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Results in Mathematics

The Regularity Properties and Blow-up of Solutions for Nonlocal Wave Equations and Applications

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Abstract. In this paper, the Cauchy problem for linear and nonlinear wave equations is studied.The equation involves an abstract operator A in a Hilbert space H and a convolution term. Here, assuming sufficient smoothness on the initial data and on coefficients, the existence, uniqueness, regularity properties, and blow-up of solutions are established in terms of fractional powers of a given sectorial operator A. We obtain the regularity properties of a wide class of wave equations by choosing a space H and an operator A that appear in the field of physics.

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The main objective of this article is to study the existence, uniqueness, regularity, and blow-up properties of the initial value problem (IVP) for the abstract wave equatıon (WE)

$$u_{tt} - a \ast \Delta u + Au = f\left(u\right), \; \left(x, \, t\right) \in \mathbb{R}_T^n = \mathbb{R}^n \times \left(0, T\right), \tag{1.1}$$

$$u\left(x, 0\right) = \varphi\left(x\right), \; u_t\left(x, 0\right) = \psi\left(x\right) \text{ for a.e. } x \in \mathbb{R}^n, \tag{1.2}$$

where $T \in (0, \; \infty]$, A is a linear and $f(u)$ is a nonlinear operator in a Hilbert space H, a is a complex-valued function on $\mathbb{R}^n$, and $\ast$ denotes convolution. Here, $\varphi\left(x\right)$ and $\psi\left(x\right)$ are the given H-valued initial functions.

The qualitative behaviours of a wide class of wave equations can be found, e.g. in [2,4–10], [17–19] and [29–32]. Wave-type equations occur in a wide variety of physical systems, such as the propagation of waves in elastic rods, hydro-dynamical processes in plasma, and in materials science, which describes spinodal decomposition and the absence of mechanical stresses (see [1,11,14, 19–21,24]). Note that abstract hyperbolic equations were studied, e.g., in [2, 12,22,23].

Unlike these studies, in this paper both linear and nonlinear abstract wave equations are considered. The L^p-well-posedness of the Cauchy problem (1.1)–(1.2) depends crucially on the presence of the linear operator A and nonlinear function $f(u)$. We determine the class of operator A and function f to guarantee the existence, uniqueness, regularity properties, and blow up of solutions (1.1)–(1.2) in terms of fractional powers of the operator A. By assigning a concrete space for H and an appropriate operator A we can obtain a variety of wave equations that occur in applications. As an example, we can let $H = l_2$ and choose A_1 as a matrix of finite or infinite dimension, i.e.,

$$A_1 = [a_{ij}], \; i, j = 1, 2, ..., N, \; N \in \mathbb{N}, \; D(A) = \; l_2^{\sigma}$$
$$= \left\{ u = \{u_j\}, \; j = 1, 2, ..., \infty, \; \|u\|_{l_2^{\sigma}} = \left(\sum_{j=1}^{\infty} 2^{\sigma j} |u_j|^2 \right)^{\frac{1}{2}} < \infty \right\}$$
$$\text{for } N = \infty, \tag{1.3}$$

where $\mathbb{N}$ denotes the set of natural numbers and a_{ij} are real numbers (see, e.g., [27, §1.18] for the space l_2^{σ}). Therefore, as a corollary of our main result, we obtain the existence, uniqueness, regularity properties, and blow-up of the following IVP:

$$\partial_t^2 u - a * \Delta u + (A_1^2 + \omega)u = f(u), \; (x,t) \in \mathbb{R}_T^n, \; i = 1, 2, .., N,$$
$$u_i(x, 0) = \varphi_i(x), \; \partial_t u_i(x, 0) = \psi_i(x) \; \text{ for a.e. } x \in \mathbb{R}^n \tag{1.4}$$

in mixed $L^{\mathbf{p}}(\mathbb{R}_T^n; l_2)$, where $\mathbf{p} = (p, p, 2)$.

As a second example we can choose $H = L^2(0, 1)$ and A_2 a degenerate differential operator in $L^2(0, 1)$ with nonlocal boundary conditions

$$D(A_2) = \left\{ u \in W_{\gamma}^{[2],2}(0, 1), \; \alpha_k u^{[\nu_k]}(0) + \beta_k u^{[\nu_k]}(1) = 0, \; k = 1, 2 \right\},$$
$$A_2 u = b_1(y) u^{[2]} + b_2(y) u^{[1]}, \; x \in \mathbb{R}^n, \; y \in (0, 1), \; \nu_k \in \{0, 1\}, \tag{1.5}$$

where $u^{[i]} = \left(y^{\gamma} \frac{d}{dy} \right)^i u$ for $0 \leq \gamma < \frac{1}{2}$, $b_1 = b_1(y)$ is a continous function, $b_2 = b_2(y)$ is a bounded function in $y \in [0, 1]$ for a.e. $x \in \mathbb{R}^n$, α_k, β_k are complex numbers, and $W_{\gamma}^{[2],2}(0, 1)$ is a weighted Sobolev space defined by

$$W_{\gamma}^{[2],2}(0, 1) = \{ \; u : u \in L^2(0, 1), \; u^{[2]} \in L^2(0, 1),$$
$$\|u\|_{W_{\gamma}^{[2],2}} = \|u\|_{L^2} + \left\| u^{[2]} \right\|_{L^2} < \infty.$$

Moreover, our main result implies the $L^{\mathbf{p}}(\Omega)$ -regularity property of a nonlocal mixed problem for the degenerate PDE

$$\partial_t^2 u - a * \Delta u + (A_2^2 + \omega)u = f\left(u\right), (x,t) \in \mathbb{R}_T^n, \tag{1.6}$$

$$\alpha_k u^{[\nu_k]}\left(x,0,t\right) + \beta_k u^{[\nu_k]}\left(x,1,t\right) = 0,\ k = 1,2, \tag{1.7}$$

$$u\left(x,y,0\right) = \varphi\left(x,y\right),\ u_t\left(x,y,0\right) = \psi\left(x,y\right),$$
$$\left(x,\ y\right) \in \mathbb{R}^n \times \left(0,1\right),\ t \in \left(0,T\right),\ u = u\left(x,y,t\right), \tag{1.8}$$

where the mixed norm is defined as

$$\|f\|_{L^{\mathbf{p}}(\not\leq)} = \left(\int_{\mathbb{R}^n}\int_0^T\left(\int_0^1\left|f\left(x,y,t\right)\right|^2 dy\right)^{\frac{p}{2}} dx\ dt\right)^{\frac{1}{p}} < \infty.$$

Note that if we let $H = \mathbb{C}$ and let the operator A be a complex-valued function, then one can obtain the previous results in the literature.

The traditional methods of the classical theory for wave equations is very limited in its ability to handle abstract wave equations. Here, a L^p -estimate containing fractional degrees of A is shown for the solution. Therefore, to overcome these difficulties we implement more powerful tools of abstract harmonic analysis, operator theory, interpolation of Banach spaces, and embedding theorems of Sobolev-Lions spaces.

1. Definitions and Background

In order to state our results precisely, we introduce some notation and some function spaces.

Let E be a Banach space and let $L^p\left(\Omega;E\right)$ denote the space of strongly measurable E-valued functions that are defined on a measurable subset $\Omega \subset \mathbb{R}^n$ with the norm

$$\|f\|_p = \|f\|_{L^p(\Omega;E)} = \left(\int_\Omega \|f\left(x\right)\|_E^p\, dx\right)^{\frac{1}{p}},\ 1 \leq p < \infty,$$
$$\|f\|_{L^\infty(\Omega;E)} = ess\sup_{x\in\Omega}\|f\left(x\right)\|_E.$$

Let E_1 and E_2 be two Banach spaces, and let $\left(E_1,E_2\right)_{\theta,p}$ for $\theta \in (0,1)$, $p \in [1,\infty]$ denote the real interpolation spaces defined by the K-method [27, §1.3.2.], Let E_1 and E_2 be two Banach spaces, and $B\left(E_1,E_2\right)$ denote the space of all bounded linear operators from E_1 to E_2. For $E_1 = E_2 = E$ that space will be denoted by $B\left(E\right)$.

Here,

$$S_\psi = \left\{\lambda \in \mathbb{C},\ \left|\arg\lambda\right| \leq \phi,\ 0 \leq \phi < \pi\right\}.$$

A closed linear operator A is said to be ψ-sectorial in a Banach space E with bound $M > 0$ if $D(A)$ and $R(A)$ are dense on E, $N(A) = \{0\}$ and

$$\left\| (A + \lambda I)^{-1} \right\|_{B(E)} \leq M |\lambda|^{-1}$$

for all $\lambda \in S_\phi$, $0 \leq \phi < \pi$, where I is the identity operator in E, and $D(A)$ and $R(A)$ denote the domain and range of the operator A, respectively. It is known that (see, e.g., [27, §1.15.1]) there exist fractional powers A^θ of a sectorial operator A. Let $E\left(A^\theta\right)$ denote the space $D\left(A^\theta\right)$ with the graphical norm

$$\|u\|_{E(A^\theta)} = \left(\|u\|^p + \left\| A^\theta u \right\|^p \right)^{\frac{1}{p}}, \; 1 \leq p < \infty, \; 0 < \theta < \infty.$$

A sectorial operator $A(\xi)$ is said to be uniformly sectorial in E for $\xi \in \mathbb{R}^n$, if $D(A(\xi))$ is independent of ξ and the following uniform estimate

$$\left\| (A + \lambda I)^{-1} \right\|_{B(E)} \leq M |\lambda|^{-1}$$

holds for all $\lambda \in S_\phi$.

A linear operator $A = A(\xi)$ belongs to $\sigma(M_0, \omega, E)$ (see [23, § 11.2]) if $D(A)$, $R(A)$ are dense on E, $N(A) = \{0\}$, $D(A(\xi))$ is independent of $\xi \in \mathbb{R}^n$ and for $\mathrm{Re}\lambda > \omega$ the uniform estimate holds

$$\left\| \left(A(\xi) - \lambda^2 I \right)^{-1} \right\|_{B(E)} \leq M_0 \left| \mathrm{Re}\lambda - \omega \right|^{-1}.$$

Remark 1.1. It is known (see, e.g., [22, § 1.6], Theorem 6.3) that if $A \in \sigma(M_0, \omega, E)$ and $0 \leq \alpha < 1$, then A generates a bounded group operator $U_A(t)$ satisfying

$$\|U_A(t)\|_{B(E)} \leq M e^{\omega|t|}, \; \|A^\alpha U_A(t)\|_{B(E)} \leq M |t|^{-\alpha}, \; t \in [0, T]. \tag{2.1}$$

Let $1 \leq p \leq q < \infty$. A function $\Psi \in L^\infty(\mathbb{R}^n)$ is called a Fourier multiplier from $L^p(\mathbb{R}^n; E)$ to $L^q(\mathbb{R}^n; E)$ if the map $P\colon u \to \mathbb{F}^{-1}\Psi(\xi)\mathbb{F}u$ for $u \in S(\mathbb{R}^n; E)$ is well defined and extends to a bounded linear operator

$$P\colon L^p(\mathbb{R}^n; E) \to L^q(\mathbb{R}^n; E).$$

Let E be a Banach space and let $S = S(\mathbb{R}^n; E)$ denote E-valued Schwartz class, i.e., the space of all E-valued rapidly decreasing smooth functions on $\mathbb{R}^n$ equipped with its usual topology generated by seminorms. Let $S(\mathbb{R}^n; \mathbb{C})$ be denoted by S. Let $S'(\mathbb{R}^n; E)$ denote the space of all continuous linear functions from S into E equipped with the bounded convergence topology. Recall that $S(\mathbb{R}^n; E)$ is norm dense in $L^p(\mathbb{R}^n; E)$ when $1 \leq p < \infty$.

The Fourier transformation of the operator function $B(x)$ with domain $D(B)$ independent on $x \in \mathbb{R}^n$ is a generalized function defined as

$$\hat{A}(\xi)\, u(\varphi) = A(x)\, u(\hat{\varphi}) \;\text{ for } u \in S'(\mathbb{R}^n; E(B)), \; \varphi \in S(R^n).$$

(For details see, e.g., [2, Section 3].)

