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Results in Mathematics

A local estimate for the mean curvature flow

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Abstract. We establish a pointwise estimate of $|A|$ along the mean curvature flow in terms of the initial geometry and the $|HA|$ bound. As corollaries we obtain the blowup rate estimate of $|HA|$ and an extension theorem with respect to $|HA|$.

Mathematics Subject Classification. 53E10.

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1. Introduction

Let $\mathbf{x}_0 : \Sigma^n \to \mathbb{R}^{n+1}$ be a complete smooth immersed hypersurface without boundary and a family of immersions $\mathbf{x}(x, t) : \Sigma^n \times [0, T) \to \mathbb{R}^{n+1}$ be a solution to the equation

$$\partial_t \mathbf{x} = -H\mathbf{n}, \quad \mathbf{x}(0) = \mathbf{x}_0,$$

which is called a mean curvature flow with the singular time T. If Σ is a closed embedded hypersurface, then the flow develops a singularity at $T < \infty$ and $\sup_{\Sigma_t} |A| \to \infty$ as $t \to T$ according to Huisken [1].

Since the finite-time singularity for a compact mean curvature flow is characterized by the blowup of the second fundamental form, it is of great interest to express this criterion in terms of some simpler quantity. A natural conjecture is the blowup of the mean curvature H, which is proposed as an open problem in [2]. The case of $n = 2$ was confirmed by Li–Wang [3]. However, in [4] Stolarski showed that for general cases $n \geq 7$ the mean curvature does not necessarily blow up at the finite singular time.

Hence we turn to consider some alternative conditions for general dimensions $n \geq 2$ which may be stronger than the mean curvature bound. In [5] Cooper proved the HA tensor also blows up at time T. In [5–7], Cooper and Le-Sesum proved that the mean curvature blows up under the assumption of some slow blowup rate of the second fundamental form. Some extension results under integral conditions also can be seen in Le–Sesum [8] and Xu–Ye–Zhao [9].

Note that similar blowup and extension results have been studied for Ricci flow as well. In [10] Hamilton proved that the Riemann curvature tensor blows up at the finite singular time. In [11] Sesum proved the blowup of the Ricci curvature. In [12–14], Wang, Chen-Wang and Kotschwar-Munteanu-Wang arrived at estimates on curvature growth in terms of the Ricci curvature.

The explict local estimate in Kotschwar–Munteanu–Wang [14] has some precedent on a gradient shrinking soliton in [15] that a bound on Ric implies a polynomial growth bound on Rm. The feasibility lies in the observation that the second order derivatives of Ric appear as time-derivative of Rm, i.e.,

$$\partial_t Rm = c\nabla^2 Ric,$$

which helps to yield a differential inequality on integrations. This equation follows from the fact that Ric describes the metric evolution along a Ricci flow. In a similar way HA describes the metric evolution of the mean curvature flow and plays the role of Ric by

$$\partial_t |A|^2 = 2\big(\nabla^2 H \cdot A + H \cdot \mathrm{tr}(A^3)\big).$$

In the present paper, we follow the techniques on integration estimates from [14] and establish the following local L^∞ estimate of A in terms of the initial geometry and the $|HA|$ bound along the flow.

Theorem 1.1. *Fix $x_0 \in \mathbb{R}^{n+1}$ and $r > 0$. Let $\mathbf{x} : \Sigma^n \times [t_0, t_1] \to \mathbb{R}^{n+1}$ be a complete smooth mean curvature flow satisfying the uniform bound*

$$\sup_{B(x_0,r)\cap\Sigma_t} |HA|(\cdot, t) \le K(t), \quad \forall\, t \in [t_0, t_1].$$

Then for any $q > n+2$ there exist positive constants $C = C(n, r, t_1-t_0, q, K(t_0))$ and $c = c(n, q)$ such that for any $t \in [t_0, t_1]$

$$\sup_{B(x_0,r/2)\cap\Sigma_t} |A| \le C\Big(1 + \|A\|^q_{L^q(B(x_0,2r)\cap\Sigma_{t_0})}\Big)^c \Big(1 + Vol_{g(t_0)}(B_{2r,t})\Big)^c \left(\int_{t_0}^{t} e^{\int_{t_0}^{s} cK} ds\right)^c,$$

where $B_{2r,t} = B(x_0, 2r + n^{1/4}\int_{t_0}^{t}\sqrt{K}) \cap \Sigma_{t_0}$.

This local estimate provides a new proof of the blowup of $|HA|$ in [5] and extends the estimates for the Ricci flow in terms of Ric in [12–14] to an estimate for the mean curvature flow in terms of $|HA|$.

One of its direct corollaries is the following extension theorem as well as a blowup estimate of $|HA|$ at the first finite singular time. This result generalizes Theorem 1.2 of [7] and Theorem 5.1 of [5] and can be seen as another version of Theorem 1.1 of [12] and Theorem 2 of [14].

Theorem 1.2. *There exists a positive constant* $\epsilon = \epsilon(n)$ *satisfying the following properties. Let* $\mathbf{x} : \Sigma^n \times [0, T) \to \mathbb{R}^{n+1}$ *be a complete smooth mean curvature flow with* $T < \infty$*. Suppose each time slice* Σ_t *has bounded second fundamental form. If*

$$\sup_{\Sigma_t} |HA| \le \frac{\epsilon}{T-t}, \quad \forall\, t \in [0, T),$$

then

$$\limsup_{t\to T} \sup_{\Sigma_t} |A|(\cdot, t) \le C(n, T, \Sigma_0) < \infty,$$

which implies the flow can be extended past time T*. Conversely, if the flow blows up at time* T*, then*

$$\limsup_{t\to T} \left((T-t) \sup_{\Sigma_t} |HA| \right) \ge \epsilon.$$

The organization of this paper is as follows. In Sect. 2 we recall some basic results on mean curvature flow. In Sect. 3 we develop L^p estimate in terms of initial data and $|HA|$ bound, following the argument in [14]. In Sect. 4 we establish the L^∞ estimate by Moser iteration as in [8] and finish the extension theorem. In Sect. 5 we estimate the blowup rate of $|HA|$, using the L^∞ estimate and the blowup estimate of $|A|$.

