Results Math (2023) 78:88

1422-6383/23/030001-10
published online February 28, 2023
https://doi.org/10.1007/s00025-023-01870-1

Results in Mathematics

Remarks on a Question of Bourin for Positive Semidefinite Matrices

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Abstract. Let A and B be positive semidefinite matrices. For $t \in \left[\frac{3}{4}, 1\right]$ and for every unitarily invariant norm, it is shown that

$$|||A^tB^{1-t} + B^tA^{1-t}||| \leq 2^{2\left(t-\frac{3}{4}\right)}|||A + B|||$$

and for $t \in \left[0, \frac{1}{4}\right]$,

$$|||A^tB^{1-t} + B^tA^{1-t}||| \leq 2^{2\left(\frac{1}{4}-t\right)}|||A + B|||.$$

These norm inequalities are sharper than an earlier norm inequality due to Alakhrass and closely related to an open question of Bourin. In fact, they lead to an affirmative solution of Bourin's question for $t = \frac{1}{4}$ and $\frac{3}{4}$, which is a result due to Hayajneh and Kittaneh (Int J Math 32 (2150043):7, 2021).

Mathematics Subject Classification. Primary 15A60; Secondary 15B57, 47A30, 47B15.

Keywords. Unitarily invariant norm, trace norm, positive semidefinite matrix, Bourin's question, inequality.

1. Introduction

Throughout this paper, all matrices are assumed to be complex square matrices of the same size. Let A and B be positive semidefinite matrices. For $t \in [0, 1]$, let

$$b_t = A^tB^{1-t} + B^tA^{1-t}.$$

In this paper, $|||.|||$ denotes any unitarily invariant norm on the space of matrices. Among the most important unitarily invariant norms are the usual operator (or the spectral) norm $\|.\|_\infty$ and the Schatten $p-$norm $\|.\|_p$ for $1 \leq p < \infty$.

In [10], and in his work on the subadditivity of concave functions of positive semidefinite matrices, J.C. Bourin asked the following question.

Question 1.1. *Let A and B be any two positive semidefinite matrices, and let $t \in [0,1]$. Is it true that*

$$|||b_t||| \le |||A+B|||?$$

Question 1.1 has been answered for the Hilbert-Schmidt norm under the condition $t \in \left[\frac{1}{4}, \frac{3}{4}\right]$, see [5,19], and [21].

In [13], Hayajneh and Kittaneh proved the following norm inequality for all unitarily invariant norms:

$$|||b_t||| \le |||A||| + |||B|||,$$

where A and B are positive semidefinite matrices and $t \in [0,1]$. Consequently,

$$\|b_t\|_1 \le \|A+B\|_1$$

for $t \in [0,1]$. This gives an affirmative answer to Question 1.1 for the trace norm.

In [18], the authors proved the following stronger norm inequality for all unitarily invariant norms:

$$|||b_t||| \le \sqrt{|||A+B|||\,(|||A||| + |||B|||)},$$

where A and B are positive semidefinite matrices and $t \in [0,1]$.

They also gave an affirmative answer to Question 1.1 in the case of $t = \frac{1}{4}$ and $t = \frac{3}{4}$ for all unitarily invariant norms. Clearly, Question 1.1 is true for $t = 0$ and $t = 1$.

Moreover, a partial answer to this question in the wider class of unitarily invariant norms has been given in [13] by proving that

$$|||\mathrm{Re}\ b_t||| \le |||A+B|||$$

and

$$|||\mathrm{Im}\ b_t||| \le |||A+B|||.$$

In [1], Alakhrass proved the following norm inequality for all unitarily invariant norms:

$$|||b_t||| \le 2^{\left|\frac{1}{2}-t\right|}|||A+B|||, \tag{1.1}$$

where A and B are positive semidefinite matrices and $t \in [0,1]$.

