Results Math (2024) 79:56

1422-6383/24/020001-30
published online January 10, 2024
https://doi.org/10.1007/s00025-023-02084-1

Results in Mathematics

Riemannian Warped Product Maps

Kiran Meena, Bayram Şahin, and Hemangi Madhusudan Shah

Abstract. In this paper, we introduce Riemannian warped product map as a generalization of warped product isometric immersion and Riemannian warped product submersion with examples and obtain some characterizations. First, we define Riemannian warped product map and find conditions for a Riemannian map to be Riemannian warped product map. We show that Riemannian warped product map is a composition of Riemannian warped product submersion followed by warped product isometric immersion locally. In addition, we show that Riemannian warped product map satisfies the generalized eikonal equation which is a well known bridge between geometrical and physical optics. We also find necessary and sufficient conditions for the fibers, range space of the derivative map of Riemannian warped product map and horizontal distributions to be totally geodesic and minimal. Further, we give some fundamental geometric properties for the study of such smooth maps. Precisely, we construct Gauss formula (second fundamental form), Weingarten formula and tension field. We obtain necessary and sufficient conditions for a Riemannian warped product map to be totally geodesic, harmonic and umbilical. Comparatively, we analyse the obtained results with the existing results for a Riemannian map between Riemannian manifolds.

Mathematics Subject Classification. 53B20, 53C42, 53C43.

Keywords. Riemannian warped product manifold, second fundamental form, Riemannian map, totally geodesic map, harmonic map, umbilical map.

1. Introduction

The notion of warped product manifolds is a generalization of the Riemannian product manifolds [3]. In general relativity, warped product manifolds have been used to construct Schwarzschild and Robertson–Walker cosmological models [24]. In addition, warped product manifolds were used to obtain new families of Hamiltonian-stationary Lagrangian submanifolds [11]. Moore gave sufficient conditions for an isometric immersion into Euclidean space to decompose into a product immersion [19]. Nash concluded that every Riemannian manifold hence warped product manifold can be embedded to some Euclidean space [20]. For the differential geometry of submanifolds of warped product manifolds, we refer to [6,12,13].

Let $\varphi_i : M_i \to N_i$ be isometric immersions between Riemannian manifolds for $i = 1, 2, \ldots, k$. Suppose $\rho_i : N_i \to \mathbb{R}^+$ and $f_i := \rho_i \circ \varphi_i : M_i \to \mathbb{R}^+$ for $i = 1, 2, \ldots, k-1$ be smooth functions. Then the smooth map $\varphi : M_1 \times_{f_1} M_2 \times_{f_2} \cdots \times_{f_{k-1}} M_k \to N_1 \times_{\rho_1} N_2 \times_{\rho_2} \cdots \times_{\rho_{k-1}} N_k$ between warped product manifolds such that $\varphi(p_1, p_2, \ldots, p_k) = (\varphi_1(p_1), \varphi_2(p_2), \ldots, \varphi_k(p_k))$ is also an isometric immersion, called *warped product isometric immersion*. For the fundamental studies of warped product immersions we refer to [5–10,22,31].

The geometry of Riemannian submersions and their applications have been discussed widely by Falcitelli et al. [16]. It is known that Riemannian submersions have been used to construct some Riemannian manifolds with positive or non-negative sectional curvature and Einstein manifolds. In addition, Riemannian submersions have many applications in physics, Yang-Mills theory, Kaluza-Klein theory, supergravity and superstring theories etc. Recently, warped product Riemannian submersions were studied by Erken et al. [14,15]. Let $\varphi_i : M_i \to N_i$ be Riemannian submersions between Riemannian manifolds for $i = 1, 2, \ldots, k$. Suppose $\rho_i : N_i \to \mathbb{R}^+$ and $f_i := \rho_i \circ \varphi_i : M_i \to \mathbb{R}^+$ for $i = 1, 2, \ldots, k-1$ be smooth functions. Then the smooth map $\varphi : M_1 \times_{f_1} M_2 \times_{f_2} \cdots \times_{f_{k-1}} M_k \to N_1 \times_{\rho_1} N_2 \times_{\rho_2} \cdots \times_{\rho_{k-1}} N_k$ between warped product manifolds such that $\varphi(p_1, p_2, \ldots, p_k) = (\varphi_1(p_1), \varphi_2(p_2), \ldots, \varphi_k(p_k))$ is also a Riemannian submersion, called *Riemannian warped product submersion*.

In 1992, a generalization of the notions of an isometric immersion and Riemannian submersion namely "*Riemannian map*" was introduced by Fischer [17] satisfying the generalized eikonal equation whose applications are well studied in geometry and physics. Importantly, Fischer proposed an approach to build a quantum model of nature using Riemannian maps. He pointed out an interesting relationship between Riemannian maps, harmonic maps and Lagrangian field theory on the mathematical side, and Maxwell's equation, Shrödinger's equation and their proposed generalization on the physical side. In the last decade, Şahin investigated the geometry of Riemannian maps widely [29].

Now, it will be interesting to introduce the new area of research namely "*Riemannian warped product map*" as a generalization of warped product isometric immersion and Riemannian warped product submersion. In Sect. 2, we recall some basic geometric concepts about the Riemannian maps and Riemannian warped product manifolds. In Sect. 3, we define Riemannian warped product map with examples and obtain some characterizations. We show that Riemannian warped product map satisfies the generalized eikonal equation. We also find necessary and sufficient conditions for the fibers, range space of the derivative map of Riemannian warped product map and horizontal distributions to be totally geodesic and minimal. In Sect. 4, we calculate the second fundamental form (Gauss formula) followed by a necessary and sufficient condition for a Riemannian warped product map to be totally geodesic. In Sect. 5, we calculate tension field followed by a necessary and sufficient condition for a Riemannian warped product map to be harmonic. In the last section, we construct Weingarten formula followed by a necessary and sufficient condition for a Riemannian warped product map to be umbilical.

2. Preliminaries

In this section, we survey the notion of Riemannian map between Riemannian manifolds and its fundamental geometric properties. Later on we recall the notion of Riemannian warped product manifolds.

2.1. Riemannian maps

Let $\varphi : (M^m, g_M) \to (N^n, g_N)$ be a smooth map between Riemannian manifolds such that $0 < rank\varphi \leq \min\{m, n\}$ and let φ_* be its differential map. We denote the kernel space of φ_* at $p \in M$ by $\mathcal{V}_p = ker\varphi_{*p}$ and its orthogonal complement space in the tangent space T_pM by $\mathcal{H}_p = (ker\varphi_{*p})^\perp$. Thus we have

$$T_pM = (ker\varphi_{*p}) \oplus (ker\varphi_{*p})^\perp = \mathcal{V}_p \oplus \mathcal{H}_p.$$

Similarly, we denote the range space of φ_* at $\varphi(p) \in N$ by $range\varphi_{*p}$ and its orthogonal complement space in the tangent space $T_{\varphi(p)}N$ by $(range\varphi_{*p})^\perp$. If $rank\varphi < \min\{m, n\}$, we have $(range\varphi_*)^\perp \neq \{0\}$ and hence

$$T_{\varphi(p)}N = (range\varphi_{*p}) \oplus (range\varphi_{*p})^\perp.$$

We say φ is a Riemannian map at $p \in M$ if $\varphi_{*p}|_{\mathcal{H}} : ((ker\varphi_{*p})^\perp, g_{M(p)}|_{(ker\varphi_{*p})^\perp}) \to (range\varphi_{*p}, g_{N(\varphi(p))}|_{(range\varphi_{*p})})$ is a linear isometry, i.e.

$$g_N(\varphi_*X, \varphi_*Y) = g_M(X, Y) \quad \text{for all } X, Y \in \Gamma(ker\varphi_{*p})^\perp. \tag{1}$$

Clearly, it is an isometric immersion if $ker\varphi_* = \{0\}$ and a Riemannian submersion if $(range\varphi_*)^\perp = \{0\}$.

For all vector fields X, Y on M, the O'Neill tensors A and T were defined in [23]

$$A_X Y = \mathcal{H}\nabla^M_{\mathcal{H}X}\mathcal{V}Y + \mathcal{V}\nabla^M_{\mathcal{H}X}\mathcal{H}Y, \tag{2}$$

$$T_X Y = \mathcal{H}\nabla^M_{\mathcal{V}X}\mathcal{V}Y + \mathcal{V}\nabla^M_{\mathcal{V}X}\mathcal{H}Y, \tag{3}$$

where ∇^M is the Levi-Civita connection of g_M. Here $\mathcal{V}$ and $\mathcal{H}$ denote the projections to vertical and horizontal subbundles, respectively. For any $X \in \Gamma(TM)$ the operators T_X and A_X are skew-symmetric reversing the horizontal and vertical distributions. In addition, $T_X = T_{\mathcal{V}X}, A_X = A_{\mathcal{H}X}$ and $T_U W = T_W U$ for all $U, W \in \Gamma(ker\varphi_*)$.
Now, from (2) and (3), we have

$$\nabla^M_V W = T_V W + \hat{\nabla}_V W,$$
$$\nabla^M_X V = A_X V + \mathcal{V}\nabla^M_X V$$

and

$$\nabla^M_X Y = A_X Y + \mathcal{H}\nabla^M_X Y$$

for all $X, Y \in \Gamma(ker\varphi_*)^\perp$ and $V, W \in \Gamma(ker\varphi_*)$ with $\hat{\nabla}_V W = \mathcal{V}\nabla^M_V W$.

The map φ_* can be viewed as a section of bundle $Hom(TM, \varphi^{-1}TN) \to M$, where $\varphi^{-1}TN$ is the pullback bundle whose fibers at $p \in M$ is $(\varphi^{-1}TN)_p = T_{\varphi(p)}N$. The bundle $Hom(TM, \varphi^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection $\overset{N}{\nabla}{}^{\varphi}$. Then the second fundamental form of φ is given by [21]

$$(\nabla\varphi_*)(X, Y) = (\nabla\varphi_*)(Y, X) = \overset{N}{\nabla}{}^{\varphi}_X \varphi_* Y - \varphi_*(\nabla^M_X Y) \tag{4}$$

for all $X, Y \in \Gamma(TM)$, where $\overset{N}{\nabla}{}^{\varphi}_X \varphi_* Y \circ \varphi = \nabla^N_{\varphi_* X}\varphi_* Y$.

2.2. Riemannian warped product manifolds

Let $(M_1^{m_1}, g_{M_1})$ and $(M_2^{m_2}, g_{M_2})$ be two Riemannian manifolds and f be a positive smooth function on M_1. The warped product $M := M_1 \times_f M_2$ of M_1 and M_2 is the Cartesian product $M_1 \times M_2$ with the metric $g_M = g_{M_1} + f^2 g_{M_2}$ defined by

$$g_M(X, Y) = g_{M_1}(\pi_*(X), \pi_*(Y)) + f^2(\pi(p_1))g_{M_2}(\sigma_*(X), \sigma_*(Y)),$$

where X, Y are vector fields on $M_1 \times M_2$. In addition, $\pi : M_1 \times_f M_2 \to M_1$ such that $(x, y) \to x$ and $\sigma : M_1 \times_f M_2 \to M_2$ such that $(x, y) \to y$ are the projection maps which become submersions. Moreover, we can see that the fibers $\{x\} \times M_2 = \pi^{-1}(x)$ and the leaves $M_1 \times \{y\} = \sigma^{-1}(y)$ are Riemannian submanifolds of $M = M_1 \times_f M_2$. The vectors tangent to the leaves are called horizontal and the vectors tangent to the fibers are called vertical. If $v \in T_x M_1, x \in M_1$ and $y \in M_2$, then the lift $\bar{v}$ of v to (x, y) is the unique vector of $T_{(x,y)}M_1 \times M_2 = T_{(x,y)}M$ such that $\pi_*(\bar{v}) = v$, and lift of a vector field $X \in \Gamma(TM_1)$ to $M = M_1 \times_f M_2$ is the vector field $\bar{X}$ such that $\pi_{*(x,y)}(\bar{X}_{(x,y)}) = X_x$. Thus the lift of $X \in \Gamma(TM_1)$ to $M_1 \times M_2$ is the unique element $\bar{X}$ of $\Gamma(T(M_1 \times M_2))$ which is π-related to X. Now, the set of all such horizontal lifts $\bar{X}$ is denoted by $\mathcal{L}_{\mathcal{H}}(M_1)$ and the set of all vertical lifts by $\mathcal{L}_{\mathcal{V}}(M_2)$ (for details we refer to [23] and [6]). Thus a vector field $\bar{E}$ of $M_1 \times M_2$ can be written as $\bar{E} = \bar{X} + \bar{U}$, where $\bar{X} \in \mathcal{L}_{\mathcal{H}}(M_1)$ and $\bar{U} \in \mathcal{L}_{\mathcal{V}}(M_2)$. We can easily prove that $\pi_*(\mathcal{L}_{\mathcal{H}}(M_1)) = \Gamma(TM_1)$ and $\sigma_*(\mathcal{L}_{\mathcal{V}}(M_2)) = \Gamma(TM_2)$. Clearly, $\pi_*(\bar{X}) = X \in \Gamma(TM_1)$ and $\sigma_*(\bar{U}) = U \in \Gamma(TM_2)$. In this paper, we use the same notation for a vector field and for its lift to the product manifold.

Now, we recall some basic results on warped product manifolds:

Lemma 1 (Chapter 7, Lemma 34, [24, p. 206]). *If $\tilde{f} \in \mathcal{F}(M_1)$, then the gradient of the lift $\tilde{f} \circ \pi$ of $\tilde{f}$ to $M = M_1 \times_f M_2$ is the lift of the gradient of $\tilde{f}$ on M_1.*

Lemma 2 [24]. *Let $M = M_1 \times_f M_2$ be a Riemannian warped product manifold. Then*

(*i*) $\nabla^M_{X_1} Y_1$ *is the lift of* $\nabla^{M_1}_{X_1} Y_1$,

(*ii*) $\nabla^M_{X_1} X_2 = \nabla^M_{X_2} X_1 = (X_1(f)/f) X_2$,

(*iii*) *nor* $(\nabla^M_{X_2} Y_2) = -g_M(X_2, Y_2)(\nabla^M \ln f)$,

(*iv*) *tan* $(\nabla^M_{X_2} Y_2)$ *is the lift of* $(\nabla^{M_2}_{X_2} Y_2)$,

where $X_i, Y_i \in \mathcal{L}(M_i)$. In addition, ∇^M and ∇^{M_i} are Levi-Civita connections on M and M_i respectively for $i = 1, 2$.

