Results in Mathematics

On Centroaffine Tchebychev Hypersurfaces with Constant Sectional Curvature

Xiuxiu Cheng, Zejun Hu, and Cheng Xing

Abstract. In this paper, we study the interesting open problem of classifying the locally strongly convex centroaffine Tchebychev hypersurfaces in $\mathbb{R}^{n+1}$ with constant sectional curvature. First, for arbitrary dimensions we solve the problem by assuming that the centroaffine shape operator vanishes. Second, extending the solved cases of $n = 2, 3$, we continue working with the case $n = 4$. As the result, we establish a complete classification of the flat hyperbolic centroaffine Tchebychev hypersurfaces in $\mathbb{R}^5$.

Mathematics Subject Classification. Primary 53A15; Secondary 53C24, 53C42.

Keywords. Centroaffine hypersurface, constant sectional curvature, Tchebychev hypersurface, Tchebychev vector field.

1. Introduction

Let $\mathbb{R}^{n+1}$ be the $(n + 1)$-dimensional affine space equipped with the standard flat connection D. For an immersion $x : M^n \to \mathbb{R}^{n+1}$ of an n-dimensional smooth manifold M^n, if the position vector $x(p)$ is transversal to $x_*(T_pM^n)$ at each point $p \in M^n$, we say that $x : M^n \to \mathbb{R}^{n+1}$ defines a centroaffine hypersurface with centroaffine normalization. A centroaffine hypersurface is associated with the centroaffine metric h, a difference tensor K of type $(1, 2)$, and a Tchebychev vector field defined by $T := \frac{1}{n}\text{trace}_h K$. As is well-known, (h, K) are centroaffine invariants which determine x up to centroaffine transformations of $\mathbb{R}^{n+1}$. In this paper, we study locally strongly convex centroaffine hypersurfaces, i.e., the centroaffine metric h is definite.

The centroaffine normalization of every centroaffine hypersurface induces the identity as the Weingarten operator, from the point of view of relative differential geometry any centroaffine hypersurface is a relative hypersphere (see [20], Sects. 6.3 and 7.2). Thus, in centroaffine differential geometry the usually induced Weingarten operator contains no further geometric information. In 1994, Wang [24] made a significant contribution by reasonably introducing the operator $\mathcal{T} := \hat{\nabla} T$ as the *centroaffine shape operator* of the centroaffine hypersurface $x : M^n \to \mathbb{R}^{n+1}$, where $\hat{\nabla}$ denotes the Levi-Civita connection with respect to the centroaffine metric. Then, following [17], a centroaffine hypersurface is called a *Tchebychev hypersurface* if the operator $\mathcal{T}$ is proportional to the identity isomorphism $\mathrm{id} : TM^n \to TM^n$, i.e., $\mathcal{T} = H \cdot \mathrm{id}$. Here, generally $H := (\mathrm{div}\, T)/n$ is called the *centroaffine mean curvature* of M^n.

As the *centroaffine totally umbilical hypersurfaces*, Tchebychev hypersurfaces naturally generalize the notion of affine hyperspheres in equiaffine differential geometry. Indeed, it is known that both Tchebychev hypersurfaces and affine hyperspheres have exactly the similar structure equations (cf. e.g. [4,7,24]). Because of such nice similarity, the Tchebychev hypersurfaces have been studied extensively. For references, we refer to [1,5,6,9,11,13,15,17,18,21]. Related to this similarity, we would highlight the recent result of Cheng-Hu-Vrancken [5] which proved that the ellipsoids are the only centroaffine Tchebychev hyperovaloids. Thus, after several preceding partial results of [6,15,17], it finally solves the longstanding problem of generalizing the well-known theorem of Blaschke and Deicke (cf. Theorem 3.35 in [14]) on affine hyperspheres in equiaffine differential geometry to that in centroaffine differential geometry.

Another interesting problem is the classification of non-degenerate affine hyperspheres with constant sectional curvature, that had been solved by Li-Penn [12] and Vrancken-Li-Simon [23] in the locally strongly convex case, and finally solved by Vrancken [22] in case of non-degenerate affine hyperspheres with nonzero Pick invariant. Motivated by the above mentioned similarity, we have the following natural and interesting problem:

Problem *Classify all non-degenerate centroaffine Tchebychev hypersurfaces in $\mathbb{R}^{n+1}$ with constant sectional curvature.*

Concerning this problem, there are only some partial results. As an early result, the flat centroaffine Tchebychev surfaces have been classified by Liu and Wang (see Theorem 4.2 of [17]). For arbitrary dimensions, Li and Wang [16] presented a classification of the canonical centroaffine hypersurfaces with $N(h) \leq 1$, where $N(h)$ denotes the dimension of the maximal negative definite subspace of the centroaffine metric h induced by x. Here, a centroaffine hypersurface in $\mathbb{R}^{n+1}$ is said to be *canonical* if it has flat centroaffine metric and parallel cubic form. Then, Cheng-Hu-Moruz [3] (Corollary 1.1 therein) classified all the locally strongly convex (i.e., $N(h) = 0, n$ cases) canonical centroaffine hypersurfaces in $\mathbb{R}^{n+1}$. Recently, Lalléchère et al [13] proved that flat hyperbolic centroaffine Tchebychev hypersurfaces in $\mathbb{R}^4$ must be canonical

ones. Then, applying the result of [16], one can obtain a classification of all flat hyperbolic centroaffine Tchebychev hypersurfaces in $\mathbb{R}^4$.

In this paper, we focus on the above problem for locally strongly convex centroaffine Tchebychev hypersurfaces. First, motivated by Liu and Wang's classification theorem for all centroaffine surfaces with vanishing centroaffine shape operator (see Theorem 4.1 in [17]), we prove the following theorem for higher dimensions:

Theorem 1.1. *Let $x : M^n \to \mathbb{R}^{n+1}$ $(n \geq 3)$ be a locally strongly convex centroaffine hypersurface with constant sectional curvature and vanishing centroaffine shape operator. Then, $x(M^n)$ is locally centroaffinely equivalent to one of the following hypersurfaces:*

(i) *a hyperquadric of proper affine hyperspheres centered at the origin of $\mathbb{R}^{n+1}$;*

(ii) $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n+1}^{\alpha_{n+1}} = 1$, *where either $\alpha_i > 0$ for $1 \leq i \leq n+1$; or $\alpha_j > 0$ for $2 \leq j \leq n+1$ and $\alpha_1 + \cdots + \alpha_{n+1} < 0$;*

(iii) $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n-1}^{\alpha_{n-1}} (x_n^2 + x_{n+1}^2)^{\alpha_n} \exp(\alpha_{n+1} \arctan \frac{x_n}{x_{n+1}}) = 1$, *where $\alpha_i < 0$ for $1 \leq i \leq n-1$ and $\alpha_1 + \cdots + \alpha_{n-1} + 2\alpha_n > 0$;*

(iv) $x_{n+1} = \frac{1}{2x_1}(x_2^2 + \cdots + x_{v-1}^2) - x_1(-\ln x_1 + \alpha_v \ln x_v + \cdots + \alpha_n \ln x_n)$, *where $2 \leq v \leq n+1$, $\alpha_i > 0$ for $v \leq i \leq n$, and $\alpha_v + \cdots + \alpha_n < 1$.*

Remark 1.1. Theorem 1.1 partially generalizes Theorem 1.3 of Li and Wang [16]. Moreover, by Corollary 1.1 of Cheng-Hu-Moruz [3], the examples (ii), (iii) and (iv) exhaust all locally strongly convex canonical centroaffine hypersurfaces of $\mathbb{R}^{n+1}$. The detailed calculations of these canonical hypersurfaces shall be given in Sect. 3 below.

Next, by definition, a hyperbolic centroaffine hypersurface is locally strongly convex with $N(h) = 0$. Then, motivated by that Liu and Wang [17] have classified all flat centroaffine Tchebychev surfaces in $\mathbb{R}^3$ and Lalléchère et al [13] have classified all flat hyperbolic centroaffine Tchebychev hypersurfaces in $\mathbb{R}^4$, as the second main result of this paper, we prove the following classification theorem for all flat hyperbolic centroaffine Tchebychev hypersurfaces in $\mathbb{R}^5$.

Theorem 1.2. *Let $x : M^4 \to \mathbb{R}^5$ be a flat hyperbolic centroaffine Tchebychev hypersurface. Then, $x(M^4)$ is centroaffinely equivalent to the hypersurfaces defined by:*

$$x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4} x_5^{\alpha_5} = 1,$$

where $\alpha_i > 0$ for $1 \leq i \leq 5$.

Remark 1.2. We would mention that the statement of Theorem 1 in Lalléchère et al [13] is not correct. Indeed, by Examples 1.1–1.2 and Theorem 1.3 in [16], all the hypersurfaces described in Theorem 1 of [13] except $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4} = 1$ with $\alpha_i > 0$ for $1 \leq i \leq 4$ have the property $N(h) = 1$.

This paper is organized as follows. In Sect. 2, we first review the elementary facts of centroaffine hypersurfaces in $\mathbb{R}^{n+1}$, together with recalling the general property of the Tchebychev hypersurfaces. In Sect. 3, for better understanding of our results, we give the detailed calculations of the canonical centroaffine hypersurfaces described in Theorem 1.1. In Sect. 4, we present a proof of Theorem 1.1. In Sect. 5, we construct a typical orthonormal frame on the flat hyperbolic centroaffine hypersurfaces in $\mathbb{R}^{n+1}$. Then, by applying the Codazzi equations and the Tchebychev condition with respect to the abovementioned typical orthonormal frame, we derive information of the connection coefficients and the difference tensor in Sects. 6 and 7, respectively. Finally, in Sect. 8, we complete the proof of Theorem 1.2.

2. Preliminaries

In this section, we briefly review some basic notions and facts about centroaffine hypersurfaces in $\mathbb{R}^{n+1}$. For more details, we refer to the monographs [14,19,20] and the references [8,10,24].

