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Results in Mathematics

A Note on the Spectral Analysis of Some Fourth-Order Differential Equations with a Semigroup Approach

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Abstract. In this paper, we analyze the solubility of a class of abstract fourth-order in time linear evolution equations, using the roots of the characteristic polynomial that is associated with the equations.

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1. Introduction

Let X be a separable Hilbert space and $A : D(A) \subset X \to X$ be an unbounded linear, closed, densely defined, self-adjoint and for some $a > 0$, satisfies $\langle Au, u\rangle \geqslant a\langle u, u\rangle$ or all $u \in D(A)$. We also assume that A has compact resolvent on X. It is well-known that with those hypotheses, we can define the fractional power A^α of order $0 < \alpha < 1$ according to [2] and [11], as a closed linear operator, see e.g. [10,11,13,18]. Denote by $X^\alpha = D(A^\alpha)$ for $0 \leqslant \alpha \leqslant 1$ (taking $A^0 := I$ on $X^0 := X$ when $\alpha = 0$). Recall that X^α is dense in X for all $0 \leqslant \alpha \leqslant 1$, for details see [2, Theorem 4.6.5]. The fractional power space X^α endowed with the norm $\|\cdot\|_{X^\alpha} := \|A^\alpha \cdot\|_X$ is a Banach space. With this notation, we have $X^{-\alpha} = (X^\alpha)'$ for all $\alpha > 0$, see [2,26,27] for the characterization of the negative scale.

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In this paper we consider the fourth-order linear equation in time

$$u_{tttt} + A^{\frac{2}{5}}u + aA^{\frac{1}{5}}u_{ttt} + bA^{\frac{4}{5}}u_{tt} + cAu_t = 0, \tag{1.1}$$

where $a, b, c \geqslant 0$.

We discuss here the well-posedness of Eq. (1.1) in terms of the scalar parameters a, b and c considering the theory of strongly continuous bounded linear operators on suitable phase spaces.

In recent years higher-order evolution equations in time have attracted the attention of many researchers. This can be explained by the sensitivity of the conditions of well-posedness and regularity of solutions of these equations in infinite-dimensional Banach spaces, see e.g. [3–5,8,9,22], see also the Moore-Gibson-Thompson (MGT) equations, see e.g. [1,6,7,12,15–17,19–21,24,25] and reference therein.

We also observe that it is possible to find in the literature some works dedicated to fourth-order equations in time motivated by MGT equations, see e.g. [1,7,16,19–21]; in these papers are made formulations for fourth-order equations of the MGT type, and results of blow-up of solutions, well-posedness, stability and regularity of solutions are obtained.

Our best knowledge indicates that no information is known for models of the type (1.1) concerning the well-posedness and regularity of solutions. This paper makes this point. Furthermore, abstract differential equations of order greater than two are generally ill-posed in the sense of the theory of semigroups of bounded linear operators. However, we identify cases where (1.1) is well-posed. Namely, cases in which this equation is associated with a strongly continuous semigroup and instances in which it is associated with an analytic semigroup.

We will divide our problem into two cases: $c > 0$ and $c = 0$. First of all, note that our linear problem (1.1) can be rewritten as a first-order abstract system of the form

$$U_t + S_{a,b,c}U = 0, \quad t > 0,$$

on the phase space

$$Y = X^1 \times X^{\frac{4}{5}} \times X^{\frac{3}{5}} \times X,$$

endowed with the usual inner product, where $U = \begin{bmatrix} u \\ u_t \\ u_{tt} \\ u_{ttt} \end{bmatrix}$, and the matrix operator $S_{a,b,c}$ can be seen as a unbounded linear operator defined by

$$D(S_{a,b,c}) = X^1 \times X^1 \times X^{\frac{4}{5}} \times X^{\frac{1}{5}}$$

and

$$S_{a,b,c} = \begin{bmatrix} 0 & -I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -I \\ A^{\frac{2}{5}} & cA & bA^{\frac{4}{5}} & aA^{\frac{1}{5}} \end{bmatrix}$$

Namely, the unbounded linear operator $S_{a,b,c} : D(S_{a,b,c}) \subset Y \to Y$ is closed and densely defined.

It is easily seen that $\lambda \in \rho(-S_{a,b,c})$ if and only if $\lambda^4 I + a\lambda^3 A^{\frac{1}{5}} + b\lambda^2 A^{\frac{4}{5}} + c\lambda A + A^{\frac{2}{5}}$ is bijective. Consequently, if $\lambda \in \mathbb{C}$ is such that

$$\lambda^4 + a\mu^{\frac{1}{5}}\lambda^3 + b\mu^{\frac{4}{5}}\lambda^2 + c\mu\lambda + \mu^{\frac{2}{5}} = 0$$

for some $\mu \in \sigma(A)$, then $\lambda \in \sigma(-S_{a,b,c})$. Since $\sigma(A)$ is countably infinite, we will consider the characteristic polynomials"

$$p_n(\lambda) = \lambda^4 + a\mu_n^{\frac{1}{5}}\lambda^3 + b\mu_n^{\frac{4}{5}}\lambda^2 + c\mu_n\lambda + \mu_n^{\frac{2}{5}}. \tag{1.2}$$

with $\mu_n \in \sigma(A)$. Recall that $\mu_n > 0$ and $\mu_n \to \infty$ as $n \to \infty$.