Proposition 3 [4]. *Let $\varphi_i : M_i \to N_i$ for $i = 1, 2$ be smooth functions. Then*

$$(\varphi_1 \times \varphi_2)_* x = (\varphi_{1*} x_1, \varphi_{2*} x_2),$$

where $x = (x_1, x_2) \in T_{(p_1,p_2)}(M_1 \times M_2)$.

Proposition 4 [4]. *Let π and σ be the projections of $M_1 \times M_2$ onto M_1 and M_2 respectively. Then $\lambda : T_{(p_1,p_2)}(M_1 \times M_2) \to T_{p_1} M_1 \oplus T_{p_2} M_2$ such that $x \mapsto (\pi_*, \sigma_*) x$ is an isomorphism.*

By above Proposition for $X = (X_1, X_2) \in \Gamma(T(M_1 \times M_2))$ we can write $X = X_1 + X_2$ where $X_1 \in \mathcal{L}(M_1)$ and $X_2 \in \mathcal{L}(M_2)$.

3. Characterizations of Riemannian Warped Product Maps

In this section, we introduce Riemannian warped product map between Riemannian warped product manifolds with examples and obtain some characterizations.

Proposition 5. *Let $\varphi_i : (M_i, g_{M_i}) \to (N_i, g_{N_i})$ be Riemannian maps between Riemannian manifolds for $i = 1, 2$. Suppose $\rho : N_1 \to \mathbb{R}^+$ and $f := \rho \circ \varphi_1 : M_1 \to \mathbb{R}^+$ be smooth functions. Then the map $\varphi = \varphi_1 \times \varphi_2 : (M = M_1 \times_f M_2, g_M) \to (N = N_1 \times_\rho N_2, g_N)$ between Riemannian warped product manifolds such that $(\varphi_1 \times \varphi_2)(p_1, p_2) = (\varphi_1(p_1), \varphi_2(p_2))$ is a Riemannian map. Here f is called lift of ρ.*

Proof. Let $\varphi_i : M_i \to N_i$ be two Riemannian maps between Riemannian manifolds for $i = 1, 2$. For $p = (p_1, p_2) \in M_1 \times M_2$ we have

$$\begin{aligned} T_{(p_1,p_2)}(M_1 \times M_2) &= T_{(p_1,p_2)}(M_1 \times \{p_2\}) \oplus T_{(p_1,p_2)}(\{p_1\} \times M_2) \\ &= (ker\varphi_{*p})^{\perp} \oplus (ker\varphi_{*p}), \end{aligned}$$

where

$$T_{(p_1,p_2)}(M_1 \times \{p_2\}) = \left((ker\varphi_{1*p_1})^{\perp} \times \{p_2\}\right) \oplus ((ker\varphi_{1*p_1}) \times \{p_2\})$$

and

$$T_{(p_1,p_2)}(\{p_1\} \times M_2) = \left(\{p_1\} \times (ker\varphi_{2*p_2})^{\perp}\right) \oplus (\{p_1\} \times (ker\varphi_{2*p_2})).$$

Clearly,

$$(ker\varphi_{*p})^{\perp} = \left((ker\varphi_{1*p_1})^{\perp} \times \{p_2\}\right) \oplus \left(\{p_1\} \times (ker\varphi_{2*p_2})^{\perp}\right)$$

and

$$(ker\varphi_{*p}) = ((ker\varphi_{1*p_1}) \times \{p_2\}) \oplus (\{p_1\} \times (ker\varphi_{2*p_2})).$$

Similarly, we have

$$\begin{aligned} T_{(\varphi_1(p_1),\varphi_2(p_2))}(N_1 \times N_2) =& T_{(\varphi_1(p_1),\varphi_2(p_2))}(N_1 \times \{\varphi_2(p_2)\}) \\ & \oplus T_{(\varphi_1(p_1),\varphi_2(p_2))}(\{\varphi_1(p_1)\} \times N_2) \\ =& (range\varphi_{*p})^{\perp} \oplus (range\varphi_{*p}), \end{aligned}$$

where

$$\begin{aligned} & T_{(\varphi_1(p_1),\varphi_2(p_2))}(N_1 \times \{\varphi_2(p_2)\}) \\ & \quad = \left((range\varphi_{1*p_1})^{\perp} \times \{\varphi_2(p_2)\}\right) \oplus ((range\varphi_{1*p_1}) \times \{\varphi_2(p_2)\}) \end{aligned}$$

and

$$\begin{aligned} & T_{(\varphi_1(p_1),\varphi_2(p_2))}(\{\varphi_1(p_1)\} \times N_2) \\ & \quad = \left(\{\varphi_1(p_1)\} \times (range\varphi_{2*p_2})^{\perp}\right) \oplus (\{\varphi_1(p_1)\} \times (range\varphi_{2*p_2})). \end{aligned}$$

Clearly,

$$(range\varphi_{*p})^{\perp} = \left((range\varphi_{1*p_1})^{\perp} \times \{\varphi_2(p_2)\}\right) \oplus \left(\{\varphi_1(p_1)\} \times (range\varphi_{2*p_2})^{\perp}\right)$$

and

$$(range\varphi_{*p}) = ((range\varphi_{1*p_1}) \times \{\varphi_2(p_2)\}) \oplus (\{\varphi_1(p_1)\} \times (range\varphi_{2*p_2})).$$

Now, lift of a horizontal vector $x^{\mathcal{H}} \in (ker\varphi_{i*p_i})^{\perp}$ to $(ker\varphi_{*p})^{\perp}$ is $\bar{x}^{\mathcal{H}}$ and lift of a vertical tangent vector $v^{\mathcal{V}} \in (ker\varphi_{i*p_i})$ to $(ker\varphi_{*p})$ is $\bar{v}^{\mathcal{V}}$. Then for $X = (X_1, X_2), Y = (Y_1, Y_2) \in (ker\varphi_{*p})^{\perp} = (ker\varphi_{1*p_1})^{\perp} \times (ker\varphi_{2*p_2})^{\perp}$, we have

$$\begin{aligned} g_N(\varphi_* X, \varphi_* Y) =& g_N(\varphi_*(X_1, X_2), \varphi_*(Y_1, Y_2)) \\ =& g_{N_1}(\varphi_{1*} X_1, \varphi_{1*} Y_1) + \rho^2(\varphi_1(p_1)) g_{N_2}(\varphi_{2*} X_2, \varphi_{2*} Y_2). \end{aligned}$$

Since φ_1 and φ_2 are Riemannian maps and we are denoting same notation for a vector field and its lift, using (1) in above equation, we get

$$\begin{aligned} g_N(\varphi_* X, \varphi_* Y) =& g_{M_1}(X_1, Y_1) + f^2(p_1) g_{M_2}(X_2, Y_2) \\ =& g_M(X, Y). \end{aligned}$$

This implies there is an isometry between horizontal space $(ker\varphi_*)^\perp$ and range space $range\varphi_*$. This completes the proof. □

Definition 1. Let $\varphi_i : M_i \to N_i$ be Riemannian maps between Riemannian manifolds for $i = 1, 2$. Then the map $\varphi = \varphi_1 \times \varphi_2 : (M = M_1 \times_f M_2, g_M) \to (N = N_1 \times_\rho N_2, g_N)$ between Riemannian warped product manifolds such that $(\varphi_1 \times \varphi_2)(p_1, p_2) = (\varphi_1(p_1), \varphi_2(p_2))$ is also a Riemannian map, called Riemannian warped product map.

Example 1. Let $\varphi = \varphi_1 \times \varphi_2 : (M = M_1 \times_f M_2, g_M) \to (N = N_1 \times_\rho N_2, g_N)$ be a smooth map between Riemannian warped product manifolds.

(i) If φ is a warped product isometric immersion then φ is a Riemannian warped product map with $ker\varphi_* = \{0\}$.

(ii) If φ is a Riemannian warped product submersion then φ is a Riemannian warped product map with $(range\varphi_*)^\perp = \{0\}$.

By [25] we know that distance functions are Riemannian submersions, hence Riemannian maps. Now, we give examples of special class of Riemannian warped product maps in the category of distance functions.

Example 2. Let (M_i, g_{M_i}) be Riemannian manifolds for $i = 1, 2, \ldots, k$. Let $p_i \in M_i$ be a fixed point, and d_i denote the distance functions of M_i from the fixed point p_i. Then $\varphi : (M = M_1 \times_{f_1} M_2 \times_{f_2} \cdots \times_{f_{k-1}} M_k,\ g_M) \to (N = \mathbb{R}^+ \times_{\rho_1} \mathbb{R}^+ \times_{\rho_2} \cdots \times_{\rho_{k-1}} \mathbb{R}^+,\ g_N)$ such that

$$\varphi(q_1, q_2, \ldots, q_k) = (d_1(p_1, q_1), d_2(p_2, q_2), \ldots, d_k(p_k, q_k))$$

is a Riemannian warped product map, where f_i is the lift of ρ_i for $i = 1, 2, \ldots, k-1$.

Example 3. Let (M, g_M) be a complete, non-compact Riemannian manifold without conjugate points. Then every geodesic of M is a line that is it is isometric to $\mathbb{R}$. If γ_v is a line in M, then the Busemann function $b_v : M \to \mathbb{R}$ for γ_v is defined as [25]

$$b_v(p) = \lim_{t\to\infty} (d(p, \gamma_v(t)) - t).$$

It is known that $b_v \in C^1(M)$ and $\|\nabla b_v\| = 1$, therefore Busemann function is a distance function. Hence clearly any Busemann function is a Riemannian map. Now let (M_i, g_{M_i}) be simply connected, complete manifolds without conjugate points for $i = 1, 2, \ldots, k$. Then for geodesic lines γ_i of M_i, the map $\varphi : M_1 \times_{f_1} M_2 \times_{f_2} \cdots \times_{f_{k-1}} M_k \to \mathbb{R} \times_{\rho_1} \mathbb{R} \times_{\rho_2} \cdots \times_{\rho_{k-1}} \mathbb{R}$ defined by

$$\varphi(p_1, p_2, \ldots, p_k) = (b_{\gamma_1}(p_1), b_{\gamma_2}(p_2), \ldots, b_{\gamma_1}(p_k))$$

is a Riemannian warped product map, where f_i is the lift of ρ_i for $i = 1, 2, \ldots, k-1$.

Example 4. Let $(M_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \neq 0, x_2 \neq 0, x_3 \neq 0\}, g_{M_1} = e^{2x_3} dx_1{}^2 + e^{2x_3} dx_2{}^2 + dx_3{}^2)$, $(N_1 = \{(y_1, y_2, y_3) \in \mathbb{R}^3\}, g_{N_1} = e^{2x_3} dy_1{}^2 + e^{2x_3} dy_2{}^2 + dy_3{}^2)$, $(M_2 = \{(r_1, r_2) \in \mathbb{R}^2 : r_1 \neq 0, r_2 \neq 0\}, g_{M_2} = dr_1{}^2 + dr_2{}^2)$ and $(N_2 = \{(s_1, s_2) \in \mathbb{R}^2\}, g_{N_2} = ds_1{}^2 + ds_2{}^2)$ be four Riemannian manifolds. Let $\varphi_1 : M_1 \to N_1$ such that

$$(x_1, x_2, x_3) \mapsto \left(\frac{x_1 + x_2}{\sqrt{2}}, \frac{x_1 - x_2}{\sqrt{2}}, 0\right)$$

and $\varphi_2 : M_2 \to N_2$ such that

$$(r_1, r_2) \mapsto \left(\frac{r_1 + r_2}{\sqrt{2}}, 0\right)$$

be Riemannian maps. Then the map $\varphi = \varphi_1 \times \varphi_2 : (M_1 \times_f M_2, g_M = g_{M_1} + f^2 g_{M_2}) \to (N_1 \times_\rho N_2, g_N = g_{N_1} + \rho^2 g_{N_2})$ such that

$$\varphi(x_1, x_2, x_3, r_1, r_2) = \left(\frac{x_1 + x_2}{\sqrt{2}}, \frac{x_1 - x_2}{\sqrt{2}}, 0, \frac{r_1 + r_2}{\sqrt{2}}, 0\right)$$

is a Riemannian warped product map, where $\rho : N_1 \to \mathbb{R}^+$ and $f : M_1 \to \mathbb{R}^+$ be smooth functions with $f = \rho \circ \varphi_1$.

Let $\varphi_i : M_i \to N_i$ be smooth maps between Riemannian manifolds for $i = 1, 2$ and let $\varphi : (M = M_1 \times M_2, g_M) \to (N = N_1 \times N_2, g_N)$ be a smooth map between Riemannian warped product manifolds. Define linear transformations $\mathcal{P}_{(p_1,p_2)} : T_{(p_1,p_2)}(M_1 \times M_2) \to T_{(p_1,p_2)}(M_1 \times M_2)$; $\mathcal{P}_{(p_1,p_2)} = {}^*\varphi_{*(p_1,p_2)} \circ \varphi_{*(p_1,p_2)}$ and $\mathcal{Q}_{(p_1,p_2)} : T_{(\varphi_1(p_1),\varphi_2(p_2))}(N_1 \times N_2) \to T_{(\varphi_1(p_1),\varphi_2(p_2))}(N_1 \times N_2)$; $\mathcal{Q}_{(p_1,p_2)} = {}^*\varphi_{*(p_1,p_2)} \circ \varphi_{*(p_1,p_2)}$ where $p_i \in M_i$ for $i = 1, 2$. In addition, ${}^*\varphi_*$ denotes the adjoint of φ_* (see [29]). Using these linear transformations, we obtain the following characterizations of Riemannian warped product maps:

Theorem 6. *Let $\varphi = \varphi_1 \times \varphi_2 : (M = M_1 \times_f M_2, g_M) \to (N = N_1 \times_\rho N_2, g_N)$ be a smooth map between Riemannian warped product manifolds. Then the following statements are equivalent:*

- *(i) φ is Riemannian warped product map at $(p_1, p_2) \in M$.*
- *(ii) $\mathcal{P}_{(p_1,p_2)}$ is a projection, i.e. $\mathcal{P}_{(p_1,p_2)} \circ \mathcal{P}_{(p_1,p_2)} = \mathcal{P}_{(p_1,p_2)}$.*
- *(iii) $\mathcal{Q}_{(p_1,p_2)}$ is a projection, i.e. $\mathcal{Q}_{(p_1,p_2)} \circ \mathcal{Q}_{(p_1,p_2)} = \mathcal{Q}_{(p_1,p_2)}$.*

Proof. Since the proof is similar to Theorem 80 of [29], we are omitting it. □

Now, we recall that a map $\varphi : (M, g_M) \to (N, g_N)$ between Riemannian manifolds is called subimmersion at $p \in M$ if there is an open neighborhood $\mathcal{U}$ of p, a manifold M', a submersion $\varphi_S : \mathcal{U} \to M'$, and an immersion $\varphi_I : M' \to N$ such that $\varphi|_{\mathcal{U}} = \varphi_{\mathcal{U}} = \varphi_I \circ \varphi_S$ [17,29]. We say φ is a subimmersion if it is subimmersion at each $p \in M$. It is known that $\varphi : (M, g_M) \to (N, g_N)$ is a subimmersion if and only if the rank function is locally constant, and hence constant on the connected components of M [1,27].

