

$$\sigma_p(S^*) \supset \left\{\zeta \in \mathbb{T} : \int_0^1 \frac{dx}{\Lambda(x)\mu(\overline{\zeta}, x)} < \infty\right\}.$$

To prove Theorem 2.1, we need the following lemma.

Lemma 2.1. *Let μ be a positive finite Borel measure on $\mathbb{T}$, and let $\Lambda : [0,1] \to [0,+\infty)$ be an increasing function such that $\frac{\Lambda(t)}{t^c}$ is decreasing for some $c > 0$, then*

$$K(r\zeta, r\zeta) \lesssim 1 + \int_0^r \frac{dx}{(1-x)\Lambda(1-x)P_\mu(\zeta x)}, \quad r \in (0,1), \quad \zeta \in \mathbb{T}.$$

The proof of Lemma 2.1 is inspired from [5]. For the sake of completeness, we include the proof here.

Proof. We have

$$K(z,z) = \sup\{|f(z)|^2 : \|f\|_{\Lambda,\mu} \le 1\}, \quad z \in \mathbb{D}.$$

Let $f \in \mathcal{D}_{\Lambda,\mu}$, and let $z = r \in [\frac{1}{2}, 1)$. Consider

$$\Delta_r := \left\{x + iy \in \mathbb{D} : 0 \le x \le r, \frac{x-1}{2} \le y \le \frac{1-x}{2}\right\}.$$

We have

$$\begin{aligned}
&\frac{1}{1-r}\int_{-\frac{(1-r)}{2}}^{\frac{1-r}{2}} |f(r+is)|ds \\
&\quad = \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(r+i(1-r)y)|dy \\
&\quad = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|f(iy) + \int_0^r f'(t+i(1-t)y)dt\right| dy \\
&\quad \le \int_{-\frac{1}{2}}^{\frac{1}{2}} |\langle f, K_{iy}\rangle_{\Lambda,\mu}|dy + \int_{[0,r]\times\left[-\frac{1}{2};\frac{1}{2}\right]} |f'(t+i(1-t)y)|\, dydt \\
&\quad \le \|f\|_{\Lambda,\mu}\int_{-\frac{1}{2}}^{\frac{1}{2}} \|K_{iy}\|_{\Lambda,\mu}dy + \int_{[0,r]\times\left[-\frac{1}{2};\frac{1}{2}\right]} |f'(t+i(1-t)y)|\, dydt \\
&\quad \lesssim \|f\|_{\Lambda,\mu} + \int_{[0,r]\times\left[-\frac{1}{2};\frac{1}{2}\right]} |f'(t+i(1-t)y)|\, dydt.
\end{aligned}$$

Using the following change of variables:

$$\phi : [0,r] \times \left[-\frac{1}{2}, \frac{1}{2}\right] \longrightarrow \Delta_r$$

$$(t,y) \longmapsto u+iv, \quad \text{with } \begin{cases} u = t \\ v = (1-t)y, \end{cases}$$

we obtain

$$\begin{aligned}
&\frac{1}{1-r}\int_{-\frac{(1-r)}{2}}^{\frac{1-r}{2}} |f(r+is)|ds \\
&\lesssim \|f\|_{\Lambda,\mu} + \int_{\Delta_r} |f'(u+iv)| \frac{dudv}{1-|u|} \\
&\lesssim \|f\|_{\Lambda,\mu} + (\mathcal{D}_{\Lambda,\mu}(f))^{\frac{1}{2}} \\
&\left(\int_{\Delta_r} \frac{dudv}{(1-|u|)^2\Lambda(1-|u+iv|^2)P_\mu(u+iv)}\right)^{\frac{1}{2}}.
\end{aligned}$$