In [20], and in order to study Question 1.1, the authors proved the following inequality:

$$\|b_t\|_p \le 2^{\frac{1}{2}-\frac{1}{2p}}\,\|A+B\|_p\,, \tag{1.2}$$

where A and B are positive semidefinite matrices, $t \in [0,1]$, and $p \ge 1$. In particular,

$$\|b_t\|_\infty \le 2^{\frac{1}{2}}\,\|A+B\|_\infty\,.$$

Note that for the trace norm, the inequality (1.2) is sharper than the inequality (1.1) for all $t \in [0,1]$. For the Hilbert-Schmidt norm, the inequality (1.2) is sharper than the inequality (1.1) for $t \in [0, \frac{1}{4}]$ and $t \in [\frac{3}{4}, 1]$. However, the inequality (1.1) is sharper than the inequality (1.2) for $t \in [\frac{1}{4}, \frac{3}{4}]$. But we already know that Bourin's question is true for $t \in [\frac{1}{4}, \frac{3}{4}]$ for the Hilbert-Schmidt norm.

In [12], and in order to try to answer Question 1.1, the authors proved the following inequality involving the spectral radius $r\,(.)$:

$$|||b_t||| \leq |||A+B|||\sqrt{\max\left(r\left(B^{2t-1}A^{1-2t}\right), r\left(A^{2t-1}B^{1-2t}\right)\right)},$$

where A and B are positive definite matrices and $t \in [0,1]$.

Consequently, the authors proved that if $B \leq A \leq (1+\epsilon)^2 B$, then

$$|||b_t||| \leq (1+\epsilon)\, |||A+B|||$$

for $t \in [0,1]\,, \epsilon > 0$, and for every unitarily invariant norm. Equivalently, if $\alpha \geq 1$ and if the spectrum of AB^{-1} lies in the interval $[1, \alpha]$, then

$$|||b_t||| \leq \sqrt{\alpha}|||A+B|||.$$

For a comprehensive account on related trace and norm inequalities, we refer to [2,3,7–18,20–22,25–28], and references therein.

In Section 2, we introduce a new way of proving the inequality (1.1) without using the notion of majorization.

In Section 2, we will prove the following norm inequalities for all unitarily invariant norms:

$$|||b_t||| \leq 2^{2\left(t-\frac{3}{4}\right)}|||A+B|||,$$

where A and B are positive semidefinite matrices and $t \in \left[\frac{3}{4}, 1\right]$, and

$$|||b_t||| \leq 2^{2\left(\frac{1}{4}-t\right)}|||A+B|||,$$

where A and B are positive semidefinite matrices and $t \in \left[0, \frac{1}{4}\right]$. These norm inequalities are sharper than the norm inequality (1.1) and closely related to Question 1.1. In fact, they lead to solve Question 1.1 in the case of $t = \frac{3}{4}$ and $t = \frac{1}{4}$ for all unitarily invariant norms, which is a result due to Hayajneh and Kittaneh [18].

2. Main Result

We begin with the following lemmas that will be used in proving our main result.

The following lemma can be found in [4, p. 95]. It contains the celebrated Hölder inequality for all unitarily invariant norms.

Lemma 2.1. *Let A and B be any two matrices, $p > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then for every unitarily invariant norm, we have*

$$\|\!|AB|\!\|\leq \|\!|\,|A|^p|\!\|^{\frac{1}{p}}\,\|\!|\,|B|^q|\!\|^{\frac{1}{q}}.$$

The following lemma, concerning the convexity and concavity of the power functions of positive semidefinite matrices, can be found in [23].

Lemma 2.2. *Let A and B be any two positive semidefinite matrices. Then for every unitarily invariant norm, we have*

$$\|\!|(A+B)^r|\!\| \leq 2^{r-1}\|\!|A^r + B^r|\!\| \textit{ for } r \geq 1$$

and

$$\|\!|(A+B)^r|\!\| \leq \|\!|A^r + B^r|\!\| \textit{ for } 0 \leq r \leq 1.$$

The following lemma can be found in [4, p. 258].