2. Preliminaries

Let $\mathbf{x}(p, t) : \Sigma^n \to \mathbb{R}^{n+1}$ be a family of smooth immersions. $\{(\Sigma^n, \mathbf{x}(\cdot, t)), 0 \le t < T\}$ is called a mean curvature flow if $\mathbf{x}$ satisfies

$$\partial_t \mathbf{x} = -H\mathbf{n}, \quad \forall\, t \in [0, T), \tag{2.1}$$

where we denote by $A = (h_{ij})$ the second fundamental form and by $H = g^{ij} h_{ij}$ the mean curvature. Sometimes we also write Σ_t as $\mathbf{x}(t)$ for short.

Some equations are listed here for later calculations. See [1] or [2] for details.

Lemma 2.1 (Sect. 3 of [1]). *Along the mean curvature flow,*

$$\partial_t d\mu = -H^2 d\mu,$$

$$\partial_t |A|^2 = 2(\nabla_{e_i} \nabla_{e_j} H \cdot A_{ij} + H A_{kl} A_{lm} A_{mk}), \tag{2.2}$$

$$2|\nabla H|^2 = (\Delta - \partial_t) H^2 + 2H^2 |A|^2, \tag{2.3}$$

$$2|\nabla A|^2 = (\Delta - \partial_t)|A|^2 + 2|A|^4. \tag{2.4}$$

By maximum principle the second fundamental form blows up at least at a rate of 1/2, which holds for noncompat cases as well.

Lemma 2.2 (Proposition 2.4.6 of [2]). *Suppose the flow (2.1) blows up at the finite singular time T and each time slice Σ_t has bounded second fundamtental form. Then*

$$\sup_{\Sigma_t} |A| \geq \frac{1}{\sqrt{2(T-t)}}.$$

On a hypersurface we also have the Sobolev inequality, i.e., the Michael-Simon inequality. See [1,16].

Lemma 2.3 (Lemma 5.7 of [1]). *Let f be a nonnegative Lipschitz function with compact support on a hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$. Then there exists a positive constant $c = c(n)$ such that*

$$\left(\int_\Sigma |f|^{\frac{n}{n-1}} d\mu\right)^{\frac{n-1}{n}} \leq c_n \int_\Sigma (|\nabla f| + |H| f) d\mu.$$

From L^p estimate to L^∞ estimate we require the process of Moser iteration which depends on the Michael-Simon inequality, i.e, Lemma 2.3. We conclude the following result from Lemma 5.2 in [8].

Lemma 2.4 (Moser iteration). *Let $\mathbf{x} : \Sigma^n \times [t_0, t_1] \to \mathbb{R}^{n+1}$ be a smooth mean curvature flow. Consider the differential inequality*

$$(\partial_t - \Delta) v \leq f v, \quad v \geq 0.$$

Fix $x_0 \in \mathbb{R}^{n+1}$ and $r > 0$. For any $q > n+2$ and $\beta \geq 2$ there exists a constant $C = C(n, r, t_1 - t_0, q, \beta)$ such that for any $t \in [t_0, t_1]$

$$\begin{aligned} &\|v\|_{L^\infty(D'_{t,r})} \\ &\quad \leq C\big(1 + \|f\|_{L^{q/2}(D_{t,r})}\big)^{\frac{qn^2}{\beta(q-n-2)}} \big(1 + \|H\|^{n+2}_{L^{n+2}(D_{t,r})}\big)^{\frac{qn^3}{\beta(n+2)(q-n-2)}} \|v\|_{L^\beta(D_{t,r})}, \end{aligned}$$

where

$$\begin{aligned} D_{t,r} &:= \bigcup_{t_0 \leq s \leq t} \big(B(x_0, r) \cap \Sigma_s\big), \\ D'_{t,r} &:= \bigcup_{(t_0+t)/2 \leq s \leq t} \big(B(x_0, r/2) \cap \Sigma_s\big). \end{aligned}$$

Proof. Without loss of generality, we assume $t_0 = 0$, $t = 1$ and $r = 1$. Set

$$C_0 = 1 + \|f\|_{L^q(D_{t,r})}, \quad C_1 = \big(1 + \|H\|^{n+2}_{L^{n+2}(D_{t,r})}\big)^{\frac{n}{n+2}}, \quad \nu = \frac{n+2}{2q - (n+2)},$$

where $q > \frac{n+2}{2}$. According to the proof of Lemma 5.2 of [8] we have for $\beta \geq 2$,

$$\|v\|_{L^\infty(D'_{t,r})} \leq C_b \|v\|_{L^\beta(D_{t,r})},$$

where

$$C_b = C_b(n, q, \beta, C_0, C_1) = \left(4\left(\frac{n+2}{n}\right)^{1+\nu} C_z\beta^{1+\nu}\right)^{\frac{n^2}{\beta}},$$
$$C_z = C_z(n, q, C_0, C_1) = 16 \cdot 100^{1+\nu} c_n C_a.$$

According to the proof of Lemma 4.1 of [8] we have

$$C_a = C_a(n, q, C_0, C_1) = (2c_n C_0 C_1)^{1+\nu}.$$

As a conclusion,

$$C_b = \big(C(n, q, \beta)C_z\big)^{\frac{n^2}{\beta}} = \big(C(n, q, \beta)C_a\big)^{\frac{n^2}{\beta}} = C(n, q, \beta)(C_0C_1)^{\frac{n^2(1+\nu)}{\beta}}$$
$$= C(n, q, \beta)\big(1 + \|f\|_{L^q(D_{t,r})}\big)^{\frac{2qn^2}{\beta(2q-n-2)}} \big(1 + \|H\|^{n+2}_{L^{n+2}(D_{t,r})}\big)^{\frac{2qn^3}{\beta(n+2)(2q-n-2)}}.$$

□

3. L^p estimate

Throughout this section we use c to denote a nonnegative constant depending only on n and p and we use c_n to denote a nonnegative constant depending only on n, which may change from line to line.