Note that the discriminant of (1.2) is given by

$$\begin{aligned}\Delta_n &= (-4b^3c^2)\mu_n^{\frac{22}{5}} + (-27c^4 + 18abc^3 + a^2b^2c^2)\mu_n^4 + (16b^4 - 4a^3c^3)\mu_n^{\frac{18}{5}} + (144a^2b)\mu_n^{\frac{17}{5}}\\ &+ (144bc^2 - 80ab^2c - 4a^2b^3)\mu_n^{\frac{16}{5}} + (18a^3bc - 6a^2c^2)\mu_n^{\frac{14}{5}} - (128b^2)\mu_n^{\frac{12}{5}} - (192ac)\mu_n^2\\ &- 27a^4\mu_n^{\frac{8}{5}} + 256\mu_n^{\frac{6}{5}},\end{aligned}$$

see [14, Example on the pages 192-193].

We can summarize the behavior of Δ_n when $n >> 0$ as follows (Table 1).

See Tables 1, 2, 3 below

This is a key result in our analysis since a geometric localization of the points of the spectral set of $-S_{a,b,c}$ can provide us with information about the generation of strongly continuous semigroups of bounded linear operators associated with the problem (1.1), see e.g. [23, Chapter 1, Corollary 3.8].

This work is organized as follows. In Sect. 2 we consider the case $c > 0$ and we treat three subcases $c = ab$, $c > ab$, and $c < ab$. Finally, in Sect. 3 we consider the case $c = 0$ and we treat three subcases $a = b = c = 0$, $a \neq 0$ and $b = c = 0$, $b \neq 0$ and $a = c = 0$, and $a, b \neq 0$ and $c = 0$.

The tables below summarize our results, where ✓ means that **it is possible to generate the respective strongly continuous semigroups** of bounded linear operators on Y, and × means that **it is not possible to generate the respective strongly continuous semigroups** of bounded linear operators on Y.

For $c > 0$, we have the following results (Table 2):

Table 1. Signal of the discriminant Δ_n

a	b	c	Δ_n
$\geqslant 0$	$\geqslant 0$	> 0	< 0
$\geqslant 0$	> 0	0	> 0
> 0	0	0	< 0
0	0	0	> 0

Table 2. Case $c > 0$

	Analytic semigroup	Strongly continuous semigroup
$c = ab$	$\times$	$\checkmark$
$c > ab$	$\times$	$\times$
$c < ab$	$\times$	$\checkmark$

Table 3. Case $c = 0$

	Analytic semigroup	Strongly continuous semigroup
$a = b = c = 0$	$\times$	$\times$
$a \neq 0$ and $b = c = 0$	$\times$	$\times$
$b \neq 0$ and $a = c = 0$	$\times$	$\checkmark$
$a, b \neq 0$ and $c = 0$	$\times$	$\checkmark$

For $c = 0$, we have the following results (Table 3):

2. Case $c > 0$

Since $\Delta_n < 0$ for large n when $c > 0$ we guarantee the existence of two complex conjugated roots z_n and $\overline{z_n}$ and two real roots x_n and y_n. The following lemma shows the behavior of one of the real roots for the polynomial p_n in (1.2).

Lemma 2.1. *If $c > 0$, then $y_n \to 0$ as $n \to +\infty$.*

Proof. Given $\epsilon > 0$ we have

$$p_n(-\epsilon) = \epsilon^4 - a\mu_n^{\frac{1}{5}}\epsilon^3 + b\mu_n^{\frac{4}{5}}\epsilon^2 - c\mu_n\epsilon + \mu_n^{\frac{2}{5}}.$$

It is easy to see that $p_n(-\epsilon) \to -\infty$ as $n \to +\infty$. Since $p_n(0) > 0$ for any n, we guarantee that $y_n \in (-\epsilon, 0)$ for large n; that is, $y_n \to 0$ as $n \to +\infty$.

Corollary 2.2. *If $(x_n + a\mu_n^{\frac{1}{5}}) \to K$ as $n \to +\infty$, then $Re(z_n) \to -\dfrac{K}{2}$ as $n \to +\infty$.*

Proof. It follows directly from Vieta's formulas

$$x_n + y_n + z_n + \overline{z_n} = -a\mu_n^{\frac{1}{5}}. \tag{2.1}$$

jointly with Lemma 2.1.

