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Results in Mathematics

Periodic Boundary Value Problems for Fractional Dynamic Equations on Time Scales

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Abstract. The manuscript is concerned with the existence, uniqueness, and Ulam stability of solutions of a nonlinear fractional dynamic equation involving Caputo fractional nabla derivative with the periodic boundary conditions on time scales. Based on the fixed point theory, first, we investigate the existence of a solution and then employing dynamic inequality the uniqueness result is obtained. Next, we present several results on Ulam stability. An appropriate example has been given to demonstrate the implementation of theoretical results.

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1. Introduction

The dynamic equation is used to solve the differential and difference equations together in one domain, called time scale. This new and assertive field is more general and versatile than the traditional theories of differential and difference equations, and hence it is an optimal way for accurate and malleable mathematical modeling. Traditionally, researchers have assumed that dynamical processes are either continuous or discrete and thus employed either differential or difference equations to demonstrate the mathematical description of the dynamic model. There are certain important phenomena that do not possess only continuous or only discrete data but rather hybrid data. A simple example of this type can be seen in seasonally breeding populations which leads to non overlapping generations. In such cases, solutions of dynamic equations can give the required data of the dynamic model under consideration. The study of the existence and uniqueness of solutions of various dynamic equations involving initial and boundary conditions can be seen in [1,3,15,17,29–32,34] and the references therein.

The discussion on the stability of solutions is one of the most important properties among various qualitative properties of solutions. In the existing literature, there are several stability theories, for both differential and difference equations, for instance one can see [6–10]. But the concept of Ulam stability has significant applications in various fields of mathematical analysis, this is because Ulam Stability essentially deals with the existence of an exact solution near every approximate solution and is useful in the situation when it is difficult to find the exact solution. This kind of stability for functional equations was first discussed by Ulam [33] in 1940 and Hyers [22] in 1941. Very recently, Ulam stability for differential, difference, and integral equations has been seriously studied by many researchers employing several techniques. For convenience, readers can see [11,13,15,26–28] and the references therein.

Kumar and Malik [23], by employing Banach fixed point theorem, studied the existence and stability of solutions of fractional integro-differential equations on time scales involving non-instantaneous integrable impulses and periodic boundary conditions. Further, the same authors have established the results of the existence, stability, and controllability of solutions of fractional dynamic systems on time scales [24] by using Banach fixed point theorem and the nonlinear alternative of Leray–Schauder. Applications of these results to population dynamics are also given in the same paper.

Quite recently, Bohner and Tikare [16] obtained results of Ulam stability for first-order nonlinear dynamic equations on time scales by employing the method of Picard's operator and dynamic inequalities. In [20], Gogoi et al. introduced a new approach for nabla-type fractional derivatives and integrals on the time scale domain.

In view of the usefulness of the existence, uniqueness, and Ulam stability of solutions of dynamic equations, we are motivated to study the following periodic boundary value problem for fractional dynamic equations (PBVP).

$$\begin{cases} {}^{C}\mathcal{D}^{\gamma}h(\zeta) = \mathscr{X}(\zeta, h(\zeta), {}^{C}\mathcal{D}^{\gamma}h(\zeta)), \\ h(0) = h(T) = 0, \quad T \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\zeta \in \mathcal{J} = [0, T] \cap \mathbb{T}$, $T > 0$ and $\mathscr{X} : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a ld-continuous function in its first variable and other terms are specified in Sect. 2.

This manuscript is organized as follows: In Sect. 2, we highlight the preliminaries of fractional dynamic equations on time scales. The existence and uniqueness of solutions of (1.1) is established in Sect. 3. Section 4 includes results of four types of Ulam stability of (1.1). In Sect. 5, we have presented an example for implementing all the theoretical results of the paper. Finally, the conclusion of the paper is given in Sect. 6.

2. Preliminaries

A time scale $\mathbb{T}$ is a nonempty closed subset of the set of real numbers $\mathbb{R}$ inherited from the standard topology of $\mathbb{R}$. For $\zeta \in \mathbb{T}$, we define the backward jump operator is defined as $\rho(\zeta) := \sup\{\xi \in \mathbb{T} \colon \xi < \zeta\}$. If $\mathbb{T}$ has a minimum element ζ, then we define $\rho(\zeta) = \zeta$, in this sense, we get $\inf \mathbb{T} = \sup \emptyset$. With the help of operator ρ, we classify a point $\zeta \in \mathbb{T}$ as left-scattered if $\rho(\zeta) < \zeta$ and as left-dense if $\rho(\zeta) = \zeta$. Further, the backward graininess function $\nu \colon \mathbb{T} \to [0, \infty)$ is defined by $\nu(\zeta) := \zeta - \rho(\zeta)$. We derive a new set from $\mathbb{T}$, denoted by $\mathbb{T}_\kappa$ as follows: If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_\kappa = \mathbb{T} \setminus \{m\}$. Otherwise, $\mathbb{T}_\kappa = \mathbb{T}$. This set $\mathbb{T}_\kappa$ is needed while defining ∇-derivative. The following definition is taken from [12].

Definition 2.1 (Nabla derivative). Let $h \colon \mathbb{T} \to \mathbb{R}$ be a function and $\zeta \in \mathbb{T}_\kappa$. Then we define the nabla derivative of h at the point ζ to be the number $h_\nabla(\zeta)$ (provided it exists) with the property that for each $\varepsilon > 0$ there is a neighborhood U of ζ such that

$$|h(\rho(\zeta)) - h(\theta) - h_\nabla(\zeta)[\rho(\zeta) - \theta]| \leq \varepsilon|\rho(\zeta) - \theta| \text{ for all } \theta \in U.$$

In this case, the function h is said to be nabla differentiable at $\zeta \in \mathbb{T}_\kappa$.

Theorem 2.2 [14, Theorem 8.39]. *Assume $h \colon \mathbb{T} \to \mathbb{R}$ is a function and let $\zeta \in \mathbb{T}_\kappa$. Then we have the following*:

(i) If h is continuous at ζ and ζ is left-scattered, then h is nabla differentiable at ζ with

$$h_\nabla(\zeta) = \frac{h(\zeta) - h(\rho(\zeta))}{t - \rho(t)}.$$

(ii) If ζ is left-dense, then h is nabla differentiable at ζ if and only if the limit

$$\lim_{\theta \to \zeta} \frac{h(\zeta) - h(\theta)}{\zeta - \theta}$$

exists as a finite number. In this case

$$h_\nabla(\zeta) = \lim_{\theta \to \zeta} \frac{h(\zeta) - h(\theta)}{\zeta - \theta}.$$

Definition 2.3 [14, Definition 8.43]. A function $g \colon \mathcal{J} \to \mathbb{R}$ is said to be ld-continuous if for all left-dense point of $\mathcal{J}$, g is continuous, and its right sided limit exists at all right-dense points of $\mathcal{J}$.

