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Results in Mathematics

On Solutions for Strongly Coupled Critical Elliptic Systems on Compact Riemannian Manifolds

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Abstract. In this paper, by using variational methods we investigate the existence of solutions for the following system of elliptic equations

$$\begin{cases} -\Delta_g u + a(x)u + b(x)v = \dfrac{\alpha}{2^*} f(x)u|u|^{\alpha-2}|v|^{\beta} & \text{in} \quad M, \\ -\Delta_g v + b(x)u + c(x)v \ = \dfrac{\beta}{2^*} f(x)v|v|^{\beta-2}|u|^{\alpha} & \text{in} \quad M, \end{cases}$$

where (M, g) is a smooth closed Riemannian manifold of dimension $n \geq 3, \Delta_g$ is the Laplace–Beltrami operator, a, b and c are functions Hölder continuous in M, f is a smooth function and $\alpha > 1, \beta > 1$ are two real numbers such that $\alpha + \beta = 2^*$, where $2^* = 2n/(n-2)$ denotes the critical Sobolev exponent. We get these results by assuming sufficient conditions on the function $h = \frac{\alpha}{2^*}a + \frac{2\sqrt{\alpha\beta}}{2^*}b + \frac{\beta}{2^*}c$ related to the linear geometric potential $\frac{n-2}{4(n-1)}R_g$, where R_g is the scalar curvature associated to the metric g.

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1. Introduction

Let (M, g) be a smooth closed Riemannian manifold of dimension $n \geq 3$. We are concerned with the existence of solutions of the following system:

$$\begin{cases} -\Delta_g u + a(x)u + b(x)v = \dfrac{\alpha}{2^*} f(x)u|u|^{\alpha-2}|v|^{\beta} & \text{in } M, \\ -\Delta_g v + b(x)u + c(x)v = \dfrac{\beta}{2^*} f(x)v|v|^{\beta-2}|u|^{\alpha} & \text{in } M, \end{cases} \tag{1.1}$$

where Δ_g is the Laplace–Beltrami operator, a, b and c are functions Hölder continuous in M, f is a smooth function, and $\alpha > 1, \beta > 1$ are real constants satisfying $\alpha + \beta = 2^*$, where $2^* = 2n/(n-2)$ is the critical Sobolev exponent.

Coupled systems of nonlinear equations like (1.1) are now parts of several important branches of mathematical physics. They appear in the Hartree-Fock theory for Bose–Einstein double condensates, in fiber-optic theory, in the theory of Langmuir waves in plasma physics, and in the behavior of deep water waves and freak waves in the ocean. A general reference in book form on such systems and their role in physics is by Ablowitz et al. [1].

Motivated by the varied applications the existence of solutions and their qualitative properties have been the object of study by many researchers, see for instance, [2,6,7,21] for problems in Euclidean domains and [10,11,18,20] in Riemannian context.

Next, we would like to mention some works that are strongly related to the system we propose to study. We begin with work due to Alves et al. [2], which the authors looked for positive solutions of the elliptic differential system

$$\begin{cases} -\Delta u + \tau u + \sigma v = \dfrac{\alpha}{2^*} u|u|^{\alpha-2}|v|^{\beta} & \text{in } \Omega, \\ -\Delta v + \sigma u + \mu v = \dfrac{\beta}{2^*} v|v|^{\beta-2}|u|^{\alpha} & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n, n \geq 3$, with smooth boundary and Dirichlet homogeneous boundary conditions. The main point is to compare the value of $\alpha+\beta$ with the Sobolev critical exponent. Depending on how the parameters $\tau, \mu, \sigma \in \mathbb{R}$ relate to λ_1 (the first eigenvalue of the Laplacian operator), existence and nonexistence results are provided and compared with the classical result of Brezis and Nirenberg concerning elliptic differential equations with Sobolev critical exponent. The second paper is due to Hebey [18], which the author considers elliptic systems of Yamabe-type equations

$$-\Delta_g u_i + \sum_{j=1}^{p} A_{ij}(x)u_j = u_i|u_i|^{2^*-2} \quad \text{in } M, \quad i = 1, \cdots, p,$$

where $A = (A_{ij}) : M \to M_p$ is a smooth function, $p \in \mathbb{Z}, p \geq 1$, and $M_p^s(\mathbb{R})$ denotes the vector space of symmetric $p \times p$ real matrices. Assuming sufficient conditions on the matrix A related to the linear geometric potential $\frac{n-2}{4(n-1)} R_g$, the author studies the existence of minimizing solutions for this system, the existence of high-energy solutions, blow-up theory and its compactness properties.

The system (1.1) is strongly related to the equation

$$-\Delta_g u + a(x)u = f(x)|u|^{2^*-2}u \ \text{ in } \ M,$$

which has been intensively studied in literature, see for instance, Druet [8] and Vétois [27]. When we have $a = \frac{n-2}{4(n-1)}R_g$ in the equation above, we obtain the prescribe scalar curvature equation

$$-\Delta_g u + \frac{n-2}{4(n-1)}R_g u = f(x)|u|^{2^*-2}u, \ \text{ in } \ M, \tag{1.2}$$

which is a generalization of the well-known Yamabe equation (when f is constant) whose positive solutions are such that the scalar curvature of the conformal metric $\tilde{g} = u^{2^*-2}g$ is constant. The Eq. (1.2) has been studied, for example, by Aubin [3], Aubin and Hebey [5], Escobar and Schoen [13], Hebey and Vaugon [16,17], Schoen [24], Trudinger [26] and Yamabe [28]. The study of this equation both in the classical form as in the prescribed form, motivated us in this research about the existence of solutions for system (1.1) in a compact Riemannian manifold.

Before presenting our main results, we need to introduce some notations and definitions. Throughout this work, we will denote by $H^1(M)$ the Sobolev space of all functions in $L^2(M)$ with one derivative (in the weak sense) in $L^2(M)$. We equip $H^1(M)$ with the standard $\|\cdot\|_{H^1}-$norm, that is, $\|u\|_{H^1}^2 = \|\nabla u\|_2^2 + \|u\|_2^2$, where $\|\cdot\|_q$ denotes the norm of the Lebesgue space $L^q(M)$, whenever $q \geq 1$. The norm of $L^q(M) \times L^q(M)$ will be defined by $\|(u,v)\|_q = \left(\|u\|_q^q + \|v\|_q\right)^{1/q}$.

We shall work with the space $H = H^1(M) \times H^1(M)$ endowed with the norm

$$\|(u,v)\| = \left(\|u\|_{H^1}^2 + \|v\|_{H^1}^2\right)^{1/2}.$$

In this context, we say that a pair of functions $(u,v) \in H$ is a weak solution of (1.1), if for all $(\varphi,\psi) \in H$, it holds

$$\int_M \left(\langle \nabla u, \nabla \varphi\rangle_g + \langle \nabla v, \nabla \psi\rangle_g + a(x)u\varphi + b(x)[u\psi + v\varphi] + c(x)v\psi\right) dv_g$$
$$= \int_M \frac{\alpha}{2^*} f(x)|u|^{\alpha-2}|v|^\beta u\varphi dv_g + \int_M \frac{\beta}{2^*} f(x)|v|^{\beta-2}|u|^\alpha v\psi dv_g.$$

By elliptic regularity theory (for example, see Lee and Parker [23, Theorem 4.1]), any weak solution (u,v) of (1.1), is in $C^2 \times C^2$ when a, b and c are Hölder continuous, and is in $C^\infty \times C^\infty$ when a, b and c are smooth functions.

An important relation obtained by Alves et al. [2] that we will use in this work is the following:

$$\mathcal{S}_{(\alpha,\beta)} = \left[\left(\frac{\alpha}{\beta}\right)^{\beta/\alpha+\beta} + \left(\frac{\beta}{\alpha}\right)^{\alpha/\alpha+\beta}\right] S_{\alpha+\beta}, \tag{1.3}$$

whenever $\alpha+\beta \le 2^*$, where $S_{\alpha+\beta}$ is the best Sobolev constant defined by

$$S_{\alpha+\beta} = \inf_{u \in H_0^1(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^n} |u|^{\alpha+\beta} dx\right)^{2/\alpha+\beta}},$$

and $\mathcal{S}_{(\alpha,\beta)}$ is defined by

$$\mathcal{S}_{(\alpha,\beta)} = \inf_{(u,v)\in[H_0^1(\mathbb{R}^n)]^2\setminus\{0\}} \frac{\int_{\mathbb{R}^n} \left(|\nabla u|^2 + |\nabla v|^2\right) dx}{\left(\int_{\mathbb{R}^n} |u|^{\alpha}|v|^{\beta} dx\right)^{2/\alpha+\beta}}. \tag{1.4}$$

When $\alpha+\beta = 2^*$, we denote by $S_{2^*} = K_n^{-2}$ and $\mathcal{S}_{(\alpha,\beta)} = \mathcal{S}_*$, where K_n is the sharp constant for the embedding of $H^1(\mathbb{R}^n)$ into $L^{2^*}(\mathbb{R}^n)$.

Throughout this work we assume some very general hypotheses on the functions a, b, c and f that will allow us to obtain some existence results for system (1.1) through variational methods. Precisely, we assume that the function f satisfies

$$\max_M f > 0 \tag{1.5}$$

and the functions a, b and c satisfy the following coercivity condition: there exists $C_0 > 0$ such that

$$\int_M (|\nabla_g u|^2 + |\nabla_g v|^2 + a(x)u^2 + 2b(x)uv + c(x)v^2) dv_g \\ \ge C_0 \|(u,v)\|^2, \quad \forall\ (u,v) \in H. \tag{1.6}$$

Our first result in this work can be stated as follows:

Theorem 1.1. *Let (M,g) be a closed Riemannian manifold of dimension $n \ge 3$. Let $\alpha, \beta > 1$ be two real numbers such that $\alpha+\beta = 2^*$, and let a, b and c be functions Hölder continuous in M, and $f \in C^\infty$, with a, b and c satisfying (1.6) and f satisfying (1.5), writing $h = \frac{\alpha}{2^*}a + \frac{2\sqrt{\alpha\beta}}{2^*}b + \frac{\beta}{2^*}c$. Let x_0 be some point in M such that $f(x_0) = \max_M f$. If, in addition, we assume that*

$$\begin{aligned} &(i)\ h(x_0) < \tfrac{n-2}{4(n-1)} R_g(x_0) + \tfrac{(n-4)(n-2)}{8(n-1)} \tfrac{\Delta_g f(x_0)}{f(x_0)}, \ \text{if } n \ge 4, \\ &(ii)\ h(x_0) < \tfrac{1}{8} R_g(x_0) \text{ and } h \le \tfrac{1}{8} R_g \text{ in } M, \ \text{if } n = 3. \end{aligned} \tag{1.7}$$

Then, system (1.1) has a nontrivial solution.

Theorem 1.1 will be proved using the Mountain Pass Theorem without the Palais-Smale compactness condition. A delicate part is the estimating the minimax level in order to overcome the lack of compactness of the functional associated to system (1.1) caused by the critical growth of the nonlinearities. We achieve this objective following some ideas developed in [3,4,9]. Here we face some extra difficulties due to the tight coupling of the system.

As a consequence of Theorem 1.1, we prove the following results.

Corollary 1.2. *Suppose the same assumptions of Theorem 1.1. Let x_0 be some point in M such that $f(x_0) = \max_M f$. If, in addition, we assume $b \leq 0$ and*

$$(i)\ \frac{\alpha}{2^*}a(x_0) + \frac{\beta}{2^*}c(x_0) < \frac{n-2}{4(n-1)}R_g(x_0) + \frac{(n-4)(n-2)}{8(n-1)}\frac{\Delta_g f(x_0)}{f(x_0)},\ \text{if } n \geq 4,$$
$$(ii)\ \frac{\alpha}{2^*}a(x_0) + \frac{\beta}{2^*}c(x_0) < \frac{1}{8}R_g(x_0) \text{ and } \frac{\alpha}{2^*}a + \frac{\beta}{2^*}c \leq \frac{1}{8}R_g \text{ in } M,\ \text{if } n = 3. \tag{1.8}$$

Then, system (1.1) has a pair of positive solutions.

Corollary 1.3. *Suppose the same assumptions of Theorem 1.1 and that f is constant and positive. Let x_0 be some point in M such that*

$$\begin{aligned} &(i)\ h(x_0) < \frac{n-2}{4(n-1)}R_g(x_0),\ \text{if } n \geq 4,\\ &(ii)\ h(x_0) < \frac{1}{8}R_g(x_0) \text{ and } h \leq \frac{1}{8}R_g \text{ in } M,\ \text{if } n = 3. \end{aligned} \tag{1.9}$$

Then, system (1.1) has a nontrivial solution.

For the next results, consider the functional $E_h : H \to \mathbb{R}$ given by

$$E_h(u,v) = \int_M \left(|\nabla u|_g^2 + |\nabla v|_g^2\right) dv_g + \int_M \left(au^2 + 2buv + cv^2\right) dv_g \tag{1.10}$$

and let

$$S_{f,h}^{(\alpha,\beta)} = \inf\left\{E_h(u,v) : u,v \in H^1(M) \text{ and } \int_M f(x)|u|^\alpha |v|^\beta dv_g = 1\right\}. \tag{1.11}$$

Define

$$\lambda_f(M,g) = \inf\left\{\int_M (|\nabla u|_g^2 + \frac{n-2}{4(n-1)}R_g u^2) dv_g \ :\ \int_M f(x)|u|^{2^*} dv_g = 1\right\}. \tag{1.12}$$

Remark 1.4. When f is constant and equal to $1, \lambda_f(M,g)$ is called of Yamabe invariant of the manifold (M,g), and is usually denoted by $\lambda(M,g)$. In the particular case of the unit $n-$sphere $\mathbb{S}^n$ with the standard metric is denoted by $\lambda(\mathbb{S}^n)$. It is well known that when $\lambda_f(M,g) < \frac{\lambda(\mathbb{S}^n)}{(\max_M f)^{2/2^*}}$, there exists $\varphi \in C^\infty(M)$ with $\varphi > 0$ and $\int_M f\varphi^{2^*} dv_g = 1$ such that

$$-\Delta_g \varphi + \frac{n-2}{4(n-1)}R_g\varphi = \lambda_f(M,g) f \varphi^{\frac{n+2}{n-2}}, \tag{1.13}$$

with $\lambda_f(M,g) = \int_M \left(|\nabla \varphi|_g^2 + \frac{n-2}{4(n-1)}R_g\varphi^2\right) dv_g$. It is also known that $\lambda(\mathbb{S}^n) = K_n^{-2}$, with

$$K_n = \sqrt{\frac{4}{n(n-2)\omega_n^{2/n}}},$$

where ω_n is the volume of the unit $n-$sphere (see [3,8,9]).

