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Results in Mathematics

Exponential Rationals

Maciej Sablik and Ekaterina Shulman

Abstract. Several authors studied the so called exponential polynomials, characterized as solutions to the equation

$$f(x+y)=\sum_{k=1}^{n}u_i(x)v_i(y).$$

In the present paper we deal with a more general equation

$$\sum_{j=1}^{M}P_j(x,y)f_j(a_jx+c_jy)=\sum_{k=1}^{n}u_k(x)v_k(y).$$

Here all $f_j,\ u_k,\ v_k$ are assumed to be unknown scalar functions on $\mathbb{R}^d$, while P_j are polynomials. We prove that f_j are ratios of exponential polynomials and polynomials, or sums of exponential functions multiplied by rational functions.

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1. Introduction

More than hundred years ago Stéphanos [14], Levi-Cività [7] and Stäkel [13] have shown that the only differentiable functions on $\mathbb{R}$ satisfying the equation

$$f(x+y)=\sum_{k=1}^{n}u_i(x)v_i(y), \tag{1.1}$$

with some u_i, v_i, are exponential polynomials, that is the functions of the form

$$f(x)=\sum_{k}p_k(x)e^{\lambda_k x},$$

where p_k are polynomials, $\lambda_k \in \mathbb{C}$. This result was subsequently extended to continuous and measurable functions on $\mathbb{R}$, then to functions on $\mathbb{R}^n$ (see the book by Aczél [1]), a paper of Székelyhidi [16] on Abelian groups (see also [17] and [18] by the same author), and to (continuous) functions on arbitrary (topological) semigroups by the second author [10] (with the change of exponential polynomials by matrix functions of finite-dimensional (continuous) representations). There are many interesting and important equations related to (1.1), for example the extended d'Alembert equation

$$f(x+y)+f(x-y)=\sum_{k=1}^{n} u_k(x)v_k(y)$$

considered by Rukhin in [9,19] on 2-divisible Abelian groups and Penney and Rukhin in [8] on some classes of noncommutative groups, or the equation

$$\sum_{i=1}^{n} g_i(y)f_i(xy)=\sum_{k=1}^{n} u_k(x)v_k(y), \tag{1.2}$$

for functions on arbitrary groups by the second author [12]. A reader can find many information on this subject in the book of Stetkaer [15].

In [2] Almira and the second author considered the following class of equations, restricting by continuous functions on $\mathbb{R}^d$:

$$\sum_{j=1}^{m} f_j(a_jx+c_jy)=\sum_{k=1}^{n} u_k(x)v_k(y) \text{ for all } x,y\in\mathbb{R}^d, \tag{1.3}$$

where a_j, c_j are $d\times d$-matrices. This class includes various extensions of (linearized) Skitovich-Darmois equation and other equations related to the study of probability distributions (see Feldman's [4]). It was proved in [2] that if all matrices a_j, b_j, for $1 \leqslant j \leqslant m$, and matrices $d_{i,j} := a_ic_j - a_jc$, for $i \neq j$, are invertible then each function f_j in (1.3) is an exponential polynomial. The extension of this statement to commutative hypergroups were obtained by Fechner and Székelyhidi in [3].

Let us also note that a characterization of exponential polynomials was provided by Gajda in [5]. The author introduced a class of functional equations whose solutions, in an arbitrary Abelian group, were exactly of the form $f = \sum_i m_ip_i$. Here m_i are exponential functions and p_i are polynomials.

Here we make one more step adding polynomial coefficients to summands in (1.3):

$$\sum_{j=1}^{M} P_j(x,y)f_j(a_jx+c_jy)=\sum_{k=1}^{n} u_k(x)v_k(y). \tag{1.4}$$

It should be noted that a much more general equation for functions on $\mathbb{R}$ was considered by Światak [20]:

$$\sum_{j=1}^{M} \phi_j(x,y) f_j(x + c_j(y)) = b(x,y), \tag{1.5}$$

where ϕ_j, c_j and b are continuous functions. In so general setting the solutions hardly can be found in an exact form; in [20] the results were obtained about their analytical properties. On the other hand Laczkovich [6] studied the equation

$$\sum_{j=1}^{M} \phi_j(y) f_j(x + c_j(y)) = b(y), \tag{1.6}$$

and proved that under some mild restrictions on functions ϕ_j, c_j and b the solutions f_j are exponential polynomials. Unexpectedly, the Eq. (1.4), possessing at the first glance much less degree of freedom, has a more wide class of solutions: they are the combinations of exponentials with *rational* coefficients.