The symbol $\mathcal{L}(\mathcal{J},\mathbb{R})$ is used to denote the set of all ld-continuous functions from $\mathcal{J}$ to $\mathbb{R}$. We note that the set $\mathcal{L}=\mathcal{L}(\mathcal{J},\mathbb{R})$ forms a Banach space when coupled with the supremum norm

$$\|g\| := \sup_{\zeta\in\mathcal{J}} |g(\zeta)|, \quad g\in\mathcal{L}. \tag{2.1}$$

Definition 2.4 [14, Theorem 8.47]. Let $g\colon \mathcal{J}\to\mathbb{R}$ be a ∇-integrable function. Then, for any $\zeta\in\mathcal{J}$, we have

$$\int_0^T g(\theta)\nabla\theta = \int_0^\zeta g(\theta)\nabla\theta + \int_\zeta^T g(\theta)\nabla\theta.$$

Remark 2.5. [5] Consider the coordinate-wise ld-continuous function $h_\gamma\colon \mathbb{T}\times\mathbb{T}\to\mathbb{R}$ for $\gamma\geq 0$, such that $h_0(\zeta,\zeta_0)=1$ and

$$h_{\gamma+1}(\zeta,\zeta_0) = \int_{\zeta_0}^\zeta h_\gamma(\theta,\zeta_0)\nabla\theta \text{ for all } \zeta,\zeta_0\in\mathbb{T}. \tag{2.2}$$

Furthermore, for $\alpha,\gamma>1$, one can obtain

$$\int_{\rho(\eta)}^\zeta h_{\alpha-1}(\zeta,\rho(\theta))h_{\gamma-1}(\theta,\rho(\eta))\nabla\theta = h_{\alpha+\gamma-1}(\zeta,\rho(\eta)) \tag{2.3}$$

for all $\zeta,\eta\in\mathbb{T}$ with $\eta\leq\zeta$.

Definition 2.6 [5]. Let $g\in\mathcal{L}(\mathbb{T}_\kappa,\mathbb{R})$ be such that it is Lebesgue ∇- integrable on $\mathbb{T}$. Then for $0<\gamma\leq 1$, the fractional ∇-integral (in the sense of Riemann–Liouville) is defined as

$$\mathcal{I}^\gamma_{\zeta_0} g(\zeta) := \int_{\zeta_0}^\zeta h_{\gamma-1}(\zeta,\rho(\theta))g(\theta)\nabla\theta, \quad \zeta\in U, \tag{2.4}$$

where U is a neighborhood of ζ such that $U\subset\mathbb{T}$.
Note that $\mathcal{I}^0 g(\zeta)=g(\zeta)$.

Remark 2.7. The ∇-power function $h_{\gamma-1}(\zeta,\rho(\theta))$ is different for different time scale $\mathbb{T}$.
For $\mathbb{T}=\mathbb{R}$, we have $\rho(\theta)=\theta$ and $h_{\gamma-1}(\zeta,\rho(\theta))=\frac{(\zeta-\theta)^{\gamma-1}}{\Gamma(\gamma)}$. In this case, (2.4) becomes

$$\mathcal{I}^\gamma_{\zeta_0} g(\zeta) = \int_{\zeta_0}^\zeta \frac{(\zeta-\theta)^{\gamma-1}}{\Gamma(\gamma)} g(\theta)d\theta.$$

For $\mathbb{T} = \mathbb{Z}$, we have $\rho(\theta) = \theta - 1$ and $h_{\gamma-1}(\zeta, \rho(\theta)) = \frac{(\zeta-\rho(\theta))^{\overline{\gamma-1}}}{\Gamma(\gamma)} = \binom{\zeta-\rho(\theta)}{\gamma}$, where for any $\gamma \in \mathbb{R}$, $\zeta^{\overline{\gamma}} = \frac{\Gamma(\zeta+\gamma)}{\Gamma(\zeta)}$. From (2.4), we get

$$\begin{aligned}\mathcal{I}_{0+}^{\gamma} g(\zeta) &= \int_0^{\zeta} h_{\gamma-1}(\zeta, \rho(\theta)) g(\zeta) \nabla\theta \\ &= \frac{1}{\Gamma(\gamma)} \int_0^{\zeta} (\zeta - \rho(\theta))^{\overline{\gamma-1}} g(\theta) \nabla\theta \\ &= \frac{1}{\Gamma(\gamma)} \sum_{\theta=0}^{\zeta-1} (\zeta - (\theta-1))^{\overline{\gamma-1}} g(\theta).\end{aligned}$$

For $\mathbb{T} = q^{\mathbb{N}_0}$, we have $h_{\gamma-1}(\zeta, \rho(\theta)) = \Gamma_q(\gamma) \frac{q^{\gamma}-1}{q-1} (\zeta - q\theta)_q^{\gamma-1}$, where Γ_q is a q-gamma function.

Next, based on the definition given in [18], we define the following:

Definition 2.8 (Riemann–Liouville fractional ∇-derivative). Let $g\colon \mathbb{T}_{\kappa^m} \to \mathbb{R}$ be an ld-continuous function. Then the Riemann–Liouville fractional ∇-derivative of order $\gamma \in \mathbb{R}$ is defined as follows.

$$\mathcal{D}_{0+}^{\gamma} g(\zeta) = \mathcal{D}_{0+}^{m} \mathcal{I}_{0+}^{m-\gamma} g(\zeta), \quad \zeta \in \mathcal{J},\ \gamma \geq 0,$$

where $m \in \mathbb{N}_0, m := [\gamma] + 1$. The set $\mathbb{T}_{\kappa^m}$ is obtained by cutting off m number of right-scattered minimum left end points of $\mathbb{T}$.

Definition 2.9 (Caputo fractional ∇-derivative). Let $g\colon \mathbb{T}_{\kappa^m} \to \mathbb{R}$ be an ld-continuous function such that $\nabla^m g$ exists in $\mathbb{T}_{\kappa^m}$ for $m \in \mathbb{N}_0$. Then the Caputo fractional ∇-derivative of g is defined as

$${}^{C}\mathcal{D}_{0+}^{\gamma} g(\zeta) := \int_0^{\zeta} h_{m-\gamma}(\zeta, \rho(\zeta)) \nabla^m (g(\theta)) \nabla\theta, \quad \zeta \in \mathcal{J},\ \gamma \geq 0.$$

Remark 2.10. From Definition 2.8, we see that ${}^{C}\mathcal{D}_{0+}^{\gamma} g(\zeta) = \mathcal{I}_{0+}^{m-\gamma}\ \mathcal{D}_{0+}^{\gamma}$, where $m = [\gamma] + 1$.

Theorem 2.11 [2]. *A subset $\mathcal{D}$ of $\mathrm{C}(\mathbb{T}, \mathbb{R})$ is relatively compact if and only if it is bounded and equicontinuous simultaneously, where $\mathrm{C}(\mathbb{T}, \mathbb{R})$ is the set of all continuous functions defined on $\mathbb{T}$ and taking values in $\mathbb{R}$.*

Definition 2.12 [21]. Let X and Y be two Banach spaces. A mapping $\mathscr{G}\colon X \to Y$ is completely continuous, if for a bounded subset $\mathcal{B} \subseteq X$, $\mathscr{G}(\mathcal{B})$ is relatively compact in Y.

The following Proposition is proved in [19, Proposition 3.2].

Proposition 2.13. *For $g \in \mathcal{L}([0,T]_{\mathbb{T}}, \mathbb{R})$, if $\widetilde{g}$ is an extension of g to the real line interval $[0,T]$ such that*

$$\widetilde{g}(\zeta) := \begin{cases} g(\zeta) & \text{if } \zeta \in \mathbb{T}, \\ g(\theta) & \text{if } \zeta \in (\rho(\theta), \theta) \notin \mathbb{T}, \end{cases}$$

then we get

$$\int_0^T g(\zeta)\nabla\zeta \leq \int_0^T \widetilde{g}(\zeta)d\zeta.$$

Theorem 2.14 (Krasnoselskii fixed point theorem) [25, Theorem 11.2]. *Let C be a nonempty closed, convex subset of a Banach space B. Suppose that $\mathcal{F}_1, \mathcal{F}_2 : C \to B$ be such that*

1. *$\mathcal{F}_1$ is contraction.*
2. *$\mathcal{F}_2$ is continuous and $\mathcal{F}_1(C)$ is relatively compact.*
3. *$\mathcal{F}_1[\zeta] + \mathcal{F}_2[\eta] \in C$ for all $\zeta, \eta \in C$,*

Then there is a $\overline{\zeta} \in C$ such that $\mathcal{F}_1[\overline{\zeta}] + \mathcal{F}_2[\overline{\zeta}] = \overline{\zeta}$.