Definition 1.1. Let U be an open set in a Banach space X, and let Y be a Banach space. A function $f : U \to Y$ is called (Fr échet) differentiable at $x \in U$ if there is a bounded linear operator $Df(x) : X \to Y$, called the derivative of f at a, such that

$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - Df(x)\, h\|_Y}{\|h\|_X} = 0.$$

If f is differentiable at each $x \in U$, then f is said to be differentiable. This function may also have a derivative, the second-order derivative of f, which, by the definition of derivative, will be a map

$$D^2 f :\ U \to L(X, L(X,Y)).$$

Let m be a positive integer, and let $W^{m,p}(\Omega; E)$ denote an $E-$valued Sobolev space of all functions $u \in L^p(\Omega; E)$ that have the generalized derivatives $\frac{\partial^m u}{\partial x_k^m} \in L^p(\Omega; E)$ with the norm

$$\|u\|_{W^{m,p}(\Omega;E)} = \|u\|_{L^p(\Omega;E)} + \sum_{k=1}^{n} \left\| \frac{\partial^m u}{\partial x_k^m} \right\|_{L^p(\Omega;E)} < \infty.$$

Let $W^{s,p}(\mathbb{R}^n; E)$ denote the fractional Sobolev space of order $s \in \mathbb{R}$, defined as

$$W^{s,p}(E) = W^{s,p}(\mathbb{R}^n; E) = \Bigg\{ u \in S'(\mathbb{R}^n; E),$$
$$\|u\|_{W^{s,p}(E)} = \left\| \mathbb{F}^{-1}\left(I + |\xi|^2\right)^{\frac{s}{2}} \hat{u} \right\|_{L^p(\mathbb{R}^n;E)} < \infty \Bigg\}.$$

It is clear that $W^{0,p}(\mathbb{R}^n; E) = L^p(\mathbb{R}^n; E)$. Let E_0 and E be two Banach spaces and let E_0 be continuously and densely embedded into E. Here, $W^{s,p}(\mathbb{R}^n; E_0, E)$ denotes a Sobolev- Lions- type space, i.e.,

$$W^{s,p}(\mathbb{R}^n; E_0, E) = \{u \in W^{s,p}(\mathbb{R}^n; E) \cap L^p(\mathbb{R}^n; E_0),$$
$$\|u\|_{W^{s,p}(\mathbb{R}^n;E_0,E)} = \|u\|_{L^p(\mathbb{R}^n;E_0)} + \|u\|_{W^{s,p}(\mathbb{R}^n;E)} < \infty\}.$$

In a similar way, we define the following Sobolev- Lions- type space:

$$W^{2,s,p}(\mathbb{R}_T^n; E_0, E) = \Bigg\{ u \in L^p(\mathbb{R}_T^n; E_0), \partial_t^2 u \in L^p(\mathbb{R}_T^n; E),$$
$$\mathbb{F}_x^{-1}\left(I + |\xi|^2\right)^{\frac{s}{2}} \hat{u} \in L^p(\mathbb{R}_T^n; E),\ \|u\|_{W^{2,s,p}(\mathbb{R}_T^n;E_0,E)}$$
$$= \|u\|_{L^p(\mathbb{R}_T^n;E_0)} + \left\|\partial_t^2 u\right\|_{L^p(\mathbb{R}_T^n;E)} + \left\| \mathbb{F}_x^{-1}\left(I + |\xi|^2\right)^{\frac{s}{2}} \hat{u} \right\|_{L^p(\mathbb{R}_T^n;E)} < \infty \Bigg\}.$$

Let $L_q^*(E)$ denote the space of all E-valued function space such that

$$\|u\|_{L_q^*(E)} = \left(\int_0^\infty \|u(t)\|_E^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty,\ 1 \leq q < \infty,\ \|u\|_{L_\infty^*(E)} = \sup_{0<t<\infty} \|u(t)\|_E .$$

Let $s > 0$. The Fourier-analytic representation of an E-valued Besov space on $\mathbb{R}^n$ is defined as

$$B_{p,q}^s(\mathbb{R}^n; E) = \Big\{ u \in S^{'}(\mathbb{R}^n; E),$$
$$\|u\|_{B_{p,q}^s(\mathbb{R}^n;E)} = \left\| \mathbb{F}^{-1} \sum_{k=1}^n t^{\varkappa - s} \left(1 + |\xi|^2\right)^{\frac{\varkappa}{2}} e^{-t|\xi|^2} \mathbb{F}u \right\|_{L_q^*(L^p(\mathbb{R}^n;E))},$$
$$p \in (1, \infty),\ q \in [1, \infty],\ \varkappa > s \} .$$

It should be note that the norm of a Besov space does not depend on $\varkappa$ (see, e.g., [27, § 2.3] for the case $E = \mathbb{C}$).

Let

$$X_p = L^p(\mathbb{R}^n; H),\ X_p(A^\gamma) = L^p(\mathbb{R}^n; H(A^\gamma)),\ 1 \leq p,\ q \leq \infty,$$
$$Y^{\ s,p} = Y^{\ s,p}(H) = W^{s,p}(\mathbb{R}^n; H),\ Y_q^{s,p}(H) = Y^{\ s,p}(H) \cap X_q,$$
$$\|u\|_{Y_q^{s,p}} = \|u\|_{W^{s,p}(\mathbb{R}^n;H)} + \|u\|_{X_q} < \infty,$$
$$W^{s,p}(A^\gamma) = W^{s,p}(\mathbb{R}^n; H(A^\gamma)),\ 0 < \gamma \leq 1,$$
$$Y^{s,p} = Y^{s,p}(A, H) = W^{s,p}(\mathbb{R}^n; H(A), H),\ Y^{2,s,p} = Y^{2,s,p}(A, H)$$
$$= W^{2,s,p}(\mathbb{R}_T^n; H(A), H),\ Y_q^{s,p}(A; H) = Y^{s,p}(H) \cap X_q(A),$$
$$\|u\|_{Y_q^{s,p}(A,H)} = \|u\|_{Y^{s,p}(H)} + \|u\|_{X_q(A)} < \infty,$$

Definition 1.1. For all $T > 0$, the function $u \in C^2\left([0,T]; Y_\infty^{2,s,p}(A,H)\right)$ that satisfies the equation $(1.1) - -(1.2)$ a.e. in $\mathbb{R}_T^n$ is called the continuous solution or the strong solution of the problem $(1.1) - -(1.2)$. If $T < \infty$, then $u(x,t)$ is called the local strong solution of the problem $(1.1) - (1.2)$. If $T = \infty$, then $u(x,t)$ is called the global strong solution of (1.1)–(1.2).

Sometimes we use one and the same symbol C without distinction to denote various positive constants that may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say α, we write C_α. Moreover, for $u,\ \upsilon, > 0$ the relations $u \lesssim \upsilon$, $u \approx \upsilon$ mean that there exist positive constants C, C_1, C_2 independent of u and υ such that, respectively,

$$u \leq C\upsilon,\ C_1\upsilon \leq u \leq C_2\upsilon.$$

The paper is organized as follows: In Sect. 2, some definitions and background are given. In Sect. 2, we obtain the existence of a unique solution and a priori estimates for the solution of the linearized problem (1.1)–(1.2). In Sect. 3, we show the existence and uniqueness of a local strong solution of the problem (1.1)–(1.2). In Sect. 4, the existence and uniqueness of a global strong solution of the problem (1.1)–(1.2) is derived. Section 5 is devoted to the blow-up property of the solution of (1.1)–(1.2) . In Sect. 6, we show some applications of the problem (1.1)–(1.2).

2. Estimates for the Linearized Equation

In this section, we make necessary estimates for solutions of the Cauchy problem for the nonlocal linear WE

$$u_{tt} - a * \Delta u + Au = g\left(x,t\right), \; x \in \mathbb{R}^n, \; t \in \left(0,T\right), \; T \in \left(0, \; \infty\right], \tag{2.1}$$

$$u\left(x,0\right) = \varphi\left(x\right), \; u_t\left(x,0\right) = \psi\left(x\right) \text{ for a.e. } x \in \mathbb{R}^n, \tag{2.2}$$

where A is a linear operator in a Hilbert space H and a is a complex-valued function on $\mathbb{R}^n$. Let

$$\mathbb{H}_{0p} = \left(Y^{s,p}\left(A,H\right), X_p\right)_{\frac{1}{2p},p}, \; \mathbb{H}_{1p} = \left(Y^{s,p}\left(A,H\right), X_p\right)_{\frac{1+p}{2p},p},$$

where $\left(Y^{s,p}, X_p\right)_{\theta,p}$ denotes the real interpolation space between $Y^{s,p}$ and X_p for $\theta \in (0,1)$, $p \in [1,\infty]$ (see, e.g., [27, §1.3]).

Remark 2.1. By properties of real interpolation of Banach spaces and interpolation of the intersection of spaces (see, e.g., [27, §1.3]), we obtain

$$\begin{aligned}\mathbb{H}_{0p} &= \left(Y^{s,p}\left(A,H\right)\cap X_p, X_p\right)_{\frac{1}{2p},p} = \left(Y^{s,p}\left(H\right), X_p\right)_{\frac{1}{2p},p} \cap \left(X_p\left(A\right), X_p\right)_{\frac{1}{2p},p} \\ &= W^{s\left(1-\frac{1}{2p}\right),p}\left(\mathbb{R}^n; H\right) \cap L^p\left(\mathbb{R}^n; \left(H\left(A\right), H\right)_{\frac{1}{2p},p}\right) \\ &= W^{s\left(1-\frac{1}{2p}\right),p}\left(\mathbb{R}^n; \left(H\left(A\right), H\right)_{\frac{1}{2p},p}, H\right).\end{aligned}$$

In a similar way, we have

$$\mathbb{H}_{1p} = \left(Y^{s,p}\left(A,H\right)\cap X_p, X_p\right)_{\frac{1+p}{2p},p} = W^{\frac{s(p-1)}{2p},p}\left(\mathbb{R}^n; \left(H\left(A\right), H\right)_{\frac{1+p}{2p},p}, H\right).$$

Remark 2.2. Let A be a densely defined operator on a Banach space. Let A be a sectorial operator in a Hilbert space H. In view of interpolation by the domain of sectorial operators (see, e.g., [27, §1.8.2]) we have the following relation:

$$H\left(A^{1-\theta+\varepsilon}\right) \subset \left(H\left(A\right), H\right)_{\theta,p} \subset H\left(A^{1-\theta-\varepsilon}\right)$$

for $0 < \theta < 1$ and $0 < \varepsilon < 1 - \theta$.

Note that from the result of J. Lions - J.Peetre result (see, e.g., [27, §1.8.2], we obtain the following result:

Lemma $\mathbf{A}_1$. The trace operator $u \to \frac{\partial^i u}{\partial t^i}\left(x,t\right)$ is bounded and continuous from $Y^{2,s,p}\left(A,H\right)$ onto

$$\left(Y^{s,p}\left(A,H\right), X_p\right)_{\theta_j,p}, \; \theta_j = \frac{1+jp}{2p}, \; j = 0,1.$$

Let

$$A_\xi = \left[\hat{a}\left(\xi\right)\left|\xi\right|^2 + A\right]^{\frac{1}{2}}.$$

Let A be a generator of the strongly continuous cosine operator-function in H defined by formula

$$C\left(t\right) = C_A\left(t\right) = \frac{1}{2}\left(e^{itA^{\frac{1}{2}}} + e^{-itA^{\frac{1}{2}}}\right)$$

(see, e.g., [3, §3, 12, §11]). Then, from the definition of the sine operator-function $S\left(t\right)$, we have

$$S\left(t\right) = S_A\left(t\right) = \int_0^t C\left(\sigma\right) d\sigma, \text{ i.e., } S\left(t\right) = \frac{1}{2i}A^{-\frac{1}{2}}\left(e^{itA^{\frac{1}{2}}} - e^{-itA^{\frac{1}{2}}}\right).$$

Remark 2.3. Let A be a densely defined operator in a Hilbert space H. By virtue of [3, Theorem 3.15.3.] , if A is a generator of a cosine function $C\left(t\right)$, i.e.,

$$R\left(\lambda^2, A\right) = \frac{1}{\lambda}\int_0^\infty e^{-\lambda t} C\left(t\right) dt \text{ for } \lambda > \omega,$$

then there exist ω, $M \geq 0$ such that $A \in \sigma\left(M_0, \omega, H\right)$.

Condition 2.1. Assume the following: (1) there exists $\hat{a} \in C^{(m)}\left(\mathbb{R}^n\right)$ such that

$$\hat{a}\left(\xi\right)\left|\xi\right|^2 \in S_{\psi_1}, \ \left(1 + \left|\xi\right|^2\right)^{-\left(\frac{s}{2}-2\right)}\left|D^\beta \hat{a}\left(\xi\right)\right| \leq C_0,$$
$$m = \left|\beta\right| > 1 + \frac{n}{p}, \ p \in \left(1, \infty\right) \text{ for all } \xi \in \mathbb{R}^n;$$

(2) A is ψ-sectorial in H for $\psi < \pi - \psi_1$ and A is a generator of the cosine function; (3) $A_\xi \neq 0$ for all $\xi \in \mathbb{R}^n$.

In view of Condition 2.1 and by virtue of [3, § 3] (or [12, §11]) we obtain that A_ξ is a generator of the strongly continuous cosine and sine operator function defined by

$$\eta_\pm\left(\xi\right) = e^{itA_\xi} \pm e^{-itA_\xi}, \ C\left(t\right) = C\left(\xi, t\right) = \frac{\eta_+\left(\xi\right)}{2},$$
$$S\left(t\right) = S\left(\xi, t\right) = A_\xi^{-1}\frac{\eta_-\left(\xi\right)}{2i}. \tag{2.3}$$

First we need the following lemmas:

Lemma 2.1. *Let Condition 2.1 hold. Then problem (2.1)–(2.2) has a strong solution.*

Proof. Using the Fourier transform, we get from (2.1)-(2.2)

$$\hat{u}_{tt}(\xi,t)+A_{\xi}\hat{u}(\xi,t)=\hat{g}(\xi,t),$$
$$\hat{u}(\xi,0)=\hat{\varphi}(\xi),\ \hat{u}_t(\xi,0)=\hat{\psi}(\xi), \tag{2.4}$$

where $\hat{u}(\xi,t)$ is the Fourier transform of $u(x,t)$ in x and $\hat{\varphi}(\xi)$, $\hat{\psi}(\xi)$ are the Fourier transforms of φ and ψ, respectively. By virtue of [3, § 3, 12, § 11] we obtain that A_{ξ} is a generator of a strongly continuous cosine operator function and that problem (2.4) has a unique solution for all $\xi\in\mathbb{R}^n$ that can be expressed as

$$\hat{u}(\xi,t)=C(\xi,t)\hat{\varphi}(\xi)+S(\xi,t)\hat{\psi}(\xi)+\int_0^t S(\xi,t-\tau)\hat{g}(\xi,\tau)d\tau, \tag{2.5}$$

for all $\xi\in\mathbb{R}^n$, i.e., problem (2.1)–(2.2) has a unique solution

$$u(x,t)=C_1(t)\varphi+S_1(t)\psi+Qg, \tag{2.6}$$

where $C_1(t)$, $S_1(t)$, Q are linear operator functions defined by

$$C_1(t)\varphi=\mathbb{F}^{-1}\left[C(\xi,t)\hat{\varphi}(\xi)\right],\ S_1(t)\psi=\mathbb{F}^{-1}\left[S(\xi,t)\hat{\psi}(\xi)\right],$$
$$Qg=\mathbb{F}^{-1}\tilde{Q}(\xi,t),\ \tilde{Q}(\xi,t)=\int_0^t\left[S(\xi,t-\tau)\hat{g}(\xi,\tau)\right]d\tau.$$

Now, we can show the main results of this section.