Definition 2. A map $\varphi = \varphi_1 \times \varphi_2 : (M = M_1 \times_f M_2, g_M) \to (N = N_1 \times_\rho N_2, g_N)$ between Riemannian warped product manifolds is called warped product subimmersion at $p = (p_1, p_2) \in M_1 \times M_2$ if there is an open neighborhood $\mathcal{U} = \mathcal{U}_1 \times_f \mathcal{U}_2$ of $p = (p_1, p_2)$, a warped product manifold $M_1' \times_{f'} M_2'$, a warped product submersion $\varphi_S : \mathcal{U}_1 \times_f \mathcal{U}_2 \to M_1' \times_{f'} M_2'$, and a warped product immersion $\varphi_I : M_1' \times_{f'} M_2' \to N_1 \times_\rho N_2$ such that $\varphi|_{\mathcal{U}} = \varphi_{\mathcal{U}} = \varphi_I \circ \varphi_S$, where M_1' and M_2' are Riemannian manifolds and f' is warping function on M_1'. We say φ is a warped product subimmersion if it is warped product subimmersion at each $p \in M_1 \times M_2$.

Theorem 7. *Let $\varphi = \varphi_1 \times \varphi_2 : (M = M_1 \times_f M_2, g_M) \to (N = N_1 \times_\rho N_2, g_N)$ be a Riemannian warped product map between Riemannian warped product manifolds. Then, locally, φ is the composition of a Riemannian warped product submersion followed by a warped product isometric immersion.*

Proof. Let $\mathcal{U} = \mathcal{U}_1 \times_f \mathcal{U}_2$ be an open neighborhood of $p = (p_1, p_2) \in M_1 \times M_2$, $\varphi_S : \mathcal{U}_1 \times_f \mathcal{U}_2 \to M_1' \times_{f'} M_2'$ be a warped product submersion and $\varphi_I : M_1' \times_{f'} M_2' \to N_1 \times_\rho N_2$ be a warped product immersion such that $\varphi|_{\mathcal{U}} = \varphi_{\mathcal{U}} = \varphi_I \circ \varphi_S$. From Lemma 4.3.1 of [18], it follows that $(M_1' \times_{f'} M_2', g_{M'})$ is also a Riemannian warped product manifold and $\varphi_I : (M_1' \times_{f'} M_2', g_M') \to (N_1 \times_\rho N_2, g_N)$ is a warped product isometric immersion. Therefore it is enough to show that horizontal restriction of φ_S is a warped product submersion. As in [27], for each $p = (p_1, p_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ define

$$\varphi_{S*p}|_{\mathcal{H}} : (ker\varphi_{1*p_1})^\perp \times (ker\varphi_{2*p_2})^\perp \to T_{\varphi_{S(p)}}(M_1' \times M_2')$$

as $\varphi_{S*p}|_{\mathcal{H}}X = \varphi_{S*p}X$ and

$$\varphi_{I*p}|_{\mathcal{H}} : T_{\varphi_{S(p)}}(M_1' \times M_2') \to (range\varphi_{1*p_1}) \times (range\varphi_{2*p_2})$$

as $\varphi_{I*p}|_{\mathcal{H}}Z = \varphi_{I*p}Z$. Thus we have $\varphi_{*p}|_{\mathcal{H}} = (\varphi_{I*p}|_{\mathcal{H}} \circ \varphi_{S*p}|_{\mathcal{H}}) = (\varphi_{I*p} \circ \varphi_{S*p})|_{\mathcal{H}} : ker\varphi_{*p} = (ker\varphi_{1*p_1})^\perp \times (ker\varphi_{2*p_2})^\perp \to range\varphi_{*p} = (range\varphi_{1*p_1}) \times (range\varphi_{2*p_2})$. Then for $X = (X_1, X_2)$ and $Y = (Y_1, Y_2) \in (ker\varphi_{1*p_1})^\perp \times (ker\varphi_{2*p_2})^\perp$, we get

$$g_N(\varphi_{*p}|_{\mathcal{H}}X, \varphi_{*p}|_{\mathcal{H}}Y) = g_N(\varphi_{I*p}|_{\mathcal{H}} \circ \varphi_{S*p}|_{\mathcal{H}}X, \varphi_{I*p}|_{\mathcal{H}} \circ \varphi_{S*p}|_{\mathcal{H}}Y).$$

Since $\varphi : (M = M_1 \times_f M_2, g_M) \to (N = N_1 \times_\rho N_2, g_N)$ is a Riemannian warped product map and $\varphi_I : (M_1' \times_{f'} M_2', g_M') \to (N_1 \times_\rho N_2, g_N)$ is a warped product isometric immersion, we get $g_{M'} = \varphi_I^* g_N$. Then

$$\begin{aligned} &g_N(\varphi_{*p}|_{\mathcal{H}}(X_1, X_2), \varphi_{*p}|_{\mathcal{H}}(Y_1, Y_2)) \\ &= g_N(\varphi_{I*p}|_{\mathcal{H}} \circ \varphi_{S*p}|_{\mathcal{H}}(X_1, X_2), \varphi_{I*p}|_{\mathcal{H}} \circ \varphi_{S*p}|_{\mathcal{H}}(Y_1, Y_2)), \end{aligned}$$

which implies

$$g_M|_{\mathcal{U}}((X_1, X_2), (Y_1, Y_2)) = g_{M'}(\varphi_{S*p}|_{\mathcal{H}}(X_1, X_2), \varphi_{S*p}|_{\mathcal{H}}(Y_1, Y_2)).$$

Thus $\varphi_S : (\mathcal{U}_1 \times_f \mathcal{U}_2, g_M|_{\mathcal{U}}) \to (M_1' \times_{f'} M_2', g_{M'})$ is a Riemannian warped product submersion. Hence we finish the required proof. □

Remark 1. Note that a Riemannian map between Riemannian manifolds is a composition of a Riemannian submersion followed by an isometric immersion [17] locally. It is also true that a Riemannian warped product map is a composition of a Riemannian warped product submersion followed by warped product isometric immersion locally.

Now, we give the following examples of Riemannian warped product maps as an application of the above Theorem 7:

Example 5. Consider the immersions $\varphi_1 : \mathbb{R}^+ \to \mathbb{E}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ such that

$$\varphi_1(t) = (t, t)$$

and $\varphi_2 : \mathbb{S}^1 \to \mathbb{S}^1$ such that

$$\varphi_2(s) = s.$$

Then by [22], $\varphi = \varphi_1 \times \varphi_2 : \mathbb{R}^+ \times_f \mathbb{S}^1 \to \mathbb{E}^2 \times_{f'} \mathbb{S}^1$ defined by

$$\varphi(t, s) = (t, t, s)$$

is a warped product isometric immersion with $(f' \circ \varphi_1)(t) = t = f(t)$. Also, consider the Riemannian submersions $\psi_1 : \mathbb{F}^2 = \{(t_1, t_2) \in \mathbb{R}^2 : t_1 > 0\} \to \mathbb{R}^+$ such that

$$\psi_1(t_1, t_2) = t_1$$

and $\psi_2 : \mathbb{S}^1 \to \mathbb{S}^1$ such that

$$\psi_2(s) = s.$$

Then by [14], $\psi = \psi_1 \times \psi_2 : \mathbb{F}^2 \times_{f'} \mathbb{S}^1 \to \mathbb{R}^+ \times_\rho \mathbb{S}^1$ defined by

$$\psi(t_1, t_2, s) = (t_1, s)$$

is a Riemannian warped product submersion with $(\rho \circ \psi_1)(t_1, t_2) = t_1 = f'(t_1, t_2)$. Thus $\varphi \circ \psi : \mathbb{R}^2 \times_{f'} \mathbb{R}^2 \to \mathbb{R}^2 \times_{f'} \mathbb{R}^2$ is a Riemannian warped product map.

Example 6. Consider the immersions $\varphi_i : \mathbb{R} \to \mathbb{R}^2$ for $i = 1, 2$ such that

$$\varphi_i(t) = (\cos t, \sin t).$$

Then by [22], $\varphi = \varphi_1 \times \varphi_2 : \mathbb{R} \times_f \mathbb{R} \to \mathbb{R}^2 \times_{f'} \mathbb{R}^2$ defined by

$$\varphi(t_1, t_2) = (\cos t_1, \sin t_1, \cos t_2, \sin t_2)$$

is a warped product isometric immersion with $f' \circ \varphi_1 = f$. Also, consider the Riemannian submersions $\psi_i : \mathbb{R}^2 \to \mathbb{R}$ for $i = 1, 2$ such that

$$\psi_i(x_1, x_2) = x_1.$$

Then by [14], $\psi = \psi_1 \times \psi_2 : \mathbb{R}^2 \times_{f'} \mathbb{R}^2 \to \mathbb{R} \times_\rho \mathbb{R}$ defined by

$$\psi(x_1, x_2, y_1, y_2) = (x_1, y_1)$$

is a Riemannian warped product submersion with $\rho \circ \psi_1 = f'$. Thus $\varphi \circ \psi : \mathbb{R}^2 \times_{f'} \mathbb{R}^2 \to \mathbb{R}^2 \times_{f'} \mathbb{R}^2$ is a Riemannian warped product map.

Theorem 8. *Let $\varphi = \varphi_1 \times \varphi_2 : (M = M_1 \times_f M_2, g_M) \to (N = N_1 \times_\rho N_2, g_N)$ be a Riemannian warped product map between Riemannian warped product manifolds. Then*

$$\|\varphi_*\|^2 = rank\varphi_1 + \rho^2 rank\varphi_2.$$

Proof. Define a linear transformation $G : ((ker\varphi_{*p})^\perp = (ker\varphi_{1*p_1})^\perp \times (ker\varphi_{2*p_2})^\perp, g_{Mp}) \to ((ker\varphi_{*p})^\perp = (ker\varphi_{1*p_1})^\perp \times (ker\varphi_{2*p_2})^\perp, g_{Mp})$ such that $G = {}^*\varphi_{*p} \circ \varphi_{*p}$, where ${}^*\varphi_{*p}$ is the adjoint of φ_{*p}. Then for $X = (X_1, X_2), Y = (Y_1, Y_2) \in (ker\varphi_{*p})^\perp$, we have

$$\begin{aligned} g_M(GX, Y) =& g_M({}^*\varphi_{*p} \circ \varphi_{*p} X, Y) = g_N(\varphi_{*p} X, \varphi_{*p} Y) \\ =& g_{N_1}(\varphi_{1*} X_1, \varphi_{1*} X_2) + \rho^2(\varphi_1(p_1)) g_{N_2}(\varphi_{2*} Y_1, \varphi_{2*} Y_2). \end{aligned}$$

Since φ_1 and φ_2 are Riemannian maps, using (1) in above equation, we get

$$g_M(GX, Y) = g_{M_1}(X_1, X_2) + f^2(p_1) g_{M_2}(Y_1, Y_2).$$

Now let $\{\tilde{x}_i\}_{i=1}^{m_1+m_2-r_1-r_2}, \{x_j\}_{j=r_1+1}^{m_1}$ and $\{x_k^*\}_{k=m_1-r_1+1}^{m_1-r_1+m_2-r_2}$ denote orthonormal bases of $(ker\varphi_{*p})^\perp$, $(ker\varphi_{1*p_1})^\perp$, and $(ker\varphi_{2*p_2})^\perp$, respectively, and $\tilde{x}_i = (x_j, x_k^*) \in (ker\varphi_{*p})^\perp$. Then we have

$$\begin{aligned} \|\varphi_*\|^2(p) =& \sum_{i=1}^{m_1+m_2-r_1-r_2} g_N(\varphi_{*p}\tilde{x}_i, \varphi_{*p}\tilde{x}_i) \\ =& \sum_{j=1+r_1}^{m_1} g_{N_1}(\varphi_{1*}x_j, \varphi_{1*}x_j) \\ &+ \rho^2(\varphi_1(p_1)) \sum_{k=m_1-r_1+1}^{m_2-r_2+m_1-r_1} g_{N_2}(\varphi_{2*}x_k^*, \varphi_{2*}x_k^*) \\ =& (m_1 - r_1) + \rho^2(\varphi_1(p_1))(m_2 - r_2), \end{aligned}$$

where $m_1 + m_2 - r_1 - r_2 = \dim(ker\varphi_*)^\perp, r_1 = \dim(ker\varphi_{1*}), r_2 = \dim(ker\varphi_{2*})$, $m_1 = \dim(M_1)$ and $m_2 = \dim(M_2)$. This completes the proof. □

Remark 2. We can observe that a Riemannian map φ between Riemannian manifolds satisfies $\|\varphi_*\|^2 = rank\varphi$ [17], while a Riemannian warped product map $\varphi = \varphi_1 \times \varphi_2 : M_1 \times_f M_2 \to N_1 \times_\rho M_2$ between Riemannian warped product manifolds satisfies $\|\varphi_*\|^2 = rank\varphi_1 + \rho^2 rank\varphi_2$. Hence Riemannian warped product map satisfies the generalized eikonal equation.

Lemma 9. *Let $\varphi = \varphi_1 \times \varphi_2 : (M = M_1 \times_f M_2, g_M) \to (N = N_1 \times_\rho N_2, g_N)$ be a smooth map between Riemannian warped product manifolds. Then*

(i) $\nabla^N_{\varphi_{1*}X_1} \varphi_{1*} Y_1$ *is the lift of* $\nabla^{N_1}_{\varphi_{1*}X_1} \varphi_{1*} Y_1$,

(ii) $\nabla^N_{\varphi_{1*}X_1} \varphi_{2*} X_2 = \nabla^N_{\varphi_{2*}X_2} \varphi_{1*} X_1 = (\varphi_{1*} X_1(\rho)/\rho) \varphi_{2*} X_2$,

(*iii*) *tan* $\nabla^N_{\varphi_{2*}X_2}\varphi_{2*}Y_2$ *is the lift of* $\nabla^{N_2}_{\varphi_{2*}X_2}\varphi_{2*}Y_2$,
(*iv*) *nor* $\nabla^N_{\varphi_{2*}X_2}\varphi_{2*}Y_2 = -g_N(\varphi_{2*}X_2, \varphi_{2*}Y_2)(\nabla^N \ln\rho)$,

where $\varphi_{i*}X_i, \varphi_{i*}Y_i \in \mathcal{L}(N_i)$. *In addition,* ∇^N *and* ∇^{N_i} *are the Levi-Civita connections on* N *and* N_i *respectively for* $i = 1, 2$.