Now, for $u+iv \in \Delta_r$, we have $|u+iv| \leq u+|v| \leq u + \dfrac{1-u}{2} = \dfrac{1+u}{2}$. Then $1-|u+iv| \geqslant 1-\frac{1+u}{2} = \frac{1-u}{2}$. Hence, $\Lambda(1-|u+iv|) \gtrsim \Lambda(1-u)$, and $P[\mu](u+iv) \gtrsim P[\mu](u)$. Therefore,

$$\begin{aligned}
&\frac{1}{1-r}\int_{-\frac{(1-r)}{2}}^{\frac{1-r}{2}} |f(r+is)|ds \\
&\lesssim \|f\|_{\Lambda,\mu} + (\mathcal{D}_{\Lambda,\mu}(f))^{\frac{1}{2}} \left(\int_{\Delta_r} \frac{dudv}{(1-u)^2\Lambda(1-u)P_\mu(u)}\right)^{\frac{1}{2}} \\
&= \|f\|_{\Lambda,\mu} + (\mathcal{D}_{\Lambda,\mu}(f))^{\frac{1}{2}} \left(\int_{u=0}^{r}\int_{|v|\leq\frac{1-u}{2}} dv \frac{du}{(1-u)^2\Lambda(1-u)P_\mu(u)}\right)^{\frac{1}{2}} \\
&\lesssim \|f\|_{\Lambda,\mu} + (\mathcal{D}_{\Lambda,\mu}(f))^{\frac{1}{2}} \left(\int_0^r \frac{du}{(1-u)\Lambda(1-u)P_\mu(u)}\right)^{\frac{1}{2}}.
\end{aligned}$$

Moreover, the disc $D\left(r, \frac{1-r}{4}\right)$ is included in $\left\{z = x+iy, |x-r| \lesssim \frac{1-r}{4}, |y| \lesssim \frac{1-x}{2}\right\}$. Thus

$$\begin{aligned}
|f(r)| &\lesssim \frac{1}{(1-r)^2}\int_{r-\frac{1-r}{4}}^{r+\frac{1-r}{4}} \left(\int_{y=-\frac{(1-x)}{2}}^{\frac{1-x}{2}} |f(x+iy)|dy\right) dx \\
&\lesssim \|f\|_{\Lambda,\mu} + (\mathcal{D}_{\Lambda,\mu}(f))^{\frac{1}{2}} \left(\int_0^r \frac{dx}{(1-x)\Lambda(1-x)P_\mu(x)}\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore,

$$|f(r)|^2 \lesssim \left(1 + \int_0^r \frac{dx}{(1-x)\Lambda(1-x)P_\mu(x)}\right) \|f\|^2_{\Lambda,\mu},$$

and we get

$$K(z,z) \lesssim 1 + \int_0^r \frac{dx}{(1-x)\Lambda(1-x)P_\mu(xz^*)}.$$

□

Proof of Theorem 2.1. We have $tP_\mu(\overline{\zeta}(1-t)) \gtrsim \mu(\overline{\zeta}, t)$ for any $\zeta \in \mathbb{T}$, and $t \in (0,1)$. Then

$$\int_0^1 \frac{dx}{(1-x)\Lambda(1-x)P_\mu(\overline{\zeta}x)} \lesssim \int_0^1 \frac{dy}{\Lambda(y)\mu(\overline{\zeta}, y)}, \quad \zeta \in \mathbb{T}. \tag{2}$$

Let $\zeta \in \mathbb{T}$ such that $\int_0^1 \frac{dx}{\Lambda(x)\mu(\overline{\zeta},x)} < \infty$. Combining inequality (2) with Lemma 2.1, we obtain $\sup_{0\leq r<1} K(r\overline{\zeta}, r\overline{\zeta}) < \infty$. Consider now $L_{\overline{\zeta}} : \mathcal{D}_{\Lambda,\mu} \to \mathbb{C}, f \mapsto f^*(\overline{\zeta})$. Since $L_{\overline{\zeta}}$ is continuous, it follows from Riesz representation theorem that there exists $k_{\overline{\zeta}} \in \mathcal{D}_{\Lambda,\mu}$ such that $f^*(\overline{\zeta}) = \left\langle f, k_{\overline{\zeta}} \right\rangle_{\Lambda,\mu}$. Hence, $S^* k_{\overline{\zeta}} = \zeta k_{\overline{\zeta}}$. Indeed, we have

$$\left\langle f, S^* k_{\overline{\zeta}} \right\rangle_{\Lambda,\mu} = \left\langle Sf, k_{\overline{\zeta}} \right\rangle_{\Lambda,\mu} = \overline{\zeta} f^*(\overline{\zeta}) = \left\langle f, \zeta k_\zeta \right\rangle_{\Lambda,\mu}.$$

□

3. Growth of Power of Shift Operator

Let μ be a positive finite Borel measure on $\mathbb{T}$, and let S be the shift operator acting on the Dirichlet space $\mathcal{D}_{\Lambda,\mu}$ associated with μ and a positive function Λ.