Below, we state ∇-dynamic inequality which is used in proving uniqueness of solution. This inequality is proved in [4, Corollary 3.2] for the delta case.

Theorem 2.15 *Let $g, p, h \in \mathcal{L}(\mathcal{J}, \mathbb{R})$ and p, h are two non negative functions. Then*

$$g(\zeta) \leq p(\zeta) \int_{\zeta_0}^{\zeta} g(\tau)h(\tau)\nabla\tau \text{ for all } \zeta \in \mathbb{T}_\kappa$$

implies

$$g(\zeta) \leq 0, \text{ for all } \zeta \in \mathbb{T}_\kappa.$$

3. Existence and Uniqueness Results

Definition 3.1 A function $g \in \mathcal{L} \cap L_\nabla(\mathcal{J}, \mathbb{R})$ is a solution of PBVP (1.1) if and only if $g(\zeta) \geq 0$, $\zeta \in \mathcal{J}$, and g satisfies equation and conditions in (1.1), where $L_\nabla(\mathcal{J}, \mathbb{R})$ is a class of Lebesgue ∇-integrable function from $\mathcal{J}$ to $\mathbb{R}$.

The following lemma allow us to transform the PBVP (1.1) into an integral equation, which is key to apply fixed point theory.

Lemma 3.2 *Let $1 < \gamma < 2$. Then, $g \in \mathcal{L} \cap L_\nabla(\mathcal{J}, \mathbb{R})$ is a solution of PBVP (1.1), if and only if g is a solution of the following integral equation*

$$g(\zeta) = \int_0^T G(\zeta, \theta)\mathscr{Z}(\theta, g(\theta), {}^C\mathcal{D}^\gamma g(\theta))\nabla\theta, \tag{3.1}$$

where $G(\zeta, \theta)$ is Green function defined by

$$G(\zeta, \theta) = \begin{cases} \frac{T h_{\gamma-1}(\zeta, \rho(\theta))}{T} + \frac{-\zeta h_{\gamma-1}(T, \rho(\theta))}{T} & \text{if } 0 \leq \theta < \zeta, \\ \frac{-\zeta h_{\gamma-1}(T, \rho(\theta))}{T} & \text{if } \zeta \leq \theta < T. \end{cases} \tag{3.2}$$

Proof For $1 < \gamma < 2$, in view of Definition 2.9, we have

$$ {}^{C}\mathcal{D}^{\gamma} g(\zeta) = \mathcal{I}^{2-\gamma} g^2_{\nabla}(\zeta), \quad \zeta \in \mathcal{J}. $$

Next, from Lemma 2.7 [34], we obtain

$$ \begin{aligned} \mathcal{I}^{\gamma}\ {}^{C}\mathcal{D}^{\gamma} g(\zeta) &= \mathcal{I}^{\gamma}\mathcal{I}^{2-\gamma} g^2_{\nabla}(\zeta) \\ &= \mathcal{I}^{2} g^2_{\nabla}(\zeta) \\ &= g(\zeta) + k_0 + k_1\zeta \text{ for some } k_0, k_1 \in \mathbb{R}. \end{aligned} $$

Let ${}^{C}\mathcal{D}^{\gamma} g(\zeta) = r(\zeta)$, $\zeta \in \mathcal{J}$. Then, we have

$$ g(\zeta) = \mathcal{I}^{\gamma} r(\zeta) - k_0 - k_1\zeta. \tag{3.3} $$

Now, using the boundary conditions given in (1.1), we get $k_0 = 0$, and

$$ k_1 = \frac{1}{T}\int_0^T h_{\gamma-1}(T, \beta(\theta)) r(\theta)\nabla\theta. $$

Hence, from the equation (3.3), we get

$$ \begin{aligned} g(\zeta) &= \int_0^{\zeta} h_{\gamma-1}(\zeta, \rho(\theta)) r(\theta)\nabla\theta - \frac{\zeta}{T}\int_0^T h_{\gamma-1}(T, \rho(\theta)) r(\theta)\nabla\theta \\ &= \int_0^{\zeta}\left(\frac{T h_{\gamma-1}(\zeta, \rho(\theta))}{T} + \frac{-\zeta h_{\gamma-1}(T, \rho(\theta))}{T}\right) r(\theta)\nabla\theta \\ &\quad + \int_{\zeta}^{T} \frac{\big[-\zeta h_{\gamma-1}(T, \rho(\theta))\big]}{T} r(\theta)\nabla\theta \\ &= \int_0^T G(\zeta, \theta) r(\theta)\nabla\theta. \end{aligned} $$

That is,

$$ g(\zeta) = \int_0^T G(\zeta, \theta)\ {}^{C}\mathcal{D}^{\gamma} g(\theta)\nabla\theta. $$

This together with (1.1) gives (3.1). The converse can be seen easily. □

Throughout the paper, we prove our results based on the following assumptions:

(H_1) The function $\mathscr{Z} : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a ld-continuous in its first variable and continuous in its second and third variable separately.

(H_2) For a function $\mathscr{Z}$ in (H_1), there exist positive constants $E > 0$ and F satisfying $0 < F < 1$ such that

$$ |\mathscr{Z}(\zeta, \theta_1, \psi_1) - \mathscr{Z}(\zeta, \theta_2, \psi_2)| \leq E|\theta_1 - \theta_2| + F|\psi_1 - \psi_2| $$

for $(\zeta, \theta_i, \psi_i) \in \mathcal{J} \times \mathbb{R} \times \mathbb{R}$ $(i = 1, 2)$.

(H_3) For a function $\mathscr{Z}$ in (H_1), there exists $\mathscr{P} \in \mathcal{L}$ and constants $R > 0$ and Q with $0 < Q < 1$ such that

$$ |\mathscr{Z}(\zeta, \theta, \psi)| \leq |\mathscr{P}(\zeta)| + R|\theta| + Q|\psi| $$

for $(\zeta, \theta, \psi) \in \mathcal{J} \times \mathbb{R} \times \mathbb{R}$.

(H_4) The Green function $G(\cdot,\cdot)$ is bounded piecewise continuous on $[0,T]$. Moreover, the function G satisfies

$$\int_0^\zeta |G(\zeta,\theta)|\,\nabla\theta \leq k \quad \text{and} \quad \int_\zeta^T |G(\zeta,\theta)|\,\nabla\theta \leq m,$$

where k and m are positive real constants and $0<\zeta<T$. Further,

$$\int_0^T G(\zeta,\theta)\nabla\theta = A \in \mathbb{R}.$$

To prove the existence and uniqueness results for PBVP (1.1), we shall apply Theorem 2.14. For this, first we have to go through with the following essentials.

Consider a subset of $\mathcal{L}$ defined as

$$\mathcal{M}_\alpha := \{g : \mathcal{J} \to \mathbb{R} : g(\zeta) \in \mathcal{L}, ||g|| \leq \alpha, \alpha > 0\}. \tag{3.4}$$

Clearly, $\mathcal{M}_\alpha$ is a Banach subspace of $\mathcal{L}$. Next, we define two mappings $\mathcal{F}_1 : \mathcal{M}_\alpha \to \mathcal{L}$ and $\mathcal{F}_2 : \mathcal{M}_\alpha \to \mathcal{L}$ by

$$\mathcal{F}_1[g](\zeta) := \int_0^\zeta G(\zeta,\theta)\mathscr{Z}(\theta, g(\theta), {}^C\mathcal{D}^\gamma g(\theta))\nabla\theta \tag{3.5}$$

and

$$\mathcal{F}_2[g](\zeta) := \int_\zeta^T G(\zeta,\theta)\mathscr{Z}(\theta, g(\theta), {}^C\mathcal{D}^\gamma g(\theta))\nabla\theta \tag{3.6}$$

respectively.