Theorem 3.1. *Fix $x_0 \in \mathbb{R}^{n+1}$ and $r > 0$. Let $\mathbf{x} : \Sigma^n \times [t_0, t_1] \to \mathbb{R}^{n+1}$ be a complete smooth mean curvature flow satisfying the bound*

$$\sup_{B(x_0,r)\cap\Sigma_t} |HA|(\cdot, t) \le K(t), \quad \forall\, t \in [t_0, t_1],$$

where $K(t)$ is nondecreasing. Then for any $p \ge 2$ there exist positive constants $c = c(n, p)$ such that for any $t \in [t_0, t_1]$,

$$\int_{B(x_0,r/2)\cap\Sigma_t} |A|^p \le \Bigg(K(t_0)^{-1}\int_{B(x_0,r)\cap\Sigma_{t_0}} |A|^{p+2}(t_0) + c\int_{B(x_0,r)\cap\Sigma_{t_0}} |A|^p(t_0)$$
$$+K(t_0)^{-1}r^{-(p+2)}\, Vol_{g(t_0)}(B_{r,t})\Bigg) \cdot e^{\int_{t_0}^t cK},$$

where $B_{r,t} = B(x_0, r + n^{1/4}\int_{t_0}^t \sqrt{K}) \cap \Sigma_{t_0}$.

Proof. Let $\phi(x, t)$ be a nonnegative smooth function with compact support which will be determined later. Note that $|H||A| \le K$. By Eq. (2.2) we have

$$\partial_t \int_{\Sigma_t} |A|^p\phi$$
$$\le \int_{\Sigma_t} \partial_t|A|^p\phi + \int_{\Sigma_t} |A|^p\partial_t\phi$$

$$
\begin{aligned}
&= p\int_{\Sigma_t}|A|^{p-2}\phi(\nabla_{e_i}\nabla_{e_j}H\cdot A_{ij}+HA_{kl}A_{lm}A_{mk})+\int_{\Sigma_t}|A|^p\partial_t\phi\\
&\le c\int_{\Sigma_t}|A|^{p-2}|\nabla H||\nabla A|\phi+c\int_{\Sigma_t}|A|^{p-1}|\nabla H||\nabla\phi|+c\int_{\Sigma_t}|H||A|^{p+1}\\
&\quad+\int_{\Sigma_t}|A|^p\partial_t\phi\\
&\le \frac{c}{K}\int_{\Sigma_t}|A|^p|\nabla H|^2\phi+cK\int_{\Sigma_t}|A|^{p-4}|\nabla A|^2\phi+cK\int_{\Sigma_t}|A|^p\phi\\
&\quad+cK\int_{\Sigma_t}|A|^{p-2}\phi^{-1}|\nabla\phi|^2+\int_{\Sigma_t}|A|^p\partial_t\phi.
\end{aligned}
$$

By Eq. (2.3) we have

$$
\begin{aligned}
&\int_{\Sigma_t}|A|^p|\nabla H|^2\phi\\
&\quad=\frac{1}{2}\int_{\Sigma_t}|A|^p\phi(\Delta-\partial_t)H^2+\int_{\Sigma_t}H^2|A|^{p+2}\phi\\
&\quad\le c\int_{\Sigma_t}|H||A|^{p-1}|\nabla H||\nabla A|\phi+c\int_{\Sigma_t}|H||A|^p|\nabla H||\nabla\phi|\\
&\qquad-\frac{1}{2}\partial_t\int_{\Sigma_t}H^2|A|^p\phi+\frac{1}{2}\int_{\Sigma_t}H^2\partial_t(|A|^p\phi)+\int_{\Sigma_t}H^2|A|^{p+2}\phi\\
&\quad= c\int_{\Sigma_t}|H||A|^{p-1}|\nabla H||\nabla A|\phi+c\int_{\Sigma_t}|H||A|^p|\nabla H||\nabla\phi|\\
&\qquad-\frac{1}{2}\partial_t\int_{\Sigma_t}H^2|A|^p\phi+c\int_{\Sigma_t}H^2|A|^{p-2}(\nabla_{e_i}\nabla_{e_j}H\cdot A_{ij}+HA_{kl}A_{lm}A_{mk})\phi\\
&\qquad+\frac{1}{2}\int_{\Sigma_t}H^2|A|^p\partial_t\phi+\int_{\Sigma_t}H^2|A|^{p+2}\phi\\
&\quad\le c\int_{\Sigma_t}|H||A|^{p-1}|\nabla H||\nabla A|\phi+c\int_{\Sigma_t}|H||A|^p|\nabla H||\nabla\phi|\\
&\qquad+c\int_{\Sigma_t}|H||A|^{p-1}|\nabla H|^2\phi+c\int_{\Sigma_t}H^2|A|^{p-2}|\nabla H||\nabla A|\phi+c\int_{\Sigma_t}|H|^3|A|^{p+1}\phi\\
&\qquad-\frac{1}{2}\partial_t\int_{\Sigma_t}H^2|A|^p\phi+\frac{1}{2}\int_{\Sigma_t}H^2|A|^p\partial_t\phi+c\int_{\Sigma_t}H^2|A|^{p+2}\phi\\
&\quad\le c\int_{\Sigma_t}|H||A|^{p-1}|\nabla H||\nabla A|\phi+c\int_{\Sigma_t}|H||A|^p|\nabla H||\nabla\phi|\\
&\qquad-\frac{1}{2}\partial_t\int_{\Sigma_t}H^2|A|^p\phi+\frac{1}{2}\int_{\Sigma_t}H^2|A|^p\partial_t\phi+c\int_{\Sigma_t}H^2|A|^{p+2}\phi.
\end{aligned}
$$