In the next results we deal with the case where the functions a, b and c satisfy the condition:

$$\frac{\alpha}{2^*}a(x) + \frac{2\sqrt{\alpha\beta}}{2^*}b(x) + \frac{\beta}{2^*}c(x) \leq \frac{n-2}{4(n-1)}R_g(x),\ \forall x \in M. \tag{1.14}$$

Remark 1.5. The coercivity condition (1.6) and (1.14) imply that given $\psi \in H^1(M)$ and $\xi, \zeta > 0$ such that $\left(\frac{\xi}{\zeta}\right)^2 = \frac{\alpha}{\beta}$, then

$$\begin{aligned} C_0(\xi^2+\zeta^2)\|\psi\|_{H^1}^2 &\leq (\xi^2+\zeta^2)\|\nabla\psi\|_2^2 + \int_M (a\xi^2 + 2b\xi\zeta + c\zeta^2)\psi^2 dv_g \\ &= (\xi^2+\zeta^2)\left\{\|\nabla\psi\|_2^2 + \int_M \left(\frac{\alpha}{2^*}a + \frac{2\sqrt{\alpha\beta}}{2^*}b + \frac{\beta}{2^*}c\right)\psi^2 dv_g\right\} \\ &\leq (\xi^2+\zeta^2)\left\{\|\nabla\psi\|_2^2 + \frac{n-2}{4(n-1)}\int_M R_g\psi^2 dv_g\right\}. \end{aligned}$$

Therefore, $-\Delta_g + \frac{n-2}{4(n-1)}R_g$ is also coercive. In particular we are dealing with the case where the Yamabe invariant is positive.

We can state the following result.

Theorem 1.6. *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$. Let $\alpha, \beta > 1$ be two real numbers such that $\alpha + \beta = 2^*$, and let a, b and c be functions Hölder continuous in M, and $f \in C^\infty$, with a, b and c satisfying (1.6) and (1.14), and f satisfying (1.5). Let x_0 be some point in M such that $f(x_0) = \max_M f$. If $S_{f,h}^{(\alpha,\beta)} < \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$, where $\mathcal{S}_*$ is defined in (1.4). Then, system (1.1) has a nontrivial solution.*

Complementing Theorem 1.6 and inspired by [5,13,17], we prove the following theorems:

Theorem 1.7. *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$. Let a, b and c be functions Hölder continuous in M satisfying (1.6) and (1.14). Assume that $n \geq 6$ and M is not locally conformally flat. If at a point x_0 where $f(x_0) = \max_M f$ is such that the Weyl tensor is nonvanishing (that is, $|W_g(x_0)| \neq 0$). If we assume that*

- *(i) if $\Delta_g f(x_0) = 0$ when $n = 6$, or*
- *(ii) if $\Delta_g f(x_0) = 0$ and $\left|\Delta_g^2 f(x_0)\right| / f(x_0)$ is small enough, when $n > 6$.*

Then, $S_{f,h}^{(\alpha,\beta)} < \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$. Consequently, system (1.1) has a nontrivial solution.

Theorem 1.8. *Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$. Let a, b and c be functions Hölder continuous in M satisfying (1.6) and (1.14). Assume that $n = 3, 4$ or 5, or M is locally conformally flat, when $n \geq 6$. Let*

$x_0 \in M$ be a point such that $f(x_0) = \max_M f > 0$. We have the following cases:

$$
\begin{aligned}
&(i)\ \text{if } n = 3 \text{ or, if } \Delta_g f(x_0) = 0 \text{ when } n = 4, 5;\\
&(ii)\ \text{if } \Delta_g f(x_0) = \Delta_g^2 f(x_0) = 0, \text{when } n = 6, 7;\\
&(iii)\ \text{if } \Delta_g f(x_0) = \Delta_g^2 f(x_0) = 0 \text{ and}\\
&\quad \Delta_g^3 f(x_0) = 0 \text{ or } |\nabla W_g(x_0)| = 0, \text{ when } n = 8.
\end{aligned}
$$

Then, $S_{f,h}^{(\alpha,\beta)} < \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$ unless M is conformal to the standard $\mathbb{S}^n$. Consequently, system (1.1) has a nontrivial solution. When $n > 8$ the same conclusion holds if $|\nabla W_g(x_0)| \neq 0$ and $\Delta_g^3 f(x_0) = 0$ or when $|\nabla W_g(x_0)| = 0$ if $|\nabla^2 W_g(x_0)| \neq 0$ and $\Delta_g^3 f(x_0) = \Delta_g^4 f(x_0) = 0$, or when all derivatives of W_g vanish at x_0 if $\Delta_g^m f(x_0) = 0$ for all $1 \leq m \leq \frac{n}{2} - 1$.

Corollary 1.9. *Suppose the same assumptions of Theorems 1.7 or 1.8. In addition, if $b \leq 0$ and the functions a and c satisfy*

$$\frac{\alpha}{2^*} a(x) + \frac{\beta}{2^*} c(x) \leq \frac{n-2}{4(n-1)} R_g(x), \ \forall x \in M. \tag{1.15}$$

Then, system (1.1) has a pair of positive solutions.

Corollary 1.10. *Suppose the same assumptions of Theorems 1.7 or 1.8. In addition, if we assume that $f \geq 0, b = 0$ and $a = c = \frac{n-2}{4(n-1)} R_g$. Then, system (1.1) has a nontrivial solution. Moreover, we have that*

$$S_{f,h}^{(\alpha,\beta)}(M) = \left[\left(\frac{\alpha}{\beta}\right)^{\beta/\alpha+\beta} + \left(\frac{\beta}{\alpha}\right)^{\alpha/\alpha+\beta} \right] \lambda_f(M, g). \tag{1.16}$$

Therefore, the pair $(\xi\varphi, \zeta\varphi)$ (up to rescaling) is solution for the system, for any positive solution $\varphi \in C^\infty$ of (1.13), where $\int_M f\varphi^{2^*} dv_g = 1$ and $\frac{\xi}{\zeta} = \left(\frac{\alpha}{\beta}\right)^{1/2}$.

A special case is when we consider the unit $n-$sphere $\mathbb{S}^n$ with the standard metric g_0, that is, the scalar curvature is $R_{g_0} = n(n-1)$. Note that this case is included in Theorem 1.1 when we assume the same hypotheses. Therefore the following theorem is a case special of Theorem 1.8, when $M = \mathbb{S}^n/\Gamma$.

Theorem 1.11. *Let Γ be a nontrivial finite group of isometries of $\mathbb{S}^n$ acting without fixed point on $\mathbb{S}^n$. Write $M = \mathbb{S}^n/\Gamma$, and let a, b, c and f be functions invariant under Γ and satisfying the same assumptions of Theorem 1.8. Then $S_{f,h}^{(\alpha,\beta)}(\mathbb{S}^n/\Gamma) < \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$, and therefore, system (1.1) has a nontrivial solution on $\mathbb{S}^n$.*

Remark 1.12. Note that from [13], when f is invariant under Γ and $\lambda_f(\mathbb{S}^n/\Gamma) < \frac{\lambda(\mathbb{S}^n)}{(\max_M f)^{2/2^*}}$, there is a positive solution $\varphi \in C^\infty(\mathbb{S}^n)$ to the equation

$$-\Delta_{g_0} u + \frac{n(n-2)}{4} u = f u^{2^*-1} \text{ in } \mathbb{S}^n, \tag{1.17}$$

where $\int_{\mathbb{S}^n} f\varphi^{2^*} dv_{g_0} = 1$.

Corollary 1.13. *Suppose the same assumptions of Theorem 1.11. In addition, if we assume that* $b = 0, a = c = \frac{n(n-2)}{4}$ *and* $f \geq 0$*. Then, system (1.1) has a nontrivial solution. Moreover, we have*

$$S_{f,h}^{(\alpha,\beta)}(\mathbb{S}^n) = \left[\left(\frac{\alpha}{\beta}\right)^{\beta/\alpha+\beta} + \left(\frac{\beta}{\alpha}\right)^{\alpha/\alpha+\beta}\right]\lambda_f(\mathbb{S}^n, g_0). \tag{1.18}$$

Therefore, the pair $(\xi\varphi, \zeta\varphi)$ *(up to rescaling) is solution for* system (1.1)*, where* $\varphi \in C^\infty$ *is a positive solution of Eq. (1.17).*

Corollary 1.14. *Suppose the same assumptions of Theorem 1.11. In addition, if we assume that* $b = 0, f = 1$ *and* $a = c = \frac{n(n-2)}{4}$ *then we get that* $S^{(\alpha,\beta)}(\mathbb{S}^n) = \mathcal{S}_*$ *and system (1.1) has infinitely many pair of positive solutions. Moreover, if* (u, v) *is a minimizer for* $S^{(\alpha,\beta)}(\mathbb{S}^n)$ *with* $u, v > 0$*, then up to rescaling* u *and* v *will have the following forms:*

$$u(x) = \xi_1(\rho_0 - \cos r)^{\frac{2-n}{2}} \text{ and } v(x) = \zeta_1(\rho_0 - \cos r)^{\frac{2-n}{2}} \tag{1.19}$$

where $\overline{x} \in \mathbb{S}^n$, $r = d_{g_0}(x, \overline{x}), \xi_1, \zeta_1 > 0, \rho_0 > 1$ *and* $\frac{\xi_1}{\zeta_1} = \left(\frac{\alpha}{\beta}\right)^{1/2}$.

Corollary 1.15. *Suppose the same assumptions of Theorem 1.11. In addition, if* $b \leq 0$ *and the functions* a *and* c *satisfy the following hypothesis*

$$\frac{\alpha}{2^*}a(x) + \frac{\beta}{2^*}c(x) \leq \frac{n(n-2)}{4}, \ \forall x \in M. \tag{1.20}$$

Then, system (1.1) has a pair of positive solutions on $\mathbb{S}^n$.

The paper is organized as follows. In Sect. 2 we prove an essential Sobolev inequality to prove the main results. In Sect. 3 we prove Theorem 1.1 and its consequences. In Sect. 4 we prove Theorems 1.6, 1.7 and 1.8. We dedicate Sect. 5 for the case of the sphere $\mathbb{S}^n$.

2. Some Preliminary Results

In [19], Hebey and Vaugon have established that the best constant for the Sobolev inequality is K_n^2. Precisely, they proved that there is a positive constant B such that

$$\|u\|_{2^*}^2 \leq K_n^2\|\nabla u\|_2^2 + B\|u\|_2^2, \tag{2.1}$$

for all $u \in H^1(M)$. Moreover, if $\|u\|_{2^*}^2 \leq K\|\nabla u\|_2^2 + C\|u\|_2^2$ for all $u \in H^1(M)$, where K and C are positive constants, then $K \geq K_n^2$.

Initially, we establish an inequality that will be used in the proof of the main results.

Lemma 2.1. *Let $\mathcal{S}_*$ be the constant defined in (1.4) when $\alpha+\beta=2^*$. Then, there is a positive constant B_0 such that*

$$\left(\int_M |u|^\alpha |v|^\beta dv_g\right)^{2/2^*} \le \mathcal{S}_*^{-1}\|(|\nabla u|,|\nabla v|)\|_2^2 + B_0\|(u,v)\|_2^2, \tag{2.2}$$

for all $(u,v)\in H$. Moreover, $(\mathcal{S}_*)^{-1}$ is the best constant such that the inequality holds.

Proof. Given $u,v\in H^1(M)$, since $\frac{\alpha}{2^*}+\frac{\beta}{2^*}=1$, by Hölder's inequality,

$$\int_M |u|^\alpha|v|^\beta dv_g \le \left(\int_M |u|^{2^*}dv_g\right)^{\alpha/2^*}\left(\int_M |v|^{2^*}dv_g\right)^{\beta/2^*},$$

that is,

$$\left(\int_M |u|^\alpha|v|^\beta dv_g\right)^{2/2^*} \le \left(\|u\|_{2^*}^2\right)^{\alpha/2^*}\left(\|v\|_{2^*}^2\right)^{\beta/2^*}.$$

On the other hand, by Young's inequality,

$$\begin{aligned}\left(\|u\|_{2^*}^2\right)^{\alpha/2^*}\left(\|v\|_{2^*}^2\right)^{\beta/2^*} &= \left(\varepsilon\|u\|_{2^*}^2\right)^{\alpha/2^*}\frac{\left(\|v\|_{2^*}^2\right)^{\beta/2^*}}{\varepsilon^{\alpha/2^*}}\\ &= \left(\varepsilon\|u\|_{2^*}^2\right)^{\alpha/2^*}\left(\|v\|_{2^*}^2\varepsilon^{-\alpha/\beta}\right)^{\beta/2^*}\\ &\le \frac{\alpha}{2^*}\varepsilon\|u\|_{2^*}^2 + \frac{\beta}{2^*}\varepsilon^{-\alpha/\beta}\|v\|_{2^*}^2.\end{aligned}$$

Choosing $\varepsilon = \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*}+\left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right]^{-1}\frac{2^*}{\alpha}$, by a straightforward calculation, we get

$$\frac{\alpha}{2^*}\varepsilon = \frac{\beta}{2^*}\varepsilon^{-\alpha/\beta} = \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*}+\left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right]^{-1},$$

and consequently,

$$\left(\int_M |u|^\alpha|v|^\beta dv_g\right)^{2/2^*} \le \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*}+\left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right]^{-1}\left(\|u\|_{2^*}^2+\|v\|_{2^*}^2\right). \tag{2.3}$$

Using (2.3) and the Sobolev inequality (2.1), we can find $B>0$ such that

$$\begin{aligned}&\left(\int_M |u|^\alpha|v|^\beta dv_g\right)^{2/2^*}\\ &\quad\le \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*}+\left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right]^{-1}\left(K_n^2\|(|\nabla u|,|\nabla v|)\|_2^2 + B\|(u,v)\|_2^2\right).\end{aligned}$$

Therefore, we get that

$$\left(\int_M |u|^\alpha |v|^\beta dv_g\right)^{2/2^*} \leq \mathcal{S}_*^{-1}\|(|\nabla u|, |\nabla v|)\|_2^2 + B_0\|(u,v)||_2^2$$

for all $(u,v) \in H$, where $B_0 = B\left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right]^{-1}$.