Let us prove some useful lemmas.

Lemma 3.3 *Suppose that (H_1), (H_2), and (H_4) hold. If $\frac{Ek}{1-F} < 1$, then $\mathcal{F}_1 : \mathcal{M}_\alpha \to \mathcal{L}$ defined in (3.5) is contractive.*

Proof Let ${}^C\mathcal{D}^\gamma g_i(\zeta) = r_i(\zeta)$, $\zeta \in \mathcal{J}$, $i = 1, 2$, where $g_1, g_2 \in \mathcal{M}_\alpha$. Then, in view of (3.5), we can write for $\zeta \in \mathcal{J}$,

$$\begin{aligned} |\mathcal{F}_1[g_1](\zeta) - \mathcal{F}_1[g_2](\zeta)| &= \bigg| \int_0^\zeta G(\zeta,\theta)\mathscr{Z}\big(\theta, g_1(\theta), {}^C\mathcal{D}^\gamma g_1(\theta)\big)\nabla\theta \\ &\qquad - \int_0^\zeta G(\zeta,\theta)\mathscr{Z}\big(\theta, g_2(\theta), {}^C\mathcal{D}^\gamma g_2(\theta)\big)\nabla\theta \bigg| \\ &= \bigg| \int_0^\zeta G(\zeta,\theta)\left(\mathscr{Z}\big(\theta, g_1(\theta), {}^C\mathcal{D}^\gamma g_1(\theta)\big)\right. \\ &\qquad \left. - \mathscr{Z}\big(\theta, g_2(\theta), {}^C\mathcal{D}^\gamma g_2(\theta)\big)\right)\nabla\theta \bigg| \\ &\leq \int_0^\zeta |G(\zeta,\theta)|\,\big|\mathscr{Z}\big(\theta, g_1(\theta), r_1(\theta)\big) \\ &\qquad - \mathscr{Z}\big(\theta, g_2(\theta), r_2(\theta)\big)\big|\,\nabla\theta, \end{aligned} \tag{3.7}$$

where $r_1, r_2 \in \mathcal{M}_\alpha$. But, in view of (1.1), for $\theta \in \mathcal{J}$

$$|r_1(\theta) - r_2(\theta)| = |\mathscr{Z}\big(\theta, g_1(\theta), r_1(\theta)\big) - \mathscr{Z}\big(\zeta, g_2(\zeta), r_2(\theta)\big)|$$
$$\overset{(\mathrm{H}_2)}{\leq} E|g_1(\theta) - g_2(\theta)| + F|r_1(\theta) - r_2(\theta)|.$$

This gives

$$|r_1(\theta) - r_2(\theta)| \leq \frac{E}{1-F}|g_1(\theta) - g_2(\theta)|. \tag{3.8}$$

Now, using (3.8) in (3.7), we obtain

$$\|\mathcal{F}_1[g_1] - \mathcal{F}_1[g_2]\| \leq \frac{E}{1-F}\int_0^\zeta |G(\zeta,\theta)|\,\|g_1 - g_2\|\nabla\theta$$
$$\overset{(\mathrm{H}_4)}{\leq} \frac{Ek}{1-F}\|g_1 - g_2\|.$$

Since $\frac{Ek}{1-F} < 1$, the mapping $\mathcal{F}_1 : \mathcal{M}_\alpha \to \mathcal{L}$ is contractive. □

Theorem 3.4 *Suppose that* (H_1)–(H_4) *hold. Then,* $\mathcal{F}_2 : \mathcal{M}_\alpha \to \mathcal{L}$ *defined in (3.6) is continuous and* $\mathcal{F}_2(\mathcal{M}_\alpha)$ *is relatively compact.*

Proof Let ${}^C\mathcal{D}^\gamma g_n(\zeta) = r_n(\zeta)$, $n \in \mathbb{N}$ and ${}^C\mathcal{D}^\gamma g(\zeta) = r(\zeta)$, $\zeta \in \mathcal{J}$. Consider $\mathcal{F}_2 : \mathcal{M}_\alpha \to \mathcal{L}$, the mapping defined in (3.6). We divide the proof into the following steps.

Step 1: $\mathcal{F}_2 : \mathcal{M}_\alpha \to \mathcal{L}$ is continuous.
Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}_\alpha$ which converges to g in $\mathcal{M}_\alpha$. Then, for $\zeta \in [0, T]$, we have

$$\begin{aligned}|\mathcal{F}_2[g_n](\zeta) - \mathcal{F}_2[g](\zeta)| &= \bigg|\int_\zeta^T G(\zeta,\theta)\mathscr{Z}\big(\theta, g_n(\theta), {}^C\mathcal{D}^\gamma g_n(\theta)\big)\nabla\theta \\ &\quad - \int_\zeta^T G(\zeta,\theta)\mathscr{Z}\big(\theta, g(\theta), {}^C\mathcal{D}^\gamma g(\theta)\big)\nabla\theta\bigg| \\ &= \bigg|\int_\zeta^T G(\zeta,\theta)\left(\mathscr{Z}\big(\theta, g_n(\theta), {}^C\mathcal{D}^\gamma g_n(\theta)\big)\right. \\ &\quad \left.-\mathscr{Z}\big(\theta, g(\theta), {}^C\mathcal{D}^\gamma g(\theta)\big)\right)\nabla\theta\bigg| \\ &\leq \int_0^\zeta |G(\zeta,\theta)|\left|\mathscr{Z}\big(\theta, g_n(\theta), r_n(\theta)\big)\right. \\ &\quad \left.-\mathscr{Z}\big(\theta, g(\theta), r(\theta)\big)\right|\nabla\theta,\end{aligned} \tag{3.9}$$

where $r_n, r \in \mathcal{M}_\alpha$. But, in view of (1.1), for $\theta \in \mathcal{J}$

$$\begin{aligned}|r_n(\theta) - r(\theta)| &= |\mathscr{Z}\big(\theta, g_n(\theta), r_n(\theta)\big) - \mathscr{Z}\big(\zeta, g(\zeta), r(\theta)\big)| \\ &\overset{(\mathrm{H}_2)}{\leq} E|g_n(\theta) - g(\theta)| + F|r_n(\theta) - r(\theta)|.\end{aligned}$$

This gives

$$|r_n(\theta) - r(\theta)| \leq \frac{E}{1-F}|g_n(\theta) - g(\theta)|. \tag{3.10}$$

Now, using (3.10) in (3.9), we obtain

$$\begin{aligned}\|\mathcal{F}_2[g_n] - \mathcal{F}_2[g]\| &\leq \frac{E}{1-F}\int_0^\zeta |G(\zeta,\theta)|\,\|g_n - g\|\nabla\theta \\ &\overset{(\mathrm{H}_4)}{\leq} \frac{Ek}{1-F}\|g_n - g\|.\end{aligned}$$

That is,

$$\|\mathcal{F}_2[g_n] - \mathcal{F}_2[g]\| \leq \frac{Ek}{1-F}\|g_n - g\|.$$

This yields that the right side of the above inequality approaches to 0 as g_n approaches to g. Hence $\mathcal{F}_2 : \mathcal{M}_\alpha \to \mathcal{L}$ is continuous.