Lemma 2.3. *Let A and B be any two matrices. Then for every unitarily invariant norm, we have*

$$\|\!|(ABA)^r|\!\| \leq \|\!|A^r B^r A^r|\!\| \textit{ for } r \geq 1.$$

The following lemma can be found in [4, p. 253].

Lemma 2.4. *Let A and B be any two matrices such that the product AB is normal. Then for every unitarily invariant norm, we have*

$$\|\!|AB|\!\| \leq \|\!|BA|\!\|.$$

The following lemma can be found in [6]. It contains the celebrated Heinz inequalities for all unitarily invariant norms.

Lemma 2.5. *Let A and B be positive semidefinite matrices. Then for $t \in [0,1]$ and for every unitarily invariant norm, we have*

$$2\left\|\!\left|A^{\frac{1}{2}}B^{\frac{1}{2}}\right|\!\right\| \leq \left\|\!\left|A^tB^{1-t} + A^{1-t}B^t\right|\!\right\| \leq \|\!|A+B|\!\|.$$

Convention 2.6. *For any matrix T and for every unitarily invariant norm, we have*

$$\|\!|T \oplus 0|\!\| = \|\!|T|\!\|.$$

In the following theorem, we introduce a new way of proving the inequality (1.1) without using the notion of majorization.

Theorem 2.7. *Let A and B be positive semidefinite matrices, and let $t \in [0,1]$. Then for every unitarily invariant norm, we have*

$$\|\!|b_t|\!\| \leq 2^{\left|\frac{1}{2}-t\right|}\|\!|A+B|\!\|.$$

Proof. For the end points $t = 0$ and $t = 1$, the result is obvious.

Case 1: $t \in \left(0, \frac{1}{2}\right]$. In this case, let

$$X = \begin{bmatrix} A^t & B^t \\ 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} B^{1-t} & 0 \\ A^{1-t} & 0 \end{bmatrix}.$$

For the partition $\frac{1}{p} + \frac{1}{q} = t + (1-t) = 1$ and noting that $\frac{p}{2} = \frac{1}{2t} \geq 1$ and $\frac{q}{2} = \frac{1}{2(1-t)} \leq 1$, we have

$$\begin{aligned}
|||b_t||| &= |||b_t \oplus 0||| \\
&= \left|\left|\left| \begin{bmatrix} A^t & B^t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B^{1-t} & 0 \\ A^{1-t} & 0 \end{bmatrix} \right|\right|\right| \\
&= |||XY||| \\
&\leq |||\,|X|^p|||^{\frac{1}{p}} |||\,|Y|^q|||^{\frac{1}{q}} \text{ (by Lemma 2.1)} \\
&= \left|\left|\left| (X^*X)^{\frac{p}{2}} \right|\right|\right|^{\frac{1}{p}} \left|\left|\left| (Y^*Y)^{\frac{q}{2}} \right|\right|\right|^{\frac{1}{q}} \\
&= \left|\left|\left| (XX^*)^{\frac{p}{2}} \right|\right|\right|^{\frac{1}{p}} \left|\left|\left| (Y^*Y)^{\frac{q}{2}} \right|\right|\right|^{\frac{1}{q}} \\
&= \left|\left|\left| \left(A^{2t} + B^{2t}\right)^{\frac{p}{2}} \oplus 0 \right|\right|\right|^{\frac{1}{p}} \left|\left|\left| \left(A^{2(1-t)} + B^{2(1-t)}\right)^{\frac{q}{2}} \oplus 0 \right|\right|\right|^{\frac{1}{q}} \\
&= \left|\left|\left| \left(A^{2t} + B^{2t}\right)^{\frac{p}{2}} \right|\right|\right|^{\frac{1}{p}} \left|\left|\left| \left(A^{2(1-t)} + B^{2(1-t)}\right)^{\frac{q}{2}} \right|\right|\right|^{\frac{1}{q}} \\
&\leq \left(2^{\frac{p}{2}-1}\right)^{\frac{1}{p}} |||A^{pt} + B^{pt}|||^{\frac{1}{p}} \left|\left|\left| A^{q(1-t)} + B^{q(1-t)} \right|\right|\right|^{\frac{1}{q}} \text{ (by Lemma 2.2)} \\
&= 2^{\frac{1}{2}-t} |||A + B|||^{\frac{1}{p}+\frac{1}{q}} \\
&= 2^{\frac{1}{2}-t} |||A + B|||.
\end{aligned}$$