Theorem 2.1. *Assume that Condition 2.1 holds and*

$$s>\frac{2p}{2p-1}\left(\frac{2}{q}+\frac{1}{p}\right)n \tag{2.7}$$

for $p\in[1,\infty]$ *and some* $q\in[1,2]$. *Let* $0\leq\alpha<1-\frac{1}{2p}$. *Then for* $\varphi\in X_1(A^{\alpha})\cap\mathbb{H}_{0p}$, $\psi\in X_1(A^{\alpha})\cap\mathbb{H}_{1p}$, $g(.,t)\in Y_1^{s,p}, t\in[0,T]$, *and* $g(x,.)\in L^1(0,T;Y_1^{s,p})$, $x\in\mathbb{R}^n$, *problem* (2.1) – (2.2) *has a unique solution* $u(x,t)\in C^2([0,T];X_{\infty})$, *and the following uniform estimate holds:*

$$\|A^{\alpha}u\|_{X_{\infty}}\leq C_0\Big[\|\varphi\|_{\mathbb{H}_{0p}}+\|A^{\alpha}\varphi\|_{X_1}+$$
$$\|\psi\|_{\mathbb{H}_{1p}}+\|A^{\alpha}\psi\|_{X_1}+\int_0^t\left(\|g(.,\tau)\|_{Y_1^{s,p}}+\|g(.,\tau)\|_{X_1}\right)d\tau\Big]; \tag{2.8}$$

moreover, for $\varphi\in X_1\left(A^{\frac{1}{2}+\alpha}\right)\cap\mathbb{H}_{0p}$, $\psi\in X_1\left(A^{\frac{1}{2}+\alpha}\right)\cap\mathbb{H}_{1p}$ *and* $g(.,t)\in Y_1^{s,p}\left(A^{\frac{1}{2}}\right)$ *we have*

$$\|A^{\alpha}u_t\|_{X_{\infty}}\leq C_0\Big[\|\varphi\|_{\mathbb{H}_{0p}}+\left\|A^{\frac{1}{2}+\alpha}\varphi\right\|_{X_1}$$

$$+\left\|\psi\right\|_{\mathbb{E}_{1p}}+\left\|A^{\frac{1}{2}+\alpha}\psi\right\|_{X_1}+\int\limits_0^t\left(\left\|A^{\frac{1}{2}}g\left(.,\tau\right)\right\|_{Y_1^{s,p}}+\left\|A^{\frac{1}{2}}g\left(.,\tau\right)\right\|_{X_1}\right)d\tau\Bigg],$$

uniformly in $t\in[0,T]$, where the constant $C_0>0$ depends only on A, H, and initial data.

Proof. By Lemma 2.1, the problem (2.1)–(2.2) has a solution $u(x,t)\in C^2\left([0,T];Y^{s,p}\left(A;H\right)\right)$ for $\varphi\in X_1\left(A^\alpha\right)$, $\psi\in\mathbb{H}_{1p}$, $g\left(.,t\right)\in Y_1^{s,p}$, and $g\left(x,.\right)\in L^1\left(0,T;Y_1^{s,p}\right)$. Let $N\in\mathbb{N}$ and

$$\Pi_N=\left\{\xi:\xi\in\mathbb{R}^n,\ |\xi|\leq N\right\},\ \Pi_N'=\left\{\xi:\xi\in\mathbb{R}^n,\ |\xi|\geq N\right\}.$$

From (2.6) we deduce that

$$\begin{aligned}\left\|A^\alpha u\right\|_{X_\infty}&\lesssim\left\|\mathbb{F}^{-1}C\left(\xi,t\right)A^\alpha\hat{\varphi}\left(\xi\right)\right\|_{L^\infty(\Pi_N)}\\&+\left\|\mathbb{F}^{-1}S\left(\xi,t\right)A^\alpha\hat{\psi}\left(\xi\right)\right\|_{L^\infty(\Pi_N)}+\left\|\mathbb{F}^{-1}C\left(\xi,t\right)A^\alpha\hat{\varphi}\left(\xi\right)\right\|_{L^\infty(\Pi_N')}\\&+\left\|\mathbb{F}^{-1}S\left(\xi,t\right)A^\alpha\hat{\psi}\left(\xi\right)\right\|_{L^\infty(\Pi_N')}+\frac{1}{2}\left\|\mathbb{F}^{-1}A^\alpha\tilde{Q}\left(\xi,t\right)\hat{g}\left(\xi,\tau\right)\right\|_{L^\infty(\Pi_N)}\\&+\frac{1}{2}\left\|\mathbb{F}^{-1}A^\alpha\tilde{Q}\left(\xi,t\right)\hat{g}\left(\xi,\tau\right)\right\|_{L^\infty(\Pi_N')}.\end{aligned}\tag{2.9}$$

By virtue of Remarks 2.1, 2.2 and the properties of sectorial operators, we get the following uniform estimate

$$\left\|\mathbb{F}^{-1}A^\alpha\tilde{Q}\left(\xi,t\right)\hat{g}\left(\xi,\tau\right)\right\|_{L^\infty(\Pi_N)}\leq C\left\|g\right\|_{X_1}.$$

Hence, due to the uniform boundedness of operator functions $C\left(\xi,t\right)$, $S\left(\xi,t\right)$, in view of (2.3), and by Minkowski's inequality for integrals, we get the uniform estimate

$$\begin{aligned}&\left\|\mathbb{F}^{-1}C\left(\xi,t\right)A^\alpha\hat{\varphi}\left(\xi\right)\right\|_{L^\infty(\Pi_N)}+\left\|\mathbb{F}^{-1}S\left(\xi,t\right)A^\alpha\hat{\psi}\left(\xi\right)\right\|_{L^\infty(\Pi_N)}\\&\qquad\lesssim\left[\left\|A^\alpha\varphi\right\|_{X_1}+\left\|A^\alpha\psi\right\|_{X_1}+\left\|g\right\|_{X_1}\right].\end{aligned}$$

Let

$$l<s\left(1-\frac{1}{2p}\right).$$

Moreover, from (2.6) we deduce that

$$\begin{aligned}&\left\|\mathbb{F}^{-1}C\left(\xi,t\right)A^\alpha\hat{\varphi}\left(\xi\right)\right\|_{L^\infty(\Pi_N')}+\left\|\mathbb{F}^{-1}S\left(\xi,t\right)A^\alpha\hat{\psi}\left(\xi\right)\right\|_{L^\infty}\\&\quad\lesssim\left\|\mathbb{F}^{-1}C\left(\xi,t\right)A^\alpha\hat{\varphi}\left(\xi\right)\right\|_{L^\infty}+\left\|\mathbb{F}^{-1}S\left(\xi,t\right)A^\alpha\hat{\psi}\left(\xi\right)\right\|_{L^\infty}\\&\quad+\left\|\mathbb{F}^{-1}S\left(\xi,t\right)A^\alpha\tilde{Q}\left(\xi,t\right)\hat{g}\left(\xi,\tau\right)\right\|_{L^\infty}\\&\quad\lesssim\left\|\mathbb{F}^{-1}\left(1+|\xi|^2\right)^{-\frac{l}{2}}C\left(\xi,t\right)\left(1+|\xi|^2\right)^{\frac{l}{2}}A^\alpha\hat{\varphi}\left(\xi\right)\right\|_{L^\infty}\end{aligned}$$

$$+\left\|\mathbb{F}^{-1}\left(1+|\xi|^2\right)^{-\frac{l}{2}} S(\xi,t)\left(1+|\xi|^2\right)^{\frac{l}{2}} A^{\alpha}\hat{\psi}(\xi)\right\|_{L^\infty}$$
$$+\left\|\mathbb{F}^{-1}\left(1+|\xi|^2\right)^{-\frac{l}{2}} S(\xi,t)\left(1+|\xi|^2\right)^{\frac{l}{2}} A^{\alpha}\tilde{Q}(\xi,t)\,\hat{g}(\xi,\tau)\right\|_{L^\infty}, \tag{2.10}$$

where here the space $L^\infty(\Omega;H)$ is denoted by L^∞. It is clear that

$$\frac{\partial}{\partial\xi_k}\left[\left(1+|\xi|^2\right)^{-\frac{l}{2}} A^{\alpha}C(\xi,t)\,\Phi_0(\xi)\right]$$
$$=\left(1+|\xi|^2\right)^{-\frac{l}{2}}\left[it\left(\hat{a}(\xi)\,\xi_k+\frac{\partial\hat{a}}{\partial\xi_k}|\xi|^2\right)\Phi_0(\xi)\,\eta_-(\xi)\right.$$
$$\left.A^{\alpha}\left[\hat{a}(\xi)\,|\xi|^2+A\right]^{-\frac{1}{2}}\right]$$
$$+\left(1+|\xi|^2\right)^{-\frac{l}{2}} C(\xi,t)\,A^{\alpha}\Phi_{01}(\xi)-l\xi_k\left(1+|\xi|^2\right)^{-\frac{l}{2}-1} A^{\alpha}C(\xi,t)\,\Phi_0(\xi),$$
$$\frac{\partial}{\partial\xi_k}\left[\left(1+|\xi|^2\right)^{-\frac{l}{2}} A^{\alpha}S(\xi,t)\,\Phi_1(\xi)\right]$$
$$=\left(1+|\xi|^2\right)^{-\frac{l}{2}}\left[\frac{1}{2}A_{\xi}^{--1}it\left(\hat{a}\xi_k+\frac{\partial\hat{a}}{\partial\xi_k}|\xi|^2\right)\right.$$
$$\left.\left[A_{\xi}^{-\frac{1}{2}}\eta_-(\xi)+t\eta_+(\xi)\right]A^{\alpha}\Phi_1(\xi)\right]$$
$$-l\xi_k\left(1+|\xi|^2\right)^{-\frac{l}{2}-1} A^{\alpha}S(\xi,t)\,\Phi_{11}(\xi), \tag{2.11}$$

where

$$\Phi_0(\xi)=\left[A^{1-\frac{1}{2p}-\varepsilon_0}+\left(1+|\xi|^2\right)^{s\left(1-\frac{1}{2p}\right)-\varepsilon_0}\right]^{-1},\ 0<\varepsilon_0<1-\frac{1}{2p},$$
$$\Phi_1(\xi)=\left[A^{\frac{1}{2}-\frac{1}{2p}-\varepsilon}+\left(1+|\xi|^2\right)^{s\left(\frac{1}{2}-\frac{1}{2p}\right)-\varepsilon_1}\right]^{-1},\ 0<\varepsilon_1<\frac{1}{2}-\frac{1}{2p},$$
$$\Phi_{01}(\xi)=2\xi_k s\left(1-\frac{1}{2p}-\varepsilon_0\right)\left[\left(1+|\xi|^2\right)^{s\left(1-\frac{1}{2p}\right)-\varepsilon_0-1}\right]$$
$$\times\left[A^{1-\frac{1}{2p}-\varepsilon_0}+\left(1+|\xi|^2\right)^{s\left(1-\frac{1}{2p}\right)-\varepsilon_0}\right]^{-2},$$
$$\Phi_{11}(\xi)=2\xi_k s\left(s\left(\frac{1}{2}-\frac{1}{2p}\right)-\varepsilon_1\right)\left[\left(1+|\xi|^2\right)^{s\left(\frac{1}{2}-\frac{1}{2p}\right)-\varepsilon_1-1}\right]$$
$$\times\left[A^{\frac{1}{2}-\frac{1}{2p}-\varepsilon}+\left(1+|\xi|^2\right)^{s\left(\frac{1}{2}-\frac{1}{2p}\right)-\varepsilon_1}\right]^{-2}. \tag{2.12}$$