To state the result in the exact form, let us denote by $M_d(\mathbb{R})$ the algebra of all $d \times d$-matrices with real entries, and by Inv the group of all invertible matrices in $M_d(\mathbb{R})$.

Theorem 1.1. *Let* $a_1, \dots, a_M, c_1, \dots, c_M \in M_d(\mathbb{R})$ *satisfy the conditions*

$$a_i, c_i \in Inv \text{ and } a_i^{-1}c_i - a_j^{-1}c_j \in Inv, \text{ for } i \neq j. \tag{1.7}$$

If continuous functions $f_j : \mathbb{R}^d \to \mathbb{C}$ *satisfy the functional Eq.* (1.4)

$$\sum_{j=1}^{M} f_j(a_j x + c_j y) P_j(x,y) = \sum_{k=1}^{n} u_k(x) v_k(y),$$

with some polynomials P_j *and some continuous functions* u_k, v_k*, then each* f_j *is a ratio of an exponential polynomial and a polynomial. In other words*

$$f_j(x) = \sum_{m=1}^{n} e^{\langle \lambda_m, x\rangle} r_m(x),$$

where all r_m *are rational functions.*

In fact we prove a more general result which can be considered as a version of Theorem 1.1 for vector-valued functions. Let us say that scalar-valued functions $\varphi_1, \dots, \varphi_N$ on $\mathbb{R}^d$ are *polynomially dependent up to an exponential polynomial*, if there exists a non-trivial (= non-zero) N-tuple of polynomials $p_1, \dots, p_N$ such that $p_1\varphi_1 + \cdots + p_N\varphi_N$ is an exponential polynomial.

Theorem 1.2. *Let continuous functions f_{ji} $(1 \leqslant j \leqslant n, 1 \leqslant i \leqslant r_j)$ on $\mathbb{R}^d$ satisfy the equation*

$$\sum_{i=1}^{r_1} f_{1i}(a_1x + c_1y)P_{1i}(x,y) + \cdots + \sum_{i=1}^{r_n} f_{ni}(a_nx + c_ny)P_{ni}(x,y) = \sum_{j=1}^{M} u_j(x)v_j(y), \tag{1.8}$$

where P_{ji} are polynomials on $\mathbb{R}^d \times \mathbb{R}^d$, and the matrices a_i, c_i satisfy (1.7).

Then each family $f_{k1}, \ldots, f_{kr_k}$ is polynomially dependent up to an exponential polynomial.

It is clear that Theorem 1.1 is a direct corollary of Theorem 1.2: a family that consists of one function f is polynomially dependent up to an exponential polynomial if and only if f is a ratio of an exponential polynomial and a polynomial. It should be added that apart of generality (in fact because of it) Theorem 1.2 is better adapted for the proof than Theorem 1.1.

2. Notations and Preliminary Results

We consider functions defined on $\mathbb{R}^d$, so symbols $x, y, t, \ldots$ denote elements of $\mathbb{R}^d$.

By capital letters $P, Q, F, \ldots$ we mostly denote the functions on $\mathbb{R}^d \times \mathbb{R}^d$ while for functions on $\mathbb{R}^d$ small letters are used.

Let (EP) be the set of all exponential polynomials on $\mathbb{R}^d$. Let also Λ denote the space of all finite sums of products $u(x)v(y)$, where $u, v \in C(\mathbb{R}^d)$.