Finally, if S_0 is a positive constant such that

$$\left(\int_M |u|^\alpha |v|^\beta dv_g\right)^{2/2^*} \leq S_0\|(|\nabla u|, |\nabla v|)\|_2^2 + B_1\|(u,v)||_2^2, \tag{2.4}$$

for all $(u,v) \in H$, where B_1 is some positive constant. We claim that $S_0 \geq \mathcal{S}_*^{-1}$. Indeed, given $\varphi \in H^1(M)$ and writing $u = \alpha^{1/2}\varphi$ and $v = \beta^{1/2}\varphi$, by (2.4) we have

$$\left(\alpha^{\alpha/2}\beta^{\beta/2}\right)^{2/2^*}\left(\int_M |\varphi|^{2^*} dv_g\right)^{2/2^*} \leq 2^*\left[S_0\|\nabla\varphi\|_2^2 + B_1\|\varphi\|_2^2\right],$$

which gives us

$$\begin{aligned}\left(\int_M |\varphi|^{2^*} dv_g\right)^{2/2^*} &\leq \frac{2^*}{\alpha^{\alpha/2^*}\beta^{\beta/2^*}}\left[S_0\|\nabla\varphi\|_2^2 + B_1\|\varphi\|_2^2\right]\\ &= \left[\frac{\alpha}{\alpha^{\alpha/2^*}\beta^{\beta/2^*}} + \frac{\beta}{\alpha^{\alpha/2^*}\beta^{\beta/2^*}}\right]\left(S_0\|\nabla\varphi\|_2^2 + B_1\|\varphi\|_2^2\right)\\ &= \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right] S_0\|\nabla\varphi\|_2^2 + B_2\|\varphi\|_2^2,\end{aligned}$$

for some $B_2 > 0$. Since K_n^{-2} is the best constant in the Sobolev embedding theorem (see [4,19]), we reach that

$$\left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right] S_0 \geq K_n^2,$$

and since $\mathcal{S}_* = \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right] K_n^{-2}$, we conclude the proof of the Lemma. □

An immediate consequence this result is the following inequality.

Corollary 2.2. *Let* $C = \max\{\mathcal{S}_*^{-1}, B_0\}$, *then we have*

$$\left(\int_M |u|^\alpha |v|^\beta dv_g\right)^{2/(\alpha+\beta)} \leq C\|(u,v)\|^2.$$

Another result that will be important later on is the following Brezis-Lieb type lemma.

Lemma 2.3. *Let $u_m \rightharpoonup u$ and $v_m \rightharpoonup v$ in $H^1(M)$ and let $\ell \in L^\infty(M)$. Then we have*

$$\int_M \ell(x)|u_m|^\alpha |v_m|^\beta dv_g = \int_M \ell(x)|u|^\alpha |v|^\beta dv_g + \int_M \ell(x)|u_m - u|^\alpha |v_m - v|^\beta dv_g + o_m(1),$$

where $o_m(1) \to 0$ as $m \to \infty$.

Proof. The proof is similar to [6, Lemma 2.1] and we omit it. □

3. Proof of Theorem 1.1

We begin this section by introducing some notations and definitions. First, consider the functional $I : H \to \mathbb{R}$ associated to system (1.1) given by

$$I(u, v) = \frac{1}{2}\int_M \left[|\nabla u|_g^2 + |\nabla v|_g^2 + a(x)u^2 + 2b(x)uv + c(x)v^2\right] dv_g - \frac{1}{2^*}\int_M f(x)|u|^\alpha |v|^\beta dv_g.$$

Since the functions a, b and c are Hölder continuous and f is a smooth function, we have that I is well defined and by standard arguments $I \in C^1(H, \mathbb{R})$ with

$$I'(u, v) \cdot (\varphi, \psi) = \int_M (\langle \nabla u, \nabla \varphi\rangle_g + \langle \nabla v, \nabla \psi\rangle_g + a(x)u\varphi + b(x)[u\psi + v\varphi] + c(x)v\psi)\, dv_g - \int_M \left(\frac{\alpha}{2^*} f(x)|u|^{\alpha-2}|v|^\beta u\varphi + \frac{\beta}{2^*} f(x)|v|^{\beta-2}|u|^\alpha v\psi\right) dv_g.$$

Hence, a critical point of I is a weak solution of system (1.1) and reciprocally. Moreover, by the coercivity condition (1.6), it is easy to see that I satisfies the geometry of the Mountain Pass Theorem, that is, there exist $\rho > 0$ and $R > 0$ such that

$$I(u, v) \geq \rho \quad \text{whenever} \quad \|(u, v)\| = R, \tag{3.1}$$

and there exists some $(\tilde{u}, \tilde{v}) \in H$ with $\|(\tilde{u}, \tilde{v})\| > R$ and such that $I(\tilde{u}, \tilde{v}) < 0$.

Now, for some pair $(\tilde{u}, \tilde{v})$ satisfying the second condition above, we consider the set $\Gamma = \{\gamma \in C([0, 1], H) : \gamma(0) = 0 \text{ and } \gamma(1) = (\tilde{u}, \tilde{v})\}$, and so we can define the minimax level

$$\mathrm{c} := \inf_{\gamma\in\Gamma} \sup_{0\leq t\leq 1} I(\gamma(t)) \geq \rho. \tag{3.2}$$

Next, we will estimate the level c. This will be a very delicate result.

Lemma 3.1. *Suppose that (1.7) holds, then*

$$0 < \mathrm{c} < \frac{\mathcal{S}_*^{n/2}}{nf(x_0)^{(n-2)/2}}, \tag{3.3}$$

for some par $(\tilde{u}, \tilde{v}) \in H$, *where c is defined in (3.2).*

Proof. Initially, we will verify that there exists $(\overline{u}, \overline{v}) \in H$ such that

$$\mathrm{Q}(\overline{u}, \overline{v}) < \frac{\mathcal{S}_*}{(\max_M f)^{2/2^*}}, \tag{3.4}$$

where Q is defined by

$$\mathrm{Q}(u, v) := \frac{\int_M \left(|\nabla u|_g^2 + |\nabla v|_g^2\right) dv_g + \int_M \left(a(x)u^2 + 2b(x)uv + c(x)v^2\right) dv_g}{\left(\int_M f(x)|u|^\alpha |v|^\beta dv_g\right)^{2/2^*}}$$

for $(u, v) \in H$ with $\int_M f(x)|u|^\alpha |v|^\beta dv_g > 0$.

The proof will be done considering the cases $n \geq 4$ and $n = 3$.

Let $x_0 \in M$ be a point such that $f(x_0) = \max\{f(x) : x \in M\}$. We denote by $B_\delta(x_0)$ the geodesic ball of center x_0 and radius δ, with $\delta \in (0, i_g)$, where i_g is the injective radius of (M, g). We choose δ enough small if necessary such that $f(x) > 0$ on $B_{2\delta}(x_0)$. In normal coordinates we can write the following expansions

$$\begin{aligned} h(x)\eta(r)^2 &= h(x_0) + r^\theta O(1), \\ f(x)\eta(r)^{2^*} &= f(x_0) + \frac{1}{2}\partial_{ij} f(x_0) x^i x^j + r^3 O(1), \\ \int_{\mathbb{S}^{n-1}} \sqrt{\det(g)} d\sigma &= \omega_{n-1}\left(1 - \frac{R_g(x_0)}{6n} r^2 + r^4 O(1)\right), \end{aligned} \tag{3.5}$$

where $\det(g)$ is the determinant of the components of the metric g and $h = \frac{\alpha}{2^*} a + \frac{2\sqrt{\alpha\beta}}{2^*} b + \frac{\beta}{2^*} c$, with $\theta \in (0, 1)$ such that $h \in C^{0,\theta}(M)$, and $\eta \in C_0^\infty([-2\delta, 2\delta])$, with $\eta = 1$ in $[-\delta, \delta]$ and $0 \leq \eta \leq 1$.

Now, for $n \geq 4$ and $\epsilon > 0$, we consider the following sequence of functions

$$u_\epsilon(x) = \frac{\eta(d_g(x, x_0))}{(\epsilon + d_g(x, x_0)^2)^{(n-2)/2}}. \tag{3.6}$$

For $0 < p, q < \infty$, we put $I_q^p := \displaystyle\int_0^\infty t^p (1+t)^{-q} dt$, and then it holds that

$$\begin{aligned} \frac{n-2}{n} I_n^{n/2} &= I_n^{(n-2)/2} = \frac{\omega_n}{2^{n-1}\omega_{n-1}}, \\ \frac{(n-2)^2}{2}\omega_{n-1} I_n^{n/2} &= K_n^{-2}\left(\frac{n-2}{2n}\omega_{n-1} I_n^{n/2}\right)^{2/2^*}. \end{aligned} \tag{3.7}$$

When $n=4$, from [3,4], we get

$$\begin{aligned}
&\int_M |\nabla u_\epsilon(x)|_g^2 dv_g = 2\omega_3\epsilon^{-1}\left(I_4^2 + \frac{1}{24}R_g(x_0)\epsilon\ln\epsilon + o(\epsilon\ln\epsilon)\right),\\
&\int_M h(x)u_\epsilon(x)^2 dv_g = -\frac{\omega_3}{2}h(x_0)\ln\epsilon + o(\ln\epsilon),\\
&\int_M f(x)u_\epsilon(x)^4 dv_g = \frac{\omega_3}{2}f(x_0)I_4^1\epsilon^{-2}\left(1-\frac{1}{12}R_g(x_0)\epsilon + o(\epsilon)\right).
\end{aligned} \tag{3.8}$$

Now considering $\xi,\zeta>0$ such that $\frac{\xi}{\zeta}=\sqrt{\frac{\alpha}{\beta}}$, we obtain

$$\begin{aligned}
\mathrm{Q}(\xi u_\epsilon,\zeta u_\epsilon) &= \frac{(\xi^2+\zeta^2)\int_M |\nabla u_\epsilon|_g^2 dv_g + \int_M\left[\xi^2 a(x) + 2\xi\zeta b(x) + \zeta^2 c(x)\right]u_\epsilon^2 dv_g}{(\xi^\alpha\zeta^\beta)^{2/2^*}\left(\int_M f(x)u_\epsilon^{2^*}dv_g\right)^{2/2^*}}.\\
&= \frac{(\xi^2+\zeta^2)}{(\xi^\alpha\zeta^\beta)^{2/2^*}}\frac{\int_M |\nabla u_\epsilon|_g^2 dv_g + \int_M h(x)u_\epsilon^2 dv_g}{\left(\int_M f(x)u_\epsilon^{2^*}dv_g\right)^{2/2^*}}.
\end{aligned}$$

Then, by (3.8) and (3.7), it follows that (for ϵ enough small):

$$\begin{aligned}
&\mathrm{Q}(\xi u_\epsilon,\zeta u_\epsilon)\\
&\quad= \frac{(\xi^2+\zeta^2)}{(\xi^\alpha\zeta^\beta)^{1/2}}\frac{\left[\omega_3 I_4^2 + \frac{\omega_3}{12}\epsilon\ln\epsilon\left(R_g(x_0)-6h(x_0)\right) + o(\epsilon\ln\epsilon)\right]}{\left[\frac{\omega_3}{2}f(x_0)I_4^1(1-\frac{1}{12}R_g(x)\epsilon + o(\epsilon))\right]^{1/2}}\\
&\quad= \frac{\kappa(\alpha,\beta)}{f(x_0)^{1/2}}\frac{\left[2\omega_3 I_4^2 + \frac{\omega_3}{12}\epsilon\ln\epsilon\left(R_g(x_0)-6h(x_0)\right) + o(\epsilon\ln\epsilon)\right]}{K_4^2 2\omega_3 I_4^2\left[1-\frac{1}{24}R_g(x)\epsilon + o(\epsilon)\right]}\\
&\quad= \frac{\kappa(\alpha,\beta)}{f(x_0)^{1/2}}\left\{K_4^{-2} + \frac{\frac{\omega_3}{12}\epsilon\ln\epsilon\left(R_g(x_0)-6h(x_0)\right) + o(\epsilon\ln\epsilon) + \frac{1}{24}R_g(x_0)\epsilon + o(\epsilon)}{K_4^2 2\omega_3 I_4^2\left[1-\frac{1}{24}R_g(x)\epsilon + o(\epsilon)\right]}\right\}\\
&\quad= \frac{\kappa(\alpha,\beta)}{f(x_0)^{1/2}}\left\{K_4^{-2} + \frac{\frac{\omega_3}{12}\epsilon\ln\epsilon\left(R_g(x_0)-6h(x_0)\right) + o(\epsilon\ln\epsilon)}{K_4^2 2\omega_3 I_4^2\left[1-\frac{1}{24}R_g(x)\epsilon + o(\epsilon)\right]}\right\}\\
&\quad= \frac{\kappa(\alpha,\beta)}{f(x_0)^{1/2}}\left\{K_4^{-2} + \frac{K_4^{-2}}{24 I_4^2}\epsilon\ln\epsilon\left(R_g(x_0)-6h(x_0)\right)\right.\\
&\qquad\left. + \frac{o(\epsilon\ln\epsilon)}{K_4^2 2\omega_3 I_4^2\left[1-\frac{1}{24}R_g(x)\epsilon + o(\epsilon)\right]}\right\},
\end{aligned}$$

where $\kappa(\alpha,\beta)=\left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right]$. Consequently, as $I_4^2=\int_0^\infty \frac{t^2}{(1+t)^4}dt = \frac{1}{3}$ we reach

$$\mathrm{Q}(\xi u_\epsilon,\zeta u_\epsilon) \le \frac{\mathcal{S}_*}{f(x_0)^{1/2}} + \frac{\mathcal{S}_*}{8f(x_0)^{1/2}}\epsilon\ln\epsilon\left(R_g(x_0)-6h(x_0)\right) + o(\epsilon\ln\epsilon). \tag{3.9}$$