Proof. Since $N = N_1 \times_\rho N_2$ is a Riemannian warped product manifold, by using Lemma 2 we can easily see that (i) and (ii) are hold. Now for $\varphi_{1*}X_1, \varphi_{1*}Y_1 \in \mathcal{L}(N_1)$ and $\varphi_{1*}X_2, \varphi_{1*}Y_2 \in \mathcal{L}(N_2)$, we write

$$\nabla^N_{\varphi_{2*}X_2}\varphi_{2*}Y_2 = nor\nabla^N_{\varphi_{2*}X_2}\varphi_{2*}Y_2 + tan\nabla^N_{\varphi_{2*}X_2}\varphi_{2*}Y_2.$$

Using Lemma 2, we get

$$\nabla^N_{\varphi_{2*}X_2}\varphi_{2*}Y_2 = -g_N(\varphi_{2*}X_2, \varphi_{2*}Y_2)(\nabla^N \ln\rho) + \nabla^{N_2}_{\varphi_{2*}X_2}\varphi_{2*}Y_2.$$

Thus (iii) and (iv) also hold. □

Proposition 10. *Let* $\varphi = \varphi_1 \times \varphi_2 : (M = M_1 \times_f M_2, g_M) \to (N = N_1 \times_\rho N_2, g_N)$ *be a Riemannian warped product map between Riemannian warped product manifolds. Then*

$$\varphi_{1*}X_1(\ln\rho) = X_1(\ln f) = X_1(f)/(f),$$

where $X_1 \in \Gamma(ker\varphi_{1*})^\perp$.

Proof. For $X_1 \in \Gamma(ker\varphi_{1*})^\perp$, we have

$$\varphi_{1*}X_1(\ln\rho) = g_{N_1}(\varphi_{1*}X_1, \nabla^{N_1}(\ln\rho)).$$

Since f is lift of ρ, using Lemma 1 in above equation, we get

$$\varphi_{1*}X_1(\ln\rho) = g_{N_1}(\varphi_{1*}X_1, \varphi_{1*}(\nabla^{M_1}(\ln f))).$$

Since φ_1 is a Riemannian map, we can write

$$\varphi_{1*}X_1(\ln\rho) = g_{M_1}(X_1, \nabla^{M_1}(\ln f)).$$

This completes the proof. □

Theorem 11. *Let* $\varphi = \varphi_1 \times \varphi_2 : (M = M_1 \times_f M_2, g_M) \to (N = N_1 \times_\rho N_2, g_N)$ *be a Riemannian warped product map between Riemannian warped product manifolds. Then the following statements are hold:*

(*i*) *The distribution* $ker\varphi_*$ *is totally geodesic if and only if the distributions* $ker\varphi_{1*}$ *and* $ker\varphi_{2*}$ *are totally geodesic, and* $\mathcal{H}(\nabla^M \ln f) = 0$,
(*ii*) *The distribution* $ker\varphi_*$ *is minimal if and only if either the distribution* $ker\varphi_{1*}$ *is minimal and* $H_2 = \mathcal{H}(\nabla^M \ln f)$ *or the distribution* $ker\varphi_{2*}$ *is minimal and* $H_1 = \left(\frac{r_2}{r_1}\right)\mathcal{H}(\nabla^M \ln f)$,
(*iii*) *The distribution* $(ker\varphi_*)^\perp$ *is totally geodesic if and only if the distributions* $(ker\varphi_{1*})^\perp$ *and* $(ker\varphi_{2*})^\perp$ *are totally geodesic, and* $\mathcal{V}(\nabla^M \ln f) = 0$,
(*iv*) *The distribution* $range\varphi_*$ *is minimal if and only if either the distribution* $range\varphi_{1*}$ *is minimal and* $H_4 = \nabla^N \ln\rho$ *or the distribution* $range\varphi_{2*}$ *is minimal and* $H_3 = \left(\frac{m_2 - r_2}{m_1 - r_1}\right)\nabla^N \ln\rho$,

(*v*) *The distribution* $(ker\varphi_*)^\perp$ *is minimal if and only if either the distribution* $(ker\varphi_{1*})^\perp$ *is minimal and* $H_2^\perp = \mathcal{V}(\nabla^M \ln f)$ *or the distribution* $(ker\varphi_{2*})^\perp$ *is minimal and* $H_1^\perp = \left(\frac{m_2-r_2}{m_1-r_1}\right)\mathcal{V}(\nabla^M \ln f)$,

(*vi*) *The distribution* $(range\varphi_*)^\perp$ *is minimal if and only if either the distribution* $(range\varphi_{1*})^\perp$ *is minimal and* $H_4^\perp = \nabla^N \ln\rho$ *or the distribution* $(range\varphi_{2*})^\perp$ *is minimal and* $H_3^\perp = \left(\frac{n_4}{n_3}\right)\nabla^N \ln\rho$,

where ∇^M *and* ∇^N *are Levi-Civita connections on* M *and* N *respectively. In addition* $H_1, H_2, H_3, H_4, H_1^\perp, H_2^\perp, H_3^\perp$ *and* $H_4^\perp$ *are the mean curvature vector fields of* $ker\varphi_{1*}, ker\varphi_{2*}$, $range\varphi_{1*}$, $range\varphi_{2*}$, $(ker\varphi_{1*})^\perp, (ker\varphi_{2*})^\perp$, $(range\varphi_{1*})^\perp$ *and* $(range\varphi_{2*})^\perp$ *respectively. Also here* $r_1 = \dim(ker\varphi_{1*}), r_2 = \dim(ker\varphi_{2*}), m_1 - r_1 = \dim(range\varphi_{1*})$, $m_2 - r_2 = \dim(range\varphi_{2*})$, $n_3 = \dim((range\varphi_{1*})^\perp)$ *and* $n_4 = \dim((range\varphi_{2*})^\perp)$.

Proof. Let $\{u_i\}_{i=1}^{r_1}$ and $\{u_a^*\}_{a=r_1+1}^{r_1+r_2}$ be orthonormal bases of $ker\varphi_{1*}$ and $ker\varphi_{2*}$, respectively. Then

$$\begin{aligned}\|T\|^2 &= \sum_{i,j=1}^{r_1} g_M(T(u_i,u_j),T(u_i,u_j)) + \sum_{a,b=r_1+1}^{r_1+r_2} g_M(T(u_a^*,u_b^*),T(u_a^*,u_b^*))\\ &= \sum_{i,j=1}^{r_1} g_M(T_1(u_i,u_j),T_1(u_i,u_j))\\ &\quad + \sum_{a,b=r_1+1}^{r_1+r_2} g_M(T_2(u_a^*,u_b^*) - g_M(u_a^*,u_b^*)(\mathcal{H}\nabla^M \ln f), T_2(u_a^*,u_b^*)\\ &\quad - g_M(u_a^*,u_b^*)(\mathcal{H}\nabla^M \ln f)),\end{aligned}$$

where T, T_1 and T_2 are O'Neill tensors on $ker\varphi_*, ker\varphi_{1*}$ and $ker\varphi_{2*}$ respectively [14]. Thus

$$\|T\|^2 = \|T_1\|^2 + \|T_2\|^2 + r_2\ \|\mathcal{H}(\nabla^M \ln f)\|. \tag{5}$$

In addition the mean curvature vector field of $ker\varphi_*$ is given by

$$\begin{aligned}H &= \frac{1}{r_1+r_2}\left(\sum_{i=1}^{r_1} T(u_i,u_i) + \sum_{a=r_1+1}^{r_1+r_2} T(u_a^*,u_a^*)\right)\\ &= \frac{1}{r_1+r_2}\left(\sum_{i=1}^{r_1} T_1(u_i,u_i) + \sum_{a=r_1+1}^{r_1+r_2} T_2(u_a^*,u_a^*) - g_M(u_a^*,u_a^*)(\mathcal{H}\nabla^M \ln f)\right).\end{aligned} \tag{6}$$

We know that [29]

$$H_1 = \frac{1}{r_1}\sum_{i=1}^{r_1} T_1(u_i,u_i) \tag{7}$$

and

$$H_2 = \frac{1}{r_2} \sum_{a=r_1+1}^{r_1+r_2} T_2(u_a^*, u_a^*), \tag{8}$$

where H_1 and H_2 denote the mean curvature vector fields of the distributions $ker\varphi_{1*}$ and $ker\varphi_{2*}$, respectively. Using (7) and (8) in (6), we get

$$H = \frac{1}{r_1 + r_2} \left(r_1 H_1 + r_2 (H_2 - \mathcal{H} \nabla^M \ln f) \right). \tag{9}$$

Further, let $\{x_i\}_{i=r_1+1}^{m_1}$ and $\{x_a^*\}_{a=m_1-r_1+1}^{m_1-r_1+m_2-r_2}$ be orthonormal bases of $(ker\varphi_{1*})^\perp$ and $(ker\varphi_{2*})^\perp$, respectively. Then

$$\begin{aligned} \|A\|^2 &= \sum_{i,j=r_1+1}^{m_1} g_M(A(x_i, x_j), A(x_i, x_j)) + \sum_{a,b=m_1-r_1+1}^{m_1-r_1+m_2-r_2} g_M(A(x_a^*, x_b^*), A(x_a^*, x_b^*)) \\ &= \sum_{i,j=r_1+1}^{m_1} g_M(A_1(x_i, x_j), A_1(x_i, x_j)) \\ &\quad + \sum_{a,b=m_1-r_1+1}^{m_1-r_1+m_2-r_2} g_M(A_2(x_a^*, x_b^*) - g_M(x_a^*, x_b^*)(\mathcal{V}\nabla^M \ln f), A_2(x_a^*, x_b^*) \\ &\quad - g_M(x_a^*, x_b^*)(\mathcal{V}\nabla^M \ln f)), \end{aligned}$$

where A, A_1 and A_2 are O'Neill tensors on $(ker\varphi_*)^\perp, (ker\varphi_{1*})^\perp$ and $(ker\varphi_{2*})^\perp$ respectively [14]. Thus

$$\|A\|^2 = \|A_1\|^2 + \|A_2\|^2 + (m_2 - r_2)\ \|\mathcal{V}(\nabla^M \ln f)\|. \tag{10}$$

In addition the mean curvature vector field of $range\varphi_*$ is given by

$$H' = \frac{1}{m_1 - r_1 + m_2 - r_2} \left(\sum_{i=r_1+1}^{m_1} \nabla_{x_i}^{\varphi} \varphi_{1*} x_i + \sum_{a=m_1-r_1+1}^{m_1-r_1+m_2-r_2} \nabla_{x_a^*}^{\varphi} \varphi_{2*} x_a^* \right). \tag{11}$$

Using Lemma 9 in (11), we get

$$\begin{aligned} H' =& \frac{1}{m_1 - r_1 + m_2 - r_2} \sum_{i=r_1+1}^{m_1} \nabla_{x_i}^{\varphi_1} \varphi_{1*} x_i \\ &+ \frac{1}{m_1 - r_1 + m_2 - r_2} \sum_{a=m_1-r_1+1}^{m_1-r_1+m_2-r_2} \nabla_{x_a^*}^{\varphi_2} \varphi_{2*} x_a^* - g_M(x_a^*, x_a^*) \nabla^N \ln \rho. \end{aligned} \tag{12}$$

We know that [29]

$$H_3 = \frac{1}{m_1 - r_1} \sum_{i=r_1+1}^{m_1} \nabla_{x_i}^{\varphi_1} \varphi_{1*} x_i \tag{13}$$

and

$$H_4 = \frac{1}{m_2 - r_2} \sum_{a=m_1-r_1+1}^{m_1-r_1+m_2-r_2} \nabla^{\varphi_2}_{x_a^*} \varphi_{2*} x_a^*, \tag{14}$$

where H_3 and H_4 denote the mean curvature vector fields of the distributions $range\varphi_{1*}$ and $range\varphi_{2*}$, respectively. Using (13) and (14) in (12), we get

$$H' = \frac{1}{m_1 - r_1 + m_2 - r_2} \left((m_1 - r_1) H_3 + (m_2 - r_2)(H_4 - \nabla^N \ln \rho) \right). \tag{15}$$

Also, the mean curvature vector field of $(ker\varphi_*)^\perp$ is given by

$$\begin{aligned} H^\perp &= \frac{1}{m_1 - r_1 + m_2 - r_2} \left(\sum_{i=r_1+1}^{m_1} A(x_i, x_i) + \sum_{a=m_1-r_1+1}^{m_1-r_1+m_2-r_2} A(x_a^*, x_a^*) \right) \\ &= \frac{1}{m_1 - r_1 + m_2 - r_2} \left(\sum_{i=r_1+1}^{m_1} A_1(x_i, x_i) + \sum_{a=m_1-r_1+1}^{m_1-r_1+m_2-r_2} A_2(x_a^*, x_a^*) \right. \\ &\quad \left. - g_M(x_a^*, x_a^*)(\mathcal{V}\nabla^M \ln f) \right). \end{aligned} \tag{16}$$

We know that [29]

$$H_1^\perp = \frac{1}{m_1 - r_1} \sum_{i=r_1+1}^{m_1} A_1(x_i, x_i) \tag{17}$$

and

$$H_2^\perp = \frac{1}{m_2 - r_2} \sum_{a=m_1-r_1+1}^{m_1-r_1+m_2-r_2} A_2(x_a^*, x_a^*), \tag{18}$$

where $H_1^\perp$ and $H_2^\perp$ denote the mean curvature vector fields of the distributions $(ker\varphi_{1*})^\perp$ and $(ker\varphi_{2*})^\perp$, respectively. Using (17) and (18) in (16), we get

$$H^\perp = \frac{1}{m_1 - r_1 + m_2 - r_2} \left((m_1 - r_1) H_1^\perp + (m_2 - r_2)(H_2^\perp - \mathcal{V}\nabla^M \ln f) \right). \tag{19}$$

Finally, let $\{\bar{e}_l\}_{l=1}^{n_3}$ and $\{\check{e}_s\}_{s=n_3+1}^{n_3+n_4}$ be orthonormal bases of $(range\varphi_{1*})^\perp$ and $(range\varphi_{2*})^\perp$, respectively. Then the mean curvature vector field of $(range\varphi_*)^\perp$ is

$$H'^\perp = \frac{1}{n_3 + n_4} \left(\sum_{l=1}^{n_3} \nabla^{\varphi^\perp}_{\bar{e}_l} \bar{e}_l + \sum_{s=1+n_3}^{n_3+n_4} \nabla^{\varphi^\perp}_{\check{e}_s} \check{e}_s \right). \tag{20}$$

Using Lemma 9 in (20), we get

$$H'^\perp = \frac{1}{n_3 + n_4} \left(\sum_{l=1}^{n_3} \nabla^{\varphi_1^\perp}_{\bar{e}_l} \bar{e}_l + \sum_{s=1+n_3}^{n_3+n_4} \nabla^{\varphi_2^\perp}_{\check{e}_s} \check{e}_s - g_N(\check{e}_s, \check{e}_s) \nabla^N \ln \rho \right). \tag{21}$$

We know that [29]

$$H_3^{\perp} = \frac{1}{n_3} \sum_{l=1}^{n_3} \nabla_{\bar{e}_l}^{\varphi_1^{\perp}} \bar{e}_l \tag{22}$$

and

$$H_4^{\perp} = \frac{1}{n_4} \sum_{s=1+n_3}^{n_3+n_4} \nabla_{\check{e}_s}^{\varphi_2^{\perp}} \check{e}_s, \tag{23}$$

where $H_3^{\perp}$ and $H_4^{\perp}$ denote the mean curvature vector fields of the distributions $(range\varphi_{1*})^{\perp}$ and $(range\varphi_{2*})^{\perp}$, respectively. Using (22) and (23) in (21), we get

$$H'^{\perp} = \frac{1}{n_3 + n_4} \left(n_3 \; H_3^{\perp} + n_4 (H_4^{\perp} - \nabla^N \ln \rho) \right). \tag{24}$$

Then proof follows by (5), (9), (10), (15), (19) and (24). □

4. Totally Geodesic Riemannian Warped Product Maps

In this section, we construct the Gauss formula (second fundamental form) for a Riemannian warped product map between Riemannian warped product manifolds and discuss totally geodesicity [32].