Using the resolvent properties of sectorial operators, we have

$$\left\|\left(1+|\xi|^2\right)^{\frac{l}{2}} \Phi_i(\xi)\right\|_{B(H)} \leq C,\ i=1,\ 2,$$

$$\|A^\alpha C(\xi,t)\Phi_0(\xi)\|_{B(H)} \leq C\left\|A^\alpha A^{-\left(1-\frac{1}{2p}-\varepsilon_0\right)}(\xi)\right\|_{B(H)} \leq C_0,$$

$$\|A^\alpha S(\xi,t)\Phi_1(\xi)\|_{B(H)} \leq \left\|A^{\frac{1}{2}}\eta^{-1}(\xi)\right\|_{B(H)} \left\|A^\alpha A^{-\frac{1}{2}}\Phi_1(\xi)\right\|_{B(H)}$$

$$\leq C\left\|A^\alpha A^{-\left(1-\frac{1}{2p}-\varepsilon_0\right)}(\xi)\right\|_{B(H)} \leq C_1. \tag{2.13}$$

Then by calculating $\frac{\partial}{\partial\xi_k}\Phi_0(\xi)$, $\frac{\partial}{\partial\xi_k}\Phi_1(\xi)$, we obtain

$$A^\alpha\frac{\partial}{\partial\xi_k}\Phi_0(\xi) \in B(H),\ A^\alpha\frac{\partial}{\partial\xi_k}\Phi_1(\xi) \in B(H).$$

Let us show that $G_i(.,t) \in B_{q,1}^{n\left(\frac{1}{q}+\frac{1}{p}\right)}(\mathbb{R}^n; B(H))$ for some $q\in(1,2)$ and for all $t\in[0,T]$, where

$$G_i(\xi,t) = \left(1+|\xi|^2\right)^{-\frac{l}{2}} A^\alpha C(\xi,t)\Phi_i(\xi),\ i=0,\ 1.$$

By the embedding properties of Sobolev and Besov spaces it sufficient to derive that $G_i \in W_q^\sigma(\mathbb{R}^n; B(H))$ for some $\sigma > n\left(\frac{1}{q}+\frac{1}{p}\right)$. Indeed, by contraction, Condition 2.1, and by (2.12) we get $G_i \in L^q(\mathbb{R}^n; B(H))$. For deriving the embedding relations $G_i \in W_q^\sigma(\mathbb{R}^n; B(H))$, it suffices to show that

$$\left(1+|\xi|^2\right)^{\frac{\sigma}{2}} G_i(.,t) \in L^\sigma(\mathbb{R}^n) \text{ for all } t\in[0,T].$$

Indeed, in view of (2.12), $\left(1+|\xi|^2\right)^{\frac{\sigma}{2}}\Phi_i(\xi)$ are uniformly bounded for $\xi\in\mathbb{R}^n$. By virtue of (2.3), (2.13), by assumption (2.7), and in view of Remark 2.3, we have

$$\int_{\mathbb{R}^n}\left(1+|\xi|^2\right)^{\frac{\sigma}{2}q}|G_i(\xi,t)|^q d\xi$$

$$=\int_{\mathbb{R}^n}\left(1+|\xi|^2\right)^{\frac{\sigma-l}{2}q}\|C(\xi,t)\|^q\|A^\alpha\Phi_i(\xi)\|_{B(H)}^q d\xi$$

$$\lesssim\int_{\mathbb{R}^n}\left(1+|\xi|^2\right)^{\frac{\sigma-l}{2}q}|\xi|^{-\varepsilon q}d\xi \lesssim \int_{\mathbb{R}^n}\left(1+|\xi|^2\right)^{-\left(\frac{l-\sigma}{2}\right)q}d\xi<\infty$$

for

$$s > n\left(\frac{3}{q}+\frac{1}{p}\right)\frac{2p}{2p-1}.$$

Hence by the Fourier multiplier theorems (see, e.g., [13, Theorem 4.3]) we get that the functions $G_i(\xi,t)$ are Fourier multipliers from $L^p(\mathbb{R}^n;H)$ to $L^\infty(\mathbb{R}^n;H)$. In a similar way we obtain that

$$\left(1+|\xi|^2\right)^{-\frac{s}{2}} S(\xi,t)\left(1+|\xi|^2\right)^{\frac{s}{2}} A^\alpha\hat{\psi}(\xi),$$
$$\left(1+|\xi|^2\right)^{-\frac{s}{2}} S(\xi,t)\left(1+|\xi|^2\right)^{\frac{s}{2}} A^\alpha\tilde{Q}(\xi,t)\,\hat{g}(\xi,\tau)$$

are $L^p(\mathbb{R}^n;H)\to L^\infty(\mathbb{R}^n;H)$ Fourier multipliers. Then by Minkowski's inequality for integrals, from (2.3), (2.10) -(2.12) and by Remark 2.3 we have

$$\begin{aligned}\left\|F^{-1}C(\xi,t)\,A^\alpha\hat{\varphi}(\xi)\right\|_{L^\infty}&+\left\|\mathbb{F}^{-1}S(\xi,t)\,A^\alpha\hat{\psi}(\xi)\right\|_{L^\infty}\\ &\lesssim\left\|F^{-1}C(\xi,t)\,\eta^{-2}\hat{\varphi}\right\|_{L^\infty}+\left\|\mathbb{F}^{-1}S(\xi,t)\,\eta^{-1}\hat{\psi}\right\|_{L^\infty}\\ &\lesssim\left[\|\varphi\|_{\mathbb{H}_{0p}}+\|\psi\|_{\mathbb{H}_{1p}}+\|g\|_{W^{s,p}}\right].\end{aligned}\tag{2.14}$$

Moreover, by virtue of Remark 2.1–2.3 and by reasoning as above, we have the following estimate:

$$\left\|F^{-1}A^\alpha\tilde{Q}(\xi,t)\right\|_{X_\infty}\le C\int_0^t\left(\|g(.,\tau)\|_{W^{s,p}}+\|g(.,\tau)\|_{X_1}\right)d\tau\tag{2.15}$$

uniformly in $t\in[0,T]$. Thus, from (2.6), (2.14), and (2.15) we obtain

$$\begin{aligned}\|A^\alpha u\|_{X_\infty}\le C\Big[&\|\varphi\|_{\mathbb{E}_{0p}}+\|A^\alpha\varphi\|_{X_1}\\ &+\|\psi\|_{\mathbb{H}_{1p}}+\|A^\alpha\psi\|_{X_1}+\int_0^t\left(\|g(.,\tau)\|_{Y^{s,p}}+\|g(.,\tau)\|_{X_1}\right)d\tau\Big].\end{aligned}\tag{2.16}$$

By differentiating (2.6) in a similar way, we get the second inequality

$$\begin{aligned}\|A^\alpha u_t\|_{X_\infty}\le C\Big[&\|\varphi\|_{\mathbb{H}_{0p}}+\|A^\alpha\varphi\|_{X_1}\\ &+\|A^\alpha\psi\|_{\mathbb{H}_{1p}}+\|A^\alpha\psi\|_{X_1}+\int_0^t\left(\|g(.,\tau)\|_{Y^{s,p}}+\|g(.,\tau)\|_{X_1}\right)d\tau\Big].\end{aligned}\tag{2.17}$$

Then from (2.16) and (2.17), in view of Remarks 2.1, 2.2, we obtain the estimate (2.8).

Let us now show that problem (2.1)–(2.2) has a unique solution $u\in C^{(1)}([0,T];Y^{s,p})$. Suppose that the problem (2.1) − (2.2) has two solutions u_1, $u_2\in C^{(1)}([0,T];Y^{s,p})$. Then by the linearity of (2.1), we get that $\upsilon=u_1-u_2$ is also a solution of the corresponding homogenous equation

$$u_{tt}-a*\Delta u+Au=0,\ \upsilon(x,0)=0,\ \upsilon_t(x,0)=0,\ x\in\mathbb{R}^n,\ t\in(0,T).$$

Moreover, by (2.16) we have the following estimate

$$\left\|\left\|A^{\alpha}u\right\|_{X_{\infty}}\right\|_{X_{\infty}} \leq 0.$$

Since $N(A)=\{0\}$, the above estimate implies that $\upsilon=0$, i.e., $u_1=u_2$.

Theorem 2.2. *Assume that Condition 2.1 and (2.7) are satisfied. Let* $0\leq\alpha<1-\frac{1}{2p}$*. Then for* $\varphi,\ \psi\in Y^{s,p}(A^{\alpha})$*,* $g(.,t)\in Y^{s,p}$*,* $t\in[0,T]$*, and* $g(.,t)\in L^{1}(0,T;Y^{s,p})$*,* $x\in\mathbb{R}^{n}$*, problem (2.1)–(2.2) has a unique solution* $u\in C^{2}([0,T];Y^{s,p})$ *and the following uniform estimate holds:*

$$\left\|A^{\alpha}u\right\|_{Y^{s,p}} \leq C_0\left[\left\|A^{\alpha}\varphi\right\|_{Y^{s,p}}+\left\|A^{\alpha}\psi\right\|_{Y^{s,p}}+\int_0^t\left\|g(.,\tau)\right\|_{Y^{s,p}}d\tau\right], \quad (2.18)$$

Moreover, for $\varphi,\ \psi\in Y^{s,p}\left(A^{\alpha+\frac{1}{2}}\right)$ *and* $g(x,t)\in Y^{s,p}\left(A^{\frac{1}{2}}\right)$ *we have the following estimate*

$$\left\|A^{\alpha}u_t\right\|_{Y^{s,p}}\leq C_0\left[\left\|A^{\frac{1}{2}+\alpha}\varphi\right\|_{Y^{s,p}}+\left\|A^{\frac{1}{2}+\alpha}\psi\right\|_{Y^{s,p}}+\int_0^t\left\|A^{\frac{1}{2}}g(.,\tau)\right\|_{Y^{s,p}}d\tau\right]$$

for all $t\in[0,T]$.

Proof. From (2.5) and (2.11) we get the following uniform estimate:

$$\begin{aligned}&\left(\left\|\mathbb{F}^{-1}(1+|\xi|^2)^{\frac{s}{2}}A^{\alpha}\hat{u}\right\|_{X_p}+\left\|\mathbb{F}^{-1}(1+|\xi|^2)^{\frac{s}{2}}A^{\alpha}\hat{u}_t\right\|_{X_p}\right)\\&\leq C\left\{\left\|\mathbb{F}^{-1}(1+|\xi|^2)^{\frac{s}{2}}C(\xi,t)A^{\frac{1}{2}+\alpha}\hat{\varphi}\right\|_{X_p}+\left\|\mathbb{F}^{-1}(1+|\xi|^2)^{\frac{s}{2}}A^{\frac{1}{2}+\alpha}S(\xi,t)\hat{\psi}\right\|_{X_p}\right.\\&\left.\int_0^t\left\|(1+|\xi|^2)^{\frac{s}{2}}A^{\frac{1}{2}+\alpha}\tilde{Q}(\xi,t)\hat{g}(\xi,\tau)\right\|_{X_p}d\tau\right\}.\end{aligned} \quad (2.19)$$

Using the Fourier multiplier theorem [13, Theorem 4.3] and reasoning as in Theorem 2.1 we get that $\left(1+|\xi|^2\right)^{-\frac{s}{2}}C(\xi,t)$, $\left(1+|\xi|^2\right)^{-\frac{s}{2}}S(\xi,t)$ and $\left(1+|\xi|^2\right)^{-\frac{s}{2}}A^{\alpha}S(\xi,t)$ are Fourier multipliers in $L^p(\mathbb{R}^n;H)$ uniformly with respect to $t\in[0,T]$. So, the estimate (2.19) by using the Minkowski's inequality for integrals implies (2.18).

The uniqueness of (2.1)–(2.2) is obtained by reasoning as in Theorem 2.1.

3. Local Well Posedness of IVP for Nonlinear WE

In this section, we will show the local existence and uniqueness of a solution of the nonlinear problem (1.1)–(1.2).

For this we need the following lemmas. Reasoning as in [5,15,31], we prove the following lemmas concerning the behaviour of the nonlinear term in the E-valued space $Y^{s,p}$. Here, let H be a Banach algebra.