Case 2: $t \in \left[\frac{1}{2}, 1\right)$. In this case,

$$\begin{aligned}
|||b_t||| &= |||b_t^*||| \\
&= |||b_{1-t}||| \text{ (because } \ b_t^* = b_{1-t}) \\
&\leq 2^{t-\frac{1}{2}} |||A + B||| \left(\text{since } 1 - t \in \left[0, \quad \frac{1}{2}\right] \text{ and by using Case 1} \right).
\end{aligned}$$

This completes the proof. □

To prove our main result, we state and prove the following lemma.

Lemma 2.8. *Let A and B be positive semidefinite matrices, and let $t \in \left[\frac{3}{4}, 1\right)$. Let*

$$X = \begin{bmatrix} A^{2t-1} & B^{2t-1} \\ 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} A^{1-t}B^{1-t} & 0 \\ B^{1-t}A^{1-t} & 0 \end{bmatrix}.$$

Then for every unitarily invariant norm and for $p = \frac{1}{2t-1}$ *and* $q = \frac{1}{2(1-t)}$*, we have*

$$\left|\left|\left|(XX^*)^{\frac{p}{2}}\right|\right|\right| \leq |||A+B||| \text{ and } \left|\left|\left|(Y^*Y)^{\frac{q}{2}}\right|\right|\right| \leq 2^{q\left(\frac{1}{2}-\frac{1}{q}\right)}|||A+B|||.$$

Proof. Since $t \in \left[\frac{3}{4}, 1\right)$, it follows that $\frac{p}{2} \leq 1$ and $\frac{q}{2} \geq 1$. In fact,

$$t \geq \frac{3}{4} \iff 2(2t-1) \geq 1 \iff \frac{2}{p} \geq 1 \iff \frac{p}{2} \leq 1$$

and

$$t \geq \frac{3}{4} \iff 4(1-t) \leq 1 \iff \frac{2}{q} \leq 1 \iff \frac{q}{2} \geq 1.$$

Now,

$$\begin{aligned}
\left|\left|\left|(XX^*)^{\frac{p}{2}}\right|\right|\right| &= \left|\left|\left|\left(\left(A^{2(2t-1)} + B^{2(2t-1)}\right) \oplus 0\right)^{\frac{p}{2}}\right|\right|\right| \\
&= \left|\left|\left|\left(A^{2(2t-1)} + B^{2(2t-1)}\right)^{\frac{p}{2}}\right|\right|\right| \\
&= \left|\left|\left|\left(A^{\frac{2}{p}} + B^{\frac{2}{p}}\right)^{\frac{p}{2}}\right|\right|\right| \\
&\leq \left|\left|\left|A^{\frac{2}{p}\frac{p}{2}} + B^{\frac{2}{p}\frac{p}{2}}\right|\right|\right| \text{ (by Lemma 2.2)} \\
&= |||A+B|||.
\end{aligned}$$