For $t \in \mathbb{R}^d$, we denote by R_t the shift-operator on $C(\mathbb{R}^d)$:

$$R_tf(x) = f(x+t),$$

and set

$$\Delta_tf = R_tf - f.$$

Furthermore, with any pair $(a,t) \in M_d(\mathbb{R}) \times \mathbb{R}^d$, we relate an operator $\delta_{a,t}$ on $C(\mathbb{R}^d \times \mathbb{R}^d)$ as follows:

$$\delta_{a,t}w(x,y) = w(x+at, y-t) - w(x,y).$$

Then it is easy to check that for arbitrary function $g \in C(\mathbb{R}^d)$ and a polynomial q, one has

$$\delta_{a,t}(g(x+ay)q(y)) = g(x+ay)\Delta_{-t}q(y) \tag{2.1}$$

and, more generally,

$$\delta_{a,t}(g(x+by)q(y)) = g(x+by) \cdot \Delta_{-t}q(y) + (\Delta_{(a-b)t}g)(x+by) \cdot R_{-t}q(y). \tag{2.2}$$

Indeed

$$\delta_{a,t}(g(x+by)q(y)) = g(x+at+b(y-t))q(y-t) - g(x+by)q(y)$$
$$= g(x+at+b(y-t))q(y-t) - g(x+by)q(y-t)$$
$$+g(x+by)(q(y-t)-q(y)) = (g(x+by+(a-b)t) - g(x+by))q(y-t)$$
$$+g(x+by)(q(y-t)-q(y)) = (\Delta_{(a-b)t}g)(x+by)\cdot q(y-t)$$
$$+g(x+by)\cdot\Delta_{-t}q(y).$$

In particular,

$$\delta_{a,t}(g(x+by)) = (\Delta_{(a-b)t}g)(x+by). \tag{2.3}$$

We will need the following result which is a special case of [11, Lemma 2]:

Proposition 2.1. *Let L be a finite-dimensional subspace of $C(\mathbb{R}^d)$ and let $f \in C(\mathbb{R}^d)$. Suppose that for any $y \in \mathbb{R}^d$, there is a finite-dimensional shift-invariant subspace $L(y) \subset C(\mathbb{R}^d)$ with*

$$R_y f \in L + L(y). \tag{2.4}$$

Then $f \in (EP)$.

Corollary 2.1. *If, for some $n \in \mathbb{N}$, a continuous function $f : \mathbb{R}^d \to \mathbb{C}$, satisfies the condition*

$$\Delta_{t_n}\cdots\Delta_{t_1}f(x) \in (EP) \text{ for all } t_1,\ldots,t_n \in \mathbb{R}^d, \tag{2.5}$$

then $f \in (EP)$.

Proof. Fixing $t_1,\ldots,t_{n-1}$ and denoting $\Delta_{t_{n-1}}\cdots\Delta_{t_1}f$ by g, we obtain the following relation for g:

$$g(x+t_n) - g(x) \in (EP), \tag{2.6}$$

for all $t_n \in \mathbb{R}^d$.

Since each exponential polynomial belongs to a finite-dimensional shift-invariant subspace we obtain

$$R_t g \in \mathbb{C}g + L(t), \tag{2.7}$$

for all $t \in \mathbb{R}^d$, where $L(t)$ is a finite-dimensional shift-invariant subspace. Applying Proposition 2.1 we conclude that g is an exponential polynomial. Thus

$$\Delta_{t_{n-1}}\cdots\Delta_{t_1}f(x) \in (EP), \text{ for all } t_1,\ldots,t_{n-1} \in \mathbb{R}^d.$$

Repeating these arguments $n-1$ times we get that $f \in (EP)$. □

Corollary 2.2. *Let $f \in C(\mathbb{R}^d)$ and let $c \in Inv$. Let $a_1,\ldots,a_n \in M_d(\mathbb{R})$ satisfy conditions $a_i - c \in Inv$. If, for any $t_1,\ldots,t_n$,*

$$\delta_{a_n,t_n}\cdots\delta_{a_1,t_1}(f(x+cy)) \in \Lambda, \tag{2.8}$$

then $f \in (EP)$.