Lemma 3.1. *Let $s \geq 0$, $f \in C^{[s]+1}(\mathbb{H}; H)$ with $f(0) = 0$. Then for all $u \in Y^{s,p} \cap L^{\infty}$, we have $f(u)(.) \in Y^{s,p} \cap X_{\infty}$. Moreover, there is a constant $A(M)$ depending on M such that for all $u \in Y^{s,p} \cap L^{\infty}$ with $\|u\|_{X_{\infty}} \leq M$,*

$$\|f(u)\|_{Y^{s,p}} \leq C(M) \|u)\|_{Y^{s,p}}. \tag{3.1}$$

Using Lemma 3.1 and properties of convolution operators we obtain the following corollary:

Corollary 3.1. *Let $s \geq 0$, $f \in C^{[s]+1}(\mathbb{R}; H)$ with $f(0) = 0$. Moreover, assume $\Phi \in L^{\infty}(\mathbb{R}^n; B(H))$. Then for all $u \in Y^{s,p} \cap L^{\infty}$ we have, $f(u) \in Y^{s,p} \cap X_{\infty}$. Moreover, there is a constant $A(M)$ depending on M such that for all $u \in Y^{s,p} \cap L^{\infty}$ with $\|u\|_{X_{\infty}} \leq M$,*

$$\|\Phi * f(u)\|_{Y^{s,p}} \leq C(M) \|u)\|_{Y^{s,p}}.$$

Lemma 3.2. *Let $s \geq 0$, $f \in C^{[s]+1}(\mathbb{R}; H)$. Then for every M there is a constant $K(M)$ depending on M such that for all u, $\upsilon \in Y^{s,p} \cap X_{\infty}$ with $\|u\|_{X_{\infty}} \leq M$, $\|\upsilon\|_{X_{\infty}} \leq M$, $\|u\|_{Y^{s,p}} \leq M$, $\|\upsilon\|_{Y^{s,p}} \leq M$,*

$$\|f(u) - f(\upsilon\|_{Y^{s,p}} \leq K(M) \|u - \upsilon\|_{Y^{s,p}}, \ \ \|f(u) - f(\upsilon\|_{X_{\infty}} \leq K(M) \|u - \upsilon\|_{X_{\infty}}.$$

Reasoning as in [30, Lemma 3.4] and [15, Lemma X 4] we have, respectively:

Corollary 3.2. *Let $s > \frac{n}{2}$, $f \in C^{[s]+1}(\mathbb{R}; H)$. Then for every positive M there is a constant $K(M)$ depending on M such that for all u, $\upsilon \in Y^{s,p}$ with $\|u\|_{Y^{s,p}} \leq M$, $\|\upsilon\|_{Y^{s,p}} \leq M$,*

$$\|f(u) - f(\upsilon\|_{Y^{s,p}} \leq K(M) \|u - \upsilon\|_{Y^{s,p}}.$$

Lemma 3.3. *If $s > 0$, then $Y_{\infty}^{s,p}$ is an algebra. Moreover, for f, $g \in Y_{\infty}^{s,p}$,*

$$\|fg\|_{Y^{s,p}} \leq C\left[\|f\|_{X_{\infty}} + \|g\|_{Y^{s,p}} + \|f\|_{Y^{s,p}} + \|g\|_{X_{\infty}}\right].$$

Using Corollary 3.1 and Lemma 3.3 we obtain the following results:

Lemma 3.4. *Let $s \geq 0$, $f \in C^{[s]+1}(\mathbb{R}; H)$, and $f(u) = O\left(|u|^{\gamma+1}\right)$ for $u \to 0$, $\gamma \geq 1$ a positive integer. If $u \in Y_{\infty}^{s,p}$ and $\|u\|_{X_{\infty}} \leq M$, then*

$$\|f(u)\|_{Y^{s,p}} \leq C(M)\left[\|u\|_{Y^{s,p}} \|u\|_{X_{\infty}}^{\gamma}\right],$$
$$\|f(u)\|_{X_1} \leq C(M) \|u\|_{X_p}^{p} \|u\|_{X_{\infty}}^{\gamma-1}.$$

Corollary 3.3. *Let* $s \geq 0$, $f \in C^{[s]+1}(\mathbb{R};H)$, *and* $f(u) = O\left(|u|^{\gamma+1}\right)$ *for* $u \to 0$, $\gamma \geq 1$ *a positive integer. Moreover, assume* $\Phi \in L^{\infty}(\mathbb{R}^n; B(H))$. *If* $u \in Y_{\infty}^{s,p}$ *and* $\|u\|_{X_\infty} \leq M$, *then*

$$\|\Phi * f(u)\|_{Y^{s,p}} \leq C(M)\left[\|u\|_{Y^{s,p}}\|u\|_{X_\infty}^{\gamma}\right],$$
$$\|\Phi * f(u)\|_{X_1} \leq C(M)\|u\|_{X_p}^{p}\|u\|_{X_\infty}^{\gamma-1}.$$

Lemma 3.5. *Let* $s \geq 0$, $f \in C^{[s]+1}(\mathbb{R};H)$, *and* $f(u) = O\left(|u|^{\gamma+1}\right)$ *for* $u \to 0$. *Moreover, let* $\gamma \geq 0$ *be a positive integer. If* u, $\upsilon \in Y_{\infty}^{s,p}$, $\|u\|_{Y^{s,p}} \leq M$, $\|\upsilon\|_{Y^{s,p}} \leq M$ *and* $\|u\|_{X_\infty} \leq M$, $\|\upsilon\|_{X_\infty} \leq M$, *then*

$$\|f(u) - f(\upsilon)\|_{Y^{s,p}} \leq C(M)\left[\left(\|u\|_{X_\infty} - \|\upsilon\|_{X_\infty}\right)\left(\|u\|_{Y^{s,p}} + \|\upsilon\|_{Y^{s,p}}\right)\right.$$
$$\left(\|u\|_{X_\infty} + \|\upsilon\|_{X_\infty}\right)^{\gamma-1},$$
$$\|f(u) - f(\upsilon\|_{X_1} \leq C(M)\left(\|u\|_{X_\infty} + \|\upsilon\|_{X_\infty}\right)^{\gamma-1}\left(\|u\|_{X_p} + \|\upsilon\|_{X_p}\right)\|u - \upsilon\|_{X_p}.$$

Let $\mathbb{H}_0$ *denote the real interpolation space between* $Y^{s,p}(A,H)$ *and* X_p *with* $\theta = \frac{1}{2p}$, *i.e.,*

$$\mathbb{H}_{0p} = \left(Y^{s,p}(A,H), X_p\right)_{\frac{1}{2p},p}.$$

Remark 3.0. Let $u \in Y^{2,s,p} = W^{2,s,p}(\mathbb{R}_T^n; H(A), H)$. Then by a result of J. Lions and J. Peetre (see e.g. [27, §1.8.2] the trace operator $u \to \frac{\partial^i u}{\partial t^i}(x,t)$ is bounded from $Y^{2,s,p}$ to $C\left(\mathbb{R}^n; \left(Y^{s,p}, X_p\right)_{\theta_j,p}\right)$, where

$$X_p = L^p(\mathbb{R}^n; H), \; Y^{s,p} = W^{s,p}(\mathbb{R}^n; H(A), H), \; \theta_j = \frac{1+jp}{2p}, \; j = 0, 1.$$

Moreover, if $u(x,.) \in \left(Y^{s,p}, X_p\right)_{\theta_j,p}$, then under some assumptions that will be stated in Sect. 3, $f(u) \in H$ for all x, $t \in \mathbb{R}_T^n$ and the map $u \to f(u)$ is bounded from $\left(Y^{s,p}, X_p,\right)_{\frac{1}{2p},p}$ into H. Hence, the nonlinear equation (1.1) is satisfied in the Banach space H. Here, $H(A)$ denotes a domain of A equipped with the graphical norm, $\left(Y^{s,p}, X_p\right)_{\theta,p}$ is a real interpolation space between $Y^{s,p}$ and X_p for $\theta \in (0,1)$, $p \in [1,\infty]$ (see, e.g., [27, §1.3]).

Remark 3.1. Using a result of J. Lions-I. Petree (see, e.g., [27, § 1.8]) we obtain that the map $u \to u(t_0)$, $t_0 \in [0,T]$ is continuous and surjective from $Y^{2,s,p}(A,H)$ onto $\mathbb{H}_{0p}$ and there is a constant C_1 such that

$$\|u(t_0)\|_{\mathbb{H}_{0p}} \leq C_1\|u\|_{Y^{2,s,p}(A,H)}, 1 \leq p \leq \infty. \tag{3.6}$$

Let

$$C^2\left(Y_1^{s,p}(A)\right) = C^{(2)}\left([0,T]; Y_1^{s,p}(A,H)\right), \; C^{2,s}(A,H)$$
$$= C^{(2)}\left([0,T]; Y^{s,p}(A,H)\right).$$

Condition 3.1. Assume the following:

(1) Condition 2.1 holds, $0 \leq \alpha < 1 - \frac{1}{2p}$ and

$$s > \frac{2p}{2p-1}\left(\frac{2n}{q} + \frac{1}{p}\right), q \in [1,2], p \in [1,\infty];$$

(2) the function $u \to f(u)$ is continuous from $u \in \mathbb{H}_{0p}$ into E, $f \in C^k(\mathbb{H}; H)$ with k an integer, $k \geq s > \frac{n}{p}$, and $f(u) = O\left(|u|^{\gamma+1}\right)$ for $u \to 0$, where $\gamma \geq 1$ is a positive integer.

Let

$$Y_1^{s,p}(A^\alpha; H) = Y^{s,p}(A^\alpha; H) \cap X_1(A^\alpha),\ Y^{s,p}(A^\alpha; H) = \Big\{u \in Y^{s,p}(A^\alpha; H),$$

$$\|u\|_{Y^{s,p}(A^\alpha;H)} = \|A^\alpha u\|_{X_p} + \left\|\mathbb{F}^{-1}\left(1 + |\xi|^2\right)^{\frac{s}{2}} \hat{u}\right\|_{X_p} < \infty\Big\}.$$

The main aim of this section is to prove the following results:

Theorem 3.1. *Let Condition 3.1 hold. Then there exists a constant $\delta > 0$ such that for all $\varphi \in Y_0(A^\alpha)$ and $\psi \in Y_1(A^\alpha)$ satisfying*

$$\|\varphi\|_{\mathbb{H}_{0p}} + \|A^\alpha \varphi\|_{X_1} + \|\psi\|_{\mathbb{H}_{1p}} + \|A^\alpha \psi\|_{X_1} \leq \delta, \tag{3.7}$$

problem (1.1)–(1.2) has a unique local strong solution $u \in C^2(Y_1^{s,p}(A))$. Moreover,

$$\sup_{t\in[0,T]}\left(\|u(.,t)\|_{Y_1^{s,p}(A^\alpha,H)} + \|u_t(.,t)\|_{Y_1^{s,p}(A^\alpha;E)}\right) \leq C\delta, \tag{3.8}$$

where the constant C depends only on A, H, g, f, and initial values.

Proof. By (2.5), (2.6), the problem of finding a solution u of (1.1)–(1.2) is equivalent to finding a fixed point of the mapping

$$G(u) = C_1(t)\varphi(x) + S_1(t)\psi(x) + Q(u), \tag{3.9}$$

where $C_1(t)$, $S_1(t)$ are defined by (2.6) and $Q(u)$ is the map defined by

$$Q(u) = -\int_0^t \mathbb{F}^{-1}\left[U(\xi, t-\tau)\hat{f}(u)(\xi,\tau)\right] d\tau.$$

We define the metric space

$$C(T,A) = C_\delta^2(Y_1^{s,p}(A)) = \left\{u \in C^{2,s}(A,H),\ \|u\|_{C^{2,s,p}(T,A)} \leq 5C_0\delta\right\}$$

equipped with the norm defined by

$$\|u\|_{C(T,A)} = \sup_{t\in[0,T]}\left[\|A^\alpha u(.,t)\|_{X_\infty} + \|u(.,t)\|_{Y^{s,p}} + \right.$$
$$\left.\|A^\alpha u_t(.,t)\|_{X_\infty} + \|u_t(.,t)\|_{Y^{s,p}}\right],$$

where $\delta > 0$ satisfies (3.7) and C_0 is the constant in Theorems 2.1 and 2.2. It is easy to prove that $C(T,A)$ is a complete metric space. From embedding of the Sobolev–Lions space $Y^{s,p}(A,H)$ (see, e.g., [30], Theorem 1) and the trace result (3.6) we obtain that $\|u\|_{X_\infty} \leq 1$ if we take δ small enough. For $\varphi \in Y_0(A^\alpha)$ and $\psi \in Y_1(A^\alpha)$, let

$$\|\varphi\|_{\mathbb{H}_{0p}} + \|A^\alpha \varphi\|_{X_1} + \|\psi\|_{\mathbb{H}_{1p}} + \|A^\alpha \psi\|_{X_1} = \delta.$$

We will find T and M that G is a contraction in $C^{2,s,p}(T,A)$. By Theorems 2.1, 2.2 and Corollary 3.3 $f(u) \in Y_1^{s,p}$. So, problem (1.1)–(1.2) has a solution satisfying

$$G(u)(x,t) = C_1(t)\varphi + S_1(t)\psi + Q(u), \tag{3.10}$$

where $C_1(t)$, $S_1(t)$ are defined by (2.5) and (2.6). By our assumptions, it is easy to see that the map G is well defined for $f \in C^{[s]+1}(\mathbb{H}_{0p};H)$. First, let us prove that the map G has a unique fixed point in $C(T,A)$. For this, it is sufficient to show that the operator G maps $C(T,A)$ into $C(T,A)$ and G is strictly contractive if δ is sufficiently small. In fact, by (2.7) in Theorem 2.1, Corollary 3.3, and in view of (3.7), we have

$$\begin{aligned}
\|A^\alpha G(u)\|_{X_\infty} + \|A^\alpha G_t(u)\|_{X_\infty} &\leq 2C_0 \Bigg[\|\varphi\|_{Y_0^\alpha(A^\alpha)} \\
&\quad + \|\psi\|_{Y_1^\alpha(A^\alpha)} + \int_0^t \left(\left\|\hat{f}((u))\right\|_{Y^{s,p}} + \left\|\hat{f}((u))\right\|_{X_1} \right) d\tau \Bigg] \\
&\leq 2C_0\delta + C \int_0^t \left(\|u(\tau)\|_{Y^{s,p}} \|u(\tau)\|_{X_\infty}^\gamma + \|u(\tau)\|_{X_p}^p \|u(\tau)\|_{X_\infty}^{\gamma-1} \right) d\tau \\
&\leq 2C_0\delta + C\|u\|_{C^{2,s,p}(T,A)}^{\gamma+1}.
\end{aligned} \tag{3.11}$$

On the other hand, by (2.17), Corollary 3.3, and (3.7), we get

$$\begin{aligned}
&(\|A^\alpha G(u)\|_{Y^{s,p}} + \|A^\alpha G_t(u)\|_{Y^{s,p}}) \\
&\quad \leq 2C_0 \left(\|\varphi\|_{\mathbb{E}_{0p}} + \|\psi\|_{\mathbb{E}_{1p}} + \int_0^t \left\|\hat{f}((u))\right\|_{Y^{s,p}} d\tau \right) \\
&\quad \leq 2C_0\delta + \int_0^t \left[\|u(\tau)\|_{Y^{s,p}} \|u(\tau)\|_{X_\infty}^\gamma \right] d\tau \leq 2C_0\delta + C\|u\|_{C^{2,s,p}(T,A)}^{\gamma+1}.
\end{aligned} \tag{3.12}$$

Hence combining (3.11) with (3.12), we obtain

$$\|A^\alpha G(u)\|_{Y_\infty^{s,p}} + \|A^\alpha G_t(u)\|_{Y_\infty^{s,p}} \leq 4C_0\delta + C\|u\|_{C^{2,s,p}(T,A)}^{\gamma+1}. \tag{3.13}$$

So taking δ small enough that $C\left(5C_8\delta\right)^{\gamma}<\frac{1}{5}$, by Theorems 2.1, 2.2 and (3.13), G maps $C\left(T,A\right)$ into $C\left(T,A\right)$.