Theorem 3.1. *Let μ be a positive finite Borel measure on $\mathbb{T}$, and let $\Lambda : [0,1] \to [0,+\infty)$ be an increasing function such that $\frac{\Lambda(t)}{t^c}$ is decreasing for some $c > 0$, and*

$$\int_0 \frac{dt}{\Lambda(t)} < \infty. \tag{3}$$

We have

$$||S^{*n}||^2 \asymp n\Lambda\left(\frac{1}{n}\right).$$

To prove Theorem 3.1, we require the following lemmas.

Lemma 3.1. *Let μ be a positive finite Borel measure on $\mathbb{T}$, and let $\Lambda : [0,1] \to [0,+\infty)$ be an increasing function such that $\frac{\Lambda(t)}{t^c}$ is decreasing for some $c > 0$. We have*

$$\|z^n\|^2_{\Lambda,\mu} \asymp n\Lambda\left(\frac{1}{n}\right), \quad n \geq 1.$$

Proof. Let $n \geq 1$. We have

$$\begin{aligned}
\mathcal{D}_{\Lambda,\mu}(z^n) &= \int_{z\in\mathbb{D}} n^2 \left(|z|^2\right)^{n-1} \Lambda\left(1-|z|^2\right) P[\mu](z)dA(z) \\
&= n^2 \int_{r=0}^{1} r^{2(n-1)} \Lambda\left(1-r^2\right) \int_{\theta=0}^{2\pi}\int_{t=0}^{2\pi} \frac{1-r^2}{|e^{it}-re^{i\theta}|^2}\frac{dt}{2\pi} d\mu\left(e^{i\theta}\right) dr^2 \\
&= \mu(\mathbb{T}) n^2 \int_0^1 s^{n-1}\Lambda(1-s)ds \qquad (4)\\
&\geq \mu(\mathbb{T})\left(1-\frac{2}{n}\right)^{n-1} n^2 \int_{\frac{1}{n}}^{\frac{2}{n}} \Lambda(t)dt \\
&\geq \mu(\mathbb{T})\left(1-\frac{2}{n}\right)^{n-1} n\Lambda\left(\frac{1}{n}\right).
\end{aligned}$$

Now,

$$\begin{aligned}
\int_0^1 r^{n-1}\Lambda(1-r)dr &\leq \int_0^{\frac{1}{n}} \Lambda(t)dt + \int_{\frac{1}{n}}^1 (1-t)^{n-1}\Lambda(t)dt \\
&\leq \frac{1}{n}\Lambda\left(\frac{1}{n}\right) + n^c\Lambda\left(\frac{1}{n}\right)\int_0^1 t^{n-1}(1-t)^c dt \\
&= \left[1+n^{c+1}\mathbf{B}(n,c+1)\right] n\Lambda\left(\frac{1}{n}\right),
\end{aligned}$$

where $\mathbf{B}$ is Beta function. Using equality (4), we obtain

$$\mathcal{D}_{\Lambda,\mu}(z^n) \leq \left[1+n^{c+1}\mathbf{B}(n,c+1)\right] n\Lambda\left(\frac{1}{n}\right).$$

□

In the case of $\mu = \delta_\zeta$, the Dirac measure at $\zeta \in \mathbb{T}$, the local weighted Dirichlet integral of $f \in \mathrm{Hol}(\mathbb{D})$ is given by

$$\mathcal{D}_{\Lambda,\zeta}(f) = \int_{\mathbb{D}} |f'(z)|^2 \Lambda(1-|z|^2)\frac{1-|z|^2}{|\zeta - z|^2}dA(z),$$

Suppose that $\Lambda \equiv 1$. Let $f \in \mathrm{Hol}(\mathbb{D})$, say $f(z) = \sum_{n\in\mathbb{N}} a_n z^n$, we have

$$\mathcal{D}_\zeta(f) := \mathcal{D}_{1,\zeta}(f) = \sum_{n\geq 1} \frac{1}{n(n+1)}\left|\sum_{k=1}^{n} k a_k \zeta^k\right|^2, \qquad (5)$$

see [7, Theorem 7.2.6].