Remark 2.3. It will be useful later to know that for any $\epsilon > 0$ we have $p(-a\mu_n^{\frac{1}{5}} - \epsilon)$ given by

$$\begin{aligned}&(a^2b - ac)\mu_n^{\frac{6}{5}} + (2ab - c)\epsilon\mu_n + (\epsilon^2 b)\mu_n^{\frac{4}{5}} + (a^3\epsilon)\mu_n^{\frac{3}{5}} \\ &\quad + (3a^2\epsilon^2 + 1)\mu_n^{\frac{2}{5}} + 3a\epsilon^3\mu_n^{\frac{1}{5}} + \epsilon^4.\end{aligned}$$

2.1. Subcase $ab = c$

In this subcase the polynomial p_n becomes

$$p_n(\lambda) = \lambda^4 + a\mu_n^{\frac{1}{5}}\lambda^3 + b\mu_n^{\frac{4}{5}}\lambda^2 + ab\mu_n + \mu_n^{\frac{2}{5}}.$$

We will prove that $x_n + a\mu_n^{\frac{1}{5}} \to 0$ and consequently by Corollary 2.2 we have $Re(z_n) \to 0$. From this, we will see that it is possible to prove a result of well-posedness for Eq. (1.1).

Proposition 2.4. *If $c > 0$ and $c = ab$, then $Re(z_n) \to 0$. This means that, in the best case scenario, the problem (1.1) can only generate a strongly continuous semigroup on Y.*

Proof. We will show that $x_n \to -a\mu_n^{\frac{1}{5}}$ and the other claim follows from the Hille-Yosida Theorem, see e.g. [23, Chapter 1, Theorem 3.1]. Fixing $\epsilon > 0$ it is easy to see (Remark 2.3) that $p(-a\mu_n^{\frac{1}{5}} + \epsilon) < 0$ for large n. Since $p_n(-a\mu_n^{\frac{1}{5}}) > 0$ we know that $x_n \in (-a\mu_n^{\frac{1}{5}}, -a\mu_n^{\frac{1}{5}} + \epsilon)$ for large n; that is, $x_n \to -a\mu_n^{\frac{1}{5}}$ as $n \to +\infty$.

Theorem 2.5. *If $c > 0$ and $c = ab$, then problem (1.1) is well-posedness on Y; that is, the linear operator associated with this problem is the infinitesimal generator of a strongly continuous semigroup on Y.*

Proof. Define

$$z = u_t + aA^{\frac{1}{5}}u \tag{2.2}$$

and

$$v = u_{ttt} + aA^{\frac{1}{5}}u_{tt} + bA^{\frac{4}{5}}u_t + abAu. \tag{2.3}$$

Transferring our problem to the phase space

$$Z = X \times X \times X \times X$$

endowed with the usual inner product, we define the vector $W = \begin{bmatrix} Au \\ Az \\ A^{\frac{3}{5}}z_t \\ A^{\frac{3}{5}}v \end{bmatrix}$. It is easy to see that (1.1) can be rewritten as a first-order abstract system of the form

$$W_t = \Lambda W, \quad t > 0,$$

where Λ denotes the unbounded linear operator given by

$$\Lambda = \begin{bmatrix} -aA^{\frac{1}{5}} & 1 & 0 & 0 \\ 0 & 0 & A^{\frac{2}{5}} & 0 \\ 0 & -bA^{\frac{2}{5}} & 0 & 1 \\ -I & 0 & 0 & 0 \end{bmatrix}.$$

Namely, this linear operator is closed and densely defined. Moreover, it is a maximally dissipative operator (see [23, Chapter 1, Definition 4.1]) on a space in which the norm is equivalent to the norm of space Z. This also implies that $\begin{bmatrix} u \\ u_t \\ u_{tt} \\ u_{ttt} \end{bmatrix} \in Y$.

2.2. Subcase $c > ab$

We will show that in this subcase the problem (1.1) is ill-posed, that is, the operator associated with the problem does not generate a strongly continuous semigroup of bounded linear operators.

Proposition 2.6. *If $c > 0$ and $c > ab$ then $Re(z_n) \to +\infty$. Consequently, the unbounded linear operator associated with (1.1) does not generate a strongly continuous semigroup on Y.*

Proof. It follows from Remark (2.3) that $p_n(-a\mu_n^{\frac{1}{5}} - \mu_n^{\frac{1}{10}}) < 0$ for large n. Since $p_n(\lambda) \to +\infty$ as $\lambda \to -\infty$ we guarantee that $x_n < -a\mu_n^{\frac{1}{5}} - \mu_n^{\frac{1}{10}}$. From (2.1) and Lemma 2.1 we obtain $Re(z_n) \to +\infty$. This violates a necessary condition for the generation of a strongly continuous semigroup, see e.g. [23, Chapter 1, Corollary 3.8].

2.3. Subcase $c < ab$

Note that in this subcase we have to assume that $a, b, c \neq 0$. We will see that problem (1.1) can be well-posed in the sense of the theory of semigroups of bounded linear operators. However, the semigroup associated with (1.1) cannot be analytic on Y.