Now we are going to prove that the map G is strictly contractive. Let u_1, $u_2\in C\left(T,A\right)$ be given. From (3.10) we get

$$\begin{aligned}&G\left(u_1\right)-G\left(u_2\right)\\&\quad=\int\limits_0^t\left[S\left(x,t-\tau\right)\left(\hat{f}\left(u_1\right)\left(\tau\right)-\hat{f}\left(u_2\right)\left(\tau\right)\right)\right]d\tau,\ t\in\left(0,T\right).\end{aligned}$$

By (2.7) in Theorem 2.1 and Corollary 3.3, we have

$$\begin{aligned}&\left\|A^{\alpha}\left[G\left(u_1\right)-G\left(u_2\right)\right]\right\|_{X_{\infty}}+\left\|A^{\alpha}\left[G\left(u_1\right)-G\left(u_2\right)\right]_t\right\|_{X_{\infty}}\\&\quad\leq\int\limits_0^t\left(\left\|\left[\hat{f}\left(u_1\right)-\hat{f}\left(u_2\right)\right]\right\|_{Y^{s,p}}+\left\|\left[\hat{f}\left(u_1\right)-\hat{f}\left(u_2\right)\right]\right\|_{X_1}\right)d\tau\\&\quad\leq\int\limits_0^t\Big\{\left\|u_1-u_2\right\|_{X_{\infty}}\left(\left\|u_1\right\|_{Y^{s,p}}+\left\|u_2\right\|_{Y^{s,p}}\right)\left(\left\|u_1\right\|_{X_{\infty}}+\left\|u_2\right\|_{X_{\infty}}\right)^{\gamma-1}\\&\quad+\left\|u_1-u_2\right\|_{Y^{s,p}}\left(\left\|u_1\right\|_{X_{\infty}}+\left\|u_2\right\|_{X_{\infty}}\right)^{\gamma}\\&\quad+\left(\left\|u_1\right\|_{X_{\infty}}+\left\|u_2\right\|_{X_{\infty}}\right)^{\gamma-1}\left\|u_1+u_2\right\|_{X_p}\left\|u_1-u_2\right\|_{X_p}\Big\}\\&\quad\leq C\left(\left\|u_1\right\|_{C(T,A)}+\left\|u_2\right\|_{C(T,A)}\right)^{\gamma}\left\|u_1-u_2\right\|_{C(T,A)}.\end{aligned}\tag{3.14}$$

On the other hand, by (2.17) in Theorem 2.2, Corollary 3.3, and (3.7), we get

$$\begin{aligned}&\left(\left\|A^{\alpha}\left[G\left(u_1\right)-G\left(u_2\right)\right]\right\|_{Y^{s,p}}+\left\|A^{\alpha}\left[G\left(u_1\right)-G\left(u_2\right)\right]_t\right\|_{Y^{s,p}}\right)\\&\quad\leq C\int\limits_0^t\left\|\hat{f}\left(u_1\right)\left(\tau\right)-\hat{f}\left(u_2\right)\left(\tau\right)\right\|_{Y^{s,p}}d\tau\\&\quad\leq C\int\limits_0^t\Big\{\left\|u_1-u_2\right\|_{X_{\infty}}\left(\left\|u_1\right\|_{Y^{s,p}}+\left\|u_2\right\|_{Y^{s,p}}\right)\left(\left\|u_1\right\|_{X_{\infty}}+\left\|u_2\right\|_{X_{\infty}}\right)^{\gamma-1}\\&\quad+\left\|u_1-u_2\right\|_{Y^{s,p}}\left(\left\|u_1\right\|_{X_{\infty}}+\left\|u_2\right\|_{X_{\infty}}\right)^{\gamma}\Big\}d\tau\\&\quad\leq C\left(\left\|u_1\right\|_{C(T,A)}+\left\|u_2\right\|_{C(T,A)}\right)^{\gamma}\left\|u_1-u_2\right\|_{C(T,A)}.\end{aligned}\tag{3.15}$$

Combining (3.14) with (3.15) yields

$$\begin{aligned}&\left\|G\left(u_1\right)-G\left(u_2\right)\right\|_{C(T,A)}\\&\quad\leq C\left(\left\|u_1\right\|_{C(T,A)}+\left\|u_2\right\|_{C(T,A)}\right)^{\gamma}\left\|u_1-u_2\right\|_{C(T,A)}.\end{aligned}\tag{3.16}$$

Taking δ small enough, from (3.16) we obtain that G is strictly contractive in $C(T, A)$. Using the contraction mapping principle, we get that $G(u)$ has a unique fixed point $u(x,t) \in C(T, A)$ and $u(x,t)$ is the solution of (1.1)–(1.2).

Let us show that this solution is unique in $C^{2,s}(A, H)$. Let $u_1, u_2 \in C^{2,s}(A, H)$ be two solutions of (1.1)–(1.2). Then for $u = u_1 - u_2$, we have

$$u_{tt} - a * \Delta u + Au = [f(u_1) - f(u_2)]. \tag{3.17}$$

Hence by Minkowski's inequality for integrals and by Theorem 2.2, from (3.17) we obtain

$$\|u_1 - u_2\|_{Y^{s,p}} \leq C_2(T) \int_0^t \|u_1 - u_2\|_{Y^{s,p}} \, d\tau. \tag{3.18}$$

From (3.18) and Gronwall's inequality, we have $\|u_1 - u_2\|_{Y^{s,p}} = 0$, i.e., problem (1.1)–(1.2) has a unique solution in $C^{2,s}(A, H)$.

Consider the problem (1.1)–(1.2) when $\varphi \in \mathbb{H}_{0p}$ and $\psi \in \mathbb{H}_{1p}$. Let

$$C^{(i)}\left(Y^{s,2}\right) = C^{(i)}([0,\infty); Y^{s,2}(A, H)),\ i = 0, 1, 2.$$

Condition 3.2. Assume the following: (1) Condition 2.1 holds; (2) $0 \leq \alpha < 1 - \frac{1}{2p}$, $\varphi \in \mathbb{H}_{0p}$, $\psi \in \mathbb{H}_{1p}$, and

$$s > \frac{2p}{2p-1}\left(\frac{2}{q} + \frac{1}{p}\right) n,\ q \in [1, 2],\ p \in (1, \infty);$$

(3) $f \in C^{[s]+1}(\mathbb{H}; H)$ with $f(0) = 0$.

Reasoning as in Theorem 3.1 and [13, Theorem 1.1], we have the following:

Theorem 3.2. *Let Condition 3.2 hold. Then there exists a constant $\delta > 0$ such that for all $\varphi \in \mathbb{H}_{0p}$, $\psi \in \mathbb{H}_{1p}$ satisfying*

$$\|\varphi\|_{\mathbb{H}_{0p}} + \|\psi\|_{\mathbb{H}_{1p}} \leq \delta, \tag{3.19}$$

problem (1.1)–(1.2) has a unique local strong solution $u \in C^{(2)}(Y^{s,p})$. Moreover,

$$\sup_{t\in[0,T]} \left(\|u(.,t)\|_{Y^{s,p}(A^\alpha,H)} + \|u_t(.,t)\|_{Y^{s,p}(A^\alpha;H)}\right) \leq C\delta, \tag{3.20}$$

where the constant C depends only on f and initial data.

Proof. Consider the metric space defined by

$$W_0^{s,p} = \left\{u \in C^{(2)}(Y^{s,p}),\ \|u\|_{Y^{s,p}} \leq 3C_0\delta\right\},$$

equipped with the norm

$$\|u\|_{W_0^{s,p}} = \sup_{t\in[0,T]} \left(\|u\|_{Y^{s,p}(A^\alpha;H)} + \|u_t\|_{Y^{s,p}(A^\alpha;H)}\right),$$

where $\delta > 0$ satisfies (3.19) and C_0 is the constant in Theorem 2.1. It is easy to prove that $W_0^{s,p}$ is a complete metric space. From Sobolev embedding theorem we know that $\|u\|_\infty \leq 1$ if we take δ small enough. By Theorem 2.2 and Corollary 3.1, $f(u) \in Y^{s,p}$. Thus problem (1.1) –(1.2) has a unique solution, which can be written as (3.9). We should prove that the operator $G(u)$ defined by (3.9) is strictly contractive if δ is sufficiently small. In fact, by (2.17) in Theorem 2.2 and Lemma 3.1, we get

$$\begin{aligned}
&\|A^\alpha G(u)\|_{Y^{s,p}} + \|A^\alpha G_t(u)\|_{Y^{s,p}} \\
&\quad \leq C_0 \left[\left[\|\varphi\|_{\mathbb{E}_{0p}} + \|\psi\|_{\mathbb{E}_{1p}} + \int_0^t \|K(u)(.,\tau)\|_{Y^{s,p}} d\tau \right] \right. \\
&\quad \leq C_0\delta + C_0 \int_0^t \|K(u)(.,\tau)\|_{Y^{s,p}} d\tau \\
&\quad \leq C_0\delta + C \int_0^t \|u(\tau)\|_{Y^{s,p}} d\tau \leq C_0\delta + C\|u\|_{Y^{s,p}}, \qquad (3.21)
\end{aligned}$$

where

$$K(u)(.,\tau) = S(x, t-\tau) f(u)(x,\tau).$$

Therefore, from (3.20) we have

$$\|G(u)\|_{Y^{s,p}} \leq 2C_0\delta + C\|u\|_{Y^{s,p}}. \qquad (3.22)$$

Taking δ small enough that $C(3C_0\delta)^\alpha < 1/3$, from (3.22) and from Theorems 2.1, 2.2 we get that G maps $W_0^{s,p}$ into $W_0^{s,p}$. Then reasoning as in Theorem 3.1, we obtain that $G : W_0^{s,p} \to W_0^{s,p}$ is strictly contractive. Using the contraction mapping principle, we know that $G(u)$ has a unique fixed point $u \in C^{(2)}(Y^{s,2})$ and $u(x,t)$ is the solution of problem (1.1) − (1.2). Moreover, by virtue of Theorem 2.1, from (3.19) we obtain (3.20) .

We claim that the solution of (1.1)-(1.2) is also unique in $C^{(1)}(Y^{s,2})$. In fact, let u_1 and u_2 be two solutions of the problem (1.1)–(1.2) and u_1, $u_2 \in C^{(2)}(Y^{s,2})$. Using the contraction mapping principle, we know that $G(u)$ has a unique fixed point $u \in C^{(2)}(Y^{s,2})$. Let $u = u_1 - u_2$. Then

$$u_{tt} - a * \Delta u + Au = f(u_1) - f(u_2).$$

This fact is derived in a similar way to what was done in Theorem 3.1, using Theorems 2.1, 2.2 and Gronwall's inequality.

Let

$$C^{(2,s)}(Y^{s,p}) = C^{(2)}([0,T]; Y^{s,p}(A; H)).$$

Theorem 3.3. *Let Condition 3.2 hold. Then there exist $T > 0$ such that problem (1.1)–(1.2) is well posed with solution in $C^1([0,T]; Y^{s,p}(A,H))$ for initial data $\varphi \in \mathbb{H}_{0p}$ and $\psi \in \mathbb{H}_{1p}$.*

Proof. Consider the operator $u \to f(u)$. By Corollary 3.1, $f(u)$ is locally Lipschitz on $Y^{s,p}$. Then reasoning as in Theorem 3.2 and [13, Theorem 1.1], we obtain that G: $W_0^{s,p} \to W_0^{s,p}$ is strictly contractive. Using the contraction mapping principle, we get that the operator $G(u)$ defined by (3.5) has a unique fixed point $u(x,t) \in C^{(2)}(Y^{s,p})$ and $u(x,t)$ is the solution of the problem (1.1) − (1.2). Moreover, we show that the solution $u(x,t)$ of (1.1) − (1.2) is also unique in $C^{(2)}(Y^{s,p})$. In fact, let u_1 and u_2 be two solutions of the problem (1.1) − (1.2) and u_1, $u_2 \in C^{(2)}(Y^{s,p})$. Let $u = u_1 - u_2$. Then

$$u_{tt} - a * \Delta u + Au = f(u_1) - f(u_2).$$

This fact is derived in a similar way to what was done in Theorem 3.2, by using Theorems 2.1, 2.2 and Gronwall's inequality.