Let $x : M^n \to \mathbb{R}^{n+1}$ ($n \geq 2$) be a locally strongly convex immersed centroaffine hypersurface as stated in the Introduction section. Then, for any vector fields X and Y tangent to M^n, we have the centroaffine formula of Gauss:

$$D_X x_*(Y) = x_*(\nabla_X Y) + h(X,Y)(-\varepsilon x), \tag{2.1}$$

where $\varepsilon = \pm 1$ is chosen such that h is positive definite. Associated with (2.1), $-\varepsilon x$ is called the *centroaffine normal*, ∇ and h are called the *induced (centroaffine) connection* and *centroaffine metric* induced by $-\varepsilon x$, respectively. The centroaffine hypersurface is called elliptic type (resp. hyperbolic type) if $\varepsilon = 1$ (resp. $\varepsilon = -1$). For the geometric interpretation about the type of hypersurfaces, see Sect. 2 of [10] for more comments. However, we note that Li and Wang [16] fixed x to be the centroaffine normal and thus the centroaffine metric is negative definite for elliptic type centroaffine hypersurfaces.

Denote by $\hat{\nabla}$ the Levi-Civita connection with respect to the centroaffine metric h. Then the tensor K, defined by $K(X,Y) := K_X Y := \nabla_X Y - \hat{\nabla}_X Y$, is called *difference tensor* of the centroaffine hypersurface. K is symmetric as both connections ∇ and $\hat{\nabla}$ are torsion free. We also have a totally symmetric $(0,3)$-type tensor $C := \nabla h$, called the *cubic form* of the centroaffine hypersurface. Moreover, K and C are related by

$$C(X,Y,Z) = -2h(K(X,Y),Z), \quad \forall X,Y,Z \in TM^n. \tag{2.2}$$

Then, associated to a centroaffine hypersurface $x : M^n \to \mathbb{R}^{n+1}$, we can define the *Tchebychev form* $T^\flat$ and the *Tchebychev vector field* T in implicit form by

$$T^\flat(X) = \tfrac{1}{n}\mathrm{trace}_h(K_X), \quad h(T,X) = T^\flat(X), \quad \forall X \in TM^n. \tag{2.3}$$

It should be pointed out that if T vanishes, or equivalently $\text{trace}_h(K_X) = 0$ for any tangent vector X, then $x : M^n \to \mathbb{R}^{n+1}$ reduces to be the so-called *proper* affine hypersphere centered at the origin of $\mathbb{R}^{n+1}$ (cf. p.279 of [14] or Sect. 1.15.2–1.15.3 therein).

For a centroaffine hypersurface $x : M^n \to \mathbb{R}^{n+1}$, the covariant differentiation $\hat{\nabla}T$ is of deep meaning. Indeed, Wang [24] showed that

$$h(\hat{\nabla}_X T, Y) = h(X, \hat{\nabla}_Y T), \tag{2.4}$$

which implies that the Tchebychev vector field T is a closed vector field, in the sense that the Tchebychev form $T^\flat$ is a closed form. Furthermore, Wang [24] considered the homomorphism mapping $\mathcal{T} : TM^n \to TM^n$, defined by

$$\mathcal{T}(X) := \hat{\nabla}_X T, \quad \forall X \in TM^n. \tag{2.5}$$

In [17], $\mathcal{T}$ is called the *Tchebychev operator* of $x : M^n \to \mathbb{R}^{n+1}$. Thus, (2.4) implies that the Tchebychev operator $\mathcal{T}$ is self-adjoint with respect to the centroaffine metric h. It is worth mentioning that the geometric properties derived by Wang [24] allow him to define $\mathcal{T}$ as the *centroaffine shape operator* of $x : M^n \to \mathbb{R}^{n+1}$, and therefore, the well-defined function $H := (\text{trace}_h \mathcal{T})/n = (\text{div}\, T)/n$ is called the *centroaffine mean curvature* of $x : M^n \to \mathbb{R}^{n+1}$.

Denote by $\hat{R}$ the Riemannian curvature tensor of the centroaffine metric h. Then the equations of Gauss and Codazzi are given by, respectively,

$$\hat{R}(X,Y)Z = \varepsilon(h(Y,Z)X - h(X,Z)Y) - [K_X, K_Y]Z, \tag{2.6}$$

$$(\hat{\nabla}_Z K)(X,Y) = (\hat{\nabla}_X K)(Z,Y). \tag{2.7}$$

Next, we choose an h-orthonormal tangential frame field $\{e_1, \ldots, e_n\}$ with dual frame field $\{\omega^1, \ldots, \omega^n\}$ and Levi-Civita connection forms $\{\omega_i^j\}$. We shall use the convention of indices: $1 \le i, j, k, l, m, p, q \le n$. We denote by $K_{ij}^k = h(K_{e_i} e_j, e_k)$ and $T^i = \frac{1}{n} \sum_j K_{jj}^i$ the components of K and T; and by $K_{ij,l}^k$ and $T_{,i}^j$ the components of the covariant differentiation $\hat{\nabla}K$ and $\hat{\nabla}T$ with respect to $\{e_i\}$, respectively, defined by

$$\sum_l K_{ij,l}^k \omega^l = dK_{ij}^k + \sum_l K_{lj}^k \omega_l^i + \sum_l K_{il}^k \omega_l^j + \sum_l K_{ij}^l \omega_l^k,$$

$$\sum_i T_{,i}^j \omega^i = dT^j + \sum_i T^i \omega_i^j.$$

Denote by R_{ijkl} the components of the Riemannian curvature tensor of the centroaffine metric h. Then, (2.6) and (2.7) can be written as:

$$R_{ijkl} = \varepsilon(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_m (K_{il}^m K_{jk}^m - K_{ik}^m K_{jl}^m), \tag{2.8}$$

$$K_{ij,l}^k = K_{il,j}^k. \tag{2.9}$$

Taking the contraction of the Gauss equation (2.8) twice, we derive that

$$R_{ij} = \varepsilon(n-1)\delta_{ij} + \sum_{m,k} K_{ik}^m K_{jk}^m - n\sum_m T^m K_{ij}^m, \qquad (2.10)$$

$$n(n-1)\kappa = R = n(n-1)(J+\varepsilon) - n^2\|T\|^2, \qquad (2.11)$$

where R_{ij} and R denote the components of the Ricci tensor and the scalar curvature of the centroaffine metric h, respectively, and that

$$\|T\|^2 := \sum_i (T^i)^2, \quad J := \tfrac{1}{n(n-1)} \sum_{i,j,k} (K_{ij}^k)^2.$$

As usual, J and κ are called *the centroaffine Pick invariant* and *the normalized scalar curvature* of the centroaffine hypersurface, respectively.

Similarly, the second covariant derivative $K_{ij,lp}^q$ of K is defined by

$$\sum_p K_{ij,lp}^q \omega^p = dK_{ij,l}^q + \sum_p K_{pj,l}^q \omega_p^i$$

$$+ \sum_p K_{ip,l}^q \omega_p^j + \sum_p K_{ij,p}^q \omega_p^l + \sum_p K_{ij,l}^p \omega_p^q. \qquad (2.12)$$

Then, we have the Ricci identity

$$K_{il,jp}^q = K_{il,pj}^q + \sum_m K_{ml}^q R_{mijp} + \sum_m K_{im}^q R_{mljp} + \sum_m K_{il}^m R_{mqjp}. \ (2.13)$$

As stated in the first section, a centroaffine hypersurface is called Tchebychev if its centroaffine shape operator satisfies $\mathcal{T} = H\cdot\mathrm{id}$. As an important subclass of centroaffine hypersurfaces that we have introduced earlier, Tchebychev hypersurfaces have remarkable properties. For instance, using the assumption $\mathcal{T} = H \cdot \mathrm{id}$, the equality $h(\hat{R}(X,T)T,T) = 0$ and the relation

$$(\hat{\nabla}_X \mathcal{T})(Y) - (\hat{\nabla}_Y \mathcal{T})(X) = \hat{R}(X,Y)T, \qquad (2.14)$$

we derive the following formulas:

$$\hat{\nabla}\|T\|^2 = 2HT, \quad \|T\|^2\hat{\nabla}H = T(H)T, \qquad (2.15)$$

where $\hat{\nabla}$ denotes the gradient operator with respect to the centroaffine metric h. In particular, from the second formula in (2.14) and (2.15), we further obtain that

$$\|T\|^2 \hat{R}(X,Y)T = -T(H)(h(T,Y)X - h(T,X)Y). \qquad (2.16)$$

3. Computations on Canonical Centroaffine Hypersurfaces

In [16], Li and Wang have made some calculations about the canonical centroaffine hypersurfaces in $\mathbb{R}^{n+1}$. In this section, to get the exact knowledge about locally strongly convex canonical centroaffine hypersurfaces, we compute their centroaffine invariants with more details. For convenience of the

calculations, we begin with fixing x to be the centroaffine normal and finally we decide if it is of hyperbolic type or elliptic type.

Example 3.1. Given $\alpha = (\alpha_1, \ldots, \alpha_{n+1}) \in \mathbb{R}^{n+1}$ that satisfies either (1) $\alpha_i > 0$ for $1 \leq i \leq n+1$; or (2) $\alpha_j > 0$ for $2 \leq j \leq n+1$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_{n+1} < 0$, we define

$$M_\alpha^{(1)} = \left\{ x \in \mathbb{R}^{n+1} \,|\, x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n+1}^{\alpha_{n+1}} = 1 \right\}. \tag{3.1}$$

Claim 3.1. $M_\alpha^{(1)}$ is a locally strongly convex canonical centroaffine hypersurface in $\mathbb{R}^{n+1}$. Moreover, $M_\alpha^{(1)}$ is of hyperbolic type in case (1), and of elliptic type in case (2).

Proof of Claim 3.1. Put $\beta_j = -\frac{\alpha_j}{\alpha_1}$, $2 \leq j \leq n+1$. Then, $M_\alpha^{(1)}$ can be rewritten as

$$x_2^{\beta_2} x_3^{\beta_3} \cdots x_{n+1}^{\beta_{n+1}} = x_1, \tag{3.2}$$

where, in case (1) we have $\beta_j < 0$ for $2 \leq j \leq n+1$; whereas in case (2) we have $\beta_j > 0$ for $2 \leq j \leq n+1$ and $\beta_2 + \cdots + \beta_{n+1} < 1$.