Step 2: $\mathcal{F}_2 : \mathcal{M}_\alpha \to \mathcal{L}$ is bounded.
From (3.6), we write for $\zeta \in \mathcal{J}$

$$\begin{aligned}|\mathcal{F}_2[g](\zeta)| &\leq \int_\zeta^T |G(\zeta,\theta)||\mathscr{Z}\big(\theta, g(\theta), {}^C\mathcal{D}^\gamma g(\theta)\big)|\nabla\theta \\ &= \int_\zeta^T |G(\zeta,\theta)||\mathscr{Z}\big(\theta, g(\theta), r(\theta)\big)|\nabla\theta \\ &= \int_\zeta^T |G(\zeta,\theta)||r(\theta)|\nabla\theta,\end{aligned} \tag{3.11}$$

where $r \in \mathcal{M}$. But, in view of (1.1), for $\theta \in \mathcal{J}$, we have

$$\begin{aligned}|r(\theta)| &= |\mathscr{Z}\big(\theta, g(\theta), r(\theta)\big)| \\ &\overset{(\mathrm{H}_3)}{\leq} |\mathscr{P}(\theta)| + R|g(\theta)| + Q|r(\theta)|\end{aligned}$$

This gives

$$|r(\theta)| \leq \frac{|\mathscr{P}(\theta)| + R|g(\theta)|}{1-Q}. \tag{3.12}$$

Now, using (3.12) in (3.11), and then taking the norm of (2.1), we obtain

$$\|\mathcal{F}_2[g]\| \leq \int_{\zeta}^{T} |G(\zeta,\theta)| \frac{\|\mathscr{P}\| + R\|g\|}{1-Q} \nabla\theta$$
$$\overset{(\mathrm{H}_4)}{\leq} \frac{m\|\mathscr{P}\| + R\alpha}{1-Q}.$$

That is,

$$\|\mathcal{F}_2[g]\| \leq \frac{m\|\mathscr{P}\| + R\alpha}{1-Q}.$$

Thus, $\mathcal{F}_2 : \mathcal{M} \to \mathcal{L}$ is bounded.

Step 3: $\mathcal{F}_2 : \mathcal{M} \to \mathcal{L}$ is equicontinuous.
Let $\zeta_1, \zeta_2 \in \mathscr{J}$ be such that $\zeta_1 < \zeta_2$. Then for $g \in \mathcal{M}_\alpha$, we have

$$\begin{aligned}
&|\mathcal{F}_2[g](\zeta_1) - \mathcal{F}_2[g]\zeta_2)| \\
&= \left| \int_{\zeta_1}^{T} G(\zeta_1,\theta)\mathscr{Z}\big(\theta, g(\theta), {}^{C}\mathcal{D}^{\gamma} g(\theta)\big)\nabla\theta \right. \\
&\left. - \int_{\zeta_2}^{T} G(\zeta_2,\theta)\mathscr{Z}\big(\theta, g(\theta), {}^{C}\mathcal{D}^{\gamma} g(\theta)\big)\nabla\theta \right| \\
&= \left| \int_{\zeta_1}^{T} G(\zeta_1,\theta)\nabla\theta - \int_{\zeta_2}^{T} G(\zeta_2,\theta)\nabla\theta \right| \left|\mathscr{Z}\big(\theta, g(\theta), {}^{C}\mathcal{D}^{\gamma} g(\theta)\big)\right| \\
&= \left| \int_{\zeta_1}^{T} G(\zeta_1,\theta)\nabla\theta - \int_{\zeta_2}^{T} G(\zeta_2,\theta)\nabla\theta \right| |r(\theta)| \\
&\overset{(3.12)}{\leq} \left| \int_{\zeta_1}^{T} G(\zeta_1,\theta)\nabla\theta - \int_{\zeta_2}^{T} G(\zeta_2,\theta)\nabla\theta \right| \left(\frac{|\mathscr{P}(\theta)| + R|g(\theta)|}{1-Q} \right).
\end{aligned}$$

That is,

$$|\mathcal{F}_2[g](\zeta_1) - \mathcal{F}_2[g]\zeta_2)| \leq \left(\frac{\|\mathcal{P}\| + R\alpha}{1-Q} \right) \left| \int_{\zeta_1}^{T} G(\zeta_1,\theta)\nabla\theta - \int_{\zeta_2}^{T} G(\zeta_2,\theta)\nabla\theta \right| \tag{3.13}$$

Now, using the Green function given in (3.2) and Remark 2.5, we can write, for $\zeta_1, \zeta_2 \in \mathscr{J}$

$$\begin{aligned}
\int_{\zeta_1}^{T} \zeta_1 h(T,\rho(\theta))\nabla\theta &= \int_{\zeta_1}^{T} -\zeta_1 h_{\gamma-1}(T,\rho(\theta))\nabla\theta \\
&= \frac{-\zeta_1 h_{\gamma}(T,\rho(\theta))}{T}
\end{aligned} \tag{3.14}$$

and

$$\int_{\zeta_2}^{T} \zeta_2 h(T,\rho(\theta))\nabla\theta = \frac{-\zeta_2 h_\gamma(T,\rho(\theta))}{T}. \tag{3.15}$$

Using (3.14) and (3.15) in (3.13), we get

$$|\mathcal{F}_2[g](\zeta_1) - \mathcal{F}_2[g]\zeta_2)| \leq \left(\frac{\|\mathcal{P}\| + R\alpha}{1-Q}\right)\frac{h_\gamma(T,\rho(\theta))}{T}(\zeta_1 - \zeta_2). \tag{3.16}$$

We find that the right side of (3.16) approaches to zero as ζ_1 approaches to ζ_2. This gives $\|\mathcal{F}_2[g](\zeta_1) - \mathcal{F}_2[g](\zeta_2)\| \to 0$. Thus, the mapping $\mathcal{F}_2 : \mathcal{M}_\alpha \to \mathcal{L}$ is equicontinuous. Now, since $\mathcal{F}_2(\mathcal{M}_\alpha)$ is bounded and equicontinuous, by virtue of Theorem 2.11, it infer that $\mathcal{F}_2(\mathcal{M}_\alpha)$ is relatively compact. □

Theorem 3.5 *Suppose* (H_1)–(H_4) *hold. Let* $\mathcal{M}_\alpha = \{g : \mathcal{J} \to \mathbb{R} : g(\zeta) \in \mathcal{L}, ||g|| \leq \alpha\}$, *where* α *is such that* $\frac{(m+k)\|\mathscr{P}\|}{1-Q-(m+k)R} \leq \alpha$. *Then, PBVP (1.1) has a solution in* $\mathcal{M}_\alpha$.