Proof. By (2.3), the previous condition can be written in the form

$$\Delta_{(a_n-c)t_n}\cdots\Delta_{(a_1-c)t_1}\ f(x+cy)\in\Lambda. \tag{2.9}$$

Setting

$$g=\Delta_{(a_n-c)t_n}\cdots\Delta_{(a_1-c)t_1}\ f$$

we obtain the equation

$$g(x+cy)=\sum_{j=1}^{M}u_j(x)v_j(y). \tag{2.10}$$

After the change $y=c^{-1}z$ we see that (2.10) is the Levi-Cività functional equation for the function g. Therefore, g is an exponential polynomial and (2.9) turns to

$$\Delta_{(a_n-c)t_n}\cdots\Delta_{(a_1-c)t_1}f\in(EP)\qquad\text{for all } t_1,\ldots,t_n\in\mathbb{R}^d. \tag{2.11}$$

Since all a_i-c are invertible, one can find, for each n-tuple $h_1,\ldots,h_n\in\mathbb{R}^d$, an n-tuple $t_1,\ldots,t_n$ with $h_i=(a_i-c)t_i$. Thus

$$\Delta_{h_n}\cdots\Delta_{h_1}f(x)\in(EP),\qquad\text{for all } h_1,\ldots,h_n\in\mathbb{R}^d.$$

It follows from Corollary 2.1 that f is an exponential polynomial. □

Lemma 2.1. *For every polynomial $q(x)$, there is an $h\in\mathbb{R}^d$ such that*

$$\deg(\Delta_h(q))=\deg(q)-1.$$

Proof. Aiming at the contrary suppose that $\deg(\Delta_h(q))<n-1$, for each h, where $n=\deg(q)$. Let W be a differential operator of order $n-1$ with constant coefficients, then $\Delta_h(W(q))=W(\Delta_h(q))=0$. Since h is arbitrary, $W(q)=const$. Since W is arbitrary $\deg(q)\le n-1$, a contradiction. □

3. The Proof of the Main Results

It will be convenient to deal firstly with a special case of the Eq. (1.8):

$$\sum_{i=1}^{r_1}f_{1i}(x+b_1y)P_{1i}(x,y)+\ldots+\sum_{i=1}^{r_n}f_{ni}(x+b_ny)P_{ni}(x,y)$$
$$=\sum_{j=1}^{M}u_j(x)v_j(y) \tag{3.1}$$

We will prove the following result.

Theorem 3.1. *Let $b_1,\ldots,b_n\in Inv$ and $b_k-b_j\in Inv$ when $k\neq j$. Continuous functions f_{ik} on $\mathbb{R}^d$ satisfy the relation (3.1) with some non-trivial families $(P_{ki})_{i=1}^{r_k}$, $k=1,\ldots,n$, of polynomials on $\mathbb{R}^d\times\mathbb{R}^d$ and some continuous functions u_j,v_j on $\mathbb{R}^d$ if and only if, for every k, the family $f_{k1},\ldots,f_{kr_k}$ is polynomially dependent up to an exponential polynomial.*

Proof. Part "if" is easy: if, for example, $\sum_{i=1}^{m} f_{1i}(x)p_i(x) = e(x)$, where $(p_i)_{i=1}^{m}$ is a nontrivial family of polynomials and $e \in (EP)$, then

$$\sum_{i=1}^{m} f_{1i}(x+b_1y)p_i(x+b_1y) = e(x+b_1y).$$

Since $e(x+b_1y) \in \Lambda$ and $p_i(x+b_1y)$ are polynomials on $\mathbb{R}^d \times \mathbb{R}^d$, we obtain the needed relation. So it remains to prove the implication "only if" .

For any $a \in M_d(\mathbb{R})$, we denote by $\mathcal{L}_a$ the polynomial hull of the set of all functions of the form $f(x+ay)$, where $f \in C(\mathbb{R}^d)$, that is the space of all finite sums $\sum_i f_i(x+ay)Q_i(x,y)$, where $f_i \in C(\mathbb{R}^d)$ and Q_i are polynomials on $\mathbb{R}^d \times \mathbb{R}^d$.