For $n > 4$, from [3,4], we have

$$\int_M |\nabla u_\epsilon|_g^2 dv_g = \frac{(n-2)^2}{2} I_n^{n/2} \omega_{n-1} \epsilon^{(2-n)/2} \left(1 - \frac{n+2}{6n(n-4)} R_g(x_0)\epsilon + o(\epsilon)\right),$$

$$\int_M h(x) u_\epsilon^2 dv_g = \frac{2(n-2)(n-1)}{n(n-4)} \omega_{n-1} I_n^{n/2} h(x_0) \epsilon^{(4-n)/2} + o(\epsilon^{(4-n)/2}),$$

$$\int_M f(x) u_\epsilon^{2^*} dv_g = \frac{\omega_{n-1}}{2} f(x_0) I_n^{(n-2)/2} \epsilon^{-n/2} \Big(1 - \frac{1}{2(n-2)} \left(-\frac{\Delta_g f(x_0)}{f(x_0)} + \frac{R_g(x_0)}{3}\right)\epsilon + o(\epsilon)\Big).$$

Thus, similarly to what we did above, we find that

$$\begin{aligned} \mathrm{Q}(\xi u_\epsilon, \zeta u_\epsilon) \leq & \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}} - \frac{\mathcal{S}_*}{(n-4)nf(x_0)^{2/2^*}} \\ & \times \left(\frac{(n-4)}{2}\frac{\Delta_g f(x_0)}{f(x_0)} + R_g(x_0) - \frac{4(n-1)}{(n-2)} h(x_0)\right)\epsilon \\ & + o(\epsilon). \end{aligned} \tag{3.10}$$

Now, we recall, by (1.7), that

$$h(x_0) < \frac{n-2}{4(n-1)} R_g(x_0) + \frac{(n-4)(n-2)}{8(n-1)} \frac{\Delta_g f(x_0)}{f(x_0)},$$

for $n \geq 4$. Then, by (3.9) and (3.10), it follows, for ϵ sufficient small, that (3.4) holds.

Now, we consider the case $n = 3$. As a, b and c satisfy the condition (1.6), it follows that $-\Delta_g + h$ is a coercive operator. Then we can consider $G_{x_0} : M \backslash \{x_0\} \longrightarrow \mathbb{R}$ the Green function this operator, that is,

$$-\Delta_g G_{x_0} + h G_{x_0} = \delta_{x_0},$$

where δ_{x_0} is the Dirac mass at x_0. It is well known that for x close to x_0 we can write

$$G_{x_0}(x) = \frac{1}{\omega_2 d_g(x, x_0)} + m(x_0) + o(1).$$

Next, we will use Druet's idea [9]. By using the cut-off function η, we can write G_{x_0} as follows:

$$\omega_2 G_{x_0}(x) = \frac{\eta(d_g(x, x_0))}{d_g(x, x_0)} + w_h(x), \tag{3.11}$$

where $w_h \in C^\infty_{loc}(M \backslash \{x_0\})$. In $M \backslash B_\delta(x_0)$, we have

$$-\Delta_g w_h + h w_h = \Delta_g \left(\frac{\eta}{d_g(x, x_0)}\right) - h \frac{\eta}{d_g(x, x_0)}. \tag{3.12}$$

And, in $B_\delta(x_0)$, we write in normal coordinates

$$-\Delta_g w_h + h w_h = -\frac{\partial_r(\ln(\det(g)))}{2d_g(x,x_0)^2} - h\frac{1}{d_g(x,x_0)}. \tag{3.13}$$

In particular, we have that the right side of the above equation is in $L^s(M)$ for all $1 < s < 3$, so by standard elliptic theory, $w_h \in C^0(M) \cap H^1(M)$ and moreover $w_h(x_0) = \omega_2 m(x_0)$ (for more details see Druet [9]).

As we have assumed that $h \leq \frac{1}{8}R_g$ (see (1.7)), there exists $\overline{G}_{x_0}$ the Green function of $-\Delta_g + \frac{1}{8}R_g$, and as above we can write

$$\omega_2 \overline{G}_{x_0}(x) = \frac{\eta(d_g(x,x_0))}{d_g(x,x_0)} + \overline{w}(x). \tag{3.14}$$

Now, note that

$$\begin{aligned} -\Delta_g(\overline{w} - w_h) + \frac{1}{8}R_g(\overline{w} - w_h) &= -\omega_2\Delta_g(\overline{G}_{x_0} - G_{x_0}) + \omega_2\frac{1}{8}R_g(\overline{G}_{x_0} - G_{x_0}) \\ &= \left(h - \frac{1}{8}R_g\right)\omega_2 G_{x_0} \leq 0. \end{aligned}$$

Green's Formula and the hypothesis $h \leq \frac{1}{8}R_g$ (but not equal) gives us

$$(\overline{w} - w_h)(y) = \int_M \overline{G}_y(x)\left(h(x) - \frac{1}{8}R_g(x)\right)\omega_2 G_{x_0}(x)dv_g < 0, \tag{3.15}$$

so, $\overline{w}(y) < w_h(y)$, for all $y \in M$, in particular, as $\overline{w}(x_0) = \omega_2\overline{m}(x_0) \geq 0$ it follows that $w_h(x_0) > 0$ (here $\overline{m}(x_0)$ is given by the expansion of $\overline{G}_{x_0}$ in a neighborhood of x_0, and the positive mass theorem garantee that $\overline{m}(x_0) \geq 0$, see [24, 25].

For $\epsilon > 0$ and $x \in M$, we define the function

$$v_\epsilon(x) = \epsilon^{1/4}(u_\epsilon(x) + w_h(x)),$$

where u_ϵ is the test-function defined as (3.6).

As we did in case $n \geq 4$, we estimate $\mathrm{Q}(\xi v_\epsilon, \zeta v_\epsilon)$. For this we will estimate $\int_M(|\nabla v_\epsilon|_g^2 + h v_\epsilon^2)dg$ and $\int_M f(x)v_\epsilon^6 dv_g$. First, note that

$$\begin{aligned} \int_M(|\nabla v_\epsilon|_g^2 + h v_\epsilon^2)dg &= \int_M[v_\epsilon(-\Delta_g v_\epsilon) + h v_\epsilon]dv_g \\ &= \epsilon^{1/2}\int_M[U_\epsilon^2\eta(-\Delta_g\eta) - \eta\langle\nabla\eta, \nabla U_\epsilon^2\rangle_g + h\eta^2 U_\epsilon^2]dv_g \\ &\quad + \epsilon^{1/2}\int_M \eta^2 U_\epsilon(-\Delta_g U_\epsilon)dv_g + \epsilon^{1/2} \\ &\quad \int_M(-\Delta_g w_h + h w_h)(w_h + 2\eta U_\epsilon)dv_g, \end{aligned} \tag{3.16}$$

where $U_\epsilon(x) = \frac{1}{(\epsilon + d_g(x,x_0)^2)^{1/2}}$.

Note that we can write

$$U_\epsilon^2(x)=\frac{1}{d_g(x,x_0)^2}-\frac{\epsilon}{d_g(x,x_0)^2(\epsilon+d_g(x,x_0)^2)}. \tag{3.17}$$

With that we calculate:

$$\epsilon^{1/2}\int_M U_\epsilon^2\eta(-\Delta_g\eta)dv_g=\epsilon^{1/2}\int_M\frac{\eta(-\Delta_g\eta)}{d_g(x,x_0)^2}dv_g+o(\epsilon^{1/2}), \tag{3.18}$$

$$\epsilon^{1/2}\int_M\eta\langle\nabla\eta,\nabla U_\epsilon^2\rangle_g dv_g=\epsilon^{1/2}\int_M\eta\left\langle\nabla\eta,\nabla\left(\frac{1}{d_g(x,x_0)^2}\right)\right\rangle_g dv_g+o(\epsilon^{1/2}), \tag{3.19}$$

$$\epsilon^{1/2}\int_M h\eta^2U_\epsilon^2dv_g=\epsilon^{1/2}\int_M h\frac{\eta^2}{d_g(x,x_0)^2}dv_g+o(\epsilon^{1/2}). \tag{3.20}$$

Now, as in normal coordinates the Laplacian of a radial function F can be written as follows $-\Delta_gF=\frac{1}{r^{n-1}\sqrt{\det(g)}}\partial_r(r^{n-1}\sqrt{\det(g)}\partial_rF)$, we have

$$-\Delta_gU_\epsilon=-\Delta U_\epsilon-\partial_r(\ln\sqrt{\det(g)})\partial_rU_\epsilon, \tag{3.21}$$

where $-\Delta$ is the Euclidian Laplacian. Since $-\Delta U_\epsilon=3\epsilon U_\epsilon^5$, and using (3.17), we get that

$$\begin{aligned}&\int_{B_\delta(x_0)}\eta^2U_\epsilon(-\Delta_gU_\epsilon)dv_g\\&=\int_{B_\delta(0)}U_\epsilon(-\Delta U_\epsilon-\partial_r(\ln\sqrt{\det(g)})\partial_rU_\epsilon)\sqrt{\det(g)}dx\\&=3\epsilon\int_{B_\delta(0)}U_\epsilon^6dx+O(\epsilon^{1/2})+\int_{B_\delta(x_0)}\frac{\partial_r(\ln\det(g))\partial_r(U_\epsilon^2)}{4}dv_g\\&=3\epsilon^{-1/2}\omega_2\int_0^\infty\frac{s^2}{(1+s^2)^3}ds+\int_{B_\delta(x_0)}\frac{\partial_r(\ln\det(g))}{2d_g(x,x_0)^3}dv_g+O(\epsilon^{1/2}).\end{aligned}$$

So,

$$\begin{aligned}&\int_{B_\delta(x_0)}\eta^2U_\epsilon(-\Delta_gU_\epsilon)dv_g\\&=\frac{3}{2}\omega_2I_3^{1/2}\epsilon^{-1/2}+\int_{B_\delta(x_0)}\frac{\partial_r(\ln\det(g))}{2d_g(x,x_0)^3}dv_g+O(\epsilon^{1/2}).\end{aligned} \tag{3.22}$$

Now, writing that

$$\begin{aligned}U_\epsilon(x)=&\frac{1}{d_g(x,x_0)}\\&-\frac{\epsilon}{d_g(x,x_0)(\epsilon+d_g(x,x_0)^2)^{1/2}[d_g(x,x_0)+(\epsilon+d_g(x,x_0)^2)^{1/2}]},\end{aligned} \tag{3.23}$$

we have

$$\int_{M\setminus B_\delta(x_0)} \eta^2 U_\epsilon(-\Delta_g U_\epsilon) dv_g \\ = -\int_{M\setminus B_\delta(x_0)} \frac{\eta^2}{d_g(x,x_0)} \Delta_g\left(\frac{1}{d_g(x,x_0)}\right) dv_g + O(\epsilon^{1/2}). \qquad (3.24)$$

So, we get that

$$\epsilon^{1/2}\int_M \eta^2 U_\epsilon(-\Delta_g U_\epsilon) dv_g = \frac{3}{2}\omega_2 I_3^{1/2} + \epsilon^{1/2}\int_{B_\delta(x_0)} \frac{\partial_r(\ln\det(g))}{2d_g(x,x_0)^3} dv_g \\ -\epsilon^{1/2}\int_{M\setminus B_\delta(x_0)} \frac{\eta^2}{d_g(x,x_0)} \Delta_g\left(\frac{1}{d_g(x,x_0)}\right) dv_g + o(\epsilon^{1/2}). \qquad (3.25)$$

Finally, we calculate

$$\int_M (-\Delta_g w_h + h w_h)(w_h + 2\eta U_\epsilon) dv_g \\ = \int_M (-\Delta_g w_h + h w_h)\left(w_h + \frac{2\eta}{d(x,x_0)}\right) dv_g \\ + \int_M (-\Delta_g w_h + h w_h) \\ \left(w_h + \frac{2\eta}{d_g(x,x_0)(\epsilon + d_g(x,x_0)^2)^{1/2}[d_g(x,x_0) + (\epsilon + d_{x,x_0})^{1/2}]}\right) dv_g,$$

first, by (3.11), we have

$$\int_M (-\Delta_g w_h + h w_h)\left(w_h + \frac{\eta}{d(x,x_0)}\right) dv_g \\ = \int_M (-\Delta_g w_h + h w_h)\omega_2 G_{x_0} dv_g = \omega_2 w_h(x_0). \qquad (3.26)$$

Second, we get from Eqs. (3.12) and (3.13) that

$$\int_M (-\Delta_g w_h + h w_h)\left(\frac{\eta}{d(x,x_0)}\right) dv_g \\ = -\int_{B_\delta(x_0)} \left(\frac{\partial_r(\ln(\det(g)))}{2d_g(x,x_0)^2} + \frac{h}{d_g(x,x_0)}\right) \\ \left(\frac{1}{d(x,x_0)}\right) dv_g \\ + \int_{M\setminus B_\delta(x_0)} \left(\Delta_g\left(\frac{\eta}{d_g(x,x_0)}\right) - \frac{h\eta}{d_g(x,x_0)}\right)\left(\frac{\eta}{d(x,x_0)}\right) dv_g,$$

so, we have

$$\begin{aligned}
&\int_M (-\Delta_g w_h + h w_h)\left(\frac{\eta}{d(x,x_0)}\right) dv_g \\
&\quad = -\int_M \frac{h\eta^2}{d_g(x,x_0)^2} - \int_{B_\delta(x_0)} \left(\frac{\partial_r(\ln(\det(g)))}{2d_g(x,x_0)^3}\right) dv_g \\
&\qquad + \int_{M\setminus B_\delta(x_0)} \left[\frac{\eta^2}{d_g(x,x_0)}\Delta_g\left(\frac{1}{d(x,x_0)}\right) + \frac{\eta}{d_g(x,x_0)^2}\Delta_g \eta\right] dv_g \\
&\qquad + \int_{M\setminus B_\delta(x_0)} \eta \left\langle \nabla\eta, \nabla\left(\frac{1}{d_g(x,x_0)^2}\right)\right\rangle_g dv_g
\end{aligned} \tag{3.27}$$