By Cauchy's inequality we have

$$\begin{aligned}
&\int_{\Sigma_t} |A|^p|\nabla H|^2\phi \\
&\quad \le \frac{1}{2}\int_{\Sigma_t} |A|^p|\nabla H|^2\phi + c\int_{\Sigma_t} H^2|A|^{p-2}|\nabla A|^2\phi + c\int_{\Sigma_t} H^2|A|^p\phi^{-1}|\nabla\phi|^2 \\
&\qquad -\frac{1}{2}\partial_t\int_{\Sigma_t} H^2|A|^p\phi + \frac{1}{2}\int_{\Sigma_t} H^2|A|^p\partial_t\phi + c\int_{\Sigma_t} H^2|A|^{p+2}\phi \\
&\quad \le -\frac{1}{2}\partial_t\int_{\Sigma_t} H^2|A|^p\phi + cK^2\int_{\Sigma_t} |A|^{p-4}|\nabla A|^2\phi + cK^2\int_{\Sigma_t} |A|^p\phi \\
&\qquad +\frac{1}{2}\int_{\Sigma_t} |A|^p|\nabla H|^2\phi + cK^2\int_{\Sigma_t} |A|^{p-2}\phi^{-1}|\nabla\phi|^2 + \frac{1}{2}\int_{\Sigma_t} H^2|A|^p\partial_t\phi,
\end{aligned}$$

and then

$$\begin{aligned}
&\frac{c}{K}\int_{\Sigma_t} |A|^p|\nabla H|^2\phi \\
&\quad \le -\frac{c}{K}\partial_t\int_{\Sigma_t} H^2|A|^p\phi + cK\int_{\Sigma_t} |A|^{p-4}|\nabla A|^2\phi + cK\int_{\Sigma_t} |A|^p\phi \\
&\qquad + cK\int_{\Sigma_t} |A|^{p-2}\phi^{-1}|\nabla\phi|^2 + \frac{c}{K}\int_{\Sigma_t} H^2|A|^p\partial_t\phi.
\end{aligned}$$

By Eq. (2.4) we have for $p \ge 4$,

$$\begin{aligned}
&\int_{\Sigma_t} |A|^{p-4}|\nabla A|^2\phi \\
&\quad = \frac{1}{2}\int_{\Sigma_t} |A|^{p-4}(\Delta - \partial_t)|A|^2\phi + \int_{\Sigma_t} |A|^p\phi \\
&\quad = -\frac{1}{2}\int_{\Sigma_t} \nabla(|A|^{p-4}\phi)\cdot\nabla|A|^2 - \frac{1}{2}\int_{\Sigma_t} |A|^{p-4}\phi\cdot\partial_t|A|^2 + \int_{\Sigma_t} |A|^p\phi \\
&\quad \le -(p-4)\int_{\Sigma_t} |A|^{p-4}|\nabla|A||^2\phi + \int_{\Sigma_t} |A|^{p-3}|\nabla A||\nabla\phi| \\
&\qquad - c\partial_t\int_{\Sigma_t} |A|^{p-2}\phi + c\int_{\Sigma_t} |A|^{p-2}\partial_t\phi + \int_{\Sigma_t} |A|^p\phi \\
&\quad \le -c\partial_t\int_{\Sigma_t} |A|^{p-2}\phi + \frac{1}{2}\int_{\Sigma_t} |A|^{p-4}|\nabla A|^2\phi + c\int_{\Sigma_t} |A|^{p-2}\phi^{-1}|\nabla\phi|^2 \\
&\qquad + c\int_{\Sigma_t} |A|^{p-2}\partial_t\phi + \int_{\Sigma_t} |A|^p\phi,
\end{aligned}$$

and then

$$
\begin{aligned}
&K\int_{\Sigma_t}|A|^{p-4}|\nabla A|^2\phi\\
&\quad\le -cK\partial_t\int_{\Sigma_t}|A|^{p-2}\phi+2K\int_{\Sigma_t}|A|^p\phi+cK\int_{\Sigma_t}|A|^{p-2}\phi^{-1}|\nabla\phi|^2\\
&\qquad+cK\int_{\Sigma_t}|A|^{p-2}\partial_t\phi.
\end{aligned}
$$

Combining the results above together we have for $p\ge 4$,

$$
\begin{aligned}
&\partial_t\int_{\Sigma_t}|A|^p\phi\\
&\quad\le -\frac{c}{K}\partial_t\int_{\Sigma_t}H^2|A|^p\phi+cK\int_{\Sigma_t}|A|^{p-4}|\nabla A|^2\phi+cK\int_{\Sigma_t}|A|^p\phi\\
&\qquad+cK\int_{\Sigma_t}|A|^{p-2}\phi^{-1}|\nabla\phi|^2+\frac{c}{K}\int_{\Sigma_t}H^2|A|^p\partial_t\phi+\int_{\Sigma_t}|A|^p\partial_t\phi\\
&\quad\le -\frac{c}{K}\partial_t\int_{\Sigma_t}H^2|A|^p\phi-cK\partial_t\int_{\Sigma_t}|A|^{p-2}\phi+cK\int_{\Sigma_t}|A|^p\phi\\
&\qquad+cK\int_{\Sigma_t}|A|^{p-2}\phi^{-1}|\nabla\phi|^2+\int_{\Sigma_t}|A|^p|\partial_t\phi|+cK\int_{\Sigma_t}|A|^{p-2}|\partial_t\phi|.
\end{aligned}
\tag{3.1}
$$