Now, taking local coordinates $(u_2, \ldots, u_{n+1})$ of $M_\alpha^{(1)}$ such that

$$x = (x_1, x_2, \ldots, x_{n+1}) := (e^{\beta_2 u_2 + \beta_3 u_3 + \cdots + \beta_{n+1} u_{n+1}}, e^{u_2}, \ldots, e^{u_{n+1}}),$$

we have

$$x_{u_j} = (\beta_j e^{\beta_2 u_2 + \cdots + \beta_{n+1} u_{n+1}}, 0, \ldots, 0, e^{u_j}, 0, \ldots, 0), \quad 2 \leq j \leq n,$$

$$x_{u_j u_k} = (\beta_j \beta_k e^{\beta_2 u_2 + \cdots + \beta_{n+1} u_{n+1}}, 0, \ldots, 0, \delta_{jk} e^{u_j}, 0, \ldots, 0), \quad 2 \leq j, k \leq n.$$

It follows that

$$\left[x_{u_2}, \ldots, x_{u_{n+1}}, x_{u_j u_k} \right] = (-1)^{n+2} (\beta_j \beta_k - \delta_{jk} \beta_j) e^{(1+\beta_2) u_2 + \cdots + (1+\beta_{n+1}) u_{n+1}},$$

$$\left[x_{u_2}, \ldots, x_{u_{n+1}}, x \right] = (-1)^{n+2} \left(1 - \sum_{i=2}^{n+1} \beta_i \right) e^{(1+\beta_2) u_2 + \cdots + (1+\beta_{n+1}) u_{n+1}},$$

where and later, $[\cdot]$ denotes the standard determinant in $\mathbb{R}^{n+1}$.

By using $x_{u_i u_j} = \sum_{k=2}^{n+1} \Gamma_{ij}^k x_{u_k} + h_{ij} x$, we have

$$h_{ij} = \frac{\left[x_{u_2}, x_{u_3}, \ldots, x_{u_{n+1}}, x_{u_i u_j} \right]}{\left[x_{u_2}, x_{u_3}, \ldots, x_{u_{n+1}}, x \right]} = \frac{\beta_i \beta_j - \delta_{ij} \beta_j}{1 - \sum_{k=2}^{n+1} \beta_k}, \quad 2 \leq i, j \leq n+1. \tag{3.3}$$

Thus, the centroaffine metric h is flat. Moreover, from the fact

$$\left(1 - \sum_{k=2}^{n+1} \beta_k \right) \sum_{i,j=2}^{n+1} h_{ij} z_i z_j = \left(\sum_{i=2}^{n+1} \beta_i z_i \right)^2 - \sum_{i=2}^{n+1} \beta_i z_i^2$$

$$= -\left(1 - \sum_{k=2}^{n+1} \beta_k \right) \sum_{i=2}^{n+1} \beta_i z_i^2 - \sum_{2 \leq i < j \leq n+1} \beta_i \beta_j (z_i - z_j)^2,$$

we see that the matrix (h_{ij}) is positive (resp. negative) definite, or equivalently, the hypersurface $M_\alpha^{(1)}$ is hyperbolic (resp. elliptic) in case (1) (resp. case (2)).

Next, by definition and (3.3), the difference tensor of $M_\alpha^{(1)}$ is given by

$$K_{ij}^k = \Gamma_{ij}^k = \frac{[x_{u_2}, \ldots, x_{u_i u_j}, \ldots, x_{u_{n+1}}, x]}{[x_{u_2}, \ldots, x_k, \ldots, x_{u_{n+1}}, x]}, \quad 2 \le i, j, k \le n+1, \quad (3.4)$$

where in the numerator $x_{u_i u_j}$ locates in the same position as that of x_k in the denominator. Then, straightforward calculations give that

$$K_{ij}^k = \begin{cases} \dfrac{\beta_i \beta_j}{\sum_{l=2}^{n+1} \beta_l - 1}, & i \ne j; \\[3mm] 1 + \dfrac{\beta_i(\beta_i - 1)}{\sum_{l=2}^{n+1} \beta_l - 1}, & i = j = k; \\[3mm] \dfrac{\beta_i(\beta_i - 1)}{\sum_{l=2}^{n+1} \beta_l - 1}, & i = j \ne k. \end{cases} \quad (3.5)$$

From (3.3) and (3.5), we get $\hat{\nabla} K = 0$. Thus, as claimed, $M_\alpha^{(1)}$ is a locally strongly convex canonical centroaffine hypersurface in $\mathbb{R}^{n+1}$ for both cases (1) and (2). $\qquad\square$

Example 3.2. Given $\alpha = (\alpha_1, \ldots, \alpha_{n+1}) \in \mathbb{R}^{n+1}$ with $\alpha_i < 0$ ($1 \le i \le n-1$) and $\alpha_1 + \cdots + \alpha_{n-1} + 2\alpha_n > 0$, we define

$$M_\alpha^{(2)} = \left\{ x \in \mathbb{R}^{n+1} \mid x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n-1}^{\alpha_{n-1}} (x_n^2 + x_{n+1}^2)^{\alpha_n} \exp(\alpha_{n+1} \arctan \tfrac{x_n}{x_{n+1}}) = 1 \right\}. \quad (3.6)$$

Claim 3.2. $M_\alpha^{(2)}$ is a locally strongly convex canonical centroaffine hypersurface in $\mathbb{R}^{n+1}$ of elliptic type.

Proof of Claim 3.2. Put $\beta_i = -\frac{\alpha_i}{\alpha_1}$ for $2 \le i \le n+1$. Then, $M_\alpha^{(2)}$ can be rewritten as

$$x_2^{\beta_2} \cdots x_{n-1}^{\beta_{n-1}} (x_n^2 + x_{n+1}^2)^{\beta_n} \exp(\beta_{n+1} \arctan \tfrac{x_n}{x_{n+1}}) = x_1,$$

where $\beta_i < 0$ for $2 \le i \le n-1$ and $\beta_2 + \cdots + \beta_{n-1} + 2\beta_n > 1$.

Taking local coordinates $(u_2, \ldots, u_{n+1})$ of $M_\alpha^{(2)}$ such that

$$\begin{cases} x_1 = e^{\beta_2 u_2 + \cdots + \beta_{n-1} u_{n-1} + 2\beta_n u_n + \beta_{n+1} u_{n+1}}, \\ x_2 = e^{u_2}, \quad x_3 = e^{u_3}, \quad \ldots, \quad x_{n-1} = e^{u_{n-1}}, \\ x_n = e^{u_n} \sin u_{n+1}, \quad x_{n+1} = e^{u_n} \cos u_{n+1}, \end{cases}$$

we have, letting $\langle \cdot, \cdot \rangle$ be the usual Euclidean inner product in $\mathbb{R}^{n+1}$,

$$\begin{cases} x_{u_j} = (\beta_j e^{\langle \bar{\beta}, u \rangle}, 0, \ldots, 0, e^{u_j}, 0, \ldots, 0, 0, 0), \quad 2 \le j \le n-1, \\ x_{u_n} = (2\beta_n e^{\langle \bar{\beta}, u \rangle}, 0, \ldots, 0, e^{u_n} \sin u_{n+1}, e^{u_n} \cos u_{n+1}), \\ x_{u_{n+1}} = (\beta_{n+1} e^{\langle \bar{\beta}, u \rangle}, 0, \ldots, 0, e^{u_n} \cos u_{n+1}, -e^{u_n} \sin u_{n+1}), \end{cases}$$

where $\bar{\beta} = (\beta_2, \ldots, \beta_{n-1}, 2\beta_n, \beta_{n+1})$, $u = (u_2, \ldots, u_{n+1})$. Moreover, for $2 \leq i, j \leq n-1$,

$$
\begin{cases}
x_{u_i u_j} = (\beta_i \beta_j e^{\langle \bar{\beta}, u \rangle}, 0, \ldots, 0, \delta_{ij} e^{u_i}, 0, \ldots, 0, 0, 0), \\
x_{u_i u_n} = (2\beta_i \beta_n e^{\langle \bar{\beta}, u \rangle}, 0, \ldots, 0, 0, 0), \quad x_{u_i u_{n+1}} = (\beta_i \beta_{n+1} e^{\langle \bar{\beta}, u \rangle}, 0, \ldots, 0, 0, 0), \\
x_{u_n u_n} = (4\beta_n^2 e^{\langle \bar{\beta}, u \rangle}, 0, \ldots, 0, e^{u_n} \sin u_{n+1}, e^{u_n} \cos u_{n+1}), \\
x_{u_n u_{n+1}} = (2\beta_n \beta_{n+1} e^{\langle \bar{\beta}, u \rangle}, 0, \ldots, 0, e^{u_n} \cos u_{n+1}, -e^{u_n} \sin u_{n+1}), \\
x_{u_{n+1} u_{n+1}} = (\beta_{n+1}^2 e^{\langle \bar{\beta}, u \rangle}, 0, \ldots, 0, -e^{u_n} \sin u_{n+1}, -e^{u_n} \cos u_{n+1}).
\end{cases}
$$

Thus, by $x_{u_i u_j} = \sum_{k=2}^{n+1} \Gamma_{ij}^k x_{u_k} + h_{ij} x$, as in (3.3), direct calculations give that

$$
\begin{cases}
h_{ij} = \dfrac{\beta_i \beta_j - \delta_{ij} \beta_j}{1 - 2\beta_n - \sum_{t=2}^{n-1} \beta_t}, \quad h_{in} = \dfrac{2\beta_i \beta_n}{1 - 2\beta_n - \sum_{t=2}^{n-1} \beta_t}, \\[2mm]
h_{i,n+1} = \dfrac{\beta_i \beta_{n+1}}{1 - 2\beta_n - \sum_{t=2}^{n-1} \beta_t}, \quad h_{nn} = \dfrac{4\beta_n^2 - 2\beta_n}{1 - 2\beta_n - \sum_{t=2}^{n-1} \beta_t}, \\[2mm]
h_{n,n+1} = \dfrac{2\beta_n \beta_{n+1} - \beta_{n+1}}{1 - 2\beta_n - \sum_{t=2}^{n-1} \beta_t}, \quad h_{n+1,n+1} = \dfrac{2\beta_n + \beta_{n+1}^2}{1 - 2\beta_n - \sum_{t=2}^{n-1} \beta_t},
\end{cases}
\tag{3.7}
$$

where $2 \leq i, j \leq n-1$. This implies that the centroaffine metric h is flat.