Theorem 12. *Let $\varphi = \varphi_1 \times \varphi_2 : (M = M_1 \times_f M_2, g_M) \to (N = N_1 \times_\rho N_2, g_N)$ be a Riemannian warped product map between Riemannian warped product manifolds. Then the second fundamental form of φ is*

$$\begin{aligned}(\nabla \varphi_*)(X, Y) =& (\nabla^1 \varphi_{1*})(X_1, Y_1) + (\nabla^2 \varphi_{2*})(X_2, Y_2) \\ &+ (\varphi_{1*} Y_1(\ln \rho))\varphi_{2*} X_2 + (\varphi_{1*} X_1(\ln \rho))\varphi_{2*} Y_2 \\ &- (X_1(\ln f))\varphi_{2*} Y_2 - (Y_1(\ln f))\varphi_{2*} X_2,\end{aligned} \tag{25}$$

where $X = (X_1, X_2),\ Y = (Y_1, Y_2) \in \Gamma(T(M_1 \times M_2))$ and $\varphi_i : M_i \to N_i$ are Riemannian maps between Riemannian manifolds for $i = 1, 2$. In addition, the bundle $TM_i^* \otimes \varphi_i^{-1}(TN_i)$ has connection ∇^i induced from the Levi-Civita connection ∇^{M_i} of M_i for $i = 1, 2$. This is known as Gauss formula also.

Proof. Let $\varphi_i : M_i \to N_i$ be Riemannian maps between Riemannian manifolds and TN_i be bundle over N_i for $i = 1, 2$. Now, the pullback bundle $\varphi_i^{-1}(TN_i) \to M_i$ has the fibers $(\varphi_i^{-1}(TN_i))_{p_i} = T_{\varphi_i(p_i)} N_i$ for $p_i \in M_i$. We shall use this identification without comment. The Levi-Civita connection ∇^{N_i} on N_i and the pullback connection ∇^{φ_i} are the unique linear connections on the pullback bundle $\varphi_i^{-1}(TN_i)$ such that for each $Z_i \in \Gamma(TN_i)$

$$\nabla_{X_i}^{\varphi_i} (\varphi_i^* Z_i) = \nabla_{\varphi_{i*} X_i}^{N_i} Z_i,$$

where $\varphi_i^* Z_i = Z_i \circ \varphi_i \in \Gamma(\varphi_i^{-1} N_i)$ and φ_{i*} is section of $Hom(TM_i, \varphi_i^{-1}(TN_i)) = TM_i^* \otimes \varphi_i^{-1}(TN_i) \to M_i = TM_i^* \otimes \varphi_i^{-1}(TN_i)$. In addition, the bundle $TM_i^* \otimes$

$\varphi_i^{-1}(TN_i)$ has connection ∇^i induced from the Levi-Civita connection ∇^{M_i} of M_i and the pull back connection ∇^{φ_i}. The covariant derivative of φ_{i*} called the second fundamental form of φ_i, i.e. $\nabla^i\varphi_{i*} \in \Gamma(T^*M_i \otimes T^*M_i \otimes \varphi^{-1}(TN_i))$ such that

$$(\nabla^i\varphi_{i*})(X_i, Y_i) = \nabla^{\varphi_i}_{X_i}(\varphi_{i*}(Y_i)) - \varphi_{i*}(\nabla^{M_i}_{X_i}Y_i), \tag{26}$$

where $X_i, Y_i \in \Gamma(TM_i)$. Here the map $\varphi_* = (\varphi_1 \times \varphi_2)_*$ is section of $Hom(T(M_1 \times M_2), \varphi^{-1}(T(N_1 \times N_2))) \to M_1 \times M_2$. Let the bundle $T^*(M_1 \times M_2) \otimes \varphi^{-1}(T(N_1 \times N_2))$ has connection ∇ induced from the Levi-Civita connection ∇^M of $M = M_1 \times_f M_2$ and the pull back connection ∇^φ. The covariant derivative $\nabla\varphi_*$ is the second fundamental form of φ, i.e.

$$\nabla\varphi_* \in \Gamma(T^*(M_1 \times M_2) \otimes T^*(M_1 \times M_2) \otimes \varphi^{-1}(T(N_1 \times N_2))).$$

Now for $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$, we have

$$\begin{aligned}(\nabla\varphi_*)(X, Y) &= \nabla^\varphi_X \varphi_* Y - \varphi_*(\nabla^M_X Y)\\ &= \nabla^\varphi_{(X_1,X_2)}\varphi_*(Y_1, Y_2) - \varphi_*(\nabla^M_{(X_1,X_2)}(Y_1, Y_2)).\end{aligned}$$

Now by using Proposition 4, we get

$$\begin{aligned}(\nabla\varphi_*)(X, Y) &= \nabla^\varphi_{(X_1+X_2)}(\varphi_{1*}Y_1 + \varphi_{2*}Y_2) - \varphi_*(\nabla^M_{(X_1+X_2)}(Y_1 + Y_2))\\ &= \nabla^\varphi_{X_1}\varphi_{1*}(Y_1) + \nabla^\varphi_{X_2}\varphi_{1*}(Y_1) + \nabla^\varphi_{X_1}\varphi_{2*}(Y_2)\\ &\quad + \nabla^\varphi_{X_2}\varphi_{2*}(Y_2) - \varphi_*(\nabla^M_{X_1}Y_1 + \nabla^M_{X_1}Y_2 + \nabla^M_{X_2}Y_1 + \nabla^M_{X_2}Y_2).\end{aligned} \tag{27}$$

Using Lemma 2 in (27), we get

$$\begin{aligned}(\nabla\varphi_*)(X, Y) &= \nabla^\varphi_{X_1}\varphi_{1*}(Y_1) + \nabla^\varphi_{X_2}\varphi_{1*}(Y_1) + \nabla^\varphi_{X_1}\varphi_{2*}(Y_2) + \nabla^\varphi_{X_2}\varphi_{2*}(Y_2)\\ &\quad - \varphi_*(\nabla^M_{X_1}Y_1 + (X_1(f)/f)Y_2\\ &\quad + (Y_1(f)/f)X_2 + nor(\nabla^M_{X_2}Y_2) + tan(\nabla^M_{X_2}Y_2))\\ &= \nabla^N_{\varphi_{1*}X_1}\varphi_{1*}Y_1 + \nabla^N_{\varphi_{2*}X_2}\varphi_{1*}Y_1 + \nabla^N_{\varphi_{1*}X_1}\varphi_{2*}Y_2 + \nabla^N_{\varphi_{2*}X_2}\varphi_{2*}Y_2\\ &\quad - \varphi_{1*}(\nabla^{M_1}_{X_1}Y_1) - (X_1(f)/f)\varphi_{2*}Y_2 - (Y_1(f)/f)\varphi_{2*}X_2\\ &\quad + g_M(X_2, Y_2)\varphi_*(\nabla \ln f) - \varphi_{2*}(\nabla^{M_2}_{X_2}Y_2).\end{aligned}$$

Using Lemma 9 in above equation, we get

$$\begin{aligned}(\nabla\varphi_*)(X, Y) &= \nabla^{N_1}_{\varphi_{1*}X_1}\varphi_{1*}Y_1 + (\varphi_{1*}Y_1(\rho)/\rho)\varphi_{2*}X_2 + (\varphi_{1*}X_1(\rho)/\rho)\varphi_{2*}Y_2\\ &\quad - g_N(\varphi_{2*}X_2, \varphi_{2*}Y_2)(\nabla^N \ln\rho) + \nabla^{N_2}_{\varphi_{2*}X_2}\varphi_{2*}Y_2 - \varphi_{1*}(\nabla^{M_1}_{X_1}Y_1)\\ &\quad - (X_1(f)/f)\varphi_{2*}Y_2 - (Y_1(f)/f)\varphi_{2*}X_2\\ &\quad + g_M(X_2, Y_2)\varphi_*(\nabla^M \ln f) - \varphi_{2*}(\nabla^{M_2}_{X_2}Y_2).\end{aligned} \tag{28}$$

This is the second fundamental form (Gauss formula) for a smooth map between Riemannian warped product manifolds. Since φ is Riemannian warped product map, (28) implies

$$\begin{aligned}(\nabla\varphi_*)(X,Y) =&\nabla^{N_1}_{\varphi_{1*}X_1}\varphi_{1*}Y_1 + (\varphi_{1*}Y_1(\rho)/\rho)\varphi_{2*}X_2 + (\varphi_{1*}X_1(\rho)/\rho)\varphi_{2*}Y_2\\ &- g_M(X_2,Y_2)(\nabla^N \ln\rho) + \nabla^{N_2}_{\varphi_{2*}X_2}\varphi_{2*}Y_2 - \varphi_{1*}(\nabla^{M_1}_{X_1}Y_1)\\ &- (X_1(f)/f)\varphi_{2*}Y_2 - (Y_1(f)/f)\varphi_{2*}X_2\\ &+ g_M(X_2,Y_2)\varphi_*(\nabla^M \ln f) - \varphi_{2*}(\nabla^{M_2}_{X_2}Y_2).\end{aligned} \tag{29}$$

We know that f is lift of ρ then by Lemma 1, we have

$$\varphi_*(\nabla^M \ln f) = \nabla^N \ln\rho. \tag{30}$$

Using (26) and (30) in (29), we get the required proof. □

Now, from Proposition 10, Theorem 12 and Lemma 4 of [14] we have following consequences:

Corollary 13. *Let $\varphi = \varphi_1\times\varphi_2 : (M = M_1 \times_f M_2, g_M) \to (N = N_1 \times_\rho N_2, g_N)$ be a Riemannian warped product map between Riemannian warped product manifolds. Then*

(i) $(\nabla\varphi_*)(X,Y) = -\varphi_{1*}(\mathcal{H}_1\nabla^{M_1}_{U_1}V_1) = -\varphi_{1*}(T_1(U_1,V_1)) \in \Gamma(range\varphi_{1*})$ *for* $X = (U_1,0),\ Y = (V_1,0) \in \Gamma(T(M_1\times M_2))$, *where* $U_1,V_1 \in \Gamma(ker\varphi_{1*})$.

(ii) $(\nabla\varphi_*)(X,Y) = \nabla^{N_1}_{\varphi_{1*}X_1}\varphi_{1*}Y_1 - \varphi_{1*}(\nabla^{M_1}_{X_1}Y_1) = (\nabla^1\varphi_{1*})(X_1,Y_1) \in \Gamma(range\varphi_{1*})^\perp$ *for* $X = (X_1,0),\ Y = (Y_1,0) \in \Gamma(T(M_1\times M_2))$, *where* $X_1,Y_1 \in \Gamma(ker\varphi_{1*})^\perp$.

(iii) $(\nabla\varphi_*)(X,Y) = (\nabla^1\varphi_{1*})(X_1,Y_1) - \varphi_{2*}(T_2(U_2,V_2))$ *for* $X = (X_1,U_2),\ Y = (Y_1,V_2) \in \Gamma(T(M_1\times M_2))$, *where* $X_1,Y_1 \in \Gamma(ker\varphi_{1*})^\perp$ *and* $U_2,V_2 \in \Gamma(ker\varphi_{2*})$.

(iv) $(\nabla\varphi_*)(X,Y) = (\nabla^2\varphi_{2*})(X_2,Y_2) - \varphi_{1*}(T_1(U_1,V_1)) - U_1(\ln f)\varphi_{2*}Y_2 - V_1(\ln f)\varphi_{2*}X_2$ *for* $X = (U_1,X_2),\ Y = (V_1,Y_2) \in \Gamma(T(M_1\times M_2))$, *where* $U_1,V_1 \in \Gamma(ker\varphi_{1*})$ *and* $X_2,Y_2 \in \Gamma(ker\varphi_{2*})^\perp$.

(v) $(\nabla\varphi_*)(X,Y) = -\varphi_{2*}(\mathcal{H}_2\nabla^{M_2}_{U_2}V_2) = -\varphi_{2*}(T_2(U_2,V_2)) \in \Gamma(range\varphi_{2*})$ *for* $X = (0,U_2), Y = (0,V_2) \in \Gamma(T(M_1\times M_2))$, *where* $U_2,V_2 \in \Gamma(ker\varphi_{2*})$.

(vi) $(\nabla\varphi_*)(X,Y) = (\nabla^2\varphi_{2*})(X_2,Y_2) \in \Gamma(range\varphi_{2*})^\perp$ *for* $X = (0,X_2),\ Y = (0,Y_2) \in \Gamma(T(M_1\times M_2))$, *where* $X_2,Y_2 \in \Gamma(ker\varphi_{2*})^\perp$.

(vii) $(\nabla\varphi_*)(X,Y) = -\varphi_{1*}(T_1(U_1,V_1)) - \varphi_{2*}(T_2(U_2,V_2)) \in \Gamma(range\varphi_*)$ *for* $X = (U_1,U_2), Y = (V_1,V_2) \in \Gamma(T(M_1\times M_2))$, *where* $U_i,V_i \in \Gamma(ker\varphi_{i*})$.