Consider a smooth decreasing function η, which equals 1 on $[0,r/2]$ and vanishes on $[r,\infty)$, satisfying $|\eta'|\le 3/r$. For any $0<\delta<1$ we set $\psi:=\eta^{1/\delta}$ such that

$$
|\psi'|\le\frac{3}{\delta r}\psi^{1-\delta}.
$$

Now we choose $\phi:=\psi(|x-x_0|)$. Then

$$
\begin{aligned}
&\phi^{-1}|\nabla\phi|^2\le\psi^{-1}(\psi')^2\le\frac{9}{\delta^2r^2}\phi^{1-2\delta},\\
&|\partial_t\phi|=|\psi'||\partial_t(|x-x_0|)|\le|\psi'||H|\le\frac{3}{\delta r}|H|\phi^{1-\delta}.
\end{aligned}
$$

Take $\delta=\frac{1}{p}$. By Young's inequality we have

$$
\begin{aligned}
\int_{\Sigma_t}|A|^{p-2}\phi^{-1}|\nabla\phi|^2&\le cr^{-2}\int_{\Sigma_t}|A|^{p-2}\phi^{1-\frac{2}{p}}\le c\int_{\Sigma_t}|A|^p\phi+c\int_{\Sigma_t\cap\mathrm{supp}\phi}r^{-p}\\
&\le c\int_{\Sigma_t}|A|^p\phi+cr^{-p}\mathrm{Vol}_{g(t)}\big(B(x_0,r)\cap\Sigma_t\big),
\end{aligned}
$$

and

$$\begin{aligned}\int_{\Sigma_t}|A|^{p-2}|\partial_t\phi| &\le cr^{-1}\int_{\Sigma_t}|A|^{p-1}\phi^{1-\frac{1}{p}} \le c\int_{\Sigma_t}|A|^p\phi + c\int_{\Sigma_t\cap\mathrm{supp}\phi}r^{-p}\\ &\le c\int_{\Sigma_t}|A|^p\phi + cr^{-p}\mathrm{Vol}_{g(t)}\big(B(x_0,r)\cap\Sigma_t\big),\end{aligned}$$

and

$$\begin{aligned}\int_{\Sigma_t}|A|^{p}|\partial_t\phi| &\le cr^{-1}\int_{\Sigma_t}|H||A|^{p}\phi^{1-\frac{1}{p}} \le cr^{-1}K\int_{\Sigma_t}|A|^{p-1}\phi^{1-\frac{1}{p}}\\ &\le cK\int_{\Sigma_t}|A|^p\phi + cKr^{-p}\mathrm{Vol}_{g(t)}\big(B(x_0,r)\cap\Sigma_t\big).\end{aligned}$$

Back to (3.1), we obtain

$$\begin{aligned}&\partial_t\int_{\Sigma_t}|A|^p\phi + \frac{c}{K}\partial_t\int_{\Sigma_t}H^2|A|^p\phi + cK\partial_t\int_{\Sigma_t}|A|^{p-2}\phi\\ &\quad\le cK\int_{\Sigma_t}|A|^p\phi + cKr^{-p}\mathrm{Vol}_{g(t)}\big(B(x_0,r)\cap\Sigma_t\big).\end{aligned}\tag{3.2}$$

If we set

$$U(t) = \int_{\Sigma_t}|A|^p\phi + \frac{c}{K}\int_{\Sigma_t}H^2|A|^p\phi + cK\int_{\Sigma_t}|A|^{p-2}\phi,$$

then actually it becomes

$$\begin{aligned}U' &\le -\frac{cK'}{K^2}\int_{\Sigma_t}H^2|A|^p\phi + cK'\int_{\Sigma_t}|A|^{p-2}\phi + cK\int_{\Sigma_t}|A|^p\phi\\ &\quad + cKr^{-p}\mathrm{Vol}_{g(t)}\big(B(x_0,r)\cap\Sigma_t\big)\\ &\le (K'/K + cK)U + cKr^{-p}\mathrm{Vol}_{g(t)}\big(B(x_0,r)\cap\Sigma_t\big).\end{aligned}$$

Since

$$\partial_t|x-x_0| \le |H| \le n^{1/4}\sqrt{K}$$

and

$$\partial_t d\mu = -H^2 d\mu,$$

we know

$$\begin{aligned}&B(x_0,r)\cap\Sigma_t \subset B\left(x_0, r + n^{1/4}\int_{t_0}^t\sqrt{K}\right)\cap\Sigma_{t_0} := B_{r,t},\\ &\mathrm{Vol}_{g(s)}\big(B(x_0,r)\cap\Sigma_s\big) \le \mathrm{Vol}_{g(t_0)}(B_{r,t}), \quad \forall\, s\in[t_0,t].\end{aligned}$$