Lemma 2.7. *If $a, b, c \neq 0$ and $c < ab$ then $x_n \to \dfrac{-c\mu_n^{\frac{1}{5}}}{b}$ as $n \to +\infty$.*

Proof. Firstly note that

$$p_n\left(\frac{-c\mu_n^{\frac{1}{5}}}{b}\right) = \frac{c^3(c-ab)}{b^4}\mu_n^{\frac{4}{5}} + \mu_n^{\frac{2}{5}}$$

and therefore $p_n(-cb^{-1}\mu_n^{\frac{1}{5}}) < 0$ for large n. Fixing $\epsilon > 0$ we can easily compute that

$$\begin{aligned} p_n(-cb^{-1}\mu_n^{\frac{1}{5}} - \epsilon) &= \epsilon c\mu_n + (c^4b^{-4} - ac^3b^{-3} + \epsilon^2 b)\mu_n^{\frac{4}{5}} + (4c^3b^{-3}\epsilon - 3ac^2b^{-2}\epsilon)\mu_n^{\frac{3}{5}} \\ &\quad + (1 + 6\epsilon^2c^2b^{-2} - 3\epsilon^2acb^{-1})\mu_n^{\frac{2}{5}} + (4\epsilon^3cb^{-1} - a\epsilon^2)\mu_n^{\frac{1}{5}} + \epsilon^4. \end{aligned}$$

Hence $p_n(-cb^{-1}\mu_n^{\frac{1}{5}} - \epsilon) > 0$ for large n which guarantees that

$$x_n \in (-cb^{-1}\mu_n^{\frac{1}{5}} - \epsilon, -cb^{-1}\mu_n^{\frac{1}{5}})$$

for large n; that is, $x_n \to -cb^{-1}\mu_n^{\frac{1}{5}}$ as $n \to +\infty$.

Corollary 2.8. *If $a, b, c \neq 0$ and $c < ab$ then $Re(z_n) \to (c-ab)2^{-1}b^{-1}\mu_n^{\frac{1}{5}}$. Therefore the problem (1.1) can be well-posed in the sense of the theory of semigroups of bounded linear operators. Moreover, the solution of (1.1) cannot be analytic on Y in the sense of [11].*

Proof. From Vieta's formulas and Lemma 2.7 it follows immediately that $Re(z_n) \to (c-ab)2^{-1}b^{-1}\mu_n^{\frac{1}{5}}$ and that

$$\left(\frac{Im(z_n)}{Re(z_n)}\right)^2 = \frac{4b^3\mu_n^{\frac{2}{5}}}{(c-ab)^2} \to +\infty,$$

which concludes the proof.

3. Case $c = 0$

3.1. Subcase $a = b = c = 0$

In this case, the polynomial is given by

$$p_n(\lambda) = \lambda^4 + \mu_n^{\frac{2}{5}} = 0.$$

It is easy to see that the roots of p_n are $\mu_n^{\frac{1}{10}}e^{\frac{\pi}{4}i}$, $\mu_n^{\frac{1}{10}}e^{\frac{3\pi}{4}i}$, $\mu_n^{\frac{1}{10}}e^{\frac{5\pi}{4}i}$ and $\mu_n^{\frac{1}{10}}e^{\frac{7\pi}{4}i}$. To simplify our view, we can rewrite these roots as

$$\mu_n^{\frac{1}{10}}\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right), \mu_n^{\frac{1}{10}}\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right), \mu_n^{\frac{1}{10}}\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)$$
$$\text{and } \mu_n^{\frac{1}{10}}\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right).$$

Theorem 3.1. *If $a = b = c = 0$, then problem (1.1) is ill-posed in the sense of the theory of semigroups of bounded linear operators on Y.*

Proof. Since $\mu_n \to +\infty$ our next result follows directly from the Hille-Yosida Theorem, see e.g. [23, Chapter 1, Theorem 3.1].

3.2. Subcase $a \neq 0$ and $b = c = 0$

In this case we know that $\Delta_n < 0$ for large n and therefore our polynomial

$$\lambda^4 + a\mu_n^{\frac{1}{5}}\lambda^3 + \mu_n^{\frac{2}{5}}$$

have two real roots x_n, y_n and two complex roots z_n and $\overline{z}_n$.

Lemma 3.2. *If $a \neq 0$ and $b = c = 0$ then $x_n \to -a\mu_n^{\frac{1}{5}}$ as $n \to +\infty$.*

Proof. Given $\epsilon > 0$ we have

$$p_n(-a\mu_n^{\frac{1}{5}}+\epsilon) = -\epsilon a^3\mu_n^{\frac{3}{5}} + (3a^2\epsilon^2+1)\mu_n^{\frac{2}{5}} - 3a\epsilon^3\mu_n^{\frac{1}{5}} + \epsilon^4.$$

Hence $p_n(-a\mu_n^{\frac{1}{5}}+\epsilon) < 0$ for large n. Since $p_n(-a\mu_n^{\frac{1}{5}}) > 0$ for any n, we conclude that $x_n \in (-a\mu_n^{\frac{1}{5}}, -a\mu_n^{\frac{1}{5}}+\epsilon)$ for large n; that is, $x_n \to -a\mu_n^{\frac{1}{5}}$ as $n \to +\infty$.