The solution in Theorems 3.2–3.3 can be extended to a maximal interval $[0, T_{\max})$, where finite $T_{\max}$ is characterized by the blow-up condition

$$\limsup_{T \to T_{\max}} \|u\|_{Y^{s,p}(A^\alpha;H)} = \infty.$$

Lemma 3.8. *Let Condition 3.2 hold and u be a solution of* (1.1) − (1.2). *Then there is a global solution if for all $T < \infty$, we have*

$$\sup_{t \in [0,T]} \left(\|u\|_{Y^{s,p}(A^\alpha;H)} + \|u_t\|_{Y^{s,p}(A^\alpha;H)} \right) < \infty. \tag{3.24}$$

Proof. Indeed, reasoning as in the second part of the proof of Theorem 3.1, using a continuation of the local solution of (1.1)−(1.2), and assuming contrary that (3.24) holds and $T_0 < \infty$, we obtain a contradiction, i.e., we get $T_0 = T_{\max} = \infty$.

4. Conservation of Energy and Global Existence

In this section, we prove the existence and uniqueness of the global strong solution for the problem (1.1) − (1.2). For this purpose, we are going to make a priori estimates of the strong solution of (1.1)−(1.2). Here, the scalar product of u, $\upsilon \in X_2$ will be denoted just by (u, υ). Moreover, the norm of $u \in X_2$ will be denoted by $\|u\|$.

Let

$$C^{(1)}(X^p) = C^{(1)}([0,\ T); X^p),\ C^{(2,s)}(A,H) = C^{(2)}([0,T]; Y^{s,p}(A;H)),$$

where $Y^{s,p}(A;H)$ is as defined in Sect. 2.

First, we prove the following lemma:

Lemma 4.1. *Let Condition 3.2 hold and* $0 \leq \alpha < 1 - \frac{1}{2p}$. *Assume there exist a solution* $u \in C^{(2,s)}(A, H)$ *of* (1.1)–(1.2)*. Then*

$$A^{\alpha}u, A^{\alpha}u_t \in C^{(1)}(X^p).$$

Proof. By Lemma 2.1, problem (1.1)–(1.2) is equivalent to the integral equation,

$$u(x,t) = C_1(t)\varphi + S_1(t)\psi + Q(f), \tag{4.1}$$

where $C_1(t)$, $S_1(t)$ are operator functions defined by (2.5) and (2.6), g replaced by $f(u)$, and

$$Q(f) = \int_0^t \mathbb{F}^{-1}\left[S(\xi, t-\tau)\hat{f}(u)(\xi)\right]d\tau. \tag{4.2}$$

From (4.1) we get that

$$\begin{aligned} u_t(x,t) &= \frac{d}{dt}C_1(t)\varphi + \frac{d}{dt}S_1(t)\psi \\ &+ \int_0^t \mathbb{F}^{-1}\left[C(\xi, t-\tau)\hat{f}((u)(\xi))\right]d\tau. \end{aligned} \tag{4.3}$$

Since $C_1(t)$, $S_1(t)$, and $\frac{d}{dt}S(\xi,t)$ are uniformly bounded operators in E for fixed t, by (4.2) and Fourier multiplier results in X^p spaces (see, e.g., [12, Theorem 4.3]), we have

$$\begin{aligned} \|A^{\alpha}C_1(t)\varphi\|_{X^p} &= \left\|\mathbb{F}^{-1}\left[A^{\alpha}C(\xi,t)\hat{\varphi}\right]\right\|_{X^p} \lesssim \|\varphi\|_{\mathbb{H}_{0p}} < \infty, \\ \left\|\hat{A}^{\alpha}S_1(t)\varphi\right\|_{X^p} &= \left\|\mathbb{F}^{-1}\left[\hat{A}^{\alpha}S(\xi,t)\hat{\psi}\right]\right\|_{X^p} \lesssim \|\psi\|_{\mathbb{H}_{1p}} < \infty. \end{aligned} \tag{4.4}$$

By differentiating (2.3), in a similar way we have

$$\begin{aligned} &\left\|A^{\alpha}\frac{d}{dt}C_1(t)\varphi\right\|_{X^p} = \left\|\mathbb{F}^{-1}\left[A^{\alpha}\frac{d}{dt}C(\xi,t)\hat{\varphi}\right]\right\|_{X^p} \\ &\lesssim \|\varphi\|_{\mathbb{H}_{0p}} < \infty, \\ &\left\|A^{\alpha}\frac{d}{dt}S_1(t)\varphi\right\|_{X^p} = \left\|\mathbb{F}^{-1}\left[A^{\alpha}\frac{d}{dt}S(\xi,t)\hat{\psi}\right]\right\|_{X^p} \\ &\lesssim \|\psi\|_{\mathbb{H}_{1p}} < \infty. \end{aligned} \tag{4.5}$$

For fixed t, we have $f(u) \in Y^{s,p}$. Moreover, by the assumption on A we have the uniform estimate

$$\left\|A^{\alpha}A_{\xi}^{-1}\right\|_{B(H)} \leq C_A.$$

Due to $s+r\geq 1$, from (4.2) and Fourier multiplier results in X_p we get

$$\left\| A^{\alpha}Q\left(f\right) \right\|_{X^p}\leq \left\| \mathbb{F}^{-1}\left[A^{\alpha}\int\limits_0^t S\left(\xi ,t-\tau\right) \hat{f}\left(u\right) \left(\xi\right) d\tau\right] \right\|_{X^p}$$
$$\lesssim C_A\left\| f\left(u\right) \right\|_{Y^{s,p}}<\infty . \tag{4.6}$$

Then from (4.1) and (4.3)–(4.6) we obtain the assertion.

Lemma 4.2. *Assume Condition 3.2 holds with $a=0$. Suppose a solution of $(1.1)-(1.2)$ exists in $C^{(2,s)}(A,H)$. If $\psi\in X^p$, then $u_t\in C^{(1)}(X^p)$. Moreover, if $\varphi\in X^p$, then $u\in C^{(1)}(X^p)$.*

Proof. Integrating equation (1.1) for $a=0$ twice and calculating the resulting double integral as an iterated integral, we have

$$u\left(x,t\right) =\varphi\left(x\right) +t\psi\left(x\right)$$
$$-\int\limits_0^t\left(t-\tau\right) \left(Au\right) \left(x,\tau\right) d\tau+\int\limits_0^t\left(t-\tau\right) f\left(u\right) \left(x,\tau\right) d\tau , \tag{4.7}$$

$$u_t\left(x,t\right) =\psi\left(x\right) -\int\limits_0^t\left(Au\right) \left(x,\tau\right) d\tau+\int\limits_0^t f\left(u\right) \left(x,\tau\right) d\tau . \tag{4.8}$$

From (4.8), for fixed t and τ we get $f\left(u\right) \in Y^{s,p}$ for all t. Also

$$\left\| f\left(u\right) \left(.\right) \right\|_{X^p}\lesssim\left\| \mathbb{F}^{-1}\hat{f}\left(u\right) \left(\xi\right) \right\|_{X^p}. \tag{4.9}$$

Then from (4.7)–(4.9) we obtain

$$\left\| u_t\left(.,t\right) \right\|_{X^2}\leq\left\| \psi\left(.\right) \right\|_{X^2}$$
$$+\int\limits_0^t\left\| \left(Au\right) \left(.,\tau\right) \right\|_{X^p}d+\int\limits_0^t\left\| f\left(u\right) \left(.,\tau\right) \right\|_{X^p}d\tau .$$

By the assumption on A, g and for fixed τ we have $u_t\in C^{(1)}\left(X^p\right)$,

$$\left\| Au\left(.\right) \right\|_{X^p}\lesssim\left\| \mathbb{F}^{-1}\hat{u}\left(\xi ,\tau\right) \right\|_{X^p}\lesssim\left\| u\left(.,\tau\right) \right\|_{Y^{s,p}(A)}.$$

Moreover, by Lemma 3.3 we have $u_t\in C^{(1)}\left(X^p\right)$. The second statement follows similarly from (4.7).

Let

$$\Phi\left(\sigma\right) =\int\limits_0^{\sigma}f\left(\tau\right) d\tau . \tag{4.10}$$

Condition 3.3. Assume that Condition 3.2 hold and A is a symmetric operator in H. Suppose that $s+r\geq 1$ and $a\left(x\right) =a\left(-x\right)$. Let φ, $\psi\in Y^{s,2}\left(A,H\right) \cap X_{\infty}$ and $\Phi\left(.\right) \in L^1$

Lemma 4.3. *Let Condition 3.3 hold and let* $u \in C^{(2,s)}(A,H)$ *be a solution of (1.1)–(1.2) for any* $t \in [0,\ T)$. *Then the energy*

$$E(t) = \|u_t\|^2 + (a * \Delta u, u) + (Au, u) - 2\int\limits_{\mathbb{R}^n} \Phi(u)\,dx \tag{4.11}$$

is constant.

Proof. By assumptions $\Phi(.) \in L^1$ and $Au,\ Au_t \in X^2$. Due to $a(x) = a(-x)$, we have

$$\left(\left(a * (\Delta u)_t\right), u\right) = (a * \Delta u, u_t),\ \ (Au_t, u) = (Au, u_t). \tag{4.12}$$

Hence from (4.12) we obtain

$$\frac{d}{dt}E(t) = 2(u_{tt}, u_t) + 2(a * \Delta u, u_t) + 2(Au, u_t) + 2\int\limits_{\mathbb{R}^n} \Phi_t(u)\,u_t dx =$$

$$\begin{aligned} 2(u_{tt}, u_t) - 2(a * \Delta u, u_t) + 2(Au, u_t) + 2(f(u), u_t) \\ = 2([u_{tt} - a * \Delta u + Au - f(u)], u_t) = 0, \end{aligned}$$

where (u, υ) denotes the inner product in X_2. Hence, we obtain the assertion.

5. Blow-Up in Finite Time

In this section we prove the following result:

Theorem 5.1. *Let Condition 3.3 hold and let* $u \in C^{(2,s)}(A,H)$ *be a solution of (1.1) –(1.2) for any* $t \in [0,\ T)$. *If there exist positive numbers* ν *and* t_0 *such that*

$$\sigma f(\sigma) \leq 2(1 + 2\nu)\Phi(\sigma) \ \ \textit{for all } \sigma \in \mathbb{R}, \tag{5.1}$$

and

$$E(0) = \|u_t\|^2 + (u, a * \Delta u) + (u, Au) - 2\int\limits_{\mathbb{R}^n} \Phi(u)\,dx < 0, \tag{5.2}$$

then the solution u *blows up in finite time.*

Proof. Assume that there is a global solution. Then $u,\ u_t \in X_2$ for all $t > 0$. Let

$$H(t) = \|u(t)\|^2 + b(t + t_0)^2$$

for some b and t_0 that will be determined later. We have

$$\begin{aligned} H^{(1)}(t) &= 2(u, u_t) + 2b(t + t_0), \\ H^{(2)}(t) &= 2\|u_t\|^2 + 2(u, u_{tt}) + 2b. \end{aligned} \tag{5.3}$$

From (1.1), (5.1) , and (5.2) we get

$$\begin{aligned}
(u,u_{tt}) &= (u,[a*\Delta u - Au + f(u)]) \\
&= [(u,a*\Delta u) - (u,Au) - (u,f(u))] \\
&\geq (u,a*\Delta u) - (u,Au) - 2(1+2\nu)\int\limits_{\mathbb{R}^n}\Phi(u)\,dx \\
&\geq (u,a*\Delta u) - (u,Au) \\
&+ (1+2\nu)\left[\|u_t\|^2 + (u,a*\Delta u) + (u,(Au)) - E(0)\right] = \\
&(1+2\nu)\left[\|u_t\|^2 - E(0)\right] + 2\nu\left[(u,a*\Delta u) + (u,Au)\right]. \qquad (5.4)
\end{aligned}$$

Let b be a real number such that $b \leq -E(0)$ and

$$4\nu(a*\Delta u,u) + 4\nu(u,Au) \leq -[b+E(0)]. \qquad (5.5)$$

From (5.3) and (5.5), we obtain

$$\begin{aligned}
H^{(2)}(t) &\geq 4(1+\nu)\|u_t\|^2 + 4\nu\left[a\|\nabla u\|^2 + (u,Au)\right] \\
&-2(1+2\nu)E(0) + 2b. \qquad (5.6)
\end{aligned}$$

On the other hand, in view of the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\left(H^{(1)}(t)\right)^2 &= [2(u,u_t) + 2b(t+t_0)]^2 \\
&\leq 4\left[\|u\|^2\|u_t\|^2 + b(t+t_0)^2\left(\|u\|^2 + \|u_t\|^2\right)\right] \\
&+4b^2(t+t_0)^2. \qquad (5.7)
\end{aligned}$$

Hence, by (5.3), (5.5) , and (5.7) , we obtain

$$\begin{aligned}
&H^{(2)}H - (1+\nu)\left(H^{(1)}\right)^2 \\
&\geq \left[4(1+\nu)\|u_t\|^2 + 4\nu\left(a\|\nabla u\|^2 + (u,Au)\right) + 2b\right. \\
&- 2(1+2\nu)E(0)]\left[\|u\|^2 + b(t+t_0)^2\right] - 4(1+\nu)b^2(t+t_0)^2 \\
&-4(1+\nu)\left[\|u\|^2\|u_t\|^2 + b^2(t+t_0)^2\left(\|u\|^2 + \|u_t\|^2\right)\right] \\
&= 4\nu\left(a\|\nabla u\|^2 + (u,Au)\right)H(t) + 2bH(t) - 2(1+2\nu)E(0)H(t) \\
&-4b(1+\nu)H(t) - 4b^2(1+\nu)(t+t_0)^2\|Bu_t\|^2 \\
&= -2(1+2\nu)[b+E(0)] + 4\nu a\|\nabla u\|^2 + 4\nu(u,Au) \geq 0
\end{aligned}$$

when

$$[b+E(0)] + 4\nu a\|\nabla u\|^2 + 4\nu(u,Au) \leq 0,$$

i.e., if assumption (5.2) holds. Moreover,

$$H^{(1)}(0) = 2(\varphi, \psi) + 2b(t_0) \geq 0,$$

for sufficiently large t_0. Then reasoning as in the proof of [16, Theorem1] , we get that that $H(t)$, and thus $\|u(t)\|^2$ blows up in finite time, contradicting the assumption that a global solution exists.