Also,

$$\begin{aligned}
\left|\left|\left|(Y^*Y)^{\frac{q}{2}}\right|\right|\right| &= \left|\left|\left|\left(\left(B^{1-t}A^{2(1-t)}B^{1-t} + A^{1-t}B^{2(1-t)}A^{1-t}\right) \oplus 0\right)^{\frac{q}{2}}\right|\right|\right| \\
&= \left|\left|\left|\left(B^{1-t}A^{2(1-t)}B^{1-t} + A^{1-t}B^{2(1-t)}A^{1-t}\right)^{\frac{q}{2}}\right|\right|\right| \\
&= \left|\left|\left|\left(B^{\frac{1}{2q}}A^{\frac{1}{q}}B^{\frac{1}{2q}} + A^{\frac{1}{2q}}B^{\frac{1}{q}}A^{\frac{1}{2q}}\right)^{\frac{q}{2}}\right|\right|\right| \\
&\leq 2^{\frac{q}{2}-1}\left|\left|\left|\left(B^{\frac{1}{2q}}A^{\frac{1}{q}}B^{\frac{1}{2q}}\right)^{\frac{q}{2}} + \left(A^{\frac{1}{2q}}B^{\frac{1}{q}}A^{\frac{1}{2q}}\right)^{\frac{q}{2}}\right|\right|\right| \text{ (by Lemma 2.2)} \\
&\leq 2^{\frac{q}{2}-1}\left(\left|\left|\left|\left(B^{\frac{1}{2q}}A^{\frac{1}{q}}B^{\frac{1}{2q}}\right)^{\frac{q}{2}}\right|\right|\right| + \left|\left|\left|\left(A^{\frac{1}{2q}}B^{\frac{1}{q}}A^{\frac{1}{2q}}\right)^{\frac{q}{2}}\right|\right|\right|\right) \\
&\leq 2^{\frac{q}{2}-1}\left(\left|\left|\left|B^{\frac{1}{4}}A^{\frac{1}{2}}B^{\frac{1}{4}}\right|\right|\right| + \left|\left|\left|A^{\frac{1}{4}}B^{\frac{1}{2}}A^{\frac{1}{4}}\right|\right|\right|\right) \text{ (by Lemma 2.3)} \\
&\leq 2^{\frac{q}{2}-1}\left(2\left|\left|\left|A^{\frac{1}{2}}B^{\frac{1}{2}}\right|\right|\right|\right) \text{ (by Lemma 2.4)} \\
&\leq 2^{\frac{q}{2}-1}|||A+B||| \text{ (by Lemma 2.5)} \\
&= 2^{q\left(\frac{1}{2}-\frac{1}{q}\right)}|||A+B|||.
\end{aligned}$$

This completes the proof. □

Now, we are in a position to state and prove our main result.

Theorem 2.9. *Let A and B be positive semidefinite matrices, and let $t \in \left[\frac{3}{4}, 1\right]$. Then for every unitarily invariant norm, we have*

$$|||b_t||| \leq 2^{2\left(t-\frac{3}{4}\right)}|||A+B|||. \tag{2.1}$$

Proof. The result is obvious for $t = 1$. Now, for $t \in \left[\frac{3}{4}, 1\right)$, let

$$X = \begin{bmatrix} A^{2t-1} & B^{2t-1} \\ 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} A^{1-t}B^{1-t} & 0 \\ B^{1-t}A^{1-t} & 0 \end{bmatrix}.$$

For the partition $\frac{1}{p} + \frac{1}{q} = (2t-1) + 2(1-t) = 1$, noting that $1 \leq p \leq 2$ and $2 \leq q < \infty$, we have