Moreover, putting $(h_{ij})_{n \times n} = \frac{1}{1 - 2\beta_n - \sum_{t=2}^{n-1} \beta_t} (A_{ij})_{n \times n}$, we can calculate the leading principal minors $\{D_i\}_{1 \leq i \leq n}$ of the matrix $(A_{ij})_{n \times n}$ to obtain:

$$
D_1 = \beta_2^2 - \beta_2 > 0, \quad D_2 = (1 - \beta_2 - \beta_3) \beta_2 \beta_3 > 0, \quad \ldots,
$$

$$
D_{n-2} = (1 - \beta_2 - \cdots - \beta_{n-1}) \prod_{i=2}^{n-1} (-\beta_i) > 0,
$$

$$
D_{n-1} = 2\beta_n (\beta_2 + \cdots + \beta_{n-1} + 2\beta_n - 1) \prod_{i=2}^{n-1} (-\beta_i) > 0,
\tag{3.8}
$$

$$
D_n = (4\beta_n^2 + \beta_{n+1}^2)(\beta_2 + \cdots + \beta_{n-1} + 2\beta_n - 1) \prod_{i=2}^{n-1} (-\beta_i) > 0.
$$

Hence, the matrix $(h_{ij})_{n \times n}$ is negative definite, or equivalently, $M_\alpha^{(2)}$ is of elliptic type.

Finally, direct calculations with the use of (3.4) and (3.7) show that all coefficients K_{ij}^k of the difference tensor are equal to Γ_{ij}^k which are all constant. Then, it follows that $\hat{\nabla} K = 0$ and, as claimed, $M_\alpha^{(2)}$ is a locally strongly convex canonical centroaffine hypersurface in $\mathbb{R}^{n+1}$. $\qquad\square$

Example 3.3. Given $2 \leq v \leq n+1$ and $\alpha(v) = (\alpha_v, \ldots, \alpha_n) \in \mathbb{R}^{n-v+1}$, with $\alpha_k > 0$ for $v \leq k \leq n$ and $\alpha_v + \cdots + \alpha_n < 1$, we define the graph in $\mathbb{R}^{n+1}$ by

$$
M_{\alpha(v)}^{(3)} : \begin{cases}
x_{n+1} = \frac{1}{2x_1} \sum_{i=2}^{v-1} x_i^2 + x_1 \left(\ln x_1 - \sum_{i=v}^{n} \alpha_i \ln x_i \right), & 3 \leq v \leq n; \\[2ex]
x_{n+1} = x_1 \left(\ln x_1 - \sum_{i=2}^{n} \alpha_i \ln x_i \right), & v = 2; \\[2ex]
x_{n+1} = \frac{1}{2x_1} \sum_{i=2}^{n} x_i^2 + x_1 \ln x_1, & v = n+1.
\end{cases}
\tag{3.9}
$$

Claim 3.3. $M_{\alpha(v)}^{(3)}$ with $2 \leq v \leq n+1$ are locally strongly convex canonical centroaffine hypersurfaces in $\mathbb{R}^{n+1}$ of elliptic type.

Proof of Claim 3.3. Because of similarity and simpler, the proof for both $M_{\alpha(n+1)}^{(3)}$ and $M_{\alpha(2)}^{(3)}$ are omitted. Below, we consider the cases $3 \leq v \leq n$. With the parameterization $(u_1, \ldots, u_n)$, we can express $M_{\alpha(v)}^{(3)}$ by

$$
x = \left(e^{u_1}, e^{u_1} u_2, \ldots, e^{u_1} u_{v-1}, e^{u_v}, \ldots, e^{u_n}, e^{u_1} \left(u_1 + \frac{1}{2} \sum_{t=2}^{v-1} u_t^2 - \sum_{r=v}^{n} \alpha_r u_r \right) \right).
$$

Then, as the preceding examples, by direct calculation of x_{u_i} for $1 \leq i \leq n$ and $x_{u_i u_j}$ for $1 \leq i, j \leq n$, together with the equation $x_{u_i u_j} = \sum_{k=1}^{n} \Gamma_{ij}^k x_{u_k} + h_{ij} x$, we can get

$$
\begin{cases}
h_{11} = \dfrac{1}{\sum_{s=v}^{n} \alpha_s - 1}, & h_{1i} = 0, \quad h_{1k} = \dfrac{-\alpha_k}{\sum_{s=v}^{n} \alpha_s - 1}, \\[2ex]
h_{ij} = \dfrac{\delta_{ij}}{\sum_{s=v}^{n} \alpha_s - 1}, & h_{ik} = 0, \quad h_{kl} = \dfrac{\delta_{kl} \alpha_k}{\sum_{s=v}^{n} \alpha_s - 1},
\end{cases}
\tag{3.10}
$$

where $2 \leq i, j \leq v-1$ and $v \leq k, l \leq n$. (3.10) implies that the centroaffine metric h is flat. Moreover, from the fact

$$
\left(\sum_{k=v}^{n} \alpha_k - 1 \right) \sum_{i,j=1}^{n} h_{ij} z_i z_j = \left(1 - \sum_{k=v}^{n} \alpha_k \right) z_1^2
$$

$$
+ \sum_{i=2}^{v-1} z_i^2 + \sum_{k=v}^{n} \alpha_k (z_1 - z_k)^2, \tag{3.11}
$$

we see that the matrix $(h_{ij})_{n \times n}$ is negative definite, or equivalently, the hypersurface $M_{\alpha(v)}^{(3)}$ is of elliptic type.

Moreover, direct calculations show that all $K_{ij}^k = \Gamma_{ij}^k$ are constant, which satisfy:

$$
\begin{cases}
K_{11}^1 = \dfrac{\sum_{s=v}^n \alpha_s - 2}{\sum_{s=v}^n \alpha_s - 1}, \quad K_{11}^j = 0, \quad 2 \le j \le v - 1, \\[3mm]
K_{11}^k = -\dfrac{1}{\sum_{s=v}^n \alpha_s - 1}, \quad v \le k \le n, \\[3mm]
K_{1j}^m = \delta_{jm}, \quad 2 \le j \le v - 1, \quad 1 \le m \le n,
\end{cases}
\tag{3.12}
$$

$$
\begin{cases}
K_{1k}^1 = K_{1k}^m = \dfrac{\alpha_k}{\sum_{s=v}^n \alpha_s - 1}, \quad K_{1k}^j = 0, \quad 2 \le j \le v - 1, \ v \le k, m \le n, \\[3mm]
K_{ij}^1 = K_{ij}^m = -\dfrac{\delta_{ij}}{\sum_{s=v}^n \alpha_s - 1}, \quad K_{ij}^r = 0, \ 2 \le i,j,r \le v-1, \ v \le m \le n, \\[3mm]
K_{ik}^m = 0, \quad 2 \le i \le v - 1, \quad v \le k \le n, \quad 1 \le m \le n, \\[3mm]
K_{kl}^1 = K_{kl}^m = -\dfrac{\alpha_k \delta_{kl}}{\sum_{s=v}^n \alpha_s - 1}, \quad v \le k,l,m \le n, \ m \ne k,l, \\[3mm]
K_{kl}^j = 0, \quad K_{kl}^k = \delta_{kl} - \dfrac{\alpha_k \delta_{kl}}{\sum_{s=v}^n \alpha_s - 1}, \quad 2 \le j \le n-1, \ v \le k,l \le n.
\end{cases}
\tag{3.13}
$$

It follows from (3.10), (3.12) and (3.13) that $\hat{\nabla} K = 0$. Hence, as claimed, $M_{\alpha(v)}^{(3)}$ is a locally strongly convex canonical centroaffine hypersurface in $\mathbb{R}^{n+1}$. □

Before ending this section, we give further remarks concerning the preceding examples.

Remark 3.1. Given an integer $2 \le v \le n$ and real numbers $\{\tilde{\alpha}_i\}_{1 \le i \le v-1}$ with $\tilde{\alpha}_1 < 0$, we have the centroaffine transformation of $\mathbb{R}^{n+1}$

$$
\begin{cases}
x_1 = y_1, \quad x_i = \dfrac{y_i}{\sqrt{-\tilde{\alpha}_1}}, \quad 2 \le i \le v - 1; \quad x_j = y_j, \ v \le j \le n; \\[3mm]
x_{n+1} = -\dfrac{1}{\tilde{\alpha}_1}\left(y_{n+1} + \sum_{i=2}^{v-1} \tilde{\alpha}_i y_i \right),
\end{cases}
\tag{3.14}
$$

under which the locally strongly convex canonical centroaffine hypersurface $M_{\alpha(v)}^{(3)}$, namely

$$
x_{n+1} = \frac{1}{2x_1} \sum_{i=2}^{v-1} x_i^2 - x_1 \left(-\ln x_1 + \sum_{k=v}^n \alpha_k \ln x_k \right),
\tag{3.15}
$$

is changed to be Example 1.2 of Li–Wang [16]:

$$
y_{n+1} = \frac{1}{2y_1} \sum_{i=2}^{v-1} y_i^2 - \sum_{i=2}^{v-1} \tilde{\alpha}_i y_i - y_1 \left(\tilde{\alpha}_1 \ln y_1 + \sum_{j=v}^n \tilde{\alpha}_j \ln y_j \right),
\tag{3.16}
$$

where $\tilde{\alpha}_j = -\tilde{\alpha}_1 \alpha_j, \ v \le j \le n$.

Moreover, corresponding to $\alpha_k > 0$ for $v \le k \le n$ and $\alpha_v + \cdots + \alpha_n < 1$ in (3.15), it holds that $\tilde{\alpha}_k > 0$ for $v \le k \le n$ and $\tilde{\alpha}_1 + \tilde{\alpha}_v + \cdots + \tilde{\alpha}_n < 0$ in (3.16).

In [16], more information for the canonical centroaffine hypersurface (3.16) is given. This includes the conclusion that the canonical centroaffine hypersurfaces (3.16) are of Lorentzian centroaffine metric if $\tilde{\alpha}_j > 0$ for $v \le j \le n$ and $\tilde{\alpha}_1 + \tilde{\alpha}_v + \cdots + \tilde{\alpha}_n > 0$.