Lemma 3.3. *If $a \neq 0$ and $b = c = 0$ then $y_n \to -a^{-\frac{1}{3}}\mu_n^{\frac{1}{15}}$ as $n \to +\infty$.*

Proof. Fix $\epsilon > 0$ and note that

$$p_n(-a^{-\frac{1}{3}}\mu_n^{\frac{1}{15}} - \epsilon) = -3\epsilon a^{\frac{1}{3}}\mu_n^{\frac{5}{15}} + (a^{-\frac{4}{3}} - 3\epsilon^2 a^{\frac{2}{3}})\mu_n^{\frac{4}{15}} + (4\epsilon a^{-1} - a\epsilon^3)\mu_n^{\frac{3}{15}} + 6\epsilon^2 a^{-\frac{2}{3}}\mu_n^{\frac{2}{15}} + 4\epsilon^3 a^{-\frac{1}{3}}\mu_n^{\frac{1}{15}} + \epsilon^4.$$

Thus $p_n(-a^{-\frac{1}{3}}\mu_n^{\frac{1}{15}} - \epsilon) < 0$ for large n. Since $p_n(-a^{-\frac{1}{3}}\mu_n^{\frac{1}{15}}) = a^{\frac{-4}{3}}\mu_n^{\frac{4}{15}}$, we guarantee that $p_n(-a^{-\frac{1}{3}}\mu_n^{\frac{1}{15}}) > 0$ for large n and therefore

$$y_n \in (-a^{-\frac{1}{3}}\mu_n^{\frac{1}{15}} - \epsilon, -a^{-\frac{1}{3}}\mu_n^{\frac{1}{15}})$$

when n is large. This means that $|y_n + a^{-\frac{1}{3}}\mu_n^{\frac{1}{15}}| \to 0$ as $n \to +\infty$.

Corollary 3.4. *If $a \neq 0$ and $b = c = 0$ then $Re(z_n) \to +\infty$. Consequently, problem (1.1) is ill-posed in the sense of the theory of semigroups of bounded linear operators on Y.*

3.3. Subscase $b \neq 0$ and $a = c = 0$

In this case we have the polynomial

$$\lambda^4 + b\mu_n^{\frac{4}{5}}\lambda^2 + \mu_n^{\frac{2}{5}} \tag{3.1}$$

which has $\Delta_n > 0$ for large n and $P_n = 8b\mu_n^{\frac{4}{5}} > 0$; that is, our polynomial has two pairs of non-real complex conjugate roots.

Lemma 3.5. *If $b \neq 0$ and $a = c = 0$ then all roots are imaginary for large n. Consequently, problem (1.1) can generate a strongly continuous semigroup of bounded linear operators on Y. Moreover, it cannot generate an analytic semigroup on Y in the sense of [11].*

Proof. Denoting the four roots by $z_n, \overline{z}_n, w_n, \overline{w}_n$ we know from Vieta's formulas that

$$Re(w_n) = -Re(z_n), \quad |z_n|^2|w_n|^2 = \mu_n^{\frac{2}{5}} \tag{3.2}$$

and

$$Re(w_n)(|z_n|^2 - |w_n|^2) = 0.$$

Assume that $Re(w_n) \neq 0$. Then

$$|z_n|^2 = |w_n|^2 = \mu_n^{\frac{1}{5}}. \tag{3.3}$$

Writing w_n on its polar form $w_n = r_n e^{i\theta_n}$ and using (3.1) it follows that

$$r_n^4 e^{i4\theta_n} + b\mu_n^{\frac{4}{5}} r_n^2 e^{i2\theta_n} + \mu_n^{\frac{2}{5}} = 0$$

and therefore

$$r_n^2(r_n^2 sin(4\theta_n) + b\mu_n^{\frac{4}{5}} sin(2\theta_n)) = 0. \tag{3.4}$$

From (3.1) we have $r_n = |w_n| \neq 0$ and this leads to

$$r_n^2 sin(4\theta_n) + b\mu_n^{\frac{4}{5}} sin(2\theta_n) = 0.$$

Case $sin(4\theta_n) = 0$ we have $sin(2\theta_n) = 0$, and $\theta_n = 0$ or $\theta_n = \frac{\pi}{2}$. From this, $sin(4\theta_n) \neq 0$; that is,

$$r_n^2 = \frac{-sin(2\theta_n)b\mu_n^{\frac{4}{5}}}{sin(4\theta_n)}. \tag{3.5}$$

Putting (3.3) and (3.5) together we obtain

$$-sin(2\theta_n)b\mu_n^{\frac{3}{5}} = sin(4\theta_n).$$

Thus

$$sin(2\theta_n)(b\mu_n^{\frac{3}{5}} + 2cos(2\theta_n)) = 0.$$

Since $b\mu_n^{\frac{3}{5}} + 2cos(2\theta_n) \neq 0$ for large n we conclude that $sin(2\theta_n) = 0$ and

$$\theta_n = 0 \quad \text{or} \quad \theta_n = \frac{\pi}{2} \tag{3.6}$$

which contradicts our assumption that $w_n \notin \mathbb{R}$ and $Re(w_n) \neq 0$. Therefore

$$Re(w_n) = Re(z_n) = 0$$

for large n we conclude the proof.

Theorem 3.6. *If $b \neq 0$ and $a = c = 0$ then the problem (1.1) generates a strongly continuous semigroup on Y.*

Proof. It follows similarly to Theorem 2.5.