Then for any $s \in [t_0, t]$,

$$\begin{aligned}
&\partial_s\Big(e^{-\int_{t_0}^s (K'/K+cK)}U(s)\Big)\\
&\quad\le cKe^{-\int_{t_0}^s (K'/K+cK)}r^{-p}\mathrm{Vol}_{g(t_0)}(B_{r,t})\\
&\quad\le (cK+K'/K)e^{-\int_{t_0}^s (K'/K+cK)}r^{-p}\mathrm{Vol}_{g(t_0)}(B_{r,t})\\
&\quad= \partial_s\Big(-e^{-\int_{t_0}^s (K'/K+cK)}\Big)r^{-p}\mathrm{Vol}_{g(t_0)}(B_{r,t}),
\end{aligned}$$

i.e.,

$$\partial_s\Big(e^{-\int_{t_0}^s (K'/K+cK)}\big(U(s)+r^{-p}\mathrm{Vol}_{g(t_0)}(B_{r,t})\big)\Big)\le 0,$$

which implies

$$\begin{aligned}
U(s) &\le e^{\int_{t_0}^s (K'/KcK)}\Big(U(t_0)+r^{-p}\mathrm{Vol}_{g(t_0)}(B_{r,t})\Big)\\
&= K(s)/K(t_0)\Big(U(t_0)+r^{-p}\mathrm{Vol}_{g(t_0)}(B_{r,t})\Big)e^{\int_{t_0}^s cK}, \quad \forall\, s\in[t_0,t].
\end{aligned} \tag{3.3}$$

In particular, we focus on the third term of U to see that for $p \ge 4$,

$$\begin{aligned}
&cK(t)\int_{B(x_0,r/2)\cap\Sigma_t}|A|^{p-2}\\
&\quad\le K(t)/K(t_0)\left(\int_{B(x_0,r)\cap\Sigma_{t_0}}|A|^p(t_0)+cK(t_0)\int_{B(x_0,r)\cap\Sigma_{t_0}}|A|^{p-2}(t_0)\right.\\
&\qquad\left.+\,r^{-p}\mathrm{Vol}_{g(t_0)}(B_{r,t})\right)\cdot e^{\int_{t_0}^t cK}.
\end{aligned}$$

In other words, for $p \ge 2$,

$$\begin{aligned}
\int_{B(x_0,r/2)\cap\Sigma_t}|A|^p \le &\left(K(t_0)^{-1}\int_{B(x_0,r)\cap\Sigma_{t_0}}|A|^{p+2}(t_0)+c\int_{B(x_0,r)\cap\Sigma_{t_0}}|A|^p(t_0)\right.\\
&\left.+\,K(t_0)^{-1}r^{-(p+2)}\mathrm{Vol}_{g(t_0)}(B_{r,t})\right)\cdot e^{\int_{t_0}^t cK}.
\end{aligned}$$

Similarly, we can focus on the first term instead to see that for $p \ge 4$,

$$\begin{aligned}
\int_{B(x_0,r/2)\cap\Sigma_t}|A|^p \le &\left(K(t_0)^{-1}\int_{B(x_0,r)\cap\Sigma_{t_0}}|A|^p(t_0)+c\int_{B(x_0,r)\cap\Sigma_{t_0}}|A|^{p-2}(t_0)\right.\\
&\left.+\,K(t_0)^{-1}r^{-p}\mathrm{Vol}_{g(t_0)}(B_{r,t})\right)\cdot K(t)e^{\int_{t_0}^t cK}.
\end{aligned} \tag{3.4}$$

□

4. L^∞ estimate and extension theorem

Combining Theorem 3.1 and Lemma 2.4 we obtain the following local estimate.

Theorem 4.1 (L^∞ estimate). *Fix* $x_0 \in \mathbb{R}^{n+1}$ *and* $r > 0$. *Let* $\mathbf{x} : \Sigma^n \times [t_0, t_1] \to \mathbb{R}^{n+1}$ *be a complete smooth mean curvature flow satisfying the bound*

$$\sup_{B(x_0,r)\cap\Sigma_t} |HA|(\cdot,t) \le K(t), \quad \forall\, t \in [t_0, t_1],$$

where $K(t)$ *is nondecreasing. Then for any* $q > n+2$ *there exist positive constants* $C = C(n, r, t_1 - t_0, q, K(t_0))$ *and* $c = c(n, q)$ *such that for any* $t \in [t_0, t_1]$,

$$\sup_{D'_{t,r}} |A| \le C\left(1 + \|A\|_{L^{q+2}(B(x_0,2r)\cap\Sigma_{t_0})}\right)^c \left(1 + Vol_{g(t_0)}(B_{2r,t})\right)^c \left(\int_{t_0}^t e^{\int_{t_0}^s cK} ds\right)^c,$$

where $B_{2r,t} = B(x_0, 2r + n^{1/4}\int_{t_0}^t \sqrt{K}) \cap \Sigma_{t_0}$.

Proof. Take $\beta = \frac{n+2}{2}$. Applying Lemma 2.4 to

$$(\partial_t - \Delta)|A|^2 = -2|\nabla A|^2 + 2|A|^4 \le 2|A|^2 \cdot |A|^2$$

yields

$$\begin{aligned}
&\sup_{D'_{t,r}} |A| \\
&\le C\Big(1 + \|A\|_{L^q(D_{t,r})}\Big)^{\frac{qn^2}{\beta(q-n-2)}} \Big(1 + \|A\|^{\frac{n+2}{2}}_{L^{n+2}(D_{t,r})}\Big)^{\frac{qn^3}{2\beta(n+2)(q-n-2)}} \|A\|_{L^{2\beta}(D_{t,r})} \\
&\le C\Big(1 + \|A\|_{L^q(D_{t,r})}\Big)^{\frac{2qn^2}{(n+2)(q-n-2)}} \Big(1 + \|A\|_{L^{n+2}(D_{t,r})}\Big)^{1+\frac{qn^3}{(n+2)(q-n-2)}} \\
&\le C\Big(1 + \|A\|_{L^q(D_{t,r})}\Big)^{\frac{2qn^2}{(n+2)(q-n-2)}} \Big(1 + \|A\|_{L^q(D_{t,r})} \mathrm{Vol}(D_{t,r})^{\frac{1}{n+2}-\frac{1}{q}}\Big)^{1+\frac{qn^3}{(n+2)(q-n-2)}} \\
&\le C\Big(1 + \|A\|_{L^q(D_{t,r})}\Big)^{\frac{2qn^2}{(n+2)(q-n-2)}} \Big(1 + \|A\|_{L^q(D_{t,r})}\Big)^{1+\frac{qn^3}{(n+2)(q-n-2)}} \\
&\quad \Big(1 + \mathrm{Vol}(D_{t,r})^{\frac{1}{n+2}-\frac{1}{q}}\Big)^{1+\frac{qn^3}{(n+2)(q-n-2)}} \\
&\le C\Big(1 + \|A\|_{L^q(D_{t,r})}\Big)^{1+\frac{qn^2}{q-n-2}} \Big(1 + \mathrm{Vol}_{g(t_0)}(B_{2r,t})\Big)^{\frac{q-n-2}{q(n+2)}+\frac{n^3}{(n+2)^2}},
\end{aligned} \tag{4.1}$$