Next, we recall Corollary 1.1 of Cheng-Hu-Moruz [3] which shows that the centroaffine hypersurfaces described by the preceding examples are the only locally strongly convex canonical centroaffine hypersurfaces in $\mathbb{R}^{n+1}$.

Theorem 3.1 ([3,16]). *Let $x : M^n \to \mathbb{R}^{n+1}$ ($n \ge 2$) be a locally strongly convex canonical centroaffine hypersurface. Then it is locally centroaffinely equivalent to one of the following hypersurfaces:*

(i) $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n+1}^{\alpha_{n+1}} = 1$, *where either $\alpha_i > 0$ for $1 \le i \le n+1$; or $\alpha_j > 0$ for $2 \le j \le n+1$ and $\alpha_1 + \cdots + \alpha_{n+1} < 0$;*

(ii) $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n-1}^{\alpha_{n-1}} (x_n^2 + x_{n+1}^2)^{\alpha_n} \exp(\alpha_{n+1} \arctan \frac{x_n}{x_{n+1}}) = 1$, *where $\alpha_i < 0$ for $1 \le i \le n-1$ and $\alpha_1 + \cdots + \alpha_{n-1} + 2\alpha_n > 0$;*

(iii) $x_{n+1} = \frac{1}{2x_1}(x_2^2 + \cdots + x_{v-1}^2) - x_1(-\ln x_1 + \alpha_v \ln x_v + \cdots + \alpha_n \ln x_n)$, *where $2 \le v \le n+1$, $\alpha_i > 0$ for $v \le i \le n$ and $\alpha_v + \cdots + \alpha_n < 1$.*

Remark 3.2. Theorem 3.1 complements Theorem 1.1 of [16] by giving all examples with $N(h) = n$. Since a complete proof of Theorem 3.1 is omitted in [3], hereby we would give an outline: By Theorem 6.1 in [3], all locally strongly convex canonical centroaffine surfaces in $\mathbb{R}^3$ are given by (ii) $\sim$ (v) therein. Then, from the conclusions of Theorem 1.1, Propositions 3.1 and 3.2 of [3], we see that all higher dimensional locally strongly convex canonical centroaffine hypersurfaces but that $x_{n+1} = \frac{1}{2x_1}(x_2^2 + \cdots + x_n^2) + x_1 \ln x_1$ can be produced by the Calabi products of either two lower dimensional locally strongly convex canonical centroaffine hypersurfaces, or one lower dimensional locally strongly convex canonical centroaffine hypersurface with a point, defined by (1.3) and (1.4) of [3], respectively, depending on a parameter λ. Then, by using Theorem 1.1 of [3] again, noting that as building blocks of the Calabi products, the standard immersions of the symmetric spaces $\mathbf{SL}(m, \mathbb{R})/\mathbf{SO}(m)$, $\mathbf{SL}(m, \mathbb{C})/\mathbf{SU}(m)$, $\mathbf{SU}^*(2m)/\mathbf{Sp}(m)$ and $\mathbf{E}_{6(-26)}/\mathbf{F}_4$ are not of flat centroaffine metric (cf. [2]), we can finally achieve the conclusion of Theorem 3.1.

4. Proof of Theorem 1.1

First of all, from (2.15) and (2.16), we immediately get the following lemma.

Lemma 4.1. *Let $x : M^n \to \mathbb{R}^{n+1}$ be a locally strongly convex centroaffine Tchebychev hypersurface with constant sectional curvature c. Then it holds that*

$$\|\hat{\nabla} H\|^2 = c^2 \|T\|^2, \tag{4.1}$$

where $\hat{\nabla}$ and $\|\cdot\|$ denote the gradient operator and the tensorial norm with respect to the centroaffine metric h, respectively. In particular, if the centroaffine metric is flat, i.e., $c = 0$, the centroaffine mean curvature H is a constant.

Next, before completing the proof of Theorem 1.1, with the notations of the second section and the indices convention, we are going to calculate the Laplacian ΔJ of the centroaffine Pick invariant J.

Lemma 4.2. *Let $x : M^n \to \mathbb{R}^{n+1}$ be a locally strongly convex centroaffine hypersurface. Then, with respect to a local h-orthonormal frame $\{e_1, \ldots, e_n\}$ of M^n, it holds that*

$$\begin{aligned}
\tfrac{1}{2} n(n-1)\Delta J &= \sum_{i,j,k,l} (K_{ij,l}^k)^2 + \sum_{i,j,k,l} (R_{ijkl})^2 + \sum_{i,j} (R_{ij})^2 - \varepsilon(n+1)R \\
&\quad + n \sum_{i,j,k} K_{ij}^k R_{ij} T^k + n \sum_{i,j,k} K_{ij}^k T_{,ij}^k.
\end{aligned} \tag{4.2}$$

Proof. By definition, we have

$$\tfrac{1}{2} n(n-1)\Delta J = \tfrac{1}{2}\Delta\left(\sum_{i,j,k}(K_{ij}^k)^2\right) = \sum_{i,j,k,l}(K_{ij,l}^k)^2 + \sum_{i,j,k,l} K_{ij}^k K_{ij,ll}^k. \tag{4.3}$$

By using (2.9) and (2.13), we have the calculations

$$\begin{aligned}
\sum_{i,j,k,l} K_{ij}^k K_{ij,ll}^k &= \sum_{i,j,k,l} K_{ij}^k K_{il,lj}^k + \sum_{i,j,k,l,m} K_{ij}^k K_{ml}^k R_{mijl} \\
&\quad + \sum_{i,j,k,l,m} K_{ij}^k K_{im}^k R_{mljl} + \sum_{i,j,k,l,m} K_{ij}^k K_{il}^m R_{mkjl} \\
&= n \sum_{i,j,k} K_{ij}^k T_{,ij}^k + \sum_{i,j,k,l,m} K_{ij}^k K_{ml}^k R_{mijl} \\
&\quad + \sum_{i,j,k,m} K_{ij}^k K_{im}^k R_{mj} + \sum_{i,j,k,l,m} K_{ij}^k K_{il}^m R_{mkjl}.
\end{aligned} \tag{4.4}$$

Now, from the fact

$$\begin{aligned}
\sum_{i,j,k,l,m} K_{ij}^k K_{il}^m R_{mkjl} &= -\sum_{i,j,k,l,m} K_{ij}^k K_{il}^m R_{kmjl} \\
&= -\sum_{i,j,k,l,m} K_{mj}^k K_{il}^k R_{mijl},
\end{aligned} \tag{4.5}$$

and (2.8), we obtain

$$
\begin{aligned}
&\sum_{i,j,k,l,m} K_{ij}^k K_{ml}^k R_{mijl} + \sum_{i,j,k,l,m} K_{ij}^k K_{il}^m R_{mkjl} \\
&= \sum_{i,j,k,l,m} (K_{ij}^k K_{ml}^k - K_{il}^k K_{jm}^k) R_{mijl} = \sum_{i,j,l,m} (R_{imjl})^2 - 2\varepsilon R.
\end{aligned}
\tag{4.6}
$$

Moreover, by using (2.10), we get

$$
\sum_{i,j,k,m} K_{ij}^k K_{im}^k R_{mj} = \sum_{i,j} (R_{ij})^2 + n \sum_{i,j,k} K_{ij}^k R_{ij} T^k - \varepsilon(n-1)R.
\tag{4.7}
$$

From the above calculations, we conclude that

$$
\begin{aligned}
\sum_{i,j,k,l} K_{ij}^k K_{ij,ll}^k &= \sum_{i,j,l,m} (R_{imjl})^2 + \sum_{i,j} (R_{ij})^2 - \varepsilon(n+1)R \\
&+ n \sum_{i,j,k} K_{ij}^k R_{ij} T^k + n \sum_{i,j,k} K_{ij}^k T_{,ij}^k.
\end{aligned}
\tag{4.8}
$$

The assertion of the lemma then follows from (4.3) and (4.8). $\qquad\square$

Completion of the Proof of Theorem 1.1.

Let $x : M^n \to \mathbb{R}^{n+1}$ be a locally strongly convex centroaffine hypersurface with constant sectional curvature c and vanishing centroaffine shape operator $\mathcal{T} = \hat{\nabla} T = 0$. Thus, $H = 0$ and $\|T\|$ is a constant.

If $T = 0$, then x is an equiaffine normal, thus $x : M^n \to \mathbb{R}^{n+1}$ is a proper affine hypersphere centered at the origin $O \in \mathbb{R}^{n+1}$. By the Main theorem of Vrancken-Li-Simon [23], we conclude that $x(M^n)$ is centroaffinely equivalent to either a hyperquadric of proper affine hyperspheres centered at the origin or the hypersurface $x_1 x_2 \cdots x_{n+1} = 1$ in $\mathbb{R}^{n+1}$.

If $T \neq 0$, it is seen from Lemma 4.1 and the fact $H = 0$ that $c = 0$, i.e., (M^n, h) is flat. It follows from (2.11) that J is a constant, and applying Lemma 4.2, we obtain

$$
0 = \tfrac{1}{2} n(n-1) \Delta J = \sum_{i,j,k,l} (K_{ij,l}^k)^2 + n \sum_{i,j,k} K_{ij}^k T_{,ij}^k = \sum_{i,j,k,l} (K_{ij,l}^k)^2,
$$

and therefore, $x : M^n \to \mathbb{R}^{n+1}$ is of parallel difference tensor.

Hence, $x : M^n \to \mathbb{R}^{n+1}$ is a locally strongly convex canonical centroaffine hypersurface. Now, combining with Theorem 3.1, we have completed the proof of Theorem 1.1. $\qquad\square$

5. Construction of a Typical h-orthonormal Frame

From now on, we assume that $x : M^n \to \mathbb{R}^{n+1}$ is a flat hyperbolic centroaffine hypersurface. Then, from (2.11), for any point $p \in M^n$, we have $K(p) \neq 0$.

In this section, we will construct a typical h-orthonormal frame field on M^n that should be of independent meaning. First of all, we fix an arbitrary $p \in M^n$ and construct a canonical basis of $T_p M^n$.