Proof By Lemma 3.3, $\mathcal{F}_1 : \mathcal{M}_\alpha \to \mathcal{L}$, defined in (3.5), is contractive. Also, by Theorem 3.4, $\mathcal{F}_2 : \mathcal{M}_\alpha \to \mathcal{L}$, defined in (3.6), is continuous and $\mathcal{F}_2(\mathcal{M}_\alpha)$ is relatively compact. Let ${}^C\mathcal{D}^\gamma g(\zeta) = r(\zeta)$ and ${}^C\mathcal{D}^\gamma h(\zeta) = q(\zeta)$ for $\zeta \in \mathcal{J}$. Then, for $g, h \in \mathcal{M}_\alpha$, we can write

$$\begin{aligned}|\mathcal{F}_1[g](\zeta) + \mathcal{F}_2[h](\zeta)| &\leq \int_0^\zeta |G(\zeta,\theta)|\big|\mathscr{Z}(\theta, g(\theta), {}^C\mathcal{D}^\gamma g(\theta))\big|\nabla\theta \\ &\quad + \int_\zeta^T |G(\zeta,\theta)|\big|\mathscr{Z}(\theta, h(\theta), {}^C\mathcal{D}^\gamma h(\theta))\big|\nabla\theta \\ &\leq \int_0^\zeta |G(\zeta,\theta)||r(\theta)|\nabla\theta + \int_\zeta^T |G(\zeta,\theta)||q(\theta)|\nabla\theta. \end{aligned} \tag{3.17}$$

But, in view of (1.1), for $\theta \in \mathcal{J}$, we have

$$\begin{aligned}|r(\theta)| &= |\mathscr{Z}\big(\theta, g(\theta), r(\theta)\big)| \\ &\overset{(\mathrm{H}_3)}{\leq} |\mathscr{P}(\theta)| + R|g(\theta)| + Q|r(\theta)| \\ &\leq \frac{|\mathscr{P}(\theta)| + R|g(\theta)|}{1-Q}\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}|q(\theta)| &= |\mathscr{Z}\big(\theta, h(\theta), q(\theta)\big)| \\ &\overset{(\mathrm{H}_3)}{\leq} |\mathscr{P}(\theta)| + R|h(\theta)| + Q|q(\theta)| \\ &\leq \frac{|\mathscr{P}(\theta)| + R|h(\theta)|}{1-Q}.\end{aligned} \tag{3.19}$$

Now, using (3.18) and (3.19) in (3.17), we obtain

$$\begin{aligned}\|\mathcal{F}_1[g]+\mathcal{F}_2[h]\| &\leq \int_0^\zeta |G(\zeta,\theta)|\left(\frac{\|\mathscr{P}\|+R\|g\|}{1-Q}\right)\nabla\theta \\ &\quad+\int_\zeta^T |G(\zeta,\theta)|\left(\frac{\|\mathscr{P}\|+R|h\|}{1-Q}\right)\nabla\theta \\ &\overset{(\mathrm{H}_4)}{\leq} m\left(\frac{\|\mathscr{P}\|+R\alpha}{1-Q}\right)+k\left(\frac{\|\mathscr{P}\|+R\alpha}{1-Q}\right) \\ &= \frac{(m+k)(\|\mathscr{P}\|+R\alpha)}{1-Q}\leq\alpha. \end{aligned} \tag{3.20}$$

Thus, for $g,h\in\mathcal{M}_\alpha$, $\mathcal{F}_1[g]+\mathcal{F}_2[h]\in\mathcal{M}_\alpha$. Clearly, all the hypotheses of Theorem 2.14 are satisfied. Thus, there exists a fixed point $g\in\mathcal{M}_\alpha$ such that $g=\mathcal{F}_1[g]+\mathcal{F}_2[g]$ which is a solution of PBVP (1.1). □

Theorem 3.6 (Uniqueness) *If the functions $\mathscr{Z}$ and G satisfies conditions given in (H_1)–(H_4), then (1.1) has unique solution.*

Proof Let $g_1,g_2\in\mathcal{M}_\alpha$ be two solutions of PBVP (1.1) and ${}^C\mathcal{D}^\gamma g_i(\zeta)=r_i(\zeta)$ for $\zeta\in\mathcal{J}$, $i=1,2$. Then, we have

$$\begin{aligned}|g_1(\zeta)-g_2(\zeta)| &\leq \int_0^T |G(\zeta,\theta)|\big|\mathscr{Z}(\theta,g_1(\theta),{}^C\mathcal{D}^\gamma g_1(\theta))-\mathscr{Z}(\theta,g_2(\theta),{}^C\mathcal{D}^\gamma g_2(\theta))\big|\nabla\theta \\ &\leq \int_0^T |G(\zeta,\theta)|\,|\mathscr{Z}(\theta,g_1(\theta),r_1(\theta))-\mathscr{Z}(\theta,g_2(\theta),r_2(\theta))|\,\nabla\theta. \end{aligned} \tag{3.21}$$

In view of (1.1), for $\theta\in\mathcal{J}$, we have

$$\begin{aligned}|r_1(\theta)-r_2(\theta)| &= |\mathscr{Z}(\theta,g_1(\theta),r_1(\theta))-\mathscr{Z}(\theta,g_2(\theta),r_2(\theta))| \\ &\overset{(\mathrm{H}_2)}{\leq} E|g_1(\theta)-g_2(\theta)|+F|r_1(\theta)-r_2(\theta)|.\end{aligned}$$

This gives

$$|r_1(\theta)-r_2(\theta)|\leq\frac{E}{1-F}|g_1(\theta)-g_2(\theta)|. \tag{3.22}$$

Now, using (3.22) in (3.21), we obtain

$$|g_1(\zeta)-g_2(\zeta)|\leq\frac{E}{1-F}\int_0^T |G(\zeta,\theta)||g_1(\theta)-g_2(\theta)|\nabla\theta. \tag{3.23}$$

Applying the inequality given in Theorem 2.15 to (3.23) gives that $|g_1(\zeta)-g_2(\zeta)|\leq 0$ and hence $g_1(\zeta)=g_2(\zeta)$ for all $\zeta\in\mathcal{J}$. This proves the uniqueness of solution of (1.1). □

4. Ulam Stability Results for (1.1)

In this section, we are analyzing Ulam stability of PBVP (1.1).

Definition 4.1 We say that PBVP (1.1) has Hyers–Ulam stability (HUS) if there exists $N_{\mathscr{Z}} > 0$ such that for each $\varepsilon > 0$ and for each $g \in \mathcal{M}_\alpha$ satisfying

$$|{}^C\mathcal{D}^\gamma g(\zeta) - \mathscr{Z}(\zeta, g(\zeta), {}^C\mathcal{D}^\gamma g(\zeta))| \leq \varepsilon \text{ for all } \zeta \in \mathcal{J}_\kappa, \tag{4.1}$$

there exists a solution $h \in \mathcal{M}_\alpha$ of PBVP (1.1) such that

$$|g(\zeta) - h(\zeta)| \leq N_{\mathscr{Z}}\varepsilon \text{ for all } \zeta \in \mathcal{J}.$$

Such $N_{\mathscr{Z}} > 0$ is said as HUS constant.

Definition 4.2 We say that PBVP (1.1) has generalized Hyers–Ulam stability (GHUS) if there exists a positive continuous function $\mathcal{H}_{\mathscr{Z}}$ with $\mathcal{H}_{\mathscr{Z}}(0) = 0$ such that for each $g \in \mathcal{M}_\alpha$ satisfying (4.1), there exists a solution $h \in \mathcal{M}_\alpha$ of PBVP (1.1) such that

$$|g(\zeta) - h(\zeta)| \leq \mathcal{H}_{\mathscr{Z}}(\varepsilon) \text{ for all } \zeta \in \mathcal{J}.$$

Definition 4.3 Let $\mathcal{K}$ be a family of positive, nondecreasing ld-continuous real-valued function defined on $\mathcal{J}$. We say that PBVP (1.1) has Hyers–Ulam–Rassias stability (HURS) of type $\mathcal{K}$ if for each $\Psi \in \mathcal{K}$ and $\varepsilon > 0$, there exists $N_{\mathscr{Z},\Psi} > 0$ such that for each $g \in \mathcal{M}_\alpha$ satisfying

$$|{}^C\mathcal{D}^\gamma g(\zeta) - \mathscr{Z}(\zeta, g(\zeta), {}^C\mathcal{D}^\gamma g(\zeta))| \leq \varepsilon\Psi(\zeta) \text{ for all } \zeta \in \mathcal{J}_\kappa, \tag{4.2}$$

there exists a solution $h \in \mathcal{M}_\alpha$ of PBVP (1.1) such that

$$|g(\zeta) - h(\zeta)| \leq \varepsilon N_{\mathscr{Z},\Psi}\Psi(\zeta) \text{ for all } \zeta \in \mathcal{J}.$$

Such $N_{\mathscr{Z},\Psi} > 0$ is said as HURS constant.