For the rest of the paper, we suppose that $\int_0 \frac{dt}{\Lambda(t)} < \infty$. Therefore, according to Lemma 2.1, the reproducing kernel of $\mathcal{D}_{\Lambda,\zeta}$ satisfies $\sup_{0\leq r<1} K_{\Lambda,\zeta}(r\zeta, r\zeta) < \infty$. Then the following space

$$\mathcal{D}^0(\Lambda,\zeta) = \{f \in \mathcal{D}_{\Lambda,\zeta} : \; f^*(\zeta) = 0\},$$

is closed in $\mathcal{D}_{\Lambda,\zeta}$. In order to extend formula (5) to $\mathcal{D}_{\Lambda,\zeta}$, we endow the space $\mathcal{D}^0(\Lambda,\zeta)$ with the following norm:

$$||f||^2_{0,\Lambda,\zeta} := \mathcal{D}_{\Lambda,\zeta}(f), \quad f \in \mathrm{Hol}(\mathbb{D}).$$

Additionally, we consider the weighted Bergman space $\mathcal{A}^2_{(1-|z|^2)\Lambda(1-|z|^2)}$ equipped with the following norm:

$$||f||^2 = \int_{\mathbb{D}} |f(z)|^2(1-|z|^2)\Lambda(1-|z|^2)dA(z), \quad f \in \mathrm{Hol}(\mathbb{D}).$$

Lemma 3.2. *Let μ be a positive finite Borel measure on $\mathbb{T}$, and let $\Lambda : [0,1] \to [0,+\infty)$ be an increasing function such that $\frac{\Lambda(t)}{t^c}$ is decreasing for some $c > 0$, and*

$$\int_0 \frac{dt}{\Lambda(t)} < \infty.$$

For $f \in \mathcal{D}_{\Lambda,\zeta}$ write $f(z) = \sum_{n\in\mathbb{N}} a_n z^n$. We have

$$\mathcal{D}_{\Lambda,\zeta}(f) = \sum_{n\geq 1} ||z^{n-1}||^2 \left|\sum_{k=1}^{n} k a_k \zeta^k\right|^2.$$

Proof. Without loss of generality, we may assume that $\zeta = 1$. We make the following identification

$$\mathbf{T}\colon \mathcal{D}^0(\Lambda,1) \longrightarrow \mathcal{A}^2_{(1-|z|^2)\Lambda(1-|z|^2)}, \;\; f \mapsto \mathbf{T}f(z) = f'(z)/(z-1).$$

The operator $\mathbf{T}$ is a surjective isometry. Indeed, let $f \in \mathcal{D}^0(\Lambda,1)$, and denote $||f||_0 := ||f||_{0,\Lambda,1}$, we have

$$\begin{aligned} ||\mathbf{T}f||^2 &= \int_{\mathbb{D}} \left|\frac{f'(z)}{z-1}\right|^2 (1-|z|^2)\Lambda(1-|z|^2)dA(z) \\ &= \int_{\mathbb{D}} |f'(z)|^2 \frac{1-|z|^2}{|1-z|^2}\Lambda(1-|z|^2)dA(z) \\ &= ||f||_0^2, \end{aligned}$$

then $\mathbf{T}$ is an isometry. To prove that $\mathbf{T}$ is surjective, we consider

$$\mathbf{V}\colon \mathbb{C}[z] \longrightarrow \mathcal{D}^0(\Lambda,1), \;\; p \mapsto \mathbf{V}p(z) = \int_z^1 (\lambda-1)p(\lambda)d\lambda,$$

where $\mathbb{C}[z]$ is the set of polynomials. Let $p \in \mathbb{C}[z]$, we have

$$\begin{aligned} ||\mathbf{V}p||_0^2 &= \int_{\mathbb{D}} |\left(\mathbf{V}p\right)'(z)|^2 \frac{1-|z|^2}{|1-z|^2}\Lambda(1-|z|^2)dA(z) \\ &= \int_{\mathbb{D}} |p(z)|^2(1-|z|^2)\Lambda(1-|z|^2)dA(z) \\ &= ||p||^2, \end{aligned}$$

and

$$\mathbf{TV}p(z) = \frac{(\mathbf{V}p)'(z)}{z-1} = p(z), \quad z \in \mathbb{D}.$$