(viii) $(\nabla\varphi_*)(X,Y) = (\nabla^1\varphi_{1*})(X_1,Y_1) + (\nabla^2\varphi_{2*})(X_2,Y_2) \in \Gamma(range\varphi_*)^\perp$ *for* $X = (X_1,X_2),\ Y = (Y_1,Y_2) \in \Gamma(T(M_1\times M_2))$, *where* $X_i,Y_i \in \Gamma(ker\varphi_{i*})^\perp$.

(ix) $(\nabla\varphi_*)(X,Y) = (\nabla^1\varphi_{1*})(X_1,Y_1) - \varphi_{2*}(\mathcal{H}_2\nabla^{M_2}_{U_2}Y_2)$ *for* $X = (X_1,U_2),\ Y = (Y_1,Y_2) \in \Gamma(T(M_1\times M_2))$, *where* $X_1,Y_1 \in \Gamma(ker\varphi_{1*})^\perp$, $Y_2 \in \Gamma(ker\varphi_{2*})^\perp$ *and* $U_2 \in \Gamma(ker\varphi_{2*})$.

(x) $(\nabla\varphi_*)(X,Y) = (\nabla^1\varphi_{1*})(X_1,Y_1) - \varphi_{2*}(A_2(X_2,V_2))$ *for* $X = (X_1,X_2)$, $Y = (Y_1,V_2) \in \Gamma(T(M_1\times M_2))$, *where* $X_1,Y_1 \in \Gamma(ker\varphi_{1*})^\perp$, $X_2 \in \Gamma(ker\varphi_{2*})^\perp$ *and* $V_2 \in \Gamma(ker\varphi_{2*})$.

(*xi*) $(\nabla\varphi_*)(X,Y) = (\nabla^2\varphi_{2*})(X_2,Y_2) - \varphi_{1*}(\mathcal{H}_1\nabla_{U_1}Y_1) - U_1(\ln f)\varphi_{2*}Y_2$ *for* $X=(U_1,X_2),\ Y=(Y_1,Y_2)\in\Gamma(T(M_1\times M_2))$, *where* $Y_1\in\Gamma(ker\varphi_{1*})^\perp$, $X_2,Y_2\in\Gamma(ker\varphi_{2*})^\perp$ *and* $U_1\in\Gamma(ker\varphi_{1*})$.

(*xii*) $(\nabla\varphi_*)(X,Y) = (\nabla^2\varphi_{2*})(X_2,Y_2) - V_1(\ln f)\varphi_{2*}X_2 - \varphi_{1*}(A_1(X_1,V_1))$ *for* $X=(X_1,X_2),\ Y=(V_1,Y_2)\in\Gamma(T(M_1\times M_2))$, *where* $X_1\in\Gamma(ker\varphi_{1*})^\perp$, $X_2,Y_2\in\Gamma(ker\varphi_{2*})^\perp$ *and* $V_1\in\Gamma(ker\varphi_{1*})$.

Remark 3. In Lemma 3.1 of [26], Şahin showed that for a Riemannian map φ between Riemannian manifolds $(\nabla\varphi_*)(X,Y)\in\Gamma(range\varphi_*)^\perp$ if $X,Y\in\Gamma(ker\varphi_*)^\perp$. Similarly in the (*viii*) statement of Corollary 13, we get that for a Riemannian warped product map between Riemannian warped product manifolds $(\nabla\varphi_*)(X,Y)\in\Gamma(range\varphi_*)^\perp$ for $X=(X_1,X_2),\ Y=(Y_1,Y_2)$, where $X_i,Y_i\in\Gamma(ker\varphi_{i*})^\perp$.

Now, we give the definition of totally geodesic map between Riemannian warped product manifolds.

Definition 3. Let $\varphi=\varphi_1\times\varphi_2:(M=M_1\times_f M_2,g_M)\to(N=N_1\times_\rho N_2,g_N)$ be a Riemannian warped product map between Riemannian warped product manifolds. Then φ is called totally geodesic if $(\nabla\varphi_*)(X,Y)=0$ for all $X=(X_1,X_2),\ Y=(Y_1,Y_2)\in\Gamma(T(M_1\times M_2))$.

Theorem 14. *Let* $\varphi=\varphi_1\times\varphi_2:(M=M_1\times_f M_2,g_M)\to(N=N_1\times_\rho N_2,g_N)$ *be a Riemannian warped product map between Riemannian warped product manifolds. Then* φ *is totally geodesic if and only if*

(*i*) φ_1 *is totally geodesic, and*
(*ii*) φ_2 *is totally geodesic, and*
(*iii*) f *is constant on* $ker\varphi_{1*}$.

Proof. The proof follows by Theorem 12 and Corollary 13. □

5. Harmonic Riemannian Warped Product Maps

In this section, we calculate the tension field for a Riemannian warped product map between Riemannian warped product manifolds and discuss harmonicity [2,29].

First, we give the definition of the tension field for a smooth map between Riemannian warped product manifolds [2].

Definition 4. Let $\varphi=\varphi_1\times\varphi_2:(M=M_1\times_f M_2,g_M)\to(N=N_1\times_\rho N_2,g_N)$ be a smooth map between Riemannian warped product manifolds. Then the tension field $\tau(\varphi)$ of φ is trace of the second fundamental form of φ with respect to g_M, i.e.

$$\tau(\varphi)=trace(\nabla\varphi_*)=trace(\nabla\varphi_*)((X_1,X_2),(Y_1,Y_2))=\sum_{i=1}^{m_1}\sum_{a=m_1+1}^{m_1+m_2}(\nabla\varphi_*)((e_i,e_a),(e_i,e_a)),$$

where $(X_1, X_2), (Y_1, Y_2) \in \Gamma(T(M_1 \times M_2))$ and $\{e_i\}_{i=1}^{m_1}$, $\{e_a\}_{a=m_1+1}^{m_1+m_2}$ are orthonormal bases of $T_{p_1}M_1$ and $T_{p_2}M_2$, respectively. The tension field of φ is a vector field along φ, i.e. $\tau(\varphi) \in \Gamma_\varphi(T(N_1 \times N_2))$.

Lemma 15. *Let $\varphi = \varphi_1 \times \varphi_2 : (M = M_1 \times_f M_2, g_M) \to (N = N_1 \times_\rho N_2, g_N)$ be a Riemannian warped product map between Riemannian warped product manifolds. Then*

$$\tau(\varphi) = -r_1\ \varphi_{1*}(H_1) - r_2\ \varphi_{2*}(H_2) + (m_1 - r_1)H_3 + (m_2 - r_2)H_4,$$

where $r_1 = \dim(ker\varphi_{1*}), r_2 = \dim(ker\varphi_{2*}), m_1 - r_1 = \dim(range\varphi_{1*})$ and $m_2 - r_2 = \dim(range\varphi_{2*})$. In addition H_1, H_2, H_3 and H_4 are the mean curvature vector fields of $ker\varphi_{1*}, ker\varphi_{2*}$, $range\varphi_{1*}$ and $range\varphi_{2*}$, respectively.

Proof. Let $\{u_1, u_2, \ldots, u_{r_1}\}, \{x_{r_1+1}, x_{r_1+2}, \ldots, x_{m_1}\}, \{u^*_{r_1+1}, u^*_{r_1+2}, \ldots, u^*_{r_1+r_2}\}$ and $\{x^*_{m_1-r_1+1}, x^*_{m_1-r_1+2}, \ldots, x^*_{m_1-r_1+m_2-r_2}\}$ be orthonormal bases of $ker\varphi_{1*}, (ker\varphi_{1*})^\perp$, $ker\varphi_{2*}$ and $(ker\varphi_{2*})^\perp$, respectively. We know that

$$\tau(\varphi) = \tau^{ker\varphi_*}(\varphi) + \tau^{(ker\varphi_*)^\perp}(\varphi). \tag{31}$$

Now

$$\tau^{ker\varphi_*}(\varphi) = \sum_{i=1}^{r_1} \sum_{a=r_1+1}^{r_1+r_2} (\nabla\varphi_*)((u_i, u^*_a), (u_i, u^*_a)), \tag{32}$$

where $(u_i, u^*_a) \in ker\varphi_{1*} \times ker\varphi_{2*} = ker\varphi_*$, and $\{u_i\}_{i=1}^{r_1}, \{u^*_a\}_{a=r_1+1}^{r_1+r_2}$ are orthonormal bases of $ker\varphi_{1*}$ and $ker\varphi_{2*}$, respectively. Using (25) in (32), we get

$$\begin{aligned}\tau^{ker\varphi_*}(\varphi) =& \sum_{i=1}^{r_1}(\nabla^1\varphi_{1*})(u_i, u_i) + \sum_{a=r_1+1}^{r_1+r_2}(\nabla^2\varphi_{2*})(u^*_a, u^*_a)\\ &+ \sum_{i=1}^{r_1}\sum_{a=r_1+1}^{r_1+r_2}(\varphi_{1*}u_i(\ln\rho))\varphi_{2*}u^*_a + \sum_{i=1}^{r_1}\sum_{a=r_1+1}^{r_1+r_2}(\varphi_{1*}u_i(\ln\rho))\varphi_{2*}u^*_a\\ &- \sum_{i=1}^{r_1}\sum_{a=r_1+1}^{r_1+r_2}\{(u_i(\ln f))\varphi_{2*}u^*_a + (u_i(\ln f))\varphi_{2*}u^*_a\}.\end{aligned}$$

Using (26) in above equation, we get

$$\tau^{ker\varphi_*}(\varphi) = -\sum_{i=1}^{r_1}(\nabla^1\varphi_{1*})(u_i, u_i) - \sum_{a=r_1+1}^{r_1+r_2}(\nabla^2\varphi_{2*})(u^*_a, u^*_a). \tag{33}$$

Using (7) and (8) in (33), we get

$$\tau^{ker\varphi_*} = -r_1\ \varphi_{1*}(H_1) - r_2\ \varphi_{2*}(H_2), \tag{34}$$

where H_1 and H_2 are the mean curvature vector fields of $ker\varphi_{1*}$ and $ker\varphi_{2*}$ respectively. On the other hand

$$\tau^{(ker\varphi_*)^\perp}(\varphi) = \sum_{i=r_1+1}^{m_1} \sum_{a=m_1-r_1+1}^{m_1-r_1+m_2-r_2} (\nabla\varphi_*)((x_i, x_a^*)(x_i, x_a^*)), \tag{35}$$

where $\{x_i\}_{i=r_1+1}^{m_1}$ and $\{x_a^*\}_{a=m_1-r_1+1}^{m_1-r_1+m_2-r_2}$ are orthonormal bases of $(ker\varphi_{1*})^\perp$ and $(ker\varphi_{2*})^\perp$, respectively. Using $(viii)$ statement of Corollary 13 in (35), we get

$$\tau^{(ker\varphi_*)^\perp}(\varphi) = \sum_{i=r_1+1}^{m_1} (\nabla^1\varphi_{1*})(x_i, x_i) + \sum_{a=m_1-r_1+1}^{m_1-r_1+m_2-r_2} (\nabla^2\varphi_{2*})(x_a^*, x_a^*).$$

Equivalently

$$\begin{aligned}\tau^{(ker\varphi_*)^\perp}(\varphi) = &\sum_{i=r_1+1}^{m_1} \sum_{p=1}^{n_1} g_N((\nabla^1\varphi_{1*})(x_i, x_i), z_p)z_p \\ &+ \sum_{a=m_1-r_1+1}^{m_1-r_1+m_2-r_2} \sum_{q=n_1+1}^{n_2} g_N((\nabla^2\varphi_{2*})(x_a^*, x_a^*), z_q')z_q',\end{aligned}$$

where $\{z_p\}_{p=1}^{n_1}$ and $\{z_q'\}_{q=n_1+1}^{n_1+n_2}$ are orthonormal bases of TN_1 and TN_2 respectively. On decomposition

$$\begin{aligned}\tau^{(ker\varphi_*)^\perp}(\varphi) = &\sum_{i=r_1+1}^{m_1} \sum_{k=r_1+1}^{m_1} g_{N_1}((\nabla^1\varphi_{1*})(x_i, x_i), \tilde{e}_k)\tilde{e}_k \\ &+ \sum_{i=r_1+1}^{m_1} \sum_{l=1}^{n_3} g_{N_1}((\nabla^1\varphi_{1*})(x_i, x_i), \bar{e}_l), \bar{e}_l) \\ &+ \sum_{a=m_1-r_1+1}^{m_1-r_1+m_2-r_2} \left\{\rho^2 \sum_{t=m_1-r_1+1}^{m_1-r_1+m_2-r_2} g_{N_2}((\nabla^2\varphi_{2*})(x_a^*, x_a^*), \hat{e}_t)\hat{e}_t\right\} \\ &+ \sum_{a=m_1-r_1+1}^{m_1-r_1+m_2-r_2} \left\{\sum_{s=1+n_3}^{n_3+n_4} \rho^2 g_{N_2}((\nabla^2\varphi_{2*})(x_a^*, x_a^*), \check{e}_s)\check{e}_s\right\},\end{aligned}$$

where $\{\tilde{e}_k\}_{k=r_1+1}^{m_1}, \{\bar{e}_l\}_{l=1}^{n_3}, \{\hat{e}_t\}_{t=m_1-r_1+1}^{m_1-r_1+m_2-r_2}$ and $\{\check{e}_s\}_{s=n_3+1}^{n_3+n_4}$ are orthonormal bases of $range\varphi_{1*}, (range\varphi_{1*})^\perp, range\varphi_{2*}$ and $(range\varphi_{2*})^\perp$, respectively. Using (26) in above equation, we get

$$\begin{aligned}\tau^{(ker\varphi_*)^\perp}(\varphi) = &\sum_{i=r_1+1}^{m_1} \sum_{l=1}^{n_3} g_{N_1}(\nabla_{x_i}^{\varphi_1}\varphi_{1*}(x_i), \bar{e}_l)\bar{e}_l \\ &+ \sum_{a=m_1-r_1+1}^{m_1-r_1+m_2-r_2} \sum_{s=1+n_3}^{n_3+n_4} \rho^2 g_{N_2}(\nabla_{x_a^*}^{\varphi_2}\varphi_{2*}(x_a^*), \check{e}_s)\check{e}_s.\end{aligned} \tag{36}$$

Using (13) and (14) in (36), we get

$$\tau^{(ker\varphi_*)^\perp}(\varphi) = (m_1 - r_1)\sum_{l=1}^{n_3} g_{N_1}(H_3, \bar{e}_l)\bar{e}_l + (m_2 - r_2)\sum_{s=1+n_3}^{n_3+n_4} \rho^2 g_{N_2}(H_4, \check{e}_s)\check{e}_s,$$

where H_3 and H_4 are the mean curvature vector fields of $range\varphi_{1*}$ and $range\varphi_{2*}$ respectively. Thus

$$\tau^{(ker\varphi_*)^\perp}(\varphi) = (m_1 - r_1)H_3 + (m_2 - r_2)H_4. \tag{37}$$

Then we get required proof by using (31), (34) and (37). □

Remark 4. In Lemma 4.2 of [27], Şahin showed that for a Riemannian map φ between Riemannian manifolds $\tau(\varphi) = -r\varphi_*(H) + (m-r)H'$, where $r = \dim(ker\varphi_*)$ and $m-r = \dim(range\varphi_*)$. Also, H and H' denote the mean curvature vector fields of $ker\varphi_*$ and $range\varphi_*$, respectively. While in Lemma 15, we get that for a Riemannian warped product map between Riemannian warped product manifolds $\tau(\varphi) = -r_1\varphi_{1*}(H_1) - r_2\varphi_{2*}(H_2) + (m_1-r_1)H_3 + (m_2-r_2)H_4$, where $r_1 = \dim(ker\varphi_{1*}), r_2 = \dim(ker\varphi_{2*}), m_1 - r_1 = \dim(range\varphi_{1*})$ and $m_2 - r_2 = \dim(range\varphi_{2*})$. Here H_1, H_2, H_3 and H_4 are the mean curvature vector fields of $ker\varphi_{1*}, ker\varphi_{2*}$, $range\varphi_{1*}$ and $range\varphi_{2*}$, respectively.