3.4. Subcase $a, b \neq 0$ and $c = 0$

In this case we have the polynomial

$$\lambda^4 + a\mu_n^{\frac{1}{5}}\lambda^3 + b\mu_n^{\frac{4}{5}}\lambda^2 + \mu_n^{\frac{2}{5}} \tag{3.7}$$

which has $\Delta_n > 0$ for large n. Moreover, $P_n = 8b\mu_n^{\frac{4}{5}} - 3a^2\mu_n^{\frac{2}{5}} > 0$ for large n, which implies that our polynomial has four complex roots $z_n, \overline{z_n}, w_n$ and $\overline{w_n}$.

Lemma 3.7. *Two of the four roots of polynomial (3.7); namely w_n and $\overline{w_n}$ satisfy*

$$|w_n| = |\overline{w_n}| \to b^{-\frac{1}{2}}\mu_n^{-\frac{1}{5}} \quad as \quad n \to +\infty.$$

Proof. From Vieta's formulas, we can easily obtain that

$$|w_n|^2|z_n|^2 = \mu_n^{\frac{2}{5}}, \tag{3.8}$$
$$Re(z_n)|w_n|^2 = -Re(w_n)|z_n|^2 \tag{3.9}$$
$$|z_n|^2 + |w_n|^2 + 4Re(z_n)Re(w_n) = b\mu_n^{\frac{4}{5}} \tag{3.10}$$

and

$$Re(w_n) + Re(z_n) = \frac{-a\mu_n^{\frac{1}{5}}}{2}. \tag{3.11}$$

Since $w_n, z_n \neq 0$ we can put (3.8) and (3.9) into Eq. (3.10) to obtain that

$$\frac{\mu_n^{\frac{2}{5}}}{|w_n|^2} + |w_n|^2 - \frac{4Re(w_n)^2\mu_n^{\frac{2}{5}}}{|w_n|^4} = b\mu_n^{\frac{4}{5}}.$$

which leads us to

$$\mu_n^{\frac{2}{5}}|w_n|^2 + |w_n|^6 - 4Re(w_n)^2\mu_n^{\frac{2}{5}} - b\mu_n^{\frac{4}{5}}|w_n|^4 = 0. \tag{3.12}$$

Now, putting (3.9) and (3.11) together we have

$$\frac{-a\mu_n^{\frac{1}{5}}|w_n|^2}{2} = Re(w_n)(|w_n|^2 - |z_n|^2). \tag{3.13}$$

Since $w_n \neq 0$ we know from (3.13) that $|w_n|^2 - |z_n|^2 \neq 0$ and consequently

$$Re(w_n) = \frac{-a\mu_n^{\frac{1}{5}}|w_n|^2}{2(|w_n|^2 - |z_n|^2)}.$$

Putting this equation together with (3.8) we have (the well-defined) equation

$$Re(w_n) = \frac{-a\mu_n^{\frac{1}{5}}|w_n|^4}{2(|w_n|^4 - \mu_n^{\frac{2}{5}})}. \tag{3.14}$$

Finally, putting (3.12) and (3.14) together we obtain

$$\mu_n^{\frac{2}{5}}|w_n|^2 + |w_n|^6 - \frac{a^2\mu_n^{\frac{4}{5}}|w_n|^8}{|w_n|^8 - 2\mu_n^{\frac{2}{5}}|w_n|^4 + \mu_n^{\frac{4}{5}}} - b\mu_n^{\frac{4}{5}}|w_n|^4 = 0.$$

From this we have

$$(\mu_n^{\frac{2}{5}}|w_n|^2 + |w_n|^6 - b\mu_n^{\frac{4}{5}}|w_n|^4)(|w_n|^8 - 2\mu_n^{\frac{2}{5}}|w_n|^4 + \mu_n^{\frac{4}{5}}) - a^2\mu_n^{\frac{4}{5}}|w_n|^8 = 0$$

and consequently

$$|w_n|^{14} - b\mu_n^{\frac{4}{5}}|w_n|^{12} - \mu_n^{\frac{2}{5}} \,\|w_n\|^{10} + (2b\mu_n^{\frac{6}{5}} - a^2\mu_n^{\frac{4}{5}})|w_n|^8 \\ - \mu_n^{\frac{4}{5}}|w_n|^6 - b\mu_n^{\frac{8}{5}}|w_n|^4 + \mu_n^{\frac{6}{5}}|w_n|^2 = 0.$$

Dividing both sides by $|w_n|^2$ we finally have

$$|w_n|^{12} - b\mu_n^{\frac{4}{5}}|w_n|^{10} - \mu_n^{\frac{2}{5}}|w_n|^8 + (2b\mu_n^{\frac{6}{5}} - a^2\mu_n^{\frac{4}{5}})|w_n|^6 - \mu_n^{\frac{4}{5}}|w_n|^4 - b\mu_n^{\frac{8}{5}}|w_n|^2 + \mu_n^{\frac{6}{5}} = 0.$$