Let $\mathcal{M}(d)$ denote the set of all monomials $\mu(x) = x_1^{m_1} \dots x_d^{m_d}$ in d variables. Note that each polynomial $Q(x,y)$ can be uniquely written in the form

$$Q(x,y) = \sum_{\mu \in \mathcal{M}(d)} \mu(x)p_\mu(y),$$

where all p_μ are polynomials. Making the change $x = z - ay$ we write $Q(x,y) = Q(z-ay,y)$ as a polynomial $S(z,y)$; decomposing this polynomial as above:

$$S(z,y) = \sum_{\mu \in \mathcal{M}(d)} \mu(z)q_\mu(y),$$

and returning to the initial variables we obtain

$$Q(x,y) = \sum_{\mu \in \mathcal{M}(d)} \mu(x+ay)q_\mu(y),$$

where all q_μ are polynomials.

Thus each function $F = \sum_i f_i(x+ay)Q_i(x,y) \in \mathcal{L}_a$ can be written in the form $\sum_k g_k(x+ay)q_k(y)$, where $g_k \in C(\mathbb{R}^d)$, q_k are polynomials.

This implies two important facts:

(i) Each subspace $\mathcal{L}_a$ of $C(\mathbb{R}^d \times \mathbb{R}^d)$ is invariant under all operators $\delta_{b,t}$.

The proof follows immediately from the equality (2.2) applied to functions $g_k(x+ay)q_k(y)$.

(ii) The family of all operators $\delta_{a,t}$, $t \in \mathbb{R}^d$, is locally nilpotent. In other words if $F \in \mathcal{L}_a$ then there is $N \in \mathbb{N}$ such that subsequently applying to F any N operators δ_{a,t_i} we obtain 0.

This follows from the equality (2.1) applied to functions $g_k(x+ay)q_k(y)$, if one takes in the account the inequality $\deg(\Delta_t q) < \deg(q)$.

Let us denote the minimal value of N by $N_a(F)$ and call it *the order* of F.

As we already mentioned, all operators $\delta_{b,t}$ preserve each subspace $\mathcal{L}_a$. It should be added that they do not increase the orders of functions in $\mathcal{L}_a$:

$$N_a(\delta_{b,t}(F)) \leqslant N_a(F), \text{ for each } F \in \mathcal{L}_a. \tag{3.2}$$

The reason is that all operators $\delta_{b,t}$ commute with $\delta_{a,s}$, whence

$$\delta_{a,t_1}\delta_{a,t_2}\dots\delta_{a,t_N}(\delta_{b,t}(F)) = \delta_{b,t}\delta_{a,t_1}\delta_{a,t_2}\dots\delta_{a,t_N}(F) = 0,$$

if $N = N_a(F)$.

Note also that the space Λ is invariant for all operators $\delta_{b,t}$.

Now we will prove the statement of Theorem 3.1 for the first collection of functions, that is we show that some non-trivial combination of functions f_{1i}, $1 \leqslant i \leqslant r_1$, with polynomial coefficients belongs to (EP). For this let us write (3.1) in the form

$$\sum_{i=1}^{r_1} f_{1i}(x+b_1y)P_{1i}(x,y) = F_2 + \dots + F_n + \Pi, \tag{3.3}$$

where each F_j — some function in $\mathcal{L}_{b_j}$, Π is a function in Λ (the notations are general, we will not change them when F_j and Π change).

As above $P_{1i}(x,y) = \sum_{\mu\in\mathcal{M}(d)} \mu(x+b_1y)q_{i\mu}(y)$, so (3.3) can be written as follows

$$\sum_{i=1}^{r_1}\sum_{\mu\in\mathcal{M}(d)} \mu(x+b_1y)q_{i\mu}(y)f_{1i}(x+b_1y) = F_2 + \dots + F_n + \Pi.$$

Let $q_{i_0\mu_0}(y)$ be one of those polynomials $q_{i\mu}$ that have the maximal degree: $deg(q_{i_0\mu_0}) = D \geqslant deg(q_{i\mu})$, for all i,μ. Using Lemma 2.1, choose $h_1,\dots,h_D$ in such a way that each subsequent application of $\Delta_{h_1},\Delta_{h_2},\dots,\Delta_{h_D}$ to $q_{i_0\mu_0}$ reduce the degree of the polynomial by 1. Thus

$$\Delta_{h_D}\dots\Delta_{h_1}q_{i_0\mu_0} = C_1 \neq 0.$$

Setting $\Delta = \Delta_{h_D}\dots\Delta_{h_1}$, we see that $\Delta q_{i\mu} = C_{i\mu}$, where $C_{i\mu}\in\mathbb{C}$ (they can be zero), for all i,μ.