Now, using the obtained in (3.18), (3.19), (3.20), (3.25) and (3.27) in the Eq. (3.16), gives us the following estimate

$$\int_M (|\nabla_g v_\epsilon|_g^2 + h v_\epsilon^2) dv_g = \frac{3}{2}\omega_2 I_3^{1/2} + \omega_2 w_h(x_0)\epsilon^{1/2} + o(\epsilon^{1/2}). \tag{3.28}$$

Now, we estimate $\int_M f(x) v_\epsilon^6 dv_g$.

$$\begin{aligned}
\int_M f(x) v_\epsilon^6 dv_g &= \epsilon^{3/2}\int_M f(x)[u_\epsilon^6 + 6u_\epsilon^5 w_h + 15u_\epsilon^4 w_h^2 + 20u_\epsilon^3 w_h^3 + 15u_\epsilon^2 w_h^4 \\
&\qquad + 6u_\epsilon w_h^5 + w_h^6] dv_g \\
&= \epsilon^{3/2}\int_M f(x)[u_\epsilon^6 + 6u_\epsilon^5 w_h + 15u_\epsilon^4 w_h^2] dv_g + o(\epsilon^{1/2}).
\end{aligned} \tag{3.29}$$

Using the expansion (3.5) in normal coordinate,

$$\begin{aligned}
\int_M f(x) u_\epsilon^6 dv_g &= f(x_0)\int_{B_\delta(x_0)} U_\epsilon^6 dx + O(1)\int_{B_\delta(x_0)} U_\epsilon^6 r^2 dx \\
&\quad + \int_{B_{2\delta}(x_0)\setminus B_\delta(x_0)} f(x) u_\epsilon^6 dv_g \\
&= \frac{\omega_2}{2} f(x_0)\epsilon^{-3/2} I_3^{1/2} + O(\epsilon^{-1/2}) + O(1),
\end{aligned}$$

so, we have

$$\int_M f(x) u_\epsilon^6 dv_g = \frac{\omega_2}{2} f(x_0)\epsilon^{-3/2} I_3^{1/2} + O(\epsilon^{-1/2}). \tag{3.30}$$

Similarly, we get

$$\int_M 6f(x) u_\epsilon^5 w_h dv_g = 3\omega_2 f(x_0) w_h(x_0)\epsilon^{-1} I_{5/2}^{1/2} + o(\epsilon^{-1/2}). \tag{3.31}$$

Also, we calculate

$$\int_M 15 f(x) u_\epsilon^4 w_h^2 dv_g = 15\omega_2 f(x_0) w_h(x_0)^2 \epsilon^{-1/2} I_2^{1/2} + o(\epsilon^{-1/2}). \tag{3.32}$$

From what was obtained in (3.30)–(3.32) and the fact that $\omega_2 I_{5/2}^{1/2} = 2\int_{\mathbb{R}^n} U_1^5 dx = 2\int_{\mathbb{R}^n}(-\Delta U_1)dx = \frac{2}{3}\omega_2$, we have that

$$\int_M f(x)v_\epsilon^6 dv_g = \frac{\omega_2}{2} f(x_0) I_3^{1/2} + 2\omega_2 w_h(x_0) f(x_0)\epsilon^{1/2} + o(\epsilon^{1/2}). \tag{3.33}$$

Now, we can calculate $\mathrm{Q}(\xi v_\epsilon, \zeta v_\epsilon)$ for ϵ enough small, by the Eqs. (3.28) and (3.33),

$$\frac{\int_M (|\nabla v_\epsilon|_g^2 + h v_\epsilon^2)dg}{\left(\int_M f(x)v_\epsilon^6 dv_g\right)^{1/3}} = \frac{\frac{3}{2}\omega_2 I_3^{1/2} + \omega_2 w_h(x_0)\epsilon^{1/2} + o(\epsilon^{1/2})}{\left(\frac{\omega_2}{2} f(x_0) I_3^{1/2} + 2\omega_2 w_h(x_0) f(x_0)\epsilon^{1/2} + o(\epsilon^{1/2})\right)^{1/3}}$$

$$= \frac{\frac{3}{2}\omega_2 I_3^{1/2} + \omega_2 w_h(x_0)\epsilon^{1/2} + o(\epsilon^{1/2})}{f(x_0)^{1/3}\left(\frac{\omega_2}{2} I_3^{1/2}\right)^{1/3}\left(1 + \frac{4w_h(x_0)}{3I_3^{1/2}}\epsilon^{1/2} + o(\epsilon^{1/2})\right)},$$

as $I_3^{1/2} = \frac{1}{3} I_3^{3/2}$ and $\left(\frac{1}{6}\omega_2 I_3^{3/2}\right)^{1/3} = \frac{K_3^2}{2}\omega_2 I_3^{3/2}$ (see (3.7)), we get that

$$\frac{\int_M (|\nabla v_\epsilon|_g^2 + h v_\epsilon^2)dg}{\left(\int_M f(x)v_\epsilon^6 dv_g\right)^{1/3}} = \frac{K_3^{-2}}{f(x_0)^{1/3}} - \frac{\omega_2 w_h(x_0)\epsilon^{1/2}}{\frac{\omega_2}{2} I_3^{3/2} + 2\omega_2 w_h(x_0)\epsilon^{1/2} + o(\epsilon^{1/2})}.$$

As $w_h(x_0) > 0$, then

$$\mathrm{Q}(\xi v_\epsilon, \zeta v_\epsilon) < \left[\left(\frac{\alpha}{\beta}\right)^{\beta/6} + \left(\frac{\beta}{\alpha}\right)^{\alpha/6}\right]\frac{K_3^{-2}}{f(x_0)^{1/3}}. \tag{3.34}$$

Therefore, we obtain (3.4), when $n = 3$.

Now, in order to prove (3.3), we define for any $t > 0$ the following functional:

$$\Phi(t) = \begin{cases} I\left(t\xi u_\epsilon, t\zeta u_\epsilon\right), & \text{when } n \geq 4 \\ I\left(t\xi v_\epsilon, t\zeta v_\epsilon\right), & \text{when } n = 3 \end{cases}$$
$$= \begin{cases} \frac{t^2}{2} X_{u_\epsilon} - \frac{t^{2^*}}{2^*} Y_{u_\epsilon}, & \text{when } n \geq 4 \\ \frac{t^2}{2} X_{v_\epsilon} - \frac{t^{2^*}}{2^*} Y_{v_\epsilon}, & \text{when } n = 3, \end{cases}$$

where $X_u = (\xi^2 + \zeta^2)\int_M |\nabla u|_g^2 dv_g + \int_M \left[\xi^2 a + 2\xi\zeta b + \zeta^2 c\right] u^2 dv_g$ and $Y_u = \xi^\alpha \zeta^\beta \int_M f(x) u^{2^*} dv_g$.

We want to find $t_0 > 0$ such that $\Phi'(t_0) = 0$, that is, such that $t_0 X - t_0^{2^*-1} Y = 0$. Hence,

$$t_0 = \begin{cases} \left(\frac{X_{u_\epsilon}}{Y_\epsilon}\right)^{(n-2)/4}, & \text{when } n \geq 4 \\ \left(\frac{X_{v_\epsilon}}{Y_{v_\epsilon}}\right)^{1/4}, & \text{when } n = 3. \end{cases}$$

Therefore, t_0 is the only critical point of Φ and since $\Phi(t) \to -\infty$ as $t \to \infty$, then t_0 is a maximum point for Φ.

Note that, by the above calculations, we get

$$\Phi(t_0) = \begin{cases} \frac{1}{n}\left(\mathrm{Q}(\xi u_\epsilon(x), \zeta u_\epsilon(x))\right)^{n/2}, \text{ when } n \geq 4 \\ \frac{1}{3}\left(\mathrm{Q}(\xi v_\epsilon, \zeta v_\epsilon)\right)^{3/2}, \text{ when } n = 3 \end{cases}$$
$$< \frac{\mathcal{S}_*^{n/2}}{nf(x_0)^{(n-2)/2}}.$$

Choose $t_1 > t_0$ large such that $\Phi(t_1) < 0$ and write $\tilde{u} = t_1\xi u_\epsilon(x)$ and $\tilde{v} = t_1\zeta u_\epsilon(x)$ when $n \geq 4$ (and $\tilde{u} = t_1\xi v_\epsilon$ and $\tilde{v} = t_1\zeta v_\epsilon$ when $n = 3$). So,

$$\begin{aligned} 0 < \mathrm{c} = \inf_{\gamma\in\Gamma}\sup_{0\leqslant t\leq 1} I(\gamma(t)) &\leq \sup_{0\leqslant t\leq 1} I\left(tt_1\xi u_\epsilon(x), tt_1\zeta u_\epsilon(x)\right) \text{ (we use } v_\epsilon \text{ if } n = 3) \\ &= \sup_{0<t\leq 1} \Phi(tt_1). \\ &\leq \Phi(t_0), \end{aligned}$$

which proves (3.3). This completes the proof. $\square$

We now have the tools for the proof of Theorem 1.1.

Proof of the Theorem 1.1. By Ekeland Variational Principle, there is a sequence $\{(u_m, v_m)\}$ in H such that

$$I(u_m, v_m) \to \mathrm{c} \quad \text{and} \quad I'(u_m, v_m) \to 0. \tag{3.35}$$

Now note that

$$\begin{aligned} &I(u_m, v_m) - \frac{1}{2^*}I'(u_m, v_m)\cdot(u_m, v_m) \\ &\quad = \frac{1}{n}\int_M \left[|\nabla u_m|_g^2 + |\nabla v_m|_g^2 + a(x)u_m^2 + 2b(x)u_m v_m + c(x)v_m^2\right] dv_g. \end{aligned}$$

Thus, by the coercivity hypothesis (1.6), we obtain that $\{(u_m, v_m)\}$ is bounded in H. Hence, there exists (u_0, v_0) in H such that, up to a subsequence,

$$\begin{aligned} &(u_m, v_m) \rightharpoonup (u_0, v_0) \text{ in } H; \\ &(u_m, v_m) \to (u_0, v_0) \text{ in } L^2(M)\times L^2(M); \\ &(u_m(x), v_m(x)) \to (u_0(x), v_0(x)) \text{ a.e in } M. \end{aligned} \tag{3.36}$$

It is easy to see that $f(x)|u_m|^{\alpha-2}u_m|v_m|^\beta$ is an uniformly bounded sequence in $L^{2^*/(2^*-1)}(M)$ and converges pointwisely to $f(x)|u_0|^{\alpha-2}u_0|v_0|^\beta$, from Lemma 4.8 in [22], we have

$$f(x)|u_m|^{\alpha-2}u_m|v_m|^\beta \rightharpoonup f(x)|u_0|^{\alpha-2}u_0|v_0|^\beta \text{ in } L^{2^*/(2^*-1)}(M). \tag{3.37}$$

Similarly we obtain the same for the sequence $f(x)|u_m|^\alpha v_m|v_m|^{\beta-2}$. As $I'(u_m, v_m)\cdot(\varphi,\psi) = o_m(1)$, for all $(\varphi,\psi) \in H$, by using (3.36), (3.37) and letting

$m \to \infty$, we reach that $I'(u_0, v_0) = 0$, that is, (u_0, v_0) is a weak solution of (1.1).

The next step is to prove that $u_0 \neq 0$ and $v_0 \neq 0$.

First, let us see that $u_0 = 0$, if and only if, $v_0 = 0$. Indeed, if $u_0 = 0$, then $-\Delta_g v_0 + c(x)v_0 = 0 \text{ in } M$. So by coercivity hypothesis (1.6), we have that $v_0 = 0$.

If $u_0 = 0$ and $v_0 = 0$, we write $\tau = \lim_{m \longrightarrow 0} \int_M \left(|\nabla u_m|_g^2 + |\nabla v_m|_g^2\right) dv_g$. Since $I'(u_m, v_m) \cdot (u_m, v_m) = o_m(1)$, then we get

$$\lim_{m\to\infty} \int_M f(x)|u_m|^\alpha |v_m|^\beta dv_g = \lim_{m\to\infty} \int_M \left(|\nabla u_m|_g^2 + |\nabla v_m|_g^2\right) dv_g = \tau.$$

On the other hand, since $I(u_m, v_m) = \mathrm{c} + o_m(1)$, then we get $\tau = n\mathrm{c}$.