Define the polynomial $q_n : \mathbb{R} \to \mathbb{R}$ as

$$q_n(x) = x^{12} - b\mu_n^{\frac{4}{5}}x^{10} - \mu_n^{\frac{2}{5}}x^8 + (2b\mu_n^{\frac{6}{5}} - a^2\mu_n^{\frac{4}{5}})x^6 - \mu_n^{\frac{4}{5}}x^4 - b\mu_n^{\frac{8}{5}}x^2 + \mu_n^{\frac{6}{5}}.$$

From

$$q_n(b^{-\frac{1}{2}}\mu_n^{-\frac{1}{5}}) = b^{-6}\mu_n^{-\frac{12}{5}} - 2b^{-4}\mu_n^{-\frac{6}{5}} - a^2b^{-3}\mu_n^{-\frac{2}{5}} - 2b^{-2}$$

we obtain that $q_n(b^{-\frac{1}{2}}\mu_n^{-\frac{1}{5}}) < 0$ for large n. It is easy to see that for any $\epsilon > 0$ we have $q_n(b^{-\frac{1}{2}}\mu_n^{-\frac{1}{5}} - \epsilon) > 0$ for large n. Thus, given $\epsilon > 0$ we have

$$|w_n| = |\overline{w_n}| \in (b^{-\frac{1}{2}}\mu_n^{-\frac{1}{5}} - \epsilon, b^{-\frac{1}{2}}\mu_n^{-\frac{1}{5}})$$

for large n, which completes the proof.

Lemma 3.8. *Assume that w_n is a root of polynomial (3.7) such that*

$$|w_n| \to b^{-\frac{1}{2}}\mu_n^{-\frac{1}{5}}$$

as $n \to +\infty$. Then $\dfrac{Re(w_n)}{Im(w_n)} \to 0$ as $n \to +\infty$.

Proof. We will denote the polar form of w_n as $r_n e^{i\theta_n}$. Remember that

$$r_n \to b^{-\frac{1}{2}}\mu_n^{-\frac{1}{5}} \quad \text{as} \quad n \to +\infty.$$

We just need to prove that $\theta_n \to \dfrac{\pi}{2}$ or $\theta_n \to \dfrac{3\pi}{2}$.

Since w_n is a root of (3.7), we have

$$r_n^2(r_n^2 cos(4\theta_n) + a\mu_n^{\frac{1}{5}}r_n cos(3\theta_n) + b\mu_n^{\frac{4}{5}}cos(2\theta_n)) = -\mu_n^{\frac{2}{5}}.$$

It is easy to see that a root from (3.7) must be different from zero. Thus

$$r_n^2 cos(4\theta_n) + a\mu_n^{\frac{1}{5}}r_n cos(3\theta_n) + b\mu_n^{\frac{4}{5}}cos(2\theta_n) = -\mu_n^{\frac{2}{5}}r_n^{-2}.$$

Since $-\mu_n^{\frac{2}{5}}r_n^{-2} \to -b\mu_n^{\frac{4}{5}}$ we conclude that

$$r_n^2 cos(4\theta_n) + a\mu_n^{\frac{1}{5}}r_n cos(3\theta_n) + b\mu_n^{\frac{4}{5}}cos(2\theta_n) \to -b\mu_n^{\frac{4}{5}}.$$

From the fact that $r_n \to b^{-\frac{1}{2}}\mu_n^{-\frac{1}{5}}$ it is easy to see that $\theta_n \to \dfrac{\pi}{2}$ or $\theta_n \to \dfrac{3\pi}{2}$ as $n \to +\infty$.

Corollary 3.9. *There exist two roots of polynomial (3.7); namely w_n and $\overline{w_n}$ such that $w_n \to b^{-\frac{1}{2}}\mu_n^{-\frac{1}{5}}i$ and $\overline{w_n} \to -b^{-\frac{1}{2}}\mu_n^{-\frac{1}{5}}i$ as $n \to +\infty$.*

Proof. It follows immediately from Lemmas 3.7 and 3.8.

Corollary 3.10. *If $a, b \neq 0$ and $c = 0$ then the problem (1.1) can generate a strongly continuous semigroup. Furthermore, the semigroup cannot be of an analytic type on Y in the sense of [11].*

Proof. From Corollary 3.9 and Vieta's formulas, we have that

$$Re(z_n) \to -\frac{a}{2}\mu_n^{\frac{1}{5}} \text{ as } n \to +\infty.$$

This means that problem (1.1) can generate a strongly continuous semigroup. Once again from Vieta's formulas we can obtain that

$$(Im(z_n))^2 \to b\mu_n^{\frac{4}{5}} - \frac{a^2}{4}\mu_n^{\frac{2}{5}} \text{ as } n \to +\infty,$$

implying that

$$\frac{(Im(z_n))^2}{(Re(z_n))^2} \to +\infty \text{ as } n \to +\infty.$$

Therefore, problem (1.1) cannot have an analytic semigroup on Y as a solution operator.

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Conflict of interest Data sharing does not apply to this article as no datasets were generated or analyzed during the current study. The authors have no relevant financial or non-financial interests to disclose.