$$\begin{aligned}
|||b_t||| &= |||b_t \oplus 0||| \\
&= \left|\left|\left| \begin{bmatrix} A^{2t-1} & B^{2t-1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A^{1-t}B^{1-t} & 0 \\ B^{1-t}A^{1-t} & 0 \end{bmatrix} \right|\right|\right| \\
&= |||XY||| \\
&\leq |||\,|X|^p|||^{\frac{1}{p}}\, |||\,|Y|^q|||^{\frac{1}{q}} \text{ (by Lemma 2.1)} \\
&= \left|\left|\left|(X^*X)^{\frac{p}{2}}\right|\right|\right|^{\frac{1}{p}} \left|\left|\left|(Y^*Y)^{\frac{q}{2}}\right|\right|\right|^{\frac{1}{q}} \\
&= \left|\left|\left|(XX^*)^{\frac{p}{2}}\right|\right|\right|^{\frac{1}{p}} \left|\left|\left|(Y^*Y)^{\frac{q}{2}}\right|\right|\right|^{\frac{1}{q}} \\
&\leq \left(|||A+B|||^{\frac{1}{p}}\right)\left(2^{\frac{1}{2}-\frac{1}{q}}|||A+B|||^{\frac{1}{q}}\right) \text{ (by Lemma 2.8)} \\
&= 2^{\frac{1}{2}-\frac{1}{q}}|||A+B|||^{\frac{1}{p}+\frac{1}{q}} \\
&= 2^{\frac{1}{2}-\frac{1}{q}}|||A+B||| \\
&= 2^{\frac{1}{2}-2(1-t)}|||A+B||| \\
&= 2^{2\left(t-\frac{3}{4}\right)}|||A+B|||.
\end{aligned}$$

This completes the proof. □

Based on Theorem 2.9, we have the following theorem.

Theorem 2.10. *Let A and B be positive semidefinite matrices, and let $t \in \left[0, \frac{1}{4}\right]$. Then for every unitarily invariant norm, we have*

$$|||b_t||| \leq 2^{2\left(\frac{1}{4}-t\right)}|||A+B|||. \tag{2.2}$$

Proof. We have

$$\begin{aligned}
|||b_t||| &= |||b_t^*||| \\
&= |||b_{1-t}||| \text{ (because } b_t^* = b_{1-t}) \\
&\leq 2^{2\left(\frac{1}{4}-t\right)}|||A+B||| \left(\text{ since } 1-t \in \left[\frac{3}{4}, 1\right] \text{ and by using Theorem 2.9}\right).
\end{aligned}$$

This completes the proof. □

Remark 2.11. The inequality (2.1) is sharper than the inequality (1.1) for $t \in \left[\frac{3}{4}, 1\right]$, and the inequality (2.2) is sharper than the inequality (1.1) for $t \in \left[0, \frac{1}{4}\right]$. In fact, for $t \in \left[\frac{3}{4}, 1\right]$, we have $2\left(t - \frac{3}{4}\right) \leq t - \frac{1}{2} \Longleftrightarrow t \leq 1$, and for $t \in \left[0, \frac{1}{4}\right]$, we have $2\left(\frac{1}{4} - t\right) \leq \frac{1}{2} - t \Longleftrightarrow 0 \leq t$.

Remark 2.12. Theorem 2.9 and Theorem 2.10 lead to an affirmative solution of Question 1.1 in the case of $t = \frac{3}{4}$ and $t = \frac{1}{4}$ for all unitarily invariant norms, which is a result due to Hayajneh and Kittaneh [18].

Author contributions All authors contributed to each part of this work equally, and they all read and approved the final manuscript.

Funding The authors have not disclosed any funding.

Data Availability Statement Not applicable.

Declarations

Conflict of interest. The authors declare that they have no competing interests.

References

[1] Alakhrass, M.: Inequalities related to Heinz mean. Linear Multilinear Algebra **64**, 1562–1569 (2016)

[2] Ando, T., Hiai, F., Okubo, K.: Trace inequalities for multiple products of two matrices. Math. Inequal. Appl. **3**, 307–318 (2000)

[3] Audenaert, K.: A norm inequality for pairs of commuting positive semidefinite matrices. Electron. J. Linear Algebra **30**, 80–84 (2015)

[4] Bhatia, R.: Matrix Analysis. Springer, New York (1997)

[5] Bhatia, R.: Trace inequalities for products of positive definite matrices. J. Math. Phys. **55**(013509), 3 (2014)

[6] Bhatia, R., Davis, C.: More matrix forms of the arithmetic-geometric mean inequality. SIAM J. Matrix Anal. **14**, 132–136 (1993)

[7] Bottazzi, T., Elencwajg, R., Larotonda, G., Varela, A.: Inequalities related to Bourin and Heinz means with a complex parameter. J. Math. Anal. Appl. **426**, 765–773 (2015)