Now, by Lemma 2.1, we know that there is a positive constant B_0 such that

$$\left(\int_M f(x)|u_m|^\alpha |v_m|^\beta dv_g\right)^{2/2^*} \\ \leq f(x_0)^{(n-2)/n} \left[\mathcal{S}_*^{-1} \left(\|\nabla u_m\|_2^2 + \|\nabla v_m\|_2^2\right) + B_0 \|(u_m, v_m)\|_2^2\right].$$

Thus, passing to the limit in the inequality above and using (3.36), we get $(n\mathrm{c})^{2/2^*} \leq f(x_0)^{(n-2)/n} \mathcal{S}_*^{-1} n\mathrm{c}$. Hence,

$$\mathrm{c} \geq \frac{\mathcal{S}_*^{n/2}}{n f(x_0)^{(n-2)/2}}.$$

But, this contradicts the estimate obtained for the level c in Lemma 3.1. Therefore, $u_0 \neq 0$ and $v_0 \neq 0$. Thus, we conclude the proof of Theorem 1.1. □

Proof of Corollary 1.2. Consider the functional $J : H \to \mathbb{R}$ defined by

$$J(u, v) = \frac{1}{2} \int_M \left[|\nabla u|_g^2 + |\nabla v|_g^2 + a(x)u^2 + 2b(x)uv + c(x)v^2\right] dv_g \\ -\frac{1}{2^*} \int_M f(x)(u^+)^\alpha (v^+)^\beta dv_g.$$

This functional satisfies the same properties of I. Using the same test functions to estimate the minimax level and using the same steps as in the previous proof, one obtains that there exists $(u_0, v_0) \in H$ a nontrivial critical point of the functional J. Now, we will prove that u_0 and v_0 are positive solutions. First, we denote by $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$. Then, since $J'(u_0, v_0) \cdot$

$(u_0^-, v_0^-) = 0$, we get

$$\begin{aligned} 0 &= \int_M \left[|\nabla u_0^-|_g^2 + |\nabla v_0^-|_g^2 + a(x)(u_0^-)^2 + b(x)[u_0 v_0^- + u_0^- v_0] + c(x)(v_0^-)^2\right] dv_g \\ &\quad - \int_M f(x) \left[\alpha(u_0^+)^{\alpha-1}(v_0^+)^{\beta} u_0^+ u_0^- + \beta(u^+)^{\alpha}(v^+)^{\beta-1} v_0^+ v_0^-\right] dv_g \\ &= \int_M \left[|\nabla u_0^-|_g^2 + |\nabla v_0^-|_g^2 + a(x)(u_0^-)^2 + b(x)[u_0 v_0^- + u_0^- v_0] + c(x)(v_0^-)^2\right] dv_g \\ &= \int_M \left[|\nabla u_0^-|_g^2 + |\nabla v_0^-|_g^2 + a(x)(u_0^-)^2 + 2b(x)u_0^- v_0^- + c(x)(v_0^-)^2\right] dv_g \\ &\quad + \int_M b(x)[u_0^+ v_0^- + u_0^- v_0^+] dv_g. \end{aligned}$$

As $b \leq 0$ and $u_0^+ v_0^- + u_0^- v_0^+ \leq 0$, we deduce that

$$\int_M \left[|\nabla u_0^-|_g^2 + |\nabla v_0^-|_g^2 + a(x)(u_0^-)^2 + 2b(x)u_0^- v_0^- + c(x)(v_0^-)^2\right] dv_g \leq 0,$$

and consequently by (1.6), we reach $u_0^- = 0$ and $v_0^- = 0$. Therefore, $u_0 \geq 0$ and $v_0 \geq 0$. By elliptic regularity theory and maximum principle follows that $u_0 > 0$ and $v_0 > 0$. □

4. Proof of Theorems 1.6, 1.7 and 1.8

In this section, we will study the case where the combination $h = \frac{\alpha}{2^*}a + \frac{2\sqrt{\alpha\beta}}{2^*}b + \frac{\beta}{2^*}c$ is less than or equal to $\frac{n-2}{4(n-1)}R_g$. We will begin by recalling some notations and definitions. Considering the functional $E_h : H \to \mathbb{R}$ given by

$$E_h(u,v) = \int_M \left(|\nabla u|_g^2 + |\nabla v|_g^2\right) dv_g + \int_M \left(au^2 + 2buv + cv^2\right) dv_g.$$

and constraint set $\Lambda_f^{\alpha,\beta} := \left\{(u,v) \in H \,:\, \int_M f(x)|u|^{\alpha}|v|^{\beta} dv_g = 1\right\}.$

Note that E_h is bounded from below on $\Lambda_f^{\alpha,\beta}$. Indeed, by the coercivity condition (1.6) and Corollary 2.2, we have

$$E_h(u,v) \geq C_0\|(u,v)\|^2 \geq C\left(\int_M |u|^{\alpha}|v|^{\beta} dv_g\right)^{2/2^*} \geq \frac{C}{f(x_0)^{2/2^*}},$$

for all $(u,v) \in \Lambda_f^{\alpha,\beta}$. Thus, we can consider

$$S_{f,h}^{(\alpha,\beta)} = \inf_{(u,v)\in\Lambda_f^{\alpha,\beta}} E_h(u,v). \tag{4.1}$$

If there exists $(u,v) \in \Lambda_f^{\alpha,\beta}$ which achieves the infimum $S_{f,h}^{(\alpha,\beta)}$, it turns out that (u,v) will be a weak solution of the following system

$$\begin{cases} -\Delta_g u + a(x)u + b(x)v = S_{f,h}^{(\alpha,\beta)} \dfrac{2\alpha}{2^*} f(x)u|u|^{\alpha-2}|v|^{\beta} \ \text{ in } \ M, \\ -\Delta_g v + b(x)u + c(x)v \ = S_{f,h}^{(\alpha,\beta)} \dfrac{2\beta}{2^*} f(x)v|v|^{\beta-2}|u|^{\alpha} \ \text{ in } \ M. \end{cases} \tag{4.2}$$

In order to achieve the existence result we need to recall some results due to Escobar–Schoen [13], Aubin–Hebey [5] and Hebey–Vaugon [17] for Prescribe scalar curvature problem, which prove that f is the scalar curvature of a conformal metric (see also [4]).

Before, let us remember that, when $\max_M f > 0$ it is known that $\lambda_f(M,g) \leq \frac{\lambda(\mathbb{S}^n)}{(\max_M f)^{2/2^*}}$, where $\lambda_f(M,g)$ is defined in (1.12), and if $\lambda_f(M,g) < \frac{\lambda(\mathbb{S}^n)}{(\max_M f)^{2/2^*}}$, then there is $\varphi \in C^\infty$ with $\varphi > 0, \int_M f(x)\varphi^{2^*} dv_g = 1$, and such that

$$\lambda_f(M,g) = \int_M \left(|\nabla \varphi|_g^2 + \frac{n-2}{4(n-1)} R_g \varphi^2 \right) dv_g,$$

that is, φ is a positive solution of the equation $-\Delta_g u + \frac{n-2}{4(n-1)} R_g u = \lambda_f(M,g) f u^{2^*-1}$. Therefore, $\hat{g} = \varphi^{2^*-2} g$ is a conformal metric to g, where $f = R_{\hat{g}}$ is the scalar curvature of the metric $\hat{g}$, and moreover, $\lambda_f(M,\hat{g}) = \lambda_f(M,g)$.

Theorem A (Escobar–Schoen [13]). *Let f be a C^∞ function with $\max_M f > 0$ on a compact riemannian manifold (M,g) not conformal to the sphere with the standard metric. Then if $n = 3$,*

$$\lambda_f(M,g) < \frac{\lambda(\mathbb{S}^n)}{(\max_M f)^{2/2^*}},$$

and consequently f is the scalar curvature of a conformal metric. The same conclusion holds for the locally conformally flat manifolds when $n \geq 4$ if at a point x_0 where f is maximal, all its derivatives up to order $n-2$ vanish.

Theorem B (Aubin–Hebey [5]). *Assume that $n \geq 6$ and (M,g) is not locally conformally flat. Let f be a smooth function with $\max_M f > 0$. If at a point x_0 where $f(x_0) = \max_M f$ is such that the Weyl tensor is nonvanishing (that is, $|W_g(x_0)| \neq 0$) and $\Delta_g f(x_0) = 0$, then if $n = 6$,*

$$\lambda_f(M,g) < \frac{\lambda(\mathbb{S}^n)}{(\max_M f)^{2/2^*}},$$

and consequently f is the scalar curvature of a conformal metric. When $n > 6$ the same conclusion holds. If in addition $\left|\Delta_g^2 f(x_0)\right| / f(x_0)$ is small enough.

Theorem C (Hebey–Vaugon [17]). *Let f be a C^∞ function satisfying $\max_M f > 0$ and $\Delta_g f(x_0) = 0$ at a point x_0 where f is maximum. Then*

$$\lambda_f(M,g) < \frac{\lambda(\mathbb{S}^n)}{(\max_M f)^{2/2^*}},$$

and consequently f is the scalar curvature of a conformal metric when $n = 4$ or 5, unless M is conformal to the standard $\mathbb{S}^n$. When $n \geq 6$ we suppose that $|W_g(x_0)| = 0$. The same conclusion holds if $\Delta_g^2 f(x_0) = 0$, when $n = 6$ or $n = 7$, and when $n = 8$ if in addition $\Delta_g^3 f(x_0) = 0$ or $|\nabla W_g(x_0)| \neq 0$. When $n > 8$ the same conclusion holds if $|\nabla W_g(x_0)| \neq 0$ and $\Delta_g^3 f(x_0) = 0$ or when $|\nabla W_g(x_0)| = 0$ if $|\nabla^2 W_g(x_0)| \neq 0$ and $\Delta_g^3 f(x_0) = \Delta_g^4 f(x_0) = 0$, or when all derivatives of W_g vanish at x_0 if $\Delta_g^m f(x_0) = 0$ for all $1 \leq m \leq \frac{n}{2} - 1$.

The next result is the first step to prove Theorems 1.7 and 1.8.

Lemma 4.1. *If $\lambda_f(M,g) < \frac{\lambda(\mathbb{S}^n)}{f(x_0)^{2/2^*}}$, then $S_{f,h}^{(\alpha,\beta)} < \dfrac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$, where $\mathcal{S}_*$ is given in (1.4).*

Proof. Since $\lambda_f(M,g) < \frac{\lambda(\mathbb{S}^n)}{f(x_0)^{2/2^*}}$, from theorems A, B and C, there exists $\varphi \in C^\infty(M)$ with $\varphi > 0$, $\int_M f(x)\varphi^{2^*} dv_g = 1$ and such that

$$\lambda_f(M,g) = \int_M \left(|\nabla \varphi|_g^2 + \frac{n-2}{4(n-1)} R_g \varphi^2 \right) dv_g < \frac{\lambda(\mathbb{S}^n)}{f(x_0)^{2/2^*}}. \tag{4.3}$$

Now, consider the following pair of functions $(w_1, w_2) \in \Lambda_f^{\alpha,\beta}$, where $w_1 = \xi \left(\xi^\alpha \zeta^\beta\right)^{-1/2^*} \varphi$ and $w_2 = \zeta \left(\xi^\alpha \zeta^\beta\right)^{-1/2^*} \varphi$, with $\frac{\xi}{\zeta} = \sqrt{\frac{\alpha}{\beta}}$, thus

$$\begin{aligned} S_{f,h}^{(\alpha,\beta)} &\leq E_h(w_1,w_2) \\ &= \frac{(\xi^2+\zeta^2)\int_M |\nabla\varphi|_g^2 dv_g + \int_M \left(\xi^2 a(x) + 2\xi\zeta b(x) + \zeta^2 c(x)\right)\varphi^2 dv_g}{(\xi^\alpha\zeta^\beta)^{2/2^*}} \\ &= \frac{(\xi^2+\zeta^2)}{(\xi^\alpha\zeta^\beta)^{2/2^*}} \left\{ \int_M |\nabla\varphi|_g^2 dv_g + \int_M \left(\frac{\alpha}{2^*} a(x) + \frac{2\sqrt{\alpha\beta}}{2^*} b(x) + \frac{\beta}{2^*} c(x) \right) \varphi^2 dv_g \right\}. \end{aligned}$$

As $h = \frac{\alpha}{2^*} a + \frac{2\sqrt{\alpha\beta}}{2^*} b + \frac{\beta}{2^*} c \leq \frac{n-2}{4(n-1)} R_g$, it follows that

$$S_{f,h}^{(\alpha,\beta)} \leq \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*} \right] \lambda_f(M,g). \tag{4.4}$$

Consequently,

$$S_{f,h}^{(\alpha,\beta)} < \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*} \right] \frac{\lambda(\mathbb{S}^n)}{f(x_0)^{2/2^*}},$$

hence $S_{f,h}^{(\alpha,\beta)} < \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$ as desired. Finishing the proof. □

We will now prove the second auxiliary result of this section.

Lemma 4.2. *If* $S_{f,h}^{(\alpha,\beta)} < \dfrac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$, *then there exists* (u,v) *in* $\Lambda_f^{\alpha,\beta}$ *such that* $E_h(u,v) = S_{f,h}^{(\alpha,\beta)}$.

Proof. Let $\{(u_m,v_m)\} \subset \Lambda_f^{\alpha,\beta}$ be a minimizing sequence for $S_{f,h}^{(\alpha,\beta)}$, that is,

$$\begin{aligned} E_h(u_m,v_m) &= \|(|\nabla u_m|,|\nabla v_m|)\|_2^2 \\ &\quad + \int_M (au_m^2 + 2bu_m v_m + cv_m^2)dv_g = S_{f,h}^{(\alpha,\beta)} + o_m(1), \end{aligned} \tag{4.5}$$

where $o_m(1) \to 0$ as $m \to \infty$. By the coercivity hypothesis (1.6), it follows that $\{(u_m,v_m)\}$ is bounded in H. Thus, there exists (u,v) in H such that, up to a subsequence, $(u_m,v_m) \rightharpoonup (u,v)$ in H, $(u_m,v_m) \to (u,v)$ in $L^2(M)\times L^2(M)$, and $(u_m(x),v_m(x)) \to (u(x),v(x))$ a.e in M. From Lemma 2.1 and (4.5), we get

$$\begin{aligned} 1 &= \left(\int_M f(x)|u_m|^\alpha|v_m|^\beta dv_g\right)^{2/2^*} \\ &\le f(x_0)^{2/2^*}\mathcal{S}_*^{-1}\|(|\nabla u_m|,|\nabla v_m|)\|_2^2 + f(x_0)^{2/2^*}B_0||(u_m,v_m)||_2^2 \\ &\le f(x_0)^{2/2^*}\mathcal{S}_*^{-1}S_{f,h}^{(\alpha,\beta)} + f(x_0)^{2/2^*}B_1\|(u_m,v_m)\|_2^2 \\ &\quad - f(x_0)^{2/2^*}\mathcal{S}_*^{-1}\int_M (au_m^2 + 2b_{u_m v_m} + cv_m^2)dv_g + o_m(1), \end{aligned}$$

where $B_0 > 0$. Letting $m \to \infty$, we obtain that

$$\begin{aligned} 1 \le\ & f(x_0)^{2/2^*}\mathcal{S}_*^{-1}S_{f,h}^{(\alpha,\beta)} + f(x_0)^{2/2^*}B_0\|(u,v)\|_2^2 - f(x_0)^{2/2^*}\mathcal{S}_*^{-1} \\ & \int_M (au^2 + 2buv + cv^2)dv_g. \end{aligned}$$

Then, since $S_{f,h}^{(\alpha,\beta)} < \dfrac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$, we find that $0 < \|(u,v)\|_2^2$, and consequently, $u \neq 0$ or $v \neq 0$.