Definition 4.4 Let $\mathcal{K}$ be a family of positive, non decreasing ld-continuous real-valued function defined on $\mathcal{J}$. We say that PBVP (1.1) has generalized Hyers–Ulam–Rassias stability (GHURS) of type $\mathcal{K}$ if for each $\Psi \in \mathcal{K}$, there exists $N_{\mathscr{Z},\Psi} > 0$ such that for each $g \in \mathcal{M}_\alpha$ satisfying

$$|{}^C\mathcal{D}^\gamma g(\zeta) - \mathscr{Z}(\zeta, g(\zeta), {}^C\mathcal{D}^\gamma g(\zeta))| \leq \Psi(\zeta) \text{ for all } \zeta \in \mathcal{J}_\kappa, \tag{4.3}$$

there exists a solution $h \in \mathcal{M}_\alpha$ of PBVP (1.1) such that

$$|g(\zeta) - h(\zeta)| \leq N_{\mathscr{Z},\Psi}\Psi(\zeta) \text{ for all } \zeta \in \mathcal{J}.$$

Such $N_{\mathscr{Z},\Psi} > 0$ is said as GHURS constant.

Remark 4.5 A function $g \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{J}, \mathbb{R})$ is s solution of (4.2) if there exists a function
$\mathscr{H} \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{J}, \mathbb{R})$ (depending on g) with the following property:

(i) $|\mathscr{H}(\zeta)| \leq \varepsilon\Psi(\zeta)$ for all $\zeta \in \mathcal{J}$,
(ii) ${}^C\mathcal{D}^\gamma g(\zeta) = \mathscr{Z}(\zeta, g(\zeta), {}^C\mathcal{D}^\gamma g(\zeta)) + \mathscr{H}(\zeta)$ for all $\zeta \in \mathcal{J}_\kappa$.

Similar arguments also hold for (4.1) and (4.3).

Theorem 4.6 *Suppose* (H_1)–(H_4) *hold true with* $\frac{AE}{1-F} < 1$. *Then, PBVP (1.1) has Hyers–Ulam–Rassias stability of type* $\mathcal{K}$.

Proof Let $g \in \mathrm{C}^1_{\mathrm{ld}}(\mathcal{J}, \mathbb{R})$ satisfy (4.2). Then, by Remark 4.5, there exists for $\mathscr{H} \in \mathrm{C}^1_{\mathrm{ld}}(\mathcal{J}, \mathbb{R})$ satisfying $|\mathscr{H}(\zeta)| \leq \varepsilon\Psi(\zeta)$ such that

$$^{C}\mathcal{D}^{\gamma} g(\zeta) = \mathscr{Z}(\zeta, g(\zeta), {}^{C}\mathcal{D}^{\gamma} g(\zeta)) + \mathscr{H}(\zeta) \text{ for all } \zeta \in \mathcal{J}_\kappa.$$

For $^{C}\mathcal{D}^{\gamma} g(\zeta) = q(\zeta)$, $\zeta \in \mathcal{J}_\kappa$ with $h \in \mathcal{M}_\alpha$, using Lemma 3.2, write

$$g(\zeta) = \int_0^T G(\zeta, \theta)(\mathscr{Z}(\theta, g(\theta), q(\theta)) + \mathcal{H}(\theta))\nabla\theta. \tag{4.4}$$

For $\Psi \in \mathcal{K}$, using Remark 4.5, from (4.4), we can write

$$\left| g(\zeta) - \int_0^T G(\zeta, \theta)\mathscr{Z}(\theta, g(\theta), q(\theta))\nabla\theta \right| \leq A\varepsilon\Psi(\zeta) \text{ for } \zeta \in \mathcal{J}_\kappa. \tag{4.5}$$

Let $h \in \mathcal{M}_\alpha$ be a solution of (1.1). Then, for $\zeta \in \mathcal{J}$, we have

$$\begin{cases} ^{C}\mathcal{D}^{\gamma} h(\zeta) = \mathscr{Z}(\zeta, h(\zeta), {}^{C}\mathcal{D}^{\gamma} h(\zeta)) \\ h(0) = h(T) = 0, \quad T \in \mathbb{R}. \end{cases} \tag{4.6}$$

If $^{C}\mathcal{D}^{\gamma} h(\zeta) = r(\zeta)$, $\zeta \in \mathcal{J}_\kappa$ with $r \in \mathcal{M}_\alpha$, then from Lemma 3.2, we have

$$h(\zeta) = \int_0^T G(\zeta, \theta)\mathscr{Z}(\theta, h(\theta), r(\theta))\nabla\theta. \tag{4.7}$$

From (4.6) and (4.7), we can write

$$\begin{aligned} |g(\zeta) - h(\zeta)| &= \left| g(\zeta) - \int_0^T G(\zeta, \theta)\mathscr{Z}(\theta, g(\theta), q(\theta))\nabla\theta + \int_0^T G(\zeta, \theta)\mathscr{Z}(\theta, g(\theta), q(\theta))\nabla\theta \right. \\ &\quad \left. - \int_0^T G(\zeta, \theta)\mathscr{Z}(\theta, h(\theta), r(\theta))\nabla\theta \right| \\ &\leq \left| g(\zeta) - \int_0^T G(\zeta, \theta)\mathscr{Z}(\theta, g(\theta), q(\theta))\nabla\theta \right| \\ &\quad + \left| \int_0^T G(\zeta, \theta)\mathscr{Z}(\theta, g(\theta), q(\theta))\nabla\theta - \int_0^T G(\zeta, \theta)\mathscr{Z}(\theta, h(\theta), r(\theta))\nabla\theta \right| \\ &\overset{(4.5)}{\leq} A\varepsilon\Psi(\zeta) + \int_0^T |G(\zeta, \theta)||\mathscr{Z}(\theta, g(\theta), q(\theta)) - \mathscr{Z}(\theta, h(\theta), r(\theta))|\nabla\theta. \end{aligned} \tag{4.8}$$

But, in view of (1.1), for $\zeta \in \mathcal{J}$

$$\begin{aligned} |r(\theta) - q(\theta)| &= |\mathscr{Z}(\theta, h(\theta), r(\theta)) - \mathscr{Z}(\theta, g(\theta), q(\theta))| \\ &\overset{(\mathrm{H}_2)}{\leq} E|h(\theta) - g(\theta)| + F|r(\theta) - q(\theta)|. \end{aligned}$$

That is,

$$|r(\theta) - q(\theta)| \leq \frac{E}{1-F}|h(\theta) - g(\theta)|.$$

From (4.8), we obtain

$$
\begin{aligned}
|g(\zeta)-h(\zeta)| &\leq \frac{AE}{1-F}|h(\theta)-g(\theta)|+A\varepsilon\Psi(\zeta)\\
&\leq \frac{A}{1-\frac{AE}{1-F}}\varepsilon\Psi(\zeta)\leq N\varepsilon\Psi(\zeta).
\end{aligned}
\tag{4.9}
$$

Thus, PBVP (1.1) has Hyers–Ulam–Rassias stability of type $\mathcal{K}$ with HURS constant $N=\frac{A(1-F)}{1-AE-F}>0$. □

Corollary 4.7 *Suppose that (H_1)–(H_4) hold true with $\frac{AE}{1-F}<1$. Then, PBVP (1.1) has generalized Hyers–Ulam–Rassias stability of type $\mathcal{K}$ with GHURS constant $N_{\mathscr{Z},\Psi}=\frac{A(1-F)}{1-AE-F}$.*