Since $(1-|z|^2)\Lambda(1-|z|^2)$ is a radial weight, we get that polynomials are dense in $\mathcal{A}^2_{(1-|z|^2)\Lambda(1-|z|^2)}$. It follows from equality (6) that $\mathbf{V}$ extends to an isometry $\tilde{\mathbf{V}}$ on $\mathcal{A}^2_{(1-|z|^2)\Lambda(1-|z|^2)}$. Using equality (6), we obtain

$$\mathbf{T}\tilde{\mathbf{V}}f = f, \quad f \in \mathcal{A}^2_{(1-|z|^2)\Lambda(1-|z|^2)}.$$

Thus, $\mathbf{T}$ is surjective. Moreover, since $(1-|z|^2)\Lambda(1-|z|^2)$ is a radial weight, we get that $\left(e_n(z) := \frac{z^n}{||z^n||}\right)$ is an orthonormal basis of $\mathcal{A}^2_{(1-|z|^2)\Lambda(1-|z|^2)}$. Hence, the following sequence

$$u_{n+1}(z) = \mathbf{V}e_n(z) = \frac{1}{||z^n||}\left(\frac{1}{n+2}(z^{n+2}-1) - \frac{1}{n+1}(z^{n+1}-1)\right), \quad z \in \mathbb{D} \text{ and } n \in \mathbb{N},$$

is an orthonormal basis of $\mathcal{D}^0(\Lambda, 1)$. Therefore, there exists a sequence $(c_n)_{n\geq 1}$ of complex numbers such that

$$f(z) = \sum_{n\geq 1} c_n u_n(z), \quad z \in \mathbb{D}.$$

Thus,

$$\mathcal{D}_{\Lambda,1}(f) = \sum_{n\geq 1} |c_n|^2.$$

According to identification (6), we obtain

$$\begin{cases} a_0 = \sum_{n\geq 1} \frac{c_n}{||z^{n-1}||n(n+1)}, \\ a_1 = \frac{-c_1}{||z^0||}, \\ a_n = \frac{1}{n}\left(\frac{c_{n-1}}{||z^{n-2}||} - \frac{c_n}{||z^{n-1}||}\right), \quad n \geq 2. \end{cases}$$

Then

$$c_n = -||z^{n-1}|| \sum_{k=1}^{n} k a_k, \ n \geq 1.$$

□

Lemma 3.3. *Let μ be a positive finite Borel measure on $\mathbb{T}$, and let $\Lambda : [0,1] \to [0, +\infty)$ be an increasing function such that $\frac{\Lambda(t)}{t^c}$ is decreasing for some $c > 0$, and*

$$\int_0 \frac{dt}{\Lambda(t)} < \infty.$$

For $f \in \mathcal{D}_{\Lambda,\zeta}$ *write* $f(z) = \sum_{n\in\mathbb{N}} a_n z^n$. *Then*

$$\mathcal{D}_{\Lambda,\zeta}(z^p f) \lesssim ||f||^2_{\Lambda,\zeta} \left(1 + \sum_{n\geq p} p^2 ||z^{n-1}||^2 \left(1 + \sum_{n\geq 1} \frac{1}{n^4||z^{n-1}||^2}\right)\right), \quad p \geq 1.$$

Proof. We have

$$z^p f(z) = \sum_{n\geq 0} a_{n-p} z^n \quad \text{with } a_{-1} = a_{-2} = ... = a_{-p} = 0, \text{ and } z \in \mathbb{D}.$$

By Theorem 3.2, we obtain

$$\begin{aligned}
\mathcal{D}_{\Lambda,\zeta}(z^p f) &= \sum_{n\geq 1} ||z^{n-1}||^2 \left|\sum_{k=1}^{n} k a_{k-p}\right|^2 \\
&= \sum_{n\geq p} ||z^{n-1}||^2 \left|\sum_{i=0}^{n-p} (i+p) a_i\right|^2 \\
&\leq \underbrace{2\sum_{n\geq p} ||z^{n-1}||^2 \left|\sum_{k=0}^{n-p} k a_k\right|^2}_{=I_1} + \underbrace{2\sum_{n\geq p} ||z^{n-1}||^2 \left|\sum_{k=0}^{n-p} p a_k\right|^2}_{I_2}.
\end{aligned}$$