We claim that $u \neq 0$ and $v \neq 0$. Moreover, $(u,v) \in \Lambda_f^{\alpha,\beta}$ is a minimizing for $S_{f,h}^{(\alpha,\beta)}$. Indeed, rewriting (4.5), we have

$$E_h(u,v) + \|(|\nabla(u_m - u)|,|\nabla(v_m - v)|)\|_2^2 = S_{f,h}^{(\alpha,\beta)} + o_m(1). \tag{4.6}$$

On the other hand, since $1 = \int_M f(x)|u_m|^\alpha|v_m|^\beta dv_g$, Lemma 2.3 gives us

$$1 = \left(\int_M f(x)|u|^\alpha|v|^\beta dv_g + \int_M f(x)|u_m - u|^\alpha|v_m - v|^\beta dv_g + o_m(1)\right)^{2/2^*}. \tag{4.7}$$

Now, note that $\int_M f(x)|u|^\alpha|v|^\beta dv_g > 0$, otherwise, by (4.7) and Lemma 2.1, we would have

$$\begin{aligned}1 &\le \left(\int_M f(x)|u_m-u|^\alpha|v_m-v|^\beta dv_g\right)^{2/2^*} + o_m(1)\\ &\le f(x_0)^{2/2^*}\mathcal{S}_*^{-1}\|(|\nabla_g(u_m-u)|,|\nabla_g(v_m-v)|)\|_2^2 + o_m(1),\end{aligned}$$

hence,

$$S_{f,h}^{(\alpha,\beta)} \le f(x_0)^{2/2^*}\mathcal{S}_*^{-1}S_{f,h}^{(\alpha,\beta)}\|(|\nabla_g(u_m-u)|,|\nabla_g(v_m-v)|)\|_2^2 + o_m(1).$$

But, using the inequality above in (4.6), we get

$$\begin{aligned}E_h(u,v) &= S_{f,h}^{(\alpha,\beta)} - \|(|\nabla(u_m-u)|,|\nabla(v_m-v)|)\|_2^2 + o_m(1)\\ &\le (f(x_0)^{2/2^*}\mathcal{S}_*^{-1}S_{f,h}^{(\alpha,\beta)} - 1)\|(|\nabla(u_m-u)|,|\nabla(v_m-v)|)\|_2^2 + o_m(1),\end{aligned}$$

again as $S_{f,h}^{(\alpha,\beta)} < \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$, we reach that $E_h(u,v) \le 0$, and so $u = v = 0$, which is a contradiction. Therefore, $\int_M f(x)|u|^\alpha|v|^\beta dv_g > 0$.

Now, returning to (4.7) we get

$$\begin{aligned}1 &= \left(\int_M f(x)|u|^\alpha|v|^\beta dv_g + \int_M f(x)|u_m-u|^\alpha|v_m-v|^\beta dv_g + o_m(1)\right)^{2/2^*}\\ &\le \left(\int_M f(x)|u|^\alpha|v|^\beta dv_g\right)^{2/2^*} + f(x_0)^{2/2^*}\left(\int_M |u_m-u|^\alpha|v_m-v|^\beta dv_g\right)^{2/2^*} + o_m(1)\\ &\le \left(\int_M f(x)|u|^\alpha|v|^\beta dv_g\right)^{2/2^*} + f(x_0)^{2/2^*}\mathcal{S}_*^{-1}\|(|\nabla(u_m-u)|,|\nabla(v_m-v)|)\|_2^2 + o_m(1),\end{aligned}$$

as $S_{f,h}^{(\alpha,\beta)} > 0$, then

$$\begin{aligned}S_{f,h}^{(\alpha,\beta)} \le\ & S_{f,h}^{(\alpha,\beta)}\left(\int_M f(x)|u|^\alpha|v|^\beta dv_g\right)^{2/2^*}\\ &+ S_{f,h}^{(\alpha,\beta)}f(x_0)^{2/2^*}\mathcal{S}_*^{-1}\|(|\nabla(u_m-u)|,\\ &|\nabla(v_m-v)|)\|_2^2 + o_m(1).\end{aligned} \tag{4.8}$$

By using (4.6), it follows that

$$\begin{aligned}E_h(u,v) \le\ & S_{f,h}^{(\alpha,\beta)}\left(\int_M f(x)|u|^\alpha|v|^\beta dv_g\right)^{2/2^*}\\ &+ (S_{f,h}^{(\alpha,\beta)}f(x_0)^{2/2^*}\mathcal{S}_*^{-1} - 1)\|(|\nabla_g(u_m-u)|,|\nabla_g(v_m-v)|)\|_2^2 + o_m(1).\end{aligned}$$

Since $S_{f,h}^{(\alpha,\beta)}f(x_0)^{2/2^*}\mathcal{S}_*^{-1} - 1 < 0$, we have

$$E_h(u,v) \le S_{f,h}^{(\alpha,\beta)}\left(\int_M f(x)|u|^\alpha|v|^\beta dv_g\right)^{2/2^*}.$$

The lower semicontinuity of E_h implies $E_h(u,v) \le \liminf E_h(u_m,v_m) = S_{f,h}^{(\alpha,\beta)}$, and hence $0 < \tau = \int_M f(x)|u|^\alpha|v|^\beta dv_g \le 1$, now writing $u_0 = \tau^{-1/2^*}u$ and $v_0 = \tau^{-1/2^*}v$, we have

$$E_h(u_0,v_0) = \frac{E_h(u,v)}{\left(\int_M f(x)|u|^\alpha|v|^\beta dv_g\right)^{2/2^*}} \le S_{f,h}^{(\alpha,\beta)},$$

with $(u_0,v_0) \in \Lambda_f^{\alpha,\beta}$. By definition of $S_{f,h}^{(\alpha,\beta)}$ it follows that $E_h(u_0,v_0) = S_{f,h}^{(\alpha,\beta)}$, so we prove that $E_h(u,v) = S_{f,h}^{(\alpha,\beta)}\tau^{2/2^*}$.

Finally, we can check that $\tau = \int_M f(x)|u|^\alpha|v|^\beta dv_g = 1$, for this, we return to (4.6) and (4.8). Then

$$\begin{aligned}1 &\le \left(\int_M f(x)|u|^\alpha|v|^\beta dv_g\right)^{2/2^*} + f(x_0)^{2/2^*}\mathcal{S}_*^{-1}\left[S_{f,h}^{(\alpha,\beta)} - E_h(u,v)\right]\\ &= \left(\int_M f(x)|u|^\alpha|v|^\beta dv_g\right)^{2/2^*}\\ &\quad + f(x_0)^{2/2^*}\mathcal{S}_*^{-1}S_{f,h}^{(\alpha,\beta)}\left[1 - \left(\int_M f(x)|u|^\alpha|v|^\beta dv_g\right)^{2/2^*}\right].\end{aligned}$$

Hence,

$$\begin{aligned}0 &\le -\left[1 - \left(\int_M f(x)|u|^\alpha|v|^\beta dv_g\right)^{2/2^*}\right]\\ &\quad + f(x_0)^{2/2^*}\mathcal{S}_*^{-1}S_{f,h}^{(\alpha,\beta)}\left[1 - \left(\int_M f(x)|u|^\alpha|v|^\beta dv_g\right)^{2/2^*}\right]\\ &= \left(-1 + f(x_0)^{2/2^*}\mathcal{S}_*^{-1}S_{f,h}^{(\alpha,\beta)}\right)\left[1 - \left(\int_M f(x)|u|^\alpha|v|^\beta dv_g\right)^{2/2^*}\right]\end{aligned}$$

As $f(x_0)^{2/2^*}S_{f,h}^{(\alpha,\beta)} < \mathcal{S}_*$, then $\int_M f(x)|u|^\alpha|v|^\beta dv_g = 1$.

Consequenlty, we get that $(u,v) \in \Lambda_f^{\alpha,\beta}$, which proves that (u,v) is a minimizer for $S_{f,h}^{(\alpha,\beta)}$. □

Now we can prove the main results of this section.

Proof of Theorem 1.6. Since $S_{f,h}^{(\alpha,\beta)} < \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$, by Lemma 4.2 there exists $(u_0,v_0) \in \Lambda_f^{\alpha,\beta}$ such that $E_h(u_0,v_0) = S_{f,h}^{(\alpha,\beta)}$. Denote by $G(u,v) = \int_M f(x)|u|^\alpha|v|^\beta dv_g - 1$, where $(u,v) \in H$. Then, there is a Lagrange multiplier λ that satisfies

$$E_h'(u_0,v_0)\cdot(\varphi,\psi) - \lambda G'(u_0,v_0)\cdot(\varphi,\psi) = 0, \text{ for all } (\varphi,\psi) \in H. \tag{4.9}$$

Taking $\varphi = u_0$ and $\psi = v_0$ above, we have that $2E_h(u_0,v_0) = 2^*\lambda$, hence $\lambda = \frac{2}{2^*}S_{f,h}^{(\alpha,\beta)} > 0$. Therefore, by (4.9), we have that (u_0,v_0) is a weak solution of the following system

$$\begin{cases} -\Delta_g u + au + bv = S^{(\alpha,\beta)}_{f,h} \frac{2\alpha}{2^*} f(x) u|u|^{\alpha-2}|v|^{\beta} \text{ in } M, \\ -\Delta_g v + bu + cv \ = S^{(\alpha,\beta)}_{f,h} \frac{2\beta}{2^*} f(x) v|v|^{\beta-2}|u|^{\alpha} \text{ in } M. \end{cases} \tag{4.10}$$

It is easy to see that the pair $((2S^{(\alpha,\beta)}_{f,h})^{1/(2^*-2)}u_0, (2S^{(\alpha,\beta)}_{f,h})^{1/(2^*-2)}v_0)$ is a weak solution of system 1.1. This completes the proof of the theorem. □

Proof of Theorems 1.7 and 1.8. From Theorem B and Theorem C together with Lemma 4.1, it follows that $S^{(\alpha,\beta)}_{f,h} < \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$. Thus the proof follows similar to Theorem 1.6. □

Let us introduce some notations before of the proof of Corollary 1.9. Let

$$\Lambda^{\alpha,\beta}_{f,+} := \left\{(u,v) \in H : \int_M f(x)(u^+)^{\alpha}(v^+)^{\beta} dv_g = 1\right\} \text{ and } S^{(\alpha,\beta)}_{f,h,+} := \inf_{(u,v)\in\Lambda^{\alpha,\beta}_{f,+}} E_h(u,v).$$

Then, if $b \leq 0$ in M, it is easy to see that $E_h(|u|,|v|) \leq E_h(u,v)$, so if $(u,v) \in \Lambda^{\alpha,\beta}_{f}$ then $(|u|,|v|) \in \Lambda^{\alpha,\beta}_{f,+}$, and therefore, we deduce that $S^{(\alpha,\beta)}_{f,h,+} \leq S^{(\alpha,\beta)}_{f,h}$. Then, by Lemma 4.1, we have $S^{(\alpha,\beta)}_{f,h,+} < \frac{\mathcal{S}_*}{f(x_0)^{2/2^*}}$. Moreover, we claim that $S^{(\alpha,\beta)}_{f,h} > 0$, indeed,

$$E_h(u,v) \geq C_0||(u,v)||^2_H \geq C\left(\int_M |u|^{\alpha}|v|^{\beta} dv_g\right)^{2/2^*} \geq C\left(\int_M (u^+)^{\alpha}(v^+)^{\beta} dv_g\right)^{2/2^*} \geq \frac{C}{f(x_0)^{2/2^*}},$$

for all $(u,v) \in \Lambda^{\alpha,\beta}_{f,+}$.