Proof The proof follows easily by taking $\varepsilon = 1$ in the proof of Theorem 4.6. □

Corollary 4.8 *Suppose that (H_1)–(H_4) hold true with $\frac{AE}{1-F}<1$. Then, PBVP (1.1) has Hyers–Ulam stability with HUS constant $N_{\mathscr{Z}}=\frac{A(1-F)}{1-AE-F}$.*

Proof The proof follows easily by taking $\psi(\zeta) \equiv 1$ in the proof of Theorem 4.6. □

Corollary 4.9 *Suppose that (H_1)–(H_4) hold true with $\frac{AE}{1-F}<1$. Then, PBVP (1.1) has generalized Hyers–Ulam stability.*

Proof Taking $\mathcal{H}_{\mathscr{Z}}(\varepsilon)=\frac{A(1-F)}{1-AE-F}\varepsilon$ and $\psi(\zeta)\equiv 1$ in the proof of Theorem 4.6, the proof follows easily. □

5. Example

Let $\mathbb{T}=[0,1]\cup[2,3]$ and $T=2$. Then $\mathcal{J}=[0,2]\cap\mathbb{T}=[0,1]\cup\{2\}$. Consider the PBVP

$$
\begin{cases}
{}^{C}\mathcal{D}^{1.5}h(\zeta)=\dfrac{e^{-4\zeta}}{8}+\dfrac{\sin|h(\zeta)|+\sin|{}^{C}\mathcal{D}^{\frac{1}{2}}h(\zeta)|}{30+e^{-3\zeta}}, & \zeta\in\mathcal{J}_{\kappa},\\
h(0)=h(2)=0.
\end{cases}
\tag{5.1}
$$

Here $\mathscr{Z}(\zeta,h(\zeta),{}^{C}\mathcal{D}^{1.5}h(\zeta))=\dfrac{e^{-4\zeta}}{8}+\dfrac{\sin|h(\zeta)|+\sin|{}^{C}\mathcal{D}^{\frac{1}{2}}h(\zeta)|}{30+e^{-3\zeta}}$ which satisfy (H_1). For $q_i\in\mathcal{L}$, $i=1,2$, let ${}^{C}\mathcal{D}^{1.5}q_i(\zeta))=r_i(\zeta)$ and for $\zeta\in\mathcal{J}$, we note that

$$
\begin{aligned}
&|\mathscr{Z}(\zeta,q_1(\zeta),r_1(\zeta))-\mathscr{Z}(\zeta,q_2(\zeta),r_2(\zeta))|\\
&\quad=\left|\frac{e^{-4\zeta}}{8}+\frac{\sin|q_1(\zeta)|+\sin|r_1(\zeta)|}{30+e^{-3\zeta}}-\frac{e^{-4\zeta}}{8}-\frac{\sin|q_2(\zeta)|+\sin|r_2(\zeta)|}{30+e^{-3\zeta}}\right|\\
&\quad=\left|\frac{\sin|q_1(\zeta)|+\sin|r_1(\zeta)|}{30+e^{-3\zeta}}-\frac{\sin|q_2(\zeta)|+\sin|r_2(\zeta)|}{30+e^{-3\zeta}}\right|\\
&\quad\leq\frac{1}{30}|q_1(\zeta)-q_2(\zeta)|+\frac{1}{30}|r_1(\zeta)-r_2(\zeta)|.
\end{aligned}
$$

That is,

$$|\mathscr{Z}(\zeta, q_1(\zeta), r_1(\zeta)) - \mathscr{Z}(\zeta, q_2(\zeta), r_2(\zeta))| \leq \frac{1}{30}|q_1 - q_2| + \frac{1}{30}|r_1 - r_2|.$$

Hence, (H_2) satisfied with $E = F = \frac{1}{30}$. Also, For $q \in \mathcal{L}$, let ${}^{C}\mathcal{D}^{1.5}q(\zeta)) = r(\zeta)$. Then for $\zeta \in \mathcal{J}$,

$$|\mathscr{Z}(\zeta, q(\zeta), r(\zeta))| \leq \frac{1}{8} + \frac{1}{30}|q(\zeta)| + \frac{1}{30}|r(\zeta)|.$$

Hence, (H_3) satisfied with $\mathcal{P} = \frac{1}{8}$, $R = \frac{1}{30}$, and $Q = \frac{1}{30}$. Now, using the above data, the inequality $\frac{Ek}{1-F} < 1$ yields $k < 29$. Again, using this value in

$$\frac{(k+m)(\|\mathcal{P}\| + R\alpha)}{1-Q} \leq \alpha, \quad \alpha > 0,$$

we obtain $m < \frac{870}{8\alpha+30}$, $\alpha > 0$. Further, keeping in mind the boundary conditions $h(0) = h(2) = 0$, using Proposition 2.13, we have

$$\begin{aligned} \left|\int_0^2 G(\zeta,\theta)\nabla\theta\right| &\leq \left|\int_0^2 G(\zeta,\theta)d\theta\right| \\ &\leq \left|\int_0^\zeta \frac{(\zeta-\theta)^{0.5}}{\Gamma(\frac{3}{2})}d\theta - \frac{\zeta}{2}\int_0^2 \frac{(\zeta-\theta)^{0.5}}{\Gamma(\frac{3}{2})}d\theta - \frac{\zeta}{2}\int_2^\zeta \frac{(\zeta-\theta)^{0.5}}{\Gamma(\frac{3}{2})}d\theta\right| \\ &\leq 1. \end{aligned}$$

This yields that (H_4) satisfied with $A = 1$. Thus, all the conditions of theorems 3.5 and 3.6 are satisfied. Hence, the PBVP (5.1) has unique solution h and by Lemma (3.2), this solution is given by

$$h(\zeta) = \int_0^2 G(\zeta,\theta)\left(\frac{e^{-4\theta}}{8} + \frac{\sin|h(\theta)| + \sin|{}^{C}\mathcal{D}^{\frac{1}{2}}h(\theta)|}{30+e^{-3\theta}}\right)\nabla\theta. \tag{5.2}$$

Further, if $g \in C^1_{ld}(\mathcal{J}, \mathbb{R})$ satisfies

$$\left|{}^{C}\mathcal{D}^{\frac{1}{2}}g(\zeta) - \frac{e^{-4\zeta}}{8} - \frac{\sin|g(\zeta)| + \sin|{}^{C}\mathcal{D}^{\frac{1}{2}}g(\zeta)|}{30+e^{-3\zeta}}\right| \leq \varepsilon,$$

then by making use of Corollary 4.8, there exists a solution h of (5.1) satisfying

$$|g(\zeta) - h(\zeta)| \leq \frac{29}{28}\varepsilon.$$

Hence, PBVP (5.1) has Hyers–Ulam stability with HUS constant $\frac{29}{28}$.

6. Conclusion

We have investigated the existence, uniqueness, and the Ulam stability of solutions of nonlinear fractional dynamic equations with periodic boundary conditions involving Caputo fractional ∇-derivative. Our approach is based on Krasnoselskii fixed point theorem, which allows breaking of the mapping and makes calculation easier. For the guarantee of uniqueness of solutions, we employ ∇-dynamic inequality. We complemented our results through a stimulative example. We believe the results presented here are employable in the mathematical modeling of hybrid continuous-discrete phenomena. Further, the involvement of fractional ∇-derivative gives significantly better accuracy in the modeling process. Investigation of qualitative properties of other nonlinear fractional dynamic equations involving various fractional derivative operators on time scales will be our future work.

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