References

[1] Abouelregal, A.E., Sedighi, H.M., Eremeyev, V.A.: Thermomagnetic behavior of a semiconductor material heated by pulsed excitation based on the fourth-order MGT photothermal model. Continuum Mech. Thermodyn. (2022). https://doi.org/10.1007/s00161-022-01170-z

[2] Amann, H.: Linear and quasilinear parabolic problems. Volume I: Abstract Linear Theory, Birkhäuser Verlag, Basel (1995)

[3] Bezerra, F.D.M., Santos, L.A.: Fractional powers approach of operators for abstract evolution equations of third order in time. J. Differ. Equ. **269**(7), 5661–5679 (2020)

[4] Bezerra, F.D.M., Santos, L.A.: Chebyshev polynomials for higher order differential equations and fractional powers. Math. Ann. (2022). https://doi.org/10.1007/s00208-022-02554-x

[5] Bezerra, F. D. M., Santos, L. A.: An extended version of the Cayley–Hamilton–Ziebur Theorem (2022). arxiv:2210.01976

[6] Caixeta, A.H., Lasiecka, I., Cavalcanti, V.N.D.: Global attractors for a third order in time nonlinear dynamics. J. Differ. Equ. **261**(1), 113–147 (2016)

[7] Dell'Oro, F., Pata, V.: On a fourth-order equation of Moore–Gibson–Thompson type. Milan J. Math. **85**, 215–234 (2017)

[8] Fattorini, H.O.: The cauchy problem. In: Encyclopedia of Mathematics and its Applications, vol. 18. Addison-Wesley Publishing Company, Reading, Massachusetts (1983)

[9] Fattorini, H.O.: Ordinary differential equations in linear topological spaces, I. J. Differ. Equ. **5**, 72–105 (1968)

[10] Hasse, M.: The Functional Calculus for Sectorial Operators. Birkhäuser Verlag (2006)

[11] Henry, D.: Geometric theory of semilinear parabolic equations. Lecture Notes in Mathematics, vol. 840. Springer-Verlag, Berlin (1981)

[12] Kaltenbacher, B., Lasiecka, I., Marchand, R.: Well-posedness and exponential decay rates for the Moore–Gibson–Thompson equation arising in high intensity ultrasound. Control Cybernet. **40**(4), 971–988 (2011)

[13] Kreĭn, S. G.: Linear differential equations in a banach space. Transl. Mathem. Monogr., 29, American Mathematical Soc. (1971)

[14] Lange, S.: Algebra, Graduate Texts in Mathematics 211, 3rd edn. Springer-Verlag, New York (2002)

[15] Liu, W., Chen, Z., Tiu, Z.: New general decay result for a fourth-order Moore–Gibson–Thompson equation with memory. Electron. Res. Arch. **28**(1), 433–457 (2020)

[16] Lizama, C., Murillo, M.: Well-posedness for a fourth-order equation of Moore–Gibson–Thompson type. Electron. J. Qual. Theory Differ. Equ. **81**, 1–18 (2021)

[17] Marchand, R., McDevitt, T., Triggiani, R.: An abstract semigroup approach to the third-order Moore–Gibson–Thompson partial differential equation arising in high-intensity ultrasound: structural decomposition, spectral analysis, exponential stability. Math. Methods Appl. Sci. **35**(15), 1896–1929 (2012)

[18] Martínez, C., Sanz, M.: Spectral mapping theorem for fractional powers in locally convex spaces, Ann. Scuola Norm. - SCI 4^e série, tome **24**(4), 685–702 (1997)

[19] Murillo-Arcila, M.: Well-posedness for the fourth-order Moore–Gibson–Thompson equation in the class of Banach-space-valued Hölder-continuous functions. Math. Methods Appl. Sci. **46**(2), 1928–1937 (2023)

[20] Mesloub, A., Zarai, A., Mesloub, F., Cherif, B.B., Abdalla, M.: The Galerkin method for fourth-order equation of the Moore–Gibson–Thompson type with integral condition. Adv. Math. Phys. (2021). https://doi.org/10.1155/2021/5532691

[21] Mesloub, F., Merah, A., Boulaaras, S.: Solution blow-up for a fractional fourth-order equation of Moore–Gibson–Thompson type with nonlinearity nonlocal in time. Math. Notes **113**(1), 72–80 (2023)

[22] Neubrander, F.: Well-posedness of higher order abstract Cauchy problems. Trans. Am. Math. Soc. **295**, 257–290 (1986)

[23] Pazy, A.: Semigroup of linear operators and applications to partial differential equations. Springer-Verlag, New York (1983)

[24] Pellicer. M., Said-Houari, B.: Wellposedness and decay rates for the cauchy problem of the Moore–Gibson–Thompson equation arising in high intensity ultrasound. Appl. Math. Optim. 1–32 (2017)

[25] Pellicer, M., Solà-Morales, J.: Optimal scalar products in the Moore–Gibson–Thompson equation. Evol. Equ. Control Theory **8**(1), 203–220 (2019)

[26] Sobolevskiĭ, P.E.: Equations of parabolic type in a Banach space. Am. Math. Soc. Transl. **49**, 1–62 (1966)

[27] Triebel, H.: Interpolation theory, function spaces, differential operators, Veb Deutscher (1978)

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