Proof of Corollary 1.9. Let $\{(u_m,v_m)\} \subset \Lambda^{\alpha,\beta}_{f,+}$ be a minimizing sequence for $S^{(\alpha,\beta)}_{f,h,+}$. Arguing similarly to Lemma 4.2, we obtain a pair $(u,v) \in \Lambda^{\alpha,\beta}_{f,+}$ such that $E_h(u,v) = S^{(\alpha,\beta)}_{f,h,+}$, with $u \neq 0$ and $v \neq 0$, where $u_m \rightharpoonup u$ and $v_m \rightharpoonup v$ in $H^1(M)$. Now, we claim that $u \geq 0$ and $v \geq 0$ in M. Indeed, if we consider $G_+(u,v) = \int_M f(x)(u^+)^{\alpha}(v^+)^{\beta} dv_g - 1$, there is a Lagrange multiplier λ such that

$$E'_h(u,v) \cdot (\varphi,\psi) - \lambda G'_+(u,v) \cdot (\varphi,\psi) = 0, \text{ for all } (\varphi,\psi) \in H. \tag{4.11}$$

Taking $\varphi = u^-$ and $\psi = v^-$ as test functions above, we have

$$2E_h(u^-,v^-) + 2\int_M b(u^+v^- + u^-v^+) dv_g = 0.$$

Since $b \leq 0$, it follows that $E_h(u^-,v^-) \leq 0$, hence $u^- = v^- = 0$. Thus, we conclude that $u \geq 0$ and $v \geq 0$. Considering $\varphi = u$ and $\psi = v$ as test functions in (4.11), we get $2E_h(u,v) = 2^*\lambda > 0$, and consequently $\lambda = \frac{2}{2^*}S^{(\alpha,\beta)}_{f,A,+} > 0$. Therefore, $((2S^{(\alpha,\beta)}_{f,h,+})^{1/(2^*-2)}u, (2S^{(\alpha,\beta)}_{f,h,+})^{1/(2^*-2)}v)$ is a weak positive solution of system (1.1), because the elliptic regularity theory gives us $u > 0$ and $v > 0$ in M. □

Proof of Corollary 1.10. Here, we assume that $b = 0, a = c = \frac{(n-2)}{4(n-1)}R_g$ and $f \geq 0$. We claim that

$$S^{(\alpha,\beta)}_{f,h} = \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right]\lambda_f(M,g).$$

Indeed, from of the proof of Lemma 4.1 (see (4.4)), it is sufficient to prove that

$$S^{(\alpha,\beta)}_{f,h} \geq \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right]\lambda_f(M,g).$$

In order to achieve this goal, let $\{(u_m, v_m)\} \subset \Lambda^{\alpha,\beta}_f$ be a minimizing sequence for $S^{(\alpha,\beta)}_{f,h}$, that is,

$$\begin{aligned}\int_M \left(|\nabla_g u_m|^2_g + |\nabla_g v_m|^2_g + \frac{n-2}{4(n-1)}R_g\left(u_m^2 + v_m^2\right)\right)dv_g\\ = S^{(\alpha,\beta)}_{f,h} + o_m(1).\end{aligned} \tag{4.12}$$

Define $w_m = t_m v_m$, where $t_m > 0$ is chosen so that

$$\int_M f(x)|u_m|^{2^*}dv_g = \int_M f(x)|w_m|^{2^*}dv_g.$$

By Young's inequality, we get that

$$\begin{aligned} t_m^\beta = \int_M f(x)|u_m|^\alpha|w_m|^\beta dv_g &\leq \frac{\alpha}{2^*}\int_M f(x)|u_m|^{2^*}dv_g + \frac{\beta}{2^*}\int_M f(x)|w_m|^{2^*}dv_g\\ &= \int_M f(x)|u_m|^{2^*}dv_g = \int_M f(x)|w_m|^{2^*}dv_g.\end{aligned} \tag{4.13}$$

Using (4.13) in (4.12), we have

$$\begin{aligned} S^{(\alpha,\beta)}_{f,h} + o_m(1) &= t_m^{2\beta/2^*}\frac{\int_M\left(|\nabla u_m|^2_g + |\nabla v_m|^2_g\right)dv_g + \int_M \frac{n-2}{4(n-1)}R_g\left(u_m^2+v_m^2\right)dv_g}{\left(\int_M f(x)|u_m|^\alpha|w_m|^\beta dv_g\right)^{2/2^*}}\\ &\geq t_m^{2\beta/2^*}\frac{\int_M\left(|\nabla u_m|^2_g + \frac{n-2}{4(n-1)}R_g u_m^2\right)dv_g}{\left(\int_M f(x)|u_m|^{2^*}dv_g\right)^{2/2^*}}\\ &\quad + t_m^{(2\beta/2^*)-2}\frac{\int_M\left(|\nabla w_m|^2_g + \frac{n-2}{4(n-1)}R_g w_m^2\right)dv_g}{\left(\int_M f(x)|w_m|^{2^*}dv_g\right)^{\frac{2}{2^*}}}\\ &\geq (t_m^{2\beta/2^*} + t_m^{(2\beta/2^*)-2})\lambda_f(M,g).\end{aligned}$$

On the other hand, it is easy to see that $t^{2\beta/2^*} + t^{(2\beta/2^*)-2} \geq \left(\sqrt{\frac{\alpha}{\beta}}\right)^{2\beta/2^*} + \left(\sqrt{\frac{\beta}{\alpha}}\right)^{2\alpha/2^*}$, for all $t > 0$. Therefore,

$$S^{(\alpha,\beta)}_{f,h} \geq \left[\left(\frac{\alpha}{\beta}\right)^{\beta/2^*} + \left(\frac{\beta}{\alpha}\right)^{\alpha/2^*}\right] \lambda_f(M, g).$$

Thus, Corollary 1.10 follows by Lemma 4.2. □

5. Case $\mathbb{S}^n$

Let $(\mathbb{S}^n, g_0)$ be the $n-$sphere, where g_0 is standard metric of $\mathbb{S}^n$. Due to the argument of Escobar and Schoen in [13] we can prove:

Lemma 5.1. *Let Γ be a nontrivial finite group of isometries of $\mathbb{S}^n$ acting without a fixed point on $\mathbb{S}^n$. Write $(M = \mathbb{S}^n/\Gamma, g)$, where g is the metric induced by $\pi : \mathbb{S}^n \longrightarrow \mathbb{S}^n/\Gamma$ covering map. Let $\overline{a}, \overline{b}, \overline{c}$ and $\overline{f}$ be functions in M satisfying the same assumptions of Theorem 1.8. Then we have that*

$$S^{(\alpha,\beta)}_{\overline{f},\overline{h}}(\mathbb{S}^n/\Gamma) < \frac{\mathcal{S}_*}{\overline{f}(x_0)^{2/2^*}}.$$

Proof. By hypotheses about Γ it is known that $M = \mathbb{S}^n/\Gamma$ is a compact Riemannian manifold locally conformally flat, which is not conformally diffeomorphic to $\mathbb{S}^n$. From Theorem C, we have $\lambda_{\overline{f}}(M, g) < \lambda(\mathbb{S}^n)/\overline{f}(x_0)^{2/2^*}$, and consequently

$$S^{(\alpha,\beta)}_{\overline{f},\overline{h}}(\mathbb{S}^n/\Gamma) < \frac{\mathcal{S}_*}{\overline{f}(x_0)^{2/2^*}}$$

as desired. □

Proof of Theorem 1.11. By Lemma 4.2 and Lemma 5.1, it follows that there exists $(\overline{u}_0, \overline{v}_0) \in H$ weak solution of system (1.1) for $(M = \mathbb{S}^n/\Gamma, g)$. Since a, b, c and f are invariant under Γ (and recall that $\pi^* g = g_0$ and $\Delta_{g_0}(u \circ \pi) = (\Delta_g u) \circ \pi$, for $u \in C^2(M)$), then writing $u_0 = \overline{u}_0 \circ \pi$ and $v_0 = \overline{v}_0 \circ \pi$ we have that $(u_0, v_0) \in H^1(\mathbb{S}^n) \times H^1(\mathbb{S}^n)$ is a weak solution of the system

$$\begin{cases} -\Delta_{g_0} u + au + bv = \dfrac{\alpha}{2^*} f(x)u|u|^{\alpha-2}|v|^{\beta} \ \ in \ \ \mathbb{S}^n, \\ -\Delta_{g_0} v + bu + cv = \dfrac{\beta}{2^*} f(x)v|v|^{\beta-2}|u|^{\alpha} \ \ in \ \ \mathbb{S}^n, \end{cases} \tag{5.1}$$

which ends the proof of the theorem. □

Proof of Corollary 1.14. As a consequence of the assumptions, from corollary 1.13 we immediately have that $S^{(\alpha,\beta)}(\mathbb{S}^n)=\mathcal{S}_*$. Let $\varphi\in C^\infty(\mathbb{S}^n)$ be a minimizer for $\lambda(\mathbb{S}^n)$, we can to see that $(\xi\varphi,\zeta\varphi)$ is a minimizer for $S^{(\alpha,\beta)}(\mathbb{S}^n)$. Indeed, notice that

$$\frac{\mathrm{Q}(\xi\varphi,\zeta\varphi)}{\left(\int_{\mathbb{S}^n}|\xi\varphi|^\alpha|\zeta\varphi|^\beta dv_{g_0}\right)^{2/2^*}}=\frac{(\xi^2+\zeta^2)}{(\xi^\alpha\zeta^\beta)^{2/2^*}}\frac{\left(\|\nabla_{g_0}\varphi\|_2^2+\frac{n(n-2)}{4}\|\varphi\|_2^2\right)}{\|\varphi\|_{2^*}^2}=\mathcal{S}_*. \tag{5.2}$$

So, $(\xi\varphi,\zeta\varphi)$ is a solution of the system

$$\begin{cases}-\Delta_{g_0}u+\frac{n(n-2)}{4}u=S^{(\alpha,\beta)}(\mathbb{S}^n)\dfrac{\alpha}{2^*}u|u|^{\alpha-2}|v|^\beta & \text{in } \mathbb{S}^n,\\ -\Delta_{g_0}v+\frac{n(n-2)}{4}v=S^{(\alpha,\beta)}(\mathbb{S}^n)\dfrac{\beta}{2^*}v|v|^{\beta-2}|u|^\alpha & \text{in } \mathbb{S}^n.\end{cases} \tag{5.3}$$

Hence the rescaling $((S^{(\alpha,\beta)}(\mathbb{S}^n))^{1/(2^*-2)}\xi\varphi,(S^{(\alpha,\beta)}(\mathbb{S}^n))^{1/(2^*-2)}\zeta\varphi)$ is solution of system (1.1). Therefore, we have infinite positive solutions for system (1.1), because for $x_0\in\mathbb{S}^n$ fixed, and any $\rho>1$, the functions

$$\varphi_{\rho,x_0}(x)=(\rho-\cos r)^{\frac{2-n}{2}} \tag{5.4}$$

are minimizer for $\lambda(\mathbb{S}^n)$, with $r=d_{g_0}(x,x_0)$ (for more details see Theorem 5.1 in [15]).

On the other hand, if (u,v) is a minimizer for $S^{(\alpha,\beta)}(\mathbb{S}^n)$, with $u,v\in C^\infty$, $u,v>0$ and $\int_{\mathbb{S}^n}u^\alpha v^\beta dg_0=1$. Let $\sigma:\mathbb{S}^n\backslash\{P_N\}\longrightarrow\mathbb{R}^n$ be the stereographic projection, where P_N is the north pole of $\mathbb{S}^n$, since $(\sigma^{-1})^*(g_0)=U^{4/(n-2)}g_e$, where $U(y)=\left(\frac{2}{1+|y|^2}\right)^{(2-n)/2}$ and g_e is the Euclidian metric. So, we have

$$\begin{aligned}\mathcal{S}_*&=\int_{\mathbb{S}^n}\left[|\nabla u|_{g_0}^2+|\nabla v|_{g_0}^2+\frac{n(n-2)}{4}(u^2+v^2)\right]dv_{g_0}\\&=\int_{\mathbb{R}^n}\left[|\nabla[(u\circ\sigma^{-1})U]|^2+|\nabla[(v\circ\sigma^{-1})U]|^2\right]dv_{g_e}\end{aligned} \tag{5.5}$$

and

$$\int_{\mathbb{S}^n}u^\alpha v^\beta dv_{g_0}=\int_{\mathbb{R}^n}[(u\circ\sigma^{-1})U]^\alpha[(v\circ\sigma^{-1})U]^\beta dv_{g_e}=1. \tag{5.6}$$

Consequently, $(\overline{u},\overline{v})$ is a minimizer for $\mathcal{S}_*$, where $\overline{u}=(u\circ\sigma^{-1})U$ and $\overline{v}=(v\circ\sigma^{-1})U$, that is,

$$\begin{cases}-\Delta\overline{u}=\mathcal{S}_*\dfrac{\alpha}{2^*}\overline{u}^{\alpha-1}\overline{v}^\beta & \text{in } \mathbb{R}^n,\\ -\Delta\overline{v}=\mathcal{S}_*\dfrac{\beta}{2^*}\overline{u}^{\alpha}\overline{v}^{\beta-1} & \text{in } \mathbb{R}^n.\end{cases} \tag{5.7}$$

From Theorem 1.3 in [12], it follows that

$$\overline{u}(y) = \xi_1 \left(\frac{\varepsilon_0}{\varepsilon_0^2 + |y - y_0|^2} \right)^{(n-2)/2} \quad \text{and } \overline{v}(y) = \zeta_1 \left(\frac{\varepsilon_0}{\varepsilon_0^2 + |y - y_0|^2} \right)^{(n-2)/2}, \tag{5.8}$$

where $y_0 \in \mathbb{R}^n, \varepsilon_0 > 0$ and $\xi_1, \zeta_1 > 0$ satisfying

$$\xi_1 n(n-2) = \mathcal{S}_* \frac{\alpha}{2^*} \xi_1^{\alpha-1} \zeta_1^{\beta},$$
$$\zeta_1 n(n-2) = \mathcal{S}_* \frac{\beta}{2^*} \xi_1^{\alpha} \zeta_1^{\beta-1},$$
$$\xi_1^{\alpha} \zeta_1^{\beta} = \left[\frac{n(n-2)}{\lambda(\mathbb{S}^n)} \right]^{\frac{n}{2}},$$

so, a simple calculation gives us

$$\xi_1^2 = \left(\frac{\alpha}{\beta} \right)^{\beta/2^*} \left[\frac{n(n-2)}{\lambda(\mathbb{S}^n)} \right]^{(n-2)/2},$$
$$\zeta_1^2 = \left(\frac{\beta}{\alpha} \right)^{\alpha/2^*} \left[\frac{n(n-2)}{\lambda(\mathbb{S}^n)} \right]^{(n-2)/2},$$
$$\overline{u} = \frac{\xi_1}{\zeta_1} \overline{v} = \left(\frac{\alpha}{\beta} \right)^{\frac{1}{2}} \overline{v}.$$

Therefore, by definition of $\overline{u}$ and $\overline{v}$ we get that $u = \frac{\xi_1}{\zeta_1} v$. Then $\xi_1^{-1} u$ is positive solution (up to a rescaling) of the equation $-\Delta_{g_0} w + \frac{n(n-2)}{4} w = w^{2^*-1}$ in $\mathbb{S}^n$. From Theorem 5.1 in [15] then up to a constant scale factor, u is of the following form, $u(x) = \xi_1 (\rho_0 - \cos r)^{\frac{2-n}{2}}$, so, $v(x) = \zeta_1 (\rho_0 - \cos r)^{\frac{2-n}{2}}$, where $r = d_{g_0}(x, x_0)$ and $\rho_0 > 1$. This complete the proof. □

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Declarations

Conflict of interest The authors declare no conflict of interest.

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