6. Applications

6.1. The Cauchy Problem for Infinite System of WEs

Consider problem (1.4). Let

$$l_2 = \left\{ u = \{u_j\},\, j = 1, 2, ...N,\ \|u\|_{l_2} = \left(\sum_{j=1}^{N} |u_j|^2 \right)^{\frac{1}{2}} < \infty \right\},$$

(see [27, § 1.18]). Let A_1 be the operator in l_2 defined by

$$A_1 = [a_{jm}],\, a_{jm} = b_j 2^{\sigma m},\, m, j = 1, 2, ..., N,\, D(A_1) = l_2^{\sigma}$$
$$= \left\{ u = \{u_j\},\ j = 1, 2, ..., N,\ \|u\|_{l_2^{\sigma}} = \left(\sum_{j=1}^{N} 2^{\sigma j} |u_j|^2 \right)^{\frac{1}{2}} < \infty \right\},$$
$$N \in \mathbb{N},\, b_j \in \mathbb{R},\, \sigma > 0.$$

Let

$$Y^{s,p,\sigma} = W^{s,p}(\mathbb{R}^n; l_2) \cap L^p(\mathbb{R}^n; l_2^{\sigma}),$$
$$W_0(l_2) = W^{s\left(1-\frac{1}{2p}\right),p}(\mathbb{R}^n; l_2) \cap L^p\left(\mathbb{R}^n; l_2^{\sigma\left(1-\frac{1}{2p}\right)}\right).$$

Let $f = \{f_m\}$, $m = 1, 2, ..., \infty$ and

$$\eta_1 = \eta_1(\xi) = \left[a|\xi|^2 + A_1 \right]^{\frac{1}{2}}.$$

Here

$$E_{ip}(l_2) = W^{s(1-\theta_i),p}(\mathbb{R}^n; l_2) \cap L^p\left(\mathbb{R}^n; l_2^{\sigma(1-\theta_i)}\right),$$

where

$$\theta_j = \frac{1 + ip}{2p},\, i = 0, 1.$$

From Theorem 3.1 we obtain the following result:

Theorem 6.1. *Assume the following: (1) assumption* (2.7) *holds,* $0 \leq \alpha < 1-\frac{1}{2p}$, $\varphi \in \mathbb{H}_{0p}(l_2)$, $\psi \in \mathbb{H}_{1p}(l_2)$ *for* $p \in [1,\infty]$*; (2)* $a \geq 0$, b_j *are nonnegative bounded numbers,* $\hat{a}(\xi) + b_j > 0$ *for* $\xi \in \mathbb{R}^n$*, and the following estimate hold*

$$\sum_{j=1}^{\infty}\left[\hat{a}\left|\xi\right|^2 + b_j\right]^{-1} \leq M \text{ for all } \xi \in \mathbb{R}^n;$$

(3) the function

$$u \rightarrow f(x,t,u) : \mathbb{R}^n \times [0,T] \times W_0(l_2) \rightarrow l_2$$

is measurable in $(x,t) \in \mathbb{R}^n \times [0,T]$ *for* $u \in W_0(l_2)$*. Moreover,* $f(x,t,u)$ *is continuous in* $u \in W_0(l_2)$ *and* $f \in C^{[s]+1}(W_0(l_2);l_2)$ *uniformly in* $x \in \mathbb{R}^n$, $t \in [0,T]$*. Then problem* (1.3) *has a unique local strong solution*

$$u \in C^{(2)}\left([0,\ T_0);Y_{\infty}^{s,p}(A_1,l_2)\right),$$

where T_0 *is a maximal time interval that is appropriately small relative to* M*. Moreover, if*

$$\sup_{t\in[0,\ T_0)}\left(\|u\|_{Y_{\infty}^{s,p}(A_1^{\alpha};l_2)} + \|u_t\|_{Y_{\infty}^{s,p}(A_1^{\alpha};l_2)}\right) < \infty,$$

then $T_0 = \infty$.

Proof. It is known that $L^p(\mathbb{R}^n;l_2)$ is a UMD space for $p \in (1,\infty)$ (se,e e.g., [25]). By Remark 2.1, the definition of $W^{s,p}(A_1,l_2)$ and real interpolation of Banach spaces (see, e.g., [27, § 1.3, 1.18]), we have

$$\mathbb{H}_{ip} = \left(W^{s,,p}(\mathbb{R}^n;l_2^{\sigma},l_2), L_p(\mathbb{R}^n;l_2)_{\theta_i,p}\right) = W^{s(1-\theta_i),p}\left(\mathbb{R}^n;l_2^{\sigma(1-\theta_i)},l_2\right)$$
$$= W^{s(1-\theta_i),p}(\mathbb{R}^n;l_2) \cap L^p\left(\mathbb{R}^n;l_2^{\sigma(1-\theta_i)}\right) = \mathbb{H}_{0i}(l_2),\ i = 0,1.$$

By assumptions (1), (2) we obtain that A_1 is sectorial in l_2, and by virtue of [3, § 3.14, 3.16], the operator $A_1^2 + \mu$ is a generator of bounded cosine function in l_2. Hence, by (4), (5), all conditions of Theorem 3.2 are satisfied, i.e., we get the conclusion.

Theorem 6.2. *Assume: (*h_1*) assumptions (1)-(3) of Theorem 6.1 are satisfied for* $p = 2$*; (*h_2*)* $f_m \in C^{[s]}(\mathbb{R};l_2)$ *with* $f(0) = 0$ *and*

$$\sum_{m=1}^{\infty} f_m(u) < \infty \text{ for all } u = \{u_m\} \in C^{(2)}\left([0,\ \infty);Y_{\infty}^{s,2}(A_1;l_2)\right);$$

(h_3*)* $B\varphi$, $B\psi \in L^2(\mathbb{R}^n;l_2)$ *and* $\Phi(\varphi) \in L^2(\mathbb{R}^n;l_2)$*; (*$h_4$*) there is some* $k > 0$ *so that*

$$\Phi(\sigma) \geq -k\left|\sigma\right|^2 \text{ for all } \sigma \in \mathbb{R} \text{ and } t \in [0,T].$$

Then: (a); there exists $T > 0$ *such that problem* (1.4) *has a global solution*

$$u \in C^{(2)}\left([0,\ \infty);Y_{\infty}^{s,2}(A_1;l_2)\right);$$

(b) if assumption 5.1 of Theorem 5.1 also holds for $H = l_2$, then the solution of (1.4) blows up in finite time.

Proof. From assumptions (h_1), (h_2) it is clear to that Condition 4.1 holds for $H = l_2$ and $r > 2 + \frac{n}{2}$. By (h_3), all other assumptions of Theorem 4.1 are satisfied. Hence, we obtain the assertion.

6.2. The Mixed Problem for Degenerate WE

Consider the problem (1.5)–(1.7). Let

$$Y^{s,p,2} = W^{s,p}\left(\mathbb{R}^n; L^2(0,1)\right) \cap L^p\left(\mathbb{R}^n; W^{[2],2}(0,1)\right),\ 1 \leq p \leq \infty.$$

Let A_2 be the operator in $L^2(0,1)$ defined by (1.6)–(1.8) and let

$$\eta_2 = \eta_2(\xi) = \left[a\left|\xi\right|^2 + \hat{A}_2(\xi)\right]^{\frac{1}{2}}.$$

Here

$$H_{ip}\left(L^2\right) = W^{[s(1-\theta_i)],p}\left(\mathbb{R}^n; L^2(0,1)\right) \cap L^p\left(\mathbb{R}^n; W^{[2(1-\theta_i)],2}(0,1)\right),$$

where

$$\theta_i = \frac{1+ip}{2p},\ i = 0,1.$$

Now we present the following result:

Condition 6.1. Assume;

(1) assumption (2.7) holds, $0 \leq \gamma < \frac{1}{2}$, and $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$;

(2) $0 \leq \alpha < 1 - \frac{1}{2p}$, $\varphi \in \mathbb{H}_{0p}\left(L^2\right)$, $\psi \in \mathbb{H}_{1p}\left(L^2\right)$ for $p \in [1,\infty]$;

(2) b_1 and b_2 are complex valued functions on $(0,1)$. Moreover, $b_1 \in C[0,1]$, $b_1(0) = b_1(1)$, $b_2 \in L_\infty(0,1)$, and $\left|b_2(x)\right| \leq C\left|b_1^{\frac{1}{2}-\mu}(x)\right|$ for $0 < \mu < \frac{1}{2}$ and for a.a. $x \in (0,1)$;

(3) $a \geq 0$ and $\eta_2(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$;

(4) the function

$$u \rightarrow f(x,t,u) : \mathbb{R}^n \times [0,T] \times W_0\left(L^2(0,1)\right) \rightarrow L^2(0,1)$$

is measurable in $(x,t) \in \mathbb{R}^n \times [0,T]$ for $u \in W_0\left(L^2(0,1)\right)$; $f(x,t,u)$. Moreover, $f(x,t,u)$ is continuous in $u \in W_0\left(L^2(0,1)\right)$ and

$$f(x,t,u) \in C^{[s]+1}\left(W_0\left(L^2(0,1)\right); L^2(0,1)\right)$$

uniformly with respect to $x \in \mathbb{R}^n$, $t \in [0,T]$.

Theorem 6.3. *Assume that Condition 6.1 is satisfied. Then problem (1.6)–(1.8) has a unique local strong solution*

$$u \in C^{(2)}\left([0,\ T_0); Y_\infty^{s,p}\left(A_2, L^2(0,1)\right)\right),$$

where T_0 is a maximal time interval that is appropriately small relative to M. Moreover, if

$$\sup_{t\in[0,\ T_0)}\left(\|u\|_{Y_{\infty}^{s,p}(A_2^{\alpha};L^2(0,1))}+\|u_t\|_{Y_{\infty}^{s,p}(A_2^{\alpha};L^2(0,1))}\right)<\infty,$$

then $T_0=\infty$.

Proof. It is known (see, e.g., [13]) that $L^2(0,1)$, is a UMD space for $p_1\in(1,\infty)$. By definition, $W^{s,p}\left(A_2,L^2(0,1)\right)$, and by real interpolation of Banach spaces (see, e.g., [27, §1.3.2.]) we have

$$\begin{aligned}\mathbb{H}_{ip}&=W^{s,,p}\left(\mathbb{R}^n;W^{[2],,2}(0,1),L^{p_1}(0,1),L^p\left(\mathbb{R}^n;L^2(0,1)\right)\right)_{\theta_i,p}\\&=W^{s(1-\theta_i),p}\left(\mathbb{R}^n;W^{[2(1-\theta_i)],2}(0,1),L^2(0,1)\right)=H_{ip}\left(L^2\right).\end{aligned}$$

In view of [26, Theorem 4.1], we obtain that the operator A_2 defined by (2.5) is uniformly sectorial in $L^2(0,1)$, and by virtue of [3, § 3.14, 3.16], the operator $A_2^2+\mu$ is a generator of the bounded cosine function in $L^2(0,1)$. Moreover, using assumptions (1), (2), we deduce that $\eta_2(\xi)\neq 0$ for all $\xi\in\mathbb{R}^n$. Hence by hypotheses (3), (4) of Condition 5.1, we get that all, hypotheses of Theorem 3.2 hold, i.e., we obtain the conclusion.

Theorem 6.4. *Assume Condition 6.1 is satisfied for* $p_1=2$. *Suppose* $f\in C^{[s]}(\mathbb{R};L^2((0,1)))$ *with* $f(0)=0$. *Moreover, let* $B\varphi$, $B\psi$, $\Phi(\varphi)\in L^2(\mathbb{R}^n\times(0,1))$ *and there exists* $k>0$ *such that*

$$\Phi(\sigma)\geq -k|\sigma|^2 \text{ for } \sigma\in\mathbb{R},\ t\in[0,T].$$

Then:

(a) there exists $T>0$ *such that problem* (1.5)−(1.7) *has a global solution*

$$u\in C^2\left([0,\ \infty);Y_{\infty}^{s,2}\right);$$

(b) if assumption (5.1) of the Theorem 5.1 also holds for $H=L^2(0,1)$, *then the solution of* (1.6) − (1.8) *blows up in finite time.*

Proof. Indeed, by assumption, all conditions of Theorem 4.1 are satisfied for $H=L^2(0,1)$, i.e., we obtain the assertion.

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