where

$$C = C(n, r, t_1 - t_0, q),$$
$$\mathrm{Vol}(D_{t,r}) := \int_{t_0}^t \mathrm{Vol}_{g(s)}(B(x_0, r) \cap \Sigma_s) ds,$$
$$B_{2r,t} = B(x_0, 2r + n^{1/4}\int_{t_0}^t \sqrt{K}) \cap \Sigma_{t_0}.$$

It is derived from Theorem 3.1 that for $q > n+2$

$$\|A\|^q_{L^q(D_{t,r})} \le \Big(K(t_0)^{-1}\|A\|^{q+2}_{L^{q+2}(B(x_0,2r)\cap\Sigma_{t_0})} + c\|A\|^q_{L^q(B(x_0,2r)\cap\Sigma_{t_0})} + K(t_0)^{-1}(2r)^{-(q+2)}\mathrm{Vol}_{g(t_0)}(B_{2r,t})\Big)\cdot\int_{t_0}^t e^{\int_{t_0}^s cK}ds,$$

where $c = c(n,q)$. Note that

$$\begin{aligned}
&K(t_0)^{-1}\|A\|^{q+2}_{L^{q+2}(B(x_0,2r)\cap\Sigma_{t_0})} + c\|A\|^q_{L^q(B(x_0,2r)\cap\Sigma_{t_0})}\\
&\quad + K(t_0)^{-1}(2r)^{-(q+2)}\mathrm{Vol}_{g(t_0)}(B_{2r,t})\\
&\le K(t_0)^{-1}\|A\|^{q+2}_{L^{q+2}(B(x_0,2r)\cap\Sigma_{t_0})} + c\|A\|^q_{L^{q+2}(B(x_0,2r)\cap\Sigma_{t_0})}\mathrm{Vol}_{g(t_0)}(B_{2r,t})^{\frac{2}{q+2}}\\
&\quad + K(t_0)^{-1}(2r)^{-(q+2)}\mathrm{Vol}_{g(t_0)}(B_{2r,t})\\
&\le C(n,r,q,K(t_0))\Big(1+\|A\|_{L^{q+2}(B(x_0,2r)\cap\Sigma_{t_0})}\Big)^{q+2}\Big(1+\mathrm{Vol}_{g(t_0)}(B_{2r,t})\Big).
\end{aligned}$$

The final coefficient is

$$C\cdot\Big(1+\|A\|_{L^{q+2}(B(x_0,2r)\cap\Sigma_0)}\Big)^{(q+2)(\frac{1}{q}+\frac{n^2}{q-n-2})} \cdot\Big(1+\mathrm{Vol}_{g(t_0)}(B_{2r,t})\Big)^{\frac{1}{q}+\frac{n^2}{q-n-2}+\frac{q-n-2}{q(n+2)}+\frac{n^3}{(n+2)^2}},$$

where $C = C(n,r,t_1-t_0,q,K(t_0))$. Back to (4.1), we have the local L^∞ estimate

$$\sup_{D'_{t,r}}|A| \le C\Big(1+\|A\|_{L^{q+2}(B(x_0,2r)\cap\Sigma_{t_0})}\Big)^c\Big(1+\mathrm{Vol}_{g(t_0)}(B_{2r})\Big)^c\left(\int_{t_0}^t e^{\int_{t_0}^s cK}ds\right)^c.$$

□

As an application of the local estimate above, one immediately gets the following extension theorem about HA.

Corollary 4.2. *Let $\mathbf{x}:\Sigma^n\times[0,T)\to\mathbb{R}^{n+1}$ be a complete smooth mean curvature flow. Suppose each time slice Σ_t has bounded $|HA|$. There exists a positive constant $C = C(n,T,K,V,E,q)$ such that if*

(1) $|HA|$ satisfies

$$\sup_{t\in[0,T)}\sup_{\Sigma_t}|HA|(\cdot,t) \le K < \infty;$$

(2) the initial data satisfies a uniform volume bound

$$\sup_{x\in\Sigma_0} Vol_{g(0)}(B(x,1+n^{1/4}T\sqrt{K})\cap\Sigma_0) \le V < \infty;$$

(3) the initial data satisfies an integral bound

$$\sup_{x\in\Sigma_0}\|A\|_{L^{q+2}(B(x,1)\cap\Sigma_0)} \le E < \infty$$

for some $q > n+2$,

then

$$\limsup_{t\to T}\sup_{\Sigma_t}|A|(\cdot,t)\le C<\infty.$$

In particular, the flow can be extended past time T.

Proof. It suffices to use $B_{2r,t}\subset B_{2r,T}$ and take $r=1$ in Theorem 4.1. □

5. Blowup estimate of $|HA|$

In this section we derive a blowup estimate of $|HA|$ from Theorem 4.1 and Lemma 2.2, which also implies a blowup estimate of mean curvature.