[8] Bourin, J.C.: Some inequalities for norms on matrices and operators. Linear Algebra Appl. **292**, 139–154 (1999)

[9] Bourin, J.C.: Matrix versions of some classical inequalities. Linear Algebra Appl. **416**, 890–907 (2006)

[10] Bourin, J.C.: Matrix subadditivity inequalities and block-matrices. Int. J. Math. **20**, 679–691 (2009)

[11] Bourin, J.C., Lee, E.Y.: Matrix inequalities from a two variables functional. Int. J. Math. **27**(1650071), 19 (2016)

[12] Darweesh, A., Hayajneh, M., Hayajneh, S., Kittaneh, F.: Norm inequalities for positive definite matrices related to a question of Bourin. Linear Multilinear Algebra (2022). https://doi.org/10.1080/03081087.2022.2091507

[13] Hayajneh, M., Hayajneh, S., Kittaneh, F.: Norm inequalities for positive semidefinite matrices and a question of Bourin. Int. J. Math. **28**(1750102), 7 (2017)

[14] Hayajneh, M., Hayajneh, S., Kittaneh, F.: On the Ando-Hiai-Okubo trace inequality. J. Operator Theory **77**, 77–86 (2017)

[15] Hayajneh, M., Hayajneh, S., Kittaneh, F.: Remarks on some norm inequalities for positive semidefinite matrices and questions of Bourin. Math. Inequal. Appl. **20**, 225–232 (2017)

[16] Hayajneh, M., Hayajneh, S., Kittaneh, F.: Norm inequalities related to the arithmetic-geometric mean inequality for positive semidefinite matrices. Positivity **22**, 1311–1324 (2018)

[17] Hayajneh, M., Hayajneh, S., Kittaneh, F.: On some classical trace inequalities and a new Hilbert-Schmidt norm inequality. Math. Inequal. Appl. **21**, 1175–1183 (2018)

[18] Hayajneh, M., Hayajneh, S., Kittaneh, F.: Norm inequalities for positive semidefinite matrices and a question of Bourin II. Int. J. Math. **32**(2150043), 7 (2021)

[19] Hayajneh, M., Hayajneh, S., Kittaneh, F.: A Hilbert-Schmidt norm Inequality for positive semidefinite matrices related to a question of Bourin. Positivity **26**, 9 (2022)

[20] Hayajneh, M., Hayajneh, S., Kittaneh, F., Lebaini, I.: Norm inequalities for positive semidefinite matrices and a question of Bourin III. Positivity **26**, 13 (2022)

[21] Hayajneh, S., Kittaneh, F.: Lieb-Thirring trace inequalities and a question of Bourin. J. Math. Phys. **54**(033504), 8 (2013)

[22] Hayajneh, S., Kittaneh, F.: Trace inequalities and a question of Bourin. Bull. Austral. Math. Soc. **88**, 384–389 (2013)

[23] Hirzallah, O., Kittaneh, F.: Non-commutative Clarkson inequalities for n-tuples of operators. Integr. Equ. Oper. Theory **60**, 369–379 (2008)

[24] Kittaneh, F.: A note on the arithmetic-geometric mean inequality for matrices. Linear Algebra Appl. **171**, 1–8 (1992)

[25] Hoa, D.T.: An the the the inequality for t-geometric means. Math. Inequal. Appl. **19**, 765–768 (2016)

[26] Horn, R.A., Johnson, C.R.: Matrix Analysis, 2nd edn. Cambridge University Press, Cambridge (2013)

[27] Lin, M.: Remarks on two recent results of Audenaert. Linear Algebra Appl. **489**, 24–29 (2016)

[28] Plevnik, L.: On a matrix trace inequality due to Ando, Hiai and Okubo. Indian J. Pure Appl. Math. **47**, 491–500 (2016)

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Received: January 11, 2023.
Accepted: February 14, 2023.