Since $||z^{n-1}|| \leq ||z^{n-1-p}||$, we get

$$I_1 \leq \mathcal{D}_{\Lambda,\zeta}(f).$$

Let $A_n = \sum_{k=0}^{n} k a_k$, we have

$$a_n = \frac{1}{n}\left(A_n - A_{n-1}\right), \quad n \geq 1,$$

and

$$\begin{aligned}
\left|\sum_{k=0}^{n} a_k\right|^2 &= \left|\sum_{k=1}^{n} \frac{1}{k}\left(A_k - A_{k-1}\right) + a_0\right|^2 \\
&= \left|\sum_{k=1}^{n} \frac{1}{k} A_k - \sum_{k=0}^{n-1} \frac{1}{k+1} A_k + a_0\right|^2 \\
&= \left|\sum_{k=1}^{n-1} \frac{1}{k(k+1)} A_k + \frac{1}{n} A_n + a_0\right|^2 \\
&\leq 4\left|\sum_{k=1}^{n-1} \frac{1}{k(k+1)} A_k\right|^2 + \frac{4}{n^2}|A_n|^2 + 2|a_0|^2.
\end{aligned}$$

Then

$$I_2 \lesssim \underbrace{\sum_{n\geq p} p^2 ||z^{n-1}||^2 \left|\sum_{k=1}^{n-p-1} \frac{1}{k(k+1)} A_k\right|^2}_{Q_1} + \underbrace{\sum_{n\geq p} p^2 \frac{||z^{n-1}||^2}{n^2} |A_{n-p}|^2}_{Q_2}$$
$$+ \underbrace{|a_0|^2 \sum_{n\geq p} p^2 ||z^{n-1}||^2}_{Q_3}.$$

It follows from the inequality $|a_0| \leq ||f||_{\Lambda,\delta_\zeta}$ that

$$Q_3 \leq ||f||_{\Lambda,\delta_\zeta} \times \sum_{n\geq p} p^2 ||z^{n-1}||^2.$$

In addition,

$$\begin{aligned} Q_2 &= \sum_{n\geq p} \frac{p^2}{n^2} ||z^{n-1}||^2 |A_{n-p}|^2 \\ &\leq \sum_{n\geq p} ||z^{n-1-p}||^2 |A_{n-p}|^2 \\ &= \mathcal{D}_{\Lambda,\zeta}(f). \end{aligned}$$

Moreover,

$$\begin{aligned} \left|\sum_{k=1}^{n} \frac{1}{k(k+1)} A_k\right|^2 &\lesssim \left(\sum_{k=1}^{n} \frac{1}{||z^{k-1}|| k^2} \times ||z^{k-1}|| |A_k|\right)^2 \\ &\leq \sum_{k=1}^{n} \frac{1}{k^4 ||z^{k-1}||^2} \times \sum_{k=1}^{n} ||z^{k-1}||^2 |A_k|^2 \\ &\lesssim \sum_{n\geq 1} \frac{1}{n^4 ||z^{n-1}||^2} \times \mathcal{D}_{\Lambda,\zeta}(f). \end{aligned}$$

Then

$$\begin{aligned} Q_1 &= \sum_{n\geq p} p^2 ||z^{n-1}||^2 \left|\sum_{k=1}^{n-p} \frac{A_k}{k(k+1)}\right|^2 \\ &\lesssim \mathcal{D}_{\Lambda,\zeta}(f) \sum_{n\geq 1} \frac{1}{n^4 ||z^{n-1}||^2} \times \sum_{n\geq p} p^2 ||z^{n-1}||^2. \end{aligned}$$

□

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Let $p \in \mathbb{N}$. By definition, $\|S^{*p}\| = \sup_{\|f\|_{\Lambda,\mu}=1} \|S^{*p}f\|_{\Lambda,\mu}$. Then we get $\|S^{*p}\|^2 \geq \|z^p\|^2_{\Lambda,\mu} \asymp p\Lambda\left(\frac{1}{p}\right)$. Now, let $n \in \mathbb{N}$. Similar to the proof of Lemma 3.1, we have

$$\|z^{n-1}\|^2 = \int_0^1 r^{n-1}(1-r)\Lambda(1-r)dr \asymp \frac{1}{n^2}\Lambda\left(\frac{1}{n}\right).$$