Now, we give the definition of harmonic map between Riemannian warped product manifolds.

Definition 5. Let $\varphi = \varphi_1 \times \varphi_2 : (M = M_1 \times_f M_2, g_M) \to (N = N_1 \times_\rho N_2, g_N)$ be a Riemannian warped product map between Riemannian warped product manifolds. Then φ is harmonic if its tension field $\tau(\varphi)$ vanishes.

Theorem 16. *Let $\varphi = \varphi_1 \times \varphi_2 : (M = M_1 \times_f M_2, g_M) \to (N = N_1 \times_\rho N_2, g_N)$ be a non-constant Riemannian warped product map between Riemannian warped product manifolds. Then any four conditions imply fifth:*

(i) *φ is harmonic.*
(ii) *The distribution $ker\varphi_{1*}$ is minimal.*
(iii) *The distribution $ker\varphi_{2*}$ is minimal.*
(iv) *The distribution $range\varphi_{1*}$ is minimal.*
(v) *The distribution $range\varphi_{2*}$ is minimal.*

Proof. We know that a distribution is minimal if and only if its mean curvature vector field vanishes. Then the proof follows by Lemma 15. □

Remark 5. The harmonicity conditions for a Riemannian map between Riemannian manifolds were given in Theorem 6.1 of [30] by Şahin. Similarly, we obtain the harmonicity conditions for a Riemannian warped product map between Riemannian warped product manifolds in Theorem 16.

6. Umbilical Riemannian Warped Product Maps

In this section, we construct Weingarten formula for a Riemannian warped product map between Riemannian warped product manifolds and discuss umbilicity.

Let $\varphi = \varphi_1 \times \varphi_2 : M = M_1 \times_f M_2 \to N = N_1 \times_\rho N_2$ be a Riemannian warped product map between Riemannian warped product manifolds and ∇^N be the Levi-Civita connection on $N = N_1 \times_\rho N_2$. From now on wards, for the sake of simplicity we denote both the Levi-Civita connection on (N, g_N) and its pullback along φ by ∇^N. Here we denote $(range\varphi_*)^\perp = (range\varphi_{1*})^\perp \times (range\varphi_{2*})^\perp$ is subbundle of $\varphi^{-1}(T(N_1 \times N_2))$ with fiber $\varphi_{1*}(T_{p_1}M_1)^\perp \times \varphi_{2*}(T_{p_2}M_2)^\perp$ the orthogonal complement of $\varphi_{1*}(T_{p_1}M_1) \times \varphi_{2*}(T_{p_2}M_2)$ for $g_N = g_{N_1} + \rho^2\, g_{N_2}$ over $p = (p_1, p_2) \in M_1 \times_f M_2$. For any vector field $X = (X_1, X_2)$ on $M = M_1 \times_f M_2$ and any section $V = (V_1, V_2)$ of $(range\varphi_*)^\perp = (range\varphi_{1*})^\perp \times (range\varphi_{2*})^\perp$, we define $\nabla^{\varphi\perp}_X V = \nabla^{\varphi\perp}_{X_1+X_2} V_1 + V_2$, which is orthogonal projection of $\nabla^N_{X_1+X_2} V_1 + V_2$ on $(range\varphi_*)^\perp = (range\varphi_{1*})^\perp \times (range\varphi_{2*})^\perp$. Then $\nabla^{\varphi\perp}$ is a linear connection on $(range\varphi_*)^\perp$ such that $\nabla^{\varphi\perp} g_N = 0$ [21].

Now we construct Weingarten formula for Riemannian warped product map. Define shape operator S_V on $range\varphi_* = range\varphi_{1*} \times range\varphi_{2*}$. Since φ is a Riemannian map, for $X = (X_1, X_2) \in \Gamma(ker\varphi_{1*}) \times \Gamma(ker\varphi_{2*}) = \Gamma(ker\varphi_*)^\perp$ and $V = (V_1, V_2) \in \Gamma(range\varphi_{1*})^\perp \times \Gamma(range\varphi_{2*})^\perp = \Gamma(range\varphi_*)^\perp$, we have [29]

$$\nabla^N_{\varphi_*(X_1+X_2)}(V_1, V_2) = -S_{(V_1,V_2)}\varphi_*(X_1, X_2) + \nabla^{\varphi\perp}_{(X_1,X_2)}(V_1, V_2).$$

Since $T_{p_1}M_1 \times T_{p_2}M_2 \cong T_{p_1}M_1 \oplus T_{p_2}M_2$, above equation can be written as

$$\nabla^N_{\varphi_* X_1 + \varphi_* X_2}(V_1 + V_2) = -S_{(V_1+V_2)}(\varphi_* X_1 + \varphi_* X_2) + \nabla^{\varphi\perp}_{(X_1+X_2)}(V_1 + V_2).$$

Since ∇ is a linear connection and S_{V_1} is a shape operator on $range\varphi_{1*}$, there is no meaning of $S_{V_1}\varphi_{2*}X_2$. Similarly we treat for S_{V_2}. Then by above equation, we have

$$\begin{aligned}&\nabla^N_{\varphi_{1*}X_1}V_1 + \nabla^N_{\varphi_{1*}X_1}V_2 + \nabla^N_{\varphi_{2*}X_2}V_1\\&+ \nabla^N_{\varphi_{2*}X_2}V_2 = -S_{V_1}\varphi_{1*}X_1 - S_{V_2}\varphi_{2*}X_2\\&+ \nabla^{\varphi\perp}_{X_1}V_1 + \nabla^{\varphi\perp}_{X_1}V_2 + \nabla^{\varphi\perp}_{X_2}V_1 + \nabla^{\varphi\perp}_{X_2}V_2.\end{aligned}$$

Using Lemma 9 in above equation, we get

$$\begin{aligned}\nabla^{N_1}_{\varphi_{1*}X_1}V_1 + \frac{(\varphi_{1*}X_1(\rho))}{\rho}V_2 + \frac{(V_1(\rho))}{\rho}\varphi_{2*}X_2 + \nabla^{N_2}_{\varphi_{2*}X_2}V_2 =& - S_{V_1}\varphi_{1*}X_1 - S_{V_2}\varphi_{2*}X_2\\&+ \nabla^{\varphi\perp}_{X_1}V_1 + \nabla^{\varphi\perp}_{X_2}V_2,\end{aligned} \tag{38}$$

where $S_{V_i}\varphi_{i*}X_i$ is tangential component (vector fields along φ_i) of $\nabla^{N_i}_{\varphi_{i*}X_i}V_i$. Observe that $\nabla^{N_i}_{\varphi_{i*}X_i}V_i$ is pullback connection of ∇^{N_i}. Here (38) is known as Weingarten formula for Riemannian warped product map.

Since $(range\varphi_*)^\perp$ is subbundle of $\varphi^{-1}(T(N\times N_2))$ and $\varphi^{-1}(T(N_1\times N_2))$ is bundle on $M = M_1\times_f M_2$, $(range\varphi_*)^\perp$ is also bundle on M. Thus

$$\varphi^{-1}(T(N_1\times N_2)) = (range\varphi_*)^\perp \oplus (range\varphi_*).$$

Lemma 17. *Let $\varphi = \varphi_1\times\varphi_2 : (M = M_1\times_f M_2, g_M) \to (N = N_1\times_\rho N_2, g_N)$ be a Riemannian warped product map between Riemannian warped product manifolds. Then ρ is a constant function on $(range\varphi_*)^\perp$.*

Proof. For $V = (V_1, V_2)\in\Gamma(range\varphi_*)^\perp$, we have

$$\begin{aligned} V(\ln\rho) =& g_N(V, \nabla^N \ln\rho)\\ =& g_{N_1}(V_1, \nabla^N\ln\rho) + \rho^2\ g_{N_2}(V_2, \nabla^N\ln\rho). \end{aligned}$$

Using Lemma 1, we get

$$V(\ln\rho) = g_{N_1}(V_1, \varphi_{1*}(\nabla^{M_1}f)).$$

Since $\varphi_{1*}(\nabla^{M_1}f)\in\Gamma(range\varphi_*)$, $V(\ln\rho) = 0$. This implies the proof. □

Proposition 18. *Let $\varphi = \varphi_1\times\varphi_2 : (M = M_1\times_f M_2, g_M) \to (N = N_1\times_\rho N_2, g_N)$ be a non-constant Riemannian warped product map between Riemannian warped product manifolds. Then*

$$g_N(S_{(V_1,V_2)}\varphi_*(X_1,X_2), \varphi_*(Y_1,Y_2)) = g_N((\nabla\varphi_*)((X_1,X_2),(Y_1,Y_2)),(V_1,V_2)),$$

where $X = (X_1,X_2)$, $Y = (Y_1,Y_2)\in\Gamma(ker\varphi_{1*})^\perp\times\Gamma(ker\varphi_{2*})^\perp$ and $V = (V_1,V_2)\in\Gamma(range\varphi_{1*})^\perp\times\Gamma(range\varphi_{2*})^\perp$.

Proof. By Weingarten formula for $X = (X_1,X_2), Y = (Y_1,Y_2)\in\Gamma(ker\varphi_{1*})^\perp\times\Gamma(ker\varphi_{2*})^\perp$ and $V = (V_1,V_2)\in\Gamma(range\varphi_*)^\perp$, we have

$$\begin{aligned} &g_N(S_{V_1}\varphi_{1*}X_1 + S_{V_2}\varphi_{2*}X_2,\ \varphi_{1*}Y_1+\varphi_{2*}Y_2)\\ &\quad = g_N(\nabla^{\varphi\perp}_{X_1}V_1 + \nabla^{\varphi\perp}_{X_2}V_2 - \nabla^N_{\varphi_{1*}X_1}V_1 - \nabla^N_{\varphi_{2*}X_2}V_2\\ &\qquad -(\varphi_{1*}X_1(\ln\rho))V_2 - (V_1(\ln\rho))\varphi_{2*}X_2,\ \varphi_{1*}Y_1+\varphi_{2*}Y_2), \end{aligned} \tag{39}$$

which implies

$$\begin{aligned} g_N(S_{V_1}\varphi_{1*}X_1 + S_{V_2}\varphi_{2*}X_2,\ \varphi_{1*}Y_1+\varphi_{2*}Y_2) =& - g_N(\nabla^{N_1}_{\varphi_{1*}X_1}V_1,\ \varphi_{1*}Y_1)\\ & - g_N(\nabla^{N_2}_{\varphi_{2*}X_2}V_2,\ \varphi_{2*}Y_2)\\ & - g_N((\varphi_{1*}X_1)(\ln\rho)V_2,\ \varphi_{1*}Y_1+\varphi_{2*}Y_2)\\ & - g_N(V_1(\ln\rho)\varphi_{2*}X_2,\ \varphi_{1*}Y_1+\varphi_{2*}Y_2)\\ =& - g_{N_1}(\nabla^{N_1}_{\varphi_{1*}X_1}V_1,\ \varphi_{1*}Y_1)\\ & - \rho^2 g_{N_2}(\nabla^{N_2}_{\varphi_{2*}X_2}V_2,\ \varphi_{2*}Y_2)\\ & - \rho^2\left\{V_1(\ln\rho)g_{N_2}(\varphi_{2*}X_2,\varphi_{2*}Y_2)\right\}. \end{aligned}$$

Using metric compatibility condition and Lemma 17 in above equation, we get

$$g_N(S_{V_1}\varphi_{1*}X_1 + S_{V_2}\varphi_{2*}X_2,\ \varphi_{1*}Y_1 + \varphi_{2*}Y_2) = g_{N_1}(\nabla^{N_1}_{\varphi_{1*}X_1}\varphi_{1*}Y_1, V_1) + \rho^2\ g_{N_2}(\nabla^{N_2}_{\varphi_{2*}X_2}\varphi_{2*}Y_2, V_2). \tag{40}$$

Using (4) in (40), we get

$$g_N(S_{V_1}\varphi_{1*}X_1 + S_{V_2}\varphi_{2*}X_2,\ \varphi_{1*}Y_1 + \varphi_{2*}Y_2) = g_{N_1}(\nabla^{\varphi_1}_{X_1}\varphi_{1*}(Y_1), V_1) + \rho^2 g_{N_2}(\nabla^{\varphi_2}_{X_2}\varphi_{2*}(Y_2), V_2).$$

Using (26) in above equation, we get

$$g_N(S_{V_1}\varphi_{1*}X_1 + S_{V_2}\varphi_{2*}X_2,\ \varphi_{1*}Y_1 + \varphi_{2*}Y_2) = g_{N_1}((\nabla^1\varphi_{1*})(X_1, Y_1), V_1) + \rho^2\ g_{N_2}((\nabla^2\varphi_{2*})(X_2, Y_2), V_2). \tag{41}$$

This implies the proof. □

Remark 6. For a Riemannian map between Riemannian manifolds Şahin obtained that $g_N(S_V\varphi_*X, \varphi_*Y) = g_N((\nabla\varphi_*)(X, Y), V)$ for $X, Y \in \Gamma(ker\varphi_*)^\perp$ and $V \in \Gamma(range\varphi_*)^\perp$ [29]. While in Proposition 18, for a Riemannian warped product map between Riemannian warped product manifolds we obtain that $g_N(S_{(V_1,V_2)}\varphi_*(X_1, X_2), \varphi_*(Y_1, Y_2)) = g_N((\nabla\varphi_*)((X_1, X_2), (Y_1, Y_2)), (V_1, V_2))$ for $X = (X_1, X_2)$, $Y = (Y_1, Y_2) \in \Gamma(ker\varphi_{1*})^\perp \times \Gamma(ker\varphi_{2*})^\perp$ and $V = (V_1, V_2) \in \Gamma(range\varphi_{1*})^\perp \times \Gamma(range\varphi_{2*})^\perp$. Now, since the second fundamental form $(\nabla\varphi_*)((X_1, X_2), (Y_1, Y_2))$ of φ is symmetric, we conclude that S_V is a symmetric linear transformation of $(range\varphi_*)$.