Lemma 5.1. *Let $x : M^n \to \mathbb{R}^{n+1}$ $(n \geq 2)$ be a flat hyperbolic centroaffine hypersurface. Then, at any $p \in M^n$, there exists an h-orthonormal basis $\{e_i\}_{i=1}^n$ of $T_p M^n$ such that the difference tensor K takes the following form at p:*

$$\begin{cases} K_{e_1} e_1 = \lambda_1 e_1, \quad K_{e_1} e_k = \mu_1 e_k, \\ K_{e_k} e_k = \mu_1 e_1 + \cdots + \mu_{k-1} e_{k-1} + \lambda_k e_k, \\ K_{e_i} e_j = \mu_i e_j, \quad 2 \leq i < j \leq n, \quad 2 \leq k \leq n, \end{cases} \tag{5.1}$$

where the coefficients satisfy the following relations:

$$\begin{cases} \mu_1 = \frac{1}{2}\left(\lambda_1 - \sqrt{\lambda_1^2 + 4}\right), \quad \lambda_k > 0, \quad \mu_k \neq 0, \quad 1 \leq k \leq n-1, \\ \mu_\ell = \frac{1}{2}\left(\lambda_\ell - \sqrt{\lambda_\ell^2 + 4(1 + \mu_1^2 + \cdots + \mu_{\ell-1}^2)}\right), \quad 2 \leq \ell \leq n-1. \end{cases} \tag{5.2}$$

Proof. The argument is now standard, for readers not familiar are referred to [4,23].

Denote by $U_p^1 M^n = \{u \in T_p M^n \,|\, h(u,u) = 1\}$ which is a compact set. Define a continuous function $f^1(u) = h(K_u u, u)$ on $U_p^1 M^n$. Since $K(p) \neq 0$, the function f^1 attains an absolute maximum at some $e_1 \in U_p^1 M^n$ such that $\lambda_1 := f^1(e_1) > 0$. This implies that $K_{e_1} e_1 = \lambda_1 e_1$. Then, considering the self-adjoint map $K_{e_1} : T_p M^n \to T_p M^n$, we obtain an h-orthonormal basis $\{e_i\}_{i=1}^n$ of $T_p M^n$, consisting of the eigenvectors of K_{e_1} with associated eigenvalues λ_1 and λ_{1j} $(2 \leq j \leq n)$ satisfying

$$K_{e_1} e_1 = \lambda_1 e_1, \quad K_{e_1} e_j = \lambda_{1j} e_j, \quad \lambda_1 \geq 2\lambda_{1j}, \quad 2 \leq j \leq n.$$

Then, taking in (2.6) $X = e_1, Y = Z = e_j$ for $j \geq 2$, as $\varepsilon = -1$, we can get

$$\lambda_{1j}^2 - \lambda_{1j}\lambda_1 - 1 = 0, \quad 2 \leq j \leq n.$$

Given that $\lambda_1 \geq 2\lambda_{1j}$, we have $\lambda_{12} = \cdots = \lambda_{1n} = \frac{1}{2}\left(\lambda_1 - \sqrt{\lambda_1^2 + 4}\right) =: \mu_1 \neq 0$.

Next, denote $T_p^2 M^n = \text{Span}\{e_2, \ldots, e_n\}$ and $U_p^2 M^n = \{u \in T_p^2 M^n \,|\, h(u,u) = 1\}$. Then, the function $f^2(u) = h(K_u u, u)$ defined on $U_p^2 M^n$ attains at an element $e_2 \in U_p^2 M^n$ so that the absolute maximum $\lambda_2 := f^2(e_2) \geq 0$. This implies that $K_{e_2} e_2 = \mu_1 e_1 + \lambda_2 e_2$.

Consider now the self-adjoint operator $\mathfrak{B}_2 : T_p^2 M^n \to T_p^2 M^n$ defined by

$$\mathfrak{B}_2 X = K_{e_2} X - h(K_{e_2} X, e_1) e_1, \quad X \in T_p^2 M^n.$$

We can choose an h-orthonormal basis of $\{e_1, e_2\}^\perp$, still denoted $\{e_i\}_{i=3}^n$, consisting of eigenvectors of $\mathfrak{B}_2$ with associated eigenvalues $\{\lambda_{2i}\}_{i=3}^n$ satisfying

$$K_{e_2} e_2 = \mu_1 e_1 + \lambda_2 e_2, \quad K_{e_2} e_j = \lambda_{2j} e_j, \quad 3 \leq j \leq n, \quad \lambda_2 \geq 2\lambda_{2j}, \quad 2 \leq j \leq n.$$

Then, taking in (2.6) $X = e_2, Y = Z = e_j$ for $j \geq 3$, we get

$$\lambda_{2j}^2 - \lambda_{2j}\lambda_2 - (1 + \mu_1^2) = 0, \quad 3 \leq j \leq n,$$

which implies $\lambda_2 > 0$ (if otherwise $\lambda_2 = 0$, then $\lambda_{2j} = h(K_{e_2}e_j, e_j) = 0$ which gives a contradiction to the above equation). Therefore, as $\lambda_2 \geq 2\lambda_{2j}$, the above equation implies that $\lambda_{23} = \cdots = \lambda_{2n} = \frac{1}{2}\left(\lambda_2 - \sqrt{\lambda_2^2 + 4(1 + \mu_1^2)}\right) =: \mu_2 \neq 0$.

Repeating the above procedure inductively, we then construct an h-orthonormal basis $\{e_i\}_{i=1}^n$ of T_pM^n for which K takes the form (5.1) with coefficients satisfying (5.2). □

Next, we are going to extend the above h-orthonormal basis $\{e_i\}_{i=1}^n$ of T_pM^n to a local h-orthonormal frame field $\{E_i\}_{i=1}^n$ on a neighbourhood U of $p \in M^n$.

Lemma 5.2. *Let $x : M^n \to \mathbb{R}^{n+1}$ be a flat hyperbolic centroaffine hypersurface. For $p \in M^n$, there exists a local h-orthonormal frame field $\{E_i\}_{i=1}^n$ on a neighbourhood U of $p \in M^n$ such that the difference tensor K takes the following form:*

$$\begin{cases} K_{E_1}E_1 = f_1 E_1, \quad K_{E_1}E_k = g_1 E_k, \\ K_{E_k}E_k = g_1 E_1 + \cdots + g_{k-1}E_{k-1} + f_k E_k, \\ K_{E_i}E_j = g_i E_j, \quad 2 \leq i < j \leq n, \quad 2 \leq k \leq n, \end{cases} \tag{5.3}$$

where, the functions $\{f_i, g_k\}$ satisfy the relations:

$$\begin{cases} g_1 = \frac{1}{2}\left(f_1 - \sqrt{f_1^2 + 4}\right), \quad f_k > 0, \quad g_k \neq 0, \quad 1 \leq k \leq n-1, \\ g_\ell = \frac{1}{2}\left(f_\ell - \sqrt{f_\ell^2 + 4(1 + g_1^2 + \cdots + g_{\ell-1}^2)}\right), \quad 2 \leq \ell \leq n-1. \end{cases} \tag{5.4}$$

Proof. We divided the proof process into three steps.

Step 1. *There exists a unit vector field E_1 on some neighborhood U' of p such that*

$$K_{E_1}E_1 = f_1 E_1, \quad K_{E_1}Y = g_1 Y, \quad \forall Y \in \{E_1\}^\perp, \tag{5.5}$$

where $f_1 > 0$, $g_1 = \frac{1}{2}\left(f_1 - \sqrt{f_1^2 + 4}\right)$.

For the h-orthonormal basis $\{e_i\}_{i=1}^n$ of T_pM^n as described in Lemma 5.1, we choose an arbitrary local differentiable h-orthonormal frame field $\{\tilde{E}_1, \ldots, \tilde{E}_n\}$ on a neighborhood $\tilde{U}$ of p such that $\tilde{E}_i(p) = e_i$ for $1 \leq i \leq n$. Then, we define a mapping

$$\varphi : \mathbb{R}^n \times \tilde{U} \to \mathbb{R}^n \quad \text{by} \quad \varphi(a_1, a_2, \ldots, a_n, q) = (b_1, b_2, \ldots, b_n),$$

where

$$b_k := \sum_{i,j=1}^n a_i a_j h(K_{\tilde{E}_i}\tilde{E}_j, \tilde{E}_k) - \lambda_1 a_k, \quad 1 \leq k \leq n, \tag{5.6}$$

are regarded as functions on $\mathbb{R}^n \times \tilde{U} : b_k = b_k(a_1, a_2, \ldots, a_n, q)$.

It is easily checked from (5.1) that $b_k(1, 0, \ldots, 0, p) = 0$ for $1 \leq k \leq n$, and

$$\frac{\partial b_k}{\partial a_j}\Big|_{(1,0,\ldots,0,p)} = \begin{cases} \lambda_1 > 0, & k = j = 1, \\ 2\mu_1 - \lambda_1 \neq 0, & 2 \leq k = j \leq n, \\ 0, & k \neq j. \end{cases} \tag{5.7}$$

Thus, $\left(\frac{\partial b_k}{\partial a_j}\right)$ at the point $(1, 0, \ldots, 0, p) \in \mathbb{R}^n \times \tilde{U}$ is invertible. By the implicit function theorem there exist differentiable functions $\{a_i(q)\}_{1 \leq i \leq n}$, defined on a neighborhood $\tilde{U}' \subset \tilde{U}$ of p such that $a_1(p) = 1$, $a_j(p) = 0$ for $2 \leq j \leq n$, and

$$b_k(a_1(q), a_2(q), \ldots, a_n(q), q) \equiv 0, \quad 1 \leq k \leq n, \quad \forall q \in \tilde{U}'. \tag{5.8}$$

Put $V = \sum_{i=1}^n a_i \tilde{E}_i$. Then $V(p) = e_1$, (5.6) and (5.8) imply that $K_V V = \lambda_1 V$.