Let now $\delta = \delta_{b_1,h_D}\dots\delta_{b_1,h_1}$; applying this operator to both parts of (3.3) and using (2.1), we obtain

$$\sum_{i=1}^{r_1}\sum_{\mu\in\mathcal{M}(d)} \mu(x+b_1y)C_{i\mu}f_{1i}(x+b_1y) = F_2 + \dots + F_n + \Pi.$$

The function

$$\psi(x) = \sum_{i=1}^{r_1}\sum_{\mu\in\mathcal{M}(d)} C_{i\mu}f_{1i}(x)\mu(x)$$

is a non-trivial polynomial combination of functions $f_{1i}(x)$. Indeed

$$\psi(x) = \sum_i f_{1i}(x)k_i(x)$$

where $k_i(x) = \sum_\mu C_{i\mu}\mu(x)$ are polynomials. Since $C_{i_0\mu_0}\neq 0$ the polynomial k_{i_0} is non-zero.

Now starting with the equality

$$\psi(x+b_1y)=F_2+\ldots+F_n+\Pi,$$

we will show that $\psi\in(EP)$.

Let $K_j=N_{b_j}(F_j)$, for $j=2,3,\ldots,n$. By definition,

$$\delta_{b_2,t_1}\delta_{b_2,t_2}\ldots\delta_{b_2,t_{K_2}}F_2=0,$$

for all $t_i\in\mathbb{R}^d$.

Since all subspaces $\mathcal{L}_{b_j}$ are invariant with respect to all operators δ_{b_2,t_1}, the operators $\delta_{b_2,t_1}\delta_{b_2,t_2}\ldots\delta_{b_2,t_{K_2}}$ transform all functions F_j, $j\neq 2$ to other functions (again F_j by our agreement) in $\mathcal{L}_{b_j}$ without increasing their orders (see (3.2)). Thus we have now:

$$\delta_{b_2,t_1}\delta_{b_2,t_2}\ldots\delta_{b_2,t_{K_2}}\psi(x+b_1y)=F_3+\ldots+F_n+\Pi.$$

Repeating the same trick we will reduce the number of F_j till in the right hand side remains only the summand Π. In other words, choosing $a_i\in M_d(\mathbb{R})$ as follows:

$$a_1=a_2=\ldots=a_{K_2}=b_2, a_{K_2+1}=\ldots=a_{K_2+K_3}=b_3,\ \text{and so on},$$

we obtain:

$$\delta_{a_1,t_1}\delta_{a_2,t_2}\ldots\delta_{a_K,t_K}\psi(x+b_1y)\in\Lambda,$$

for all $t_i\in\mathbb{R}^d$, $1\leqslant i\leqslant K$, where $K=\sum_{j=2}^n K_j$. Since $a_i-b_1\in Inv$, for all i, it follows from Corollary 2.2 that $\psi\in(EP)$. □

To deduce Theorem 1.2 from Theorem 3.1 it suffices to denote $a_j^{-1}c_j$ by b_j and set $g_{ji}(x)=f_{ji}(a_jx)$. Clearly if functions (f_{ij}) satisfy (1.8) then functions (g_{ij}) satisfy (3.1). By Theorem 3.1, all families $\{g_{ij}:1\leqslant i\leqslant r_j\}$, $j=1,\ldots,n$, are polynomially dependent up to an exponential polynomial, so the same is true for the families $\{f_{ij}:1\leqslant i\leqslant r_j\}$.

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Maciej Sablik and Ekaterina Shulman
Institute of Mathematics
University of Silesia
Bankowa 14
40 007 Katowice
Poland
e-mail: maciej.sablik@us.edu.pl;
ekaterina.shulman@us.edu.pl, shulmanka@gmail.com

Ekaterina Shulman
Vologda State University
6S. Orlova Str.
Vologda
Russia 160035

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