Then

$$\begin{aligned}\sum_n \frac{1}{n^4\|z^{n-1}\|^2} &\asymp \sum_n \frac{1}{n^2\Lambda\left(\frac{1}{n}\right)} \\ &\leq 2\sum_n \int_{1/(n+1)}^{1/n} \frac{1}{\Lambda(t)}dt \\ &\asymp \int_0 \frac{dt}{\Lambda(t)} < \infty. \end{aligned} \tag{6}$$

Furthermore,

$$\begin{aligned}\sum_{n\geq p} p^2\|z^{n-1}\|^2 &\asymp \sum_{n\geq p} \frac{p^2}{n^2}\Lambda\left(\frac{1}{n}\right) \\ &\leq p^2\Lambda\left(\frac{1}{p}\right)\sum_{n\geq p}\frac{1}{n^2} \\ &\asymp p\Lambda\left(\frac{1}{p}\right). \end{aligned} \tag{7}$$

Combining inequalities (6), (7), and Lemma 3.3, we obtain

$$\begin{aligned}\|S^{*p}\|^2 &= \sup_{\|f\|_{\Lambda,\mu}=1} \left\{|f(0)|^2 + \int_{\mathbb{T}} \mathcal{D}_{\Lambda,\zeta}(z^p f)d\mu(\zeta)\right\} \\ &\lesssim \sup_{\|f\|_{\Lambda,\mu}=1} \left\{|f(0)|^2 + p\Lambda\left(\frac{1}{p}\right)\mathcal{D}_{\Lambda,\mu}(f)\right\} \\ &\lesssim p\Lambda\left(\frac{1}{p}\right). \end{aligned}$$

4. Proofs of the Main Theorems

We recall some definitions which will be used in what follows. Let $E \subset \mathbb{T}$, let $\delta > 0$, and let $d \in [0, \infty)$. Consider

$$H^d_\delta(E) = \inf\left\{\sum_{i=1}^\infty |I_i|^d : \ E \subset \bigcup_{i=1}^\infty I_i \text{ and } |I_i| < \delta\right\},$$

where the infimum is taken over all countable intervals covering E. The Hausdorff outer measure of dimension d is given by $H^d(E) = \lim_{\delta\to 0} H^d_\delta(E)$, and the Hausdorff dimension of E is defined by

$$\dim_H(E) = \inf\{d \geq 0 : \ H^d(E) = 0\}.$$

A closed subset E of $\mathbb{T}$ is an *Ahlfors-David regular set* if there exists a measure μ supported on E such that

$$C^{-1}t^d \leq \mu(\zeta, t) \leq Ct^d$$

for all $\zeta \in \operatorname{supp}\mu$ and $t \in [0, 1]$. In this case, we have $\dim_H(E) = d$ (see [12] for more details).

Proof Theorem 1.2. Assume that there exist μ and Λ satisfying the conditions of Theorem 1.2. Consider the operator $T = S^*$ on $\mathcal{D}_{\Lambda,\mu}$. We have $\sigma_p(T) \supset E$. Indeed, for $\zeta \in E$, we have $\int_0 \frac{dt}{\Lambda(t)\mu(\overline{\zeta},t)} < \infty$, it follows from Theorem 2.1 that $\zeta \in \sigma_p(T)$. The first assertion is proved. Since $\mu(\overline{\zeta}, t)\Lambda(t) \lesssim \Lambda(t)$, condition (3) holds and the last assertion comes from Theorem 3.1.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Since E is an Ahlfors–David set, $E^* := \{\zeta \in \mathbb{T} : \overline{\zeta} \in E\}$ is too. Thus, there exists a measure μ supported on E^* such that $\mu(\zeta, t) \asymp t^d$ for any $\zeta \in E^*$, and $d = \dim_H(E)$. Let $\alpha > d$ and consider the function $\Lambda(t) = t^{1-\alpha}$, $t \in (0, 1)$. We have

$$\int_0 \frac{dt}{\Lambda(t)\mu(\overline{\zeta}, t)} \asymp \int_0 \frac{dt}{t^{1-(\alpha-d)}} < \infty, \quad \zeta \in E.$$

According to Theorem 1.2, we deduce the result.

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