Since $\|V(p)\| = \sqrt{h(V(p), V(p))} = 1$, there exists a neighborhood $U' \subset \tilde{U}'$ of p such that $V \neq 0$ on U'. Hence, the unit vector field $E_1 := V/\|V\|$ on U' with $E_1(p) = e_1$ satisfies

$$K_{E_1} E_1 = f_1 E_1, \quad f_1 = \frac{\lambda_1}{\|V\|}. \tag{5.9}$$

(5.9) implies that the distribution $\{E_1\}^\perp$ orthogonal to $\text{Span}\{E_1\}$ is K_{E_1}-invariant on U'. For any $q \in U'$, let $\{X_j\}_{2 \leq j \leq n} \subset \{E_1(q)\}^\perp$ be the orthonormal eigenvectors of $K_{E_1(q)}$ with associated eigenvalues $\{\alpha_j\}_{2 \leq j \leq n}$: $K_{E_1(q)} X_j = \alpha_j X_j$ for $2 \leq j \leq n$. Then, by using $\hat{R}(X_j, E_1(q))E_1(q) = 0$, the Gauss equation (2.6) and $\varepsilon = -1$, we derive

$$\alpha_j^2 - f_1(q)\alpha_j - 1 = 0, \quad 2 \leq j \leq n. \tag{5.10}$$

It follows that $K_{E_1}(q)$ has at most three distinct eigenvalues as below:

$$\frac{1}{2}\left(f_1(q) - \sqrt{f_1^2(q) + 4}\right), \quad f_1(q), \quad \frac{1}{2}\left(f_1(q) + \sqrt{f_1^2(q) + 4}\right).$$

On the other hand, by Lemma 5.1 and the continuity of the eigenvalue functions of K_{E_1}, we see that $\frac{1}{2}\left(f_1(q) + \sqrt{f_1^2(q) + 4}\right)$ can not be attained. We have got the conclusion of Step 1.

Step 2. *There exists a unit vector field $E_2 \in \{E_1\}^\perp$ on some neighborhood $\tilde{U} \subset U'$ of p such that*

$$K_{E_2} E_2 = g_1 E_1 + f_2 E_2, \quad K_{E_2} Y = g_2 Y, \quad \forall Y \in \{E_1, E_2\}^\perp, \tag{5.11}$$

where $g_2 = \frac{1}{2}\left(f_2 - \sqrt{f_2^2 + 4(1 + g_1^2)}\right)$.

Similar to Step 1, we begin with choosing an arbitrarily h-orthonormal vector fields $\{\tilde{E}_2', \ldots, \tilde{E}_n'\}$ of $\{E_1\}^\perp$ on some neighborhood of p, still denoted by U', such that $\tilde{E}_j'(p) = e_j$ for $2 \leq j \leq n$. Then, we define a mapping

$$\tilde{\varphi} : \mathbb{R}^{n-1} \times U' \to \mathbb{R}^{n-1} \quad by \quad \tilde{\varphi}(\tilde{a}_2, \tilde{a}_3, \ldots, \tilde{a}_n, \tilde{q}) = (\tilde{b}_2, \tilde{b}_3, \ldots, \tilde{b}_n),$$

where

$$\tilde{b}_k = \sum_{i,j=2}^{n} \tilde{a}_i \tilde{a}_j h(K_{\tilde{E}_i'} \tilde{E}_j', \tilde{E}_k') - \lambda_2 \tilde{a}_k, \quad 2 \le k \le n, \tag{5.12}$$

are regarded as functions on $\mathbb{R}^{n-1} \times U' : \tilde{b}_k = \tilde{b}_k(\tilde{a}_2, \tilde{a}_3, \ldots, \tilde{a}_n, \tilde{q})$.

As $\tilde{E}_j'(p) = e_j$, from (5.1), we see that $\tilde{b}_k(1, 0, \ldots, 0, p) = 0$ for $2 \le k \le n$ and

$$\frac{\partial \tilde{b}_k}{\partial \tilde{a}_j}\bigg|_{(1,0,\ldots,0,p)} = \begin{cases} \lambda_2 > 0, & k = j = 2, \\ 2\mu_2 - \lambda_2 \ne 0, & 3 \le k = j \le n, \\ 0, & k \ne j. \end{cases} \tag{5.13}$$

Thus, $\left(\frac{\partial \tilde{b}_k}{\partial \tilde{a}_j}\right)$ at the point $(1, 0, \ldots, 0, p) \in \mathbb{R}^{n-1} \times U'$ is invertible. By the implicit function theorem, there exist differentiable functions $\{\tilde{a}_j(\tilde{q})\}_{2 \le j \le n}$ defined on a neighborhood $\tilde{U}' \subset U'$ of p such that $\tilde{a}_2(p) = 1$, $\tilde{a}_r(p) = 0$ for $3 \le r \le n$ and

$$\tilde{b}_k(\tilde{a}_2(\tilde{q}), \ldots, \tilde{a}_n(\tilde{q}), \tilde{q}) \equiv 0, \quad 2 \le k \le n, \quad \forall q \in \tilde{U}'. \tag{5.14}$$

Put $\tilde{V} = \sum_{j=2}^{n} \tilde{a}_j \tilde{E}_j'$. Then $\tilde{V}(p) = e_2$. From (5.5), (5.12) and (5.14) we obtain

$$K_{\tilde{V}} \tilde{V} = \lambda_2 \tilde{V} + g_1 h(\tilde{V}, \tilde{V}) E_1.$$

As $\|\tilde{V}(p)\| = 1$, there exists a neighborhood $\tilde{U} \subset \tilde{U}'$ of p such that $\tilde{V} \ne 0$ on $\tilde{U}$. Then $E_2 := \tilde{V}/\|\tilde{V}\|$ defines a unit vector field over $\tilde{U}$ with $E_2(p) = e_2$ which satisfies

$$K_{E_2} E_2 = g_1 E_1 + f_2 E_2, \quad f_2 = \frac{\lambda_2}{\|\tilde{V}\|}. \tag{5.15}$$

At any point $q \in \tilde{U}$, we then consider $\mathfrak{B}_2(q) : \{E_1(q)\}^{\perp} \to \{E_1(q)\}^{\perp}$, defined by

$$\mathfrak{B}_2(q)X = K_{E_2(q)}X - h(K_{E_2(q)}X, E_1(q))E_1(q), \quad X \in \{E_1(q)\}^{\perp},$$

which is a self-adjoint transformation. Besides E_2, let $\tilde{X}_r$ be the other unit eigenvectors of $\mathfrak{B}_2(q)$ corresponding to eigenvalues $\tilde{\alpha}_r$ $(3 \le r \le n)$. Then, it holds at q:

$$K_{E_2} E_2 = g_1 E_1 + f_2 E_2, \quad K_{E_2} \tilde{X}_r = \tilde{\alpha}_r \tilde{X}_r, \quad 3 \le r \le n.$$

Now, using $\hat{R}(\tilde{X}_r, E_2(q))E_2(q) = 0$ and the Gauss equation (2.6) with $\varepsilon = -1$, we get

$$\tilde{\alpha}_r^2 - f_2 \tilde{\alpha}_r - (1 + g_1^2) = 0, \quad 3 \le r \le n. \tag{5.16}$$

This shows that $\mathfrak{B}_2(q)$ has at most three distinct eigenvalues

$$\tfrac{1}{2}\left(f_2 - \sqrt{f_2^2 + 4(1 + g_1^2)}\right), \quad f_2, \quad \tfrac{1}{2}\left(f_2 + \sqrt{f_2^2 + 4(1 + g_1^2)}\right).$$

On the other hand, from (5.1) and (5.2) that $\lambda_2 > 2\mu_2$ at p, and by continuity of the eigenvalue functions of $\mathfrak{B}_2(q)$ for $q \in \tilde{U}$, we see that $\frac{1}{2}\big(f_2 + \sqrt{f_2^2 + 4(1 + g_1^2)}\big)$ can not be attained. Thus, for any $Y \in \{E_1, E_2\}^\perp$ over $\tilde{U}$, it holds

$$K_{E_2}Y = g_2 Y, \quad g_2 = \tfrac{1}{2}\left(f_2 - \sqrt{f_2^2 + 4(1 + g_1^2)}\right). \tag{5.17}$$

This completes the proof of Step 2.

Step 3. *The completion of Lemma* 5.2*'s proof.*

After the construction of E_1 and E_2 satisfying (5.5) and (5.11) in the preceding two Steps, we can inductively prove the existence of $\{E_i\}_{3 \leq i \leq n}$ such that (5.3) and (5.4) hold.

Indeed, for any $2 \leq j \leq n - 2$, after having h-orthonormal vector fields $\{E_1, \ldots, E_j\}$ over a neighborhood $\bar{U}$ of p, which satisfy corresponding relations in (5.3) and (5.4), we choose an arbitrary h-orthonormal vector fields $\{\bar{E}_{j+1}, \ldots, \bar{E}_n\}$ of $\{E_1, \ldots, E_j\}^\perp$ on $\bar{U}$ such that $\bar{E}_i(p) = e_i$ for $j + 1 \leq i \leq n$. Then, we define a mapping

$$\bar{\varphi}_j : \mathbb{R}^{n-j} \times \bar{U} \to \mathbb{R}^{n-j} \ \ by \ \ \bar{\varphi}(\bar{a}_{j+1}, \bar{a}_{j+2}, \ldots, \bar{a}_n, \bar{q}) = (\bar{b}_{j+1}, \bar{b}_{j+2}, \ldots, \tilde{b}_n),$$

where

$$\bar{b}_k = \sum_{r,s=j+1}^{n} \bar{a}_r \bar{a}_s h(K_{\bar{E}_r}\bar{E}_s, \bar{E}_k) - \lambda_{j+1}\bar{a}_k, \quad j+1 \leq k \leq n, \tag{5.18}$$

are regarded as functions on $\mathbb{R}^{n-j} \times \bar{U} : \bar{b}_k = \bar{b}_k(\bar{a}_{j+1}, \bar{a}_{j+2}, \ldots, \bar{a}_n, \bar{q})$.