Theorem 5.1 (HA-blowup). *There exists a positive constant $\epsilon=\epsilon(n)$ satisfying the following properties. Let $\mathbf{x}:\Sigma^n\times[0,T)\to\mathbb{R}^{n+1}$ be a complete smooth mean curvature flow with $T<\infty$. Suppose each time slice Σ_t has bounded second fundamental form. If the flow blows up at time T, then*

$$\limsup_{t\to T}\left((T-t)\sup_{\Sigma_t}|HA|\right)\ge\epsilon.$$

Conversely, if

$$\sup_{\Sigma_t}|HA|\le\frac{\epsilon}{T-t},\quad \forall\, t\in[0,T),$$

then

$$\limsup_{t\to T}\sup_{\Sigma_t}|A|(\cdot,t)\le C(n,T,\Sigma_0)<\infty,$$

which implies the flow can be extended past time T.

Proof. Assume that the flow blows up at time T and there exist $t_0\in[0,T)$ and $\epsilon>0$ such that

$$\sup_{\Sigma_t}|HA|<\frac{\epsilon}{T-t},\quad \forall\, t\in[t_0,T).$$

Actually we find a smooth mean curvature flow $\mathbf{x}:\Sigma^n\times[t_0,T)\to\mathbb{R}^{n+1}$ with a $|HA|$ bound

$$K(t)=\frac{\epsilon}{T-t}.$$

For t close to T,

$$\int_{t_0}^{t}c_nK=\int_{t_0}^{t}\frac{c_n\epsilon}{T-s}ds=c_n\epsilon\log\left(\frac{T-t_0}{T-t}\right),$$

$$\int_{t_0}^{t}e^{\int_{t_0}^{s}c_nK}ds=\int_{t_0}^{t}\left(\frac{T-t_0}{T-s}\right)^{c_n\epsilon}ds=\frac{(T-t_0)^{c_n\epsilon}}{1-c_n\epsilon}\left((T-t_0)^{1-c_n\epsilon}-(T-t)^{1-c_n\epsilon}\right).$$

On the other hand, by Lemma 2.2 we know

$$\sup_{\Sigma_t}|A|\ge\frac{1}{\sqrt{2}}(T-t)^{-\frac{1}{2}}.$$

Note that Σ_{t_0} has bounded geometry. If $c_n\epsilon < 1$, then the integral $\int_{t_0}^{t} e^{\int_{t_0}^{s} c_n K} ds$ is bounded and the flow can be extended past time T by Theorem 4.1. This actually proves the second part. If $1 \le c_n\epsilon \le \frac{3}{2}$, then

$$(T-t)^{-\frac{1}{2}} \gg C(n, T, \Sigma_{t_0}, \epsilon)(T-t)^{-(c_n\epsilon - 1)}$$

as $t \to T$. In a word, the choice of $\epsilon \le \epsilon(n)$ causes a contradiction. This completes the proof of the first part. □

Remark that Theorem 5.1 certainly works for the closed cases. The type-I blowup is optimal since the standard sphere $S^n \hookrightarrow \mathbb{R}^{n+1}$ satisfies $|HA| = \frac{n}{2(T-t)}$.

Corollary 5.2 (H-blowup). *Let* $\mathbf{x} : \Sigma^n \times [0, T) \to \mathbb{R}^{n+1}$ *be a complete smooth mean curvature flow with a finite singular time* T*. Suppose each time slice* Σ_t *has bounded second fundamental form. If*

$$\limsup_{t\to T} \left((T-t)^{\lambda} \sup_{\Sigma_t} |A|\right) < \infty$$

for some $\lambda \in [\frac{1}{2}, 1)$*, then we have the blowup estimate of mean curvature*

$$\limsup_{t\to T} \left((T-t)^{1-\lambda} \sup_{\Sigma_t} |H|\right) > 0.$$

Proof. For otherwise for any $\varepsilon > 0$ one finds t_ε such that

$$\sup_{\Sigma_t} |H| \le \varepsilon (T-t)^{\lambda - 1}, \quad \forall\, t \in [t_\varepsilon, T).$$

By the assumption there exist nonnegative constants $t_1 \in [0, T)$ and

$$C := \limsup_{t\to T} \left((T-t)^{\lambda} \sup_{\Sigma_t} |A|\right) < \infty$$

such that

$$\sup_{\Sigma_t} |A| \le C(T-t)^{-\lambda}, \quad \forall\, t \in [t_1, T).$$

Hence we have

$$\sup_{\Sigma_t} |HA| \le C\varepsilon (T-t)^{-1}, \quad \forall\, t \in [\max\{t_\varepsilon, t_1\}, T).$$

Note the constant $\epsilon = \epsilon(n)$ in Theorem 5.1. Choosing ε such that $C\varepsilon < \epsilon$ causes a contradiction according to Theorem 5.1. □

Remark that by Theorem 5.1 of [5] Cooper proved the blowup of mean curvature under the same assumption in Corollary 5.2 and by Theorem 1.2 of [7] Le-Sesum proved the case of $\lambda = \frac{1}{2}$. Hence Theorem 5.1 and Corollary 5.2 can be seen as generalizations of these results.

Corollary 5.3. *Let* $\mathbf{x} : \Sigma^n \times [0, T) \to \mathbb{R}^{n+1}$ *be a complete smooth mean curvature flow with a finite singular time* T*. Suppose each time slice* Σ_t *has bounded second fundamental form. If*

$$\limsup_{t\to T}\left((T-t)^{\lambda}\sup_{\Sigma_t}|H|\right) < \infty$$

for some $\lambda \in [0, \frac{1}{2})$*, then we have the blowup estimate*

$$\limsup_{t\to T}\left((T-t)^{1-\lambda}\sup_{\Sigma_t}|A|\right) > 0.$$

In particular, $t = T$ *is a type-II singularity.*

Proof. By the same argument used in the proof of Corollary 5.2 we obtain the result. □

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