Now we give definition of umbilical map between Riemannian warped product manifolds.

Definition 6. Let $\varphi = \varphi_1 \times \varphi_2 : M = M_1 \times_f M_2 \to N = N_1 \times_\rho N_2$ be a Riemannian warped product map between Riemannian warped product manifolds. Then we say that φ is an umbilical Riemannian warped product map at $p = (p_1, p_2) \in M_1 \times_f M_2$ if

$$S_{(V_1,V_2)}\varphi_{*(X_1,X_2)} = \lambda(\varphi_*(X_1, X_2)).$$

Equivalently

$$S_{(V_1+V_2)}(\varphi_{1*}X_1 + \varphi_{2*}X_2) = \lambda(\varphi_{1*}X_1 + \varphi_{2*}X_2),$$

where $\lambda : N = N_1 \times_\rho N_2 \to \mathbb{R}$ is a smooth function such that $\lambda = (\lambda_1, \lambda_2)$ and λ_i is smooth function on N_i. We say φ is an umbilical map if it is umbilical at all $p \in M$.

Theorem 19. *Let $\varphi = \varphi_1 \times \varphi_2 : M = M_1 \times_f M_2 \to N = N_1 \times_\rho N_2$ be a Riemannian warped product map between Riemannian warped product manifolds. Then φ is umbilical if and only if*

$$(\nabla\varphi_*)((X_1, X_2), (Y_1, Y_2)) = (H_3, H_4)g_M((X_1, Y_1), (X_2, Y_2)),$$

where $X = (X_1, X_2), Y = (Y_1, Y_2) \in \Gamma(ker\varphi_{1*})^{\perp} \times \Gamma(ker\varphi_{2*})^{\perp}$ *and* $V = (V_1, V_2) \in \Gamma(range\varphi_{1*})^{\perp} \times \Gamma(range\varphi_{2*})^{\perp}$. *In addition,* H_3 *and* H_4 *are the mean curvature vector fields of* $range\varphi_{1*}$ *and* $range\varphi_{2*}$ *respectively.*

Proof. Let $\{u_1, u_2, \ldots, u_{r_1}\}, \{x_{r_1+1}, x_{r_1+2}, \ldots, x_{m_1}\}, \{u^*_{r_1+1}, u^*_{r_1+2}, \ldots, u^*_{r_1+r_2}\}$ and $\{x^*_{m_1-r_1+1}, x^*_{m_1-r_1+2}, \ldots, x^*_{m_1-r_1+m_2-r_2}\}$ be orthonormal bases of $ker\varphi_{1*}, (ker\varphi_{1*})^{\perp}$, $ker\varphi_{2*}$ and $(ker\varphi_{2*})^{\perp}$, respectively. Then the Riemannian warped product map $\varphi = \varphi_1 \times \varphi_2$ implies $\{\varphi_{1*}(x_{r_1+1}),\ \varphi_{1*}(x_{r_1+2}), \ldots, \varphi_{1*}(x_{m_1})\}$ and $\{\varphi_{2*}(x^*_{m_1-r_1+1}), \ldots,\ \varphi_{2*}(x^*_{m_1-r_1+m_2-r_2})\}$ are orthonormal bases of $range\varphi_{1*}$ and $range\varphi_{2*}$, respectively. Then

$$\begin{aligned}
&\sum_{i=r_1+1}^{m_1} \sum_{j=m_1-r_1+1}^{m_1-r_1+m_2-r_2} g_N(S_{(V_1+V_2)}\varphi_*(x_i, x^*_j), \varphi_*(x_i, x^*_j)) \\
&\quad = \sum_{i=r_1+1}^{m_1} \sum_{j=m_1-r_1+1}^{m_1-r_1+m_2-r_2} g_N(\lambda\{\varphi_{1*}(x_i) + \varphi_{2*}(x^*_j)\},\ \varphi_{1*}(x_i) + \varphi_{2*}(x^*_j)).
\end{aligned}$$

On solving above equation, we get

$$\begin{aligned}
&\sum_{i=r_1+1}^{m_1} \sum_{j=m_1-r_1+1}^{m_1-r_1+m_2-r_2} g_N(S_{V_1}(\varphi_{1*}x_i) + S_{V_2}(\varphi_{2*}x^*_j),\ \varphi_{1*}(x_i) + \varphi_{2*}(x^*_j)) \\
&\quad = \sum_{i=r_1+1}^{m_1} \sum_{j=m_1-r_1+1}^{m_1-r_1+m_2-r_2} (\lambda_1, \lambda_2)(g_{N_1}(\varphi_{1*}x_i, \varphi_{1*}x_i) + \rho^2\ g_{N_2}(\varphi_{1*}x^*_j, \varphi_{1*}x^*_j)).
\end{aligned} \tag{42}$$

Using (41) in (42), we get

$$\begin{aligned}
&\sum_{i=r_1+1}^{m_1} g_{N_1}((\nabla^1\varphi_{1*})(x_i, x_i), V_1) + \sum_{j=m_1-r_1+1}^{m_1-r_1+m_2-r_2} \rho^2\ g_{N_2}((\nabla^2\varphi_{2*})(x^*_j, x^*_j), V_2) \\
&\quad = \lambda_1(m_1 - r_1) + \lambda_2\ \rho^2\ (m_2 - r_2).
\end{aligned}$$

Using (13) and (14) in above equation, we get

$$\begin{aligned}
&(m_1 - r_1)\ g_{N_1}(H_3, V_1) + \rho^2\ (m_2 - r_2)\ g_{N_2}(H_4, V_2) \\
&\quad = \lambda_1\ (m_1 - r_1) +\ \lambda_2\ \rho^2\ (m_2 - r_2),
\end{aligned}$$

where H_3 and H_4 are the mean curvature vector fields of $range\varphi_{1*}$ and $range\varphi_{2*}$ respectively. On comparison, we get

$$\lambda_1 = g_{N_1}(H_3, V_1) \text{ and } \lambda_2 = g_{N_2}(H_4, V_2),$$

By (41) and Definition 6, we get

$$\begin{aligned}
&g_{N_1}((\nabla^1\varphi_{1*})(X_1, Y_1), V_1) + \rho^2\ g_{N_2}((\nabla^2\varphi_{2*})(X_2, Y_2), V_2) \\
&\quad = g_{N_1}(H_3, V_1) g_{M_1}(X_1, Y_1) +\ \rho^2\ g_{N_2}(H_4, V_2)\ g_{M_2}(X_2, Y_2),
\end{aligned}$$

which implies $\quad g_N((\nabla\varphi_*)((X_1, X_2)(Y_1, Y_2)), (V_1, V_2))$

$$= g_N((H_3, H_4), (V_1, V_2))g_M((X_1, Y_1), (X_2, Y_2)).$$

This implies the required proof. □

Remark 7. In Lemma 4.1 of [28], Şahin showed that a Riemannian map φ between Riemannian manifolds is umbilical if and only if $(\nabla\varphi_*)(X, Y) = H' g_M(X, Y)$ for $X, Y \in \Gamma(ker\varphi_*)^\perp$ and H' mean curvature vector field of $range\varphi_*$. While in Theorem 19, we show that a Riemannian warped product map between Riemannian warped product manifolds is umbilical if and only if $(\nabla\varphi_*)((X_1, X_2), (Y_1, Y_2)) = (H_3, H_4)g_M((X_1, Y_1), (X_2, Y_2))$ for $X = (X_1, X_2), Y = (Y_1, Y_2) \in \Gamma(ker\varphi_{1*})^\perp \times \Gamma(ker\varphi_{2*})^\perp$, and H_3 and H_4 the mean curvature vector fields of $range\varphi_{1*}$ and $range\varphi_{2*}$ respectively.

Acknowledgements

We are grateful to the referee(s) for his/her efforts to improve the quality of the paper. Kiran Meena gratefully acknowledges the financial support provided by the Department of Atomic Energy, Government of India [Offer Letter No.: HRI/133/1436 Dated 29 November 2022] and research facilities provided by Harish-Chandra Research Institute, Prayagraj, India. The first author Kiran Meena is thankful to her doctoral degree adviser Dr. Akhilesh Yadav for his motivational words.

Author contributions (i) Kiran Meena: Conceptualization, Methodology, Investigation, Writing - original draft, Editing of Revision. (ii) Bayram Şahin: Methodology, Investigation, Supervision, Validation, Review. (iii) Hemangi Madhusudan Shah: Methodology, Investigation, Editing. All the authors read and agreed to the manuscript.

Funding Kiran Meena is financial supported by the Department of Atomic Energy, Government of India [Offer Letter No.: HRI/133/1436 Dated 29 November 2022].

Declarations

Conflict of interest The authors have no conflict of interest and no financial interests for this article.

Ethics approval The submitted work is original and not submitted to more than one journal for simultaneous consideration.

Consent to participate Not applicable.

Consent for publication Not applicable.

Availability of data and materials This manuscript has no associated data.

Code availability Not applicable.

References

[1] Abraham, R., Marsden, J.E., Ratiu, T.: Manifolds, Tensor Analysis, and Applications. Springer, New York (1988)

[2] Baird, P., Wood, J.C.: Harmonic Morphisms Between Riemannian Manifolds. Clarendon Press, Oxford (2003)

[3] Bishop, R.L., O'Neill, B.: Manifolds of negative curvature. Trans. Am. Math. Soc. **145**, 1–49 (1969)

[4] Brickell, F., Clark, R.S.: Differentiable Manifolds an Introduction. AVan Nostrand Reinhold Company Ltd., New York (1970)

[5] Chen, B.Y.: Warped product immersions. J. Geom. **82**, 36–49 (2005)

[6] Chen, B.Y.: Differential Geometry of Warped Product Manifolds and Submanifolds. World Scientific Publishing Co., Pte. Ltd, Singapore (2017)

[7] Chen, B.Y.: Geometry of warped products as Riemannian submanifolds and related problems. Soochow J. Math. **28**(2), 125–156 (2002)

[8] Chen, B.Y.: On isometric minimal immersions from warped products into real space forms. Proc. Edinb. Math. Soc. **45**(3), 579–587 (2002)

[9] Chen, B.Y., Dillen, F.: Optimal inequalities for multiply warped product submanifolds. Int. Electron. J. Geom. **1**(1), 1–11 (2008)

[10] Chen, B.Y.: Warped products in real space forms. Rocky Mountain J. Math. **34**(2), 551–563 (2004)

[11] Chen, B.Y., Dillen, F.: Warped product decompositions of real space forms and Hamiltonian-stationary Lagrangian submanifolds. Nonlinear Anal. **69**(10), 3462–3494 (2008)

[12] Chen, B.Y., Wei, S.W.: Differential geometry of submanifolds of warped product manifolds $I \times_f S^{m-1}(k)$. J. Geom. **91**(1), 21–42 (2009)

[13] Chen, B.Y., Wei, S.W.: Submanifolds of warped product manifolds $I \times_f S^{m-1}(k)$ from a p-harmonic viewpoint. Bull. Transilvania Univ. Brasov **1**(50), 59–78 (2008)

[14] Erken, I.K., Murathan, C.: Riemannian warped product submersions. Results Math. **76**, 1–14 (2021)

[15] Erken, I.K., Murathan, C., Siddiqui, A.N.: Inequalities on Riemannian warped product submersions for Casorati curvatures. Mediterr. J. Math. **20**(2), 1–18 (2023)

[16] Falcitelli, M., Ianus, S., Pastore, A.M.: Riemannian Submersions and Related Topics. World Scientific, River Edge, NJ (2004)

[17] Fischer, A.E.: Riemannian maps between Riemannian manifolds. Contemp. Math. **132**, 331–366 (1992)

[18] Garcia-Rio, E., Kupeli, D.N.: Semi-Riemannian Maps and Their Applications. Kluwer Academic, Dordrecht (1999)

[19] Moore, J.D.: Isometric immersions of Riemannian products. J. Differ. Geom. **5**, 159–168 (1971)

[20] Nash, J.F.: The imbedding problem for Riemannian manifolds. Ann. Math. **63**, 20–63 (1956)

[21] Nore, T.: Second fundamental form of a map. Ann. Mat. Pura Appl. **146**(1), 281–310 (1986)

[22] Nölker, S.: Isometric immersions of warped products. Differ. Geom. Appl. **6**, 1–30 (1996)

[23] O'Neill, B.: The fundamental equations of a submersion. Michigan Math. J. **13**(4), 459–469 (1966)

[24] O'Neill, B.: Semi-Riemannian Geometry with Applications to Relativity. Academic Press, New York (1983)

[25] Petersen, P.: Riemannian Geometry, 3rd edn. Springer, London (2016)

[26] Şahin, B.: Invariant and anti-invariant Riemannian maps to Kähler manifolds. Int. J. Geom. Methods Mod. Phys. **7**(3), 337–355 (2010)

[27] Şahin, B.: Conformal Riemannian maps between Riemannian manifolds, their harmonicity and decomposition theorems. Acta Appl. Math. **109**, 829–847 (2010)

[28] Şahin, B.: Semi-invariant Riemannian maps to Kähler manifolds. Int. J. Geom. Methods Mod. Phys. **7**, 1439–1454 (2011)

[29] Şahin, B.: Riemannian Submersions, Riemannian Maps in Hermitian Geometry, and Their Applications. Academic Press, London (2017)

[30] Şahin, B.: A survey on differential geometry of Riemannian maps between Riemannian manifolds, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.), **63**(1), 151–167 (2017)

[31] Tojeiro, R.: Conformal immersions of warped products. Geom. Dedicata **128**, 17–31 (2007)

[32] Vilms, J.: Totally geodesic maps. J. Differ. Geom. **4**, 73–79 (1970)

Kiran Meena and Hemangi Madhusudan Shah
Department of Mathematics
Harish-Chandra Research Institute
Chhatnag Road, Jhunsi
Prayagraj Uttar Pradesh 211019
India
e-mail: kirankapishmeena@gmail.com;
hemangimshah@hri.res.in

Bayram Şahin
Department of Mathematics
Ege University
35100 Izmir
Turkey
e-mail: bayram.sahin@ege.edu.tr

Received: April 6, 2023.
Accepted: November 28, 2023.