As $\bar{E}_i(p) = e_i$, from (5.1), we see that $\bar{b}_k(1, 0, \ldots, 0, p) = 0$ for $j + 1 \leq k \leq n$ and

$$\frac{\partial \bar{b}_k}{\partial \bar{a}_l}\bigg|_{(1,0,\ldots,0,p)} = \begin{cases} \lambda_{j+1} > 0, & k = l = j+1, \\ 2\mu_{j+1} - \lambda_{j+1} \neq 0, & k = l \geq j+2, \\ 0, & k \neq l. \end{cases} \tag{5.19}$$

By the implicit function theorem, there exist differentiable functions $\{\bar{a}_i(\bar{q})\}_{j+1 \leq i \leq n}$, defined on a neighborhood $\bar{U}' \subset \bar{U}$ of p such that $\bar{a}_t(p) = 0$ for $j + 2 \leq t \leq n$, $\bar{a}_{j+1}(p) = 1$ and

$$\bar{b}_k(\bar{a}_{j+1}(q), \ldots, \bar{a}_n(q), q) \equiv 0, \quad j+1 \leq k \leq n, \ \ \forall q \in \bar{U}'. \tag{5.20}$$

Put $\bar{V} = \sum_{k=j+1}^{n} \bar{a}_k \bar{E}_k$. Then $\bar{V}(p) = e_{j+1}$. By (5.18) and (5.20), we can derive that

$$K_{\bar{V}}\bar{V} = \lambda_{j+1}\bar{V} + g_1 h(\bar{V}, \bar{V})E_1 + \cdots + g_j h(\bar{V}, \bar{V})E_j$$

Since $\|\bar{V}(p)\| = 1$, there exists a neighborhood $U \subset \bar{U}'$ of p such that $\bar{V} \neq 0$ on U. Then $E_{j+1} := \bar{V}/\|\bar{V}\|$ defines a unit vector field over U with $E_{j+1}(p) = e_{j+1}$ which satisfies

$$K_{E_{j+1}}E_{j+1} = g_1 E_1 + \cdots + g_j E_j + f_{j+1}E_{j+1}, \quad f_{j+1} = \tfrac{\lambda_{j+1}}{\|\bar{V}\|}. \tag{5.21}$$

At any point $q \in U$, we then consider

$$\mathfrak{B}_{j+1}(q) : \{E_1(q), \ldots, E_j(q)\}^{\perp} \to \{E_1(q), \ldots, E_j(q)\}^{\perp},$$

defined by

$$\mathfrak{B}_{j+1}(q)X = K_{E_{j+1}}X - h(K_{E_{j+1}}X, E_1)E_1 - \cdots - h(K_{E_{j+1}}X, E_j)E_j,$$

which is a self-adjoint transformation. Besides E_{j+1}, let $\bar{X}_t$ be the other unit eigenvectors of $\mathfrak{B}_{j+1}(q)$ corresponding to eigenvalues $\bar{\alpha}_t$ $(j+2 \le t \le n)$. Then, it holds at q:

$$K_{E_{j+1}}E_{j+1} = g_1 E_1 + \cdots + g_j E_j + f_{j+1}E_{j+1}, \quad K_{E_{j+1}}\bar{X}_t = \bar{\alpha}_t \tilde{X}_t, \quad j+2 \le t \le n.$$

Now, using $\hat{R}(\bar{X}_t, E_{j+1}(q))E_{j+1}(q) = 0$ and (2.6) with $\varepsilon = -1$, we get

$$\bar{\alpha}_t^2 - f_{j+1}\bar{\alpha}_t - (1 + g_1^2 + \cdots + g_j^2) = 0, \quad j+2 \le t \le n. \tag{5.22}$$

From (5.22), we see that $\mathfrak{B}_{j+1}(q)$ has at most three distinct eigenvalues

$$\frac{1}{2}\left(f_{j+1} - \sqrt{f_{j+1}^2 + 4(1 + g_1^2 + \cdots + g_j^2)}\right), \quad f_{j+1},$$
$$\frac{1}{2}\left(f_{j+1} + \sqrt{f_{j+1}^2 + 4(1 + g_1^2 \cdots + g_j^2)}\right).$$

On the other hand, from (5.1) and (5.2) that $\lambda_{j+1} > 2\mu_{j+1}$ at p, and by continuity of the eigenvalue functions of $\mathfrak{B}_{j+1}(q)$ for $q \in U$, we see that eigenvalues of the first two possibilities can be attained. Thus, for any $Y \in \{E_1, \ldots, E_{j+1}\}^{\perp}$ over U, it holds

$$K_{E_{j+1}}Y = g_{j+1}Y, \quad g_{j+1} = \frac{1}{2}\left(f_{j+1} - \sqrt{f_{j+1}^2 + 4(1 + g_1^2 + \cdots + g_j^2)}\right). \tag{5.23}$$

In summary, by induction, we can extend the vectors $\{e_1, e_2, \ldots, e_{n-1}\}$ to the h-orthonormal vector fields $\{E_1, E_2, \ldots, E_{n-1}\}$ which satisfy corresponding relations in (5.3) and (5.4). Finally, we choose a unit vector field $E_n \in \{E_1, E_2, \ldots, E_{n-1}\}^{\perp}$ so that $\{E_i\}_{i=1}^n$ forms a local h-orthonormal frame on a neighbourhood $U \subset M^n$ of p. Obviously, with respect to $\{E_i\}_{i=1}^n$, the difference tensor K takes the form in (5.3) and (5.4), where $f_i > 0$ $(1 \le i \le n-1)$ follow from its continuity and that $f_i(p) = \lambda_i > 0$.

This completes the proof of Lemma 5.2. $\qquad \square$

6. Applying the Codazzi Equations

From this section on, we consider the case $n = 4$ and assume that $x : M^4 \to \mathbb{R}^5$ is a flat hyperbolic centroaffine hypersurface. With respect to the typical h-orthogonal frame field $\{E_1, E_2, E_3, E_4\}$, described in Lemma 5.2, we shall derive by applying the Codazzi equation (2.7) the necessary information on the derivatives $\{E_i(g_j)\}$ and the connection coefficients $\{\Gamma_{ij}^k\}$ of $\hat{\nabla}$. The computations below are complicated but straightforward.

Put $\hat{\nabla}_{E_i} E_j = \sum_{k=1}^{4} \Gamma_{ij}^k E_k$, where $\Gamma_{ij}^k + \Gamma_{ik}^j = 0$ for $1 \le i, j, k \le 4$.

Then, taking in (2.7) $X = E_i$, $Y = E_j$ and $Z = E_k$ for $1 \le i, j, k \le 4$, we get

$$
\begin{aligned}
&\hat{\nabla}_{E_i} K(E_j, E_k) - K(\hat{\nabla}_{E_i} E_j, E_k) - K(E_j, \hat{\nabla}_{E_i} E_k) \\
&\quad - \hat{\nabla}_{E_j} K(E_i, E_k) + K(\hat{\nabla}_{E_j} E_i, E_k) + K(E_i, \hat{\nabla}_{E_j} E_k) = 0.
\end{aligned}
\tag{6.1}
$$

First of all, for $(i, j, k) = (1, 2, 1)$, we obtain from (6.1) and (5.3) that

$$
\begin{aligned}
&(1 + g_1^2)^2 g_1^{-2}(g_1 \Gamma_{11}^2 + E_2(g_1)) E_1 \\
&\quad - \left[g_1^{-1}(g_1^2 + 1)\Gamma_{21}^2 + g_2^{-1}(g_1^2 + 1)\Gamma_{11}^2 - g_2\Gamma_{11}^2 + E_1(g_1) \right] E_2 \\
&\quad + \left[g_2\Gamma_{11}^3 - g_1^{-1}(1 + g_1^2)\Gamma_{21}^3 \right] E_3 + \left[g_2\Gamma_{11}^4 - g_1^{-1}(1 + g_1^2)\Gamma_{21}^4 \right] E_4 = 0.
\end{aligned}
$$

Hence, we get

$$
\begin{cases}
\Gamma_{11}^3 = (g_1 g_2)^{-1}(1 + g_1^2)\Gamma_{21}^3, \quad \Gamma_{11}^4 = (g_1 g_2)^{-1}(1 + g_1^2)\Gamma_{21}^4, \\
E_1(g_1) = -g_1^{-1}(g_1^2 + 1)\Gamma_{21}^2 - g_2^{-1}(g_1^2 + 1)\Gamma_{11}^2 + g_2\Gamma_{11}^2, \quad E_2(g_1) = -g_1\Gamma_{11}^2.
\end{cases}
\tag{6.2}
$$

Similarly, letting $(i, j, k) = (1, 2, 2)$ in (6.1), and taking account of (6.2), we obtain

$$
\Gamma_{21}^3 = \Gamma_{12}^3, \quad \Gamma_{21}^4 = \Gamma_{12}^4, \quad E_1(g_2) = -2g_1\Gamma_{11}^2 - g_2\Gamma_{21}^2,
\tag{6.3}
$$

whereas letting $(i, j, k) = (1, 3, 1)$ in (6.1), and taking account of (6.2), we obtain

$$
\begin{cases}
\Gamma_{31}^3 = (g_2 g_3)^{-1} \left[g_3(g_1\Gamma_{11}^2 + g_2\Gamma_{21}^2) - (1 + g_1^2 + g_2^2 - g_3^2)\Gamma_{21}^3 \right], \\
\Gamma_{31}^2 = \Gamma_{21}^3, \quad g_2\Gamma_{31}^4 = g_3\Gamma_{21}^4, \quad E_3(g_1) = -g_2^{-1}(1 + g_1^2)\Gamma_{12}^3.
\end{cases}
\tag{6.4}
$$

Letting $(i, j, k) = (1, 3, 4)$ in (6.1), and taking account of (6.2)–(6.4), we obtain

$$
E_1(g_3) = -g_2^{-1} \left[g_1 g_3\Gamma_{11}^2 + g_2 g_3\Gamma_{21}^2 + (1 + g_1^2 + g_2^2 + g_3^2)\Gamma_{21}^3 \right].
\tag{6.5}
$$

Letting $(i, j, k) = (1, 4, 1)$ in (6.1), and taking account of (6.2), we obtain

$$
\begin{cases}
\Gamma_{41}^4 = \Gamma_{21}^2 + g_2^{-1}(g_1\Gamma_{11}^2 + g_3\Gamma_{31}^3) + g_3^{-1} f_4\Gamma_{13}^4, \\
\Gamma_{41}^3 = \Gamma_{13}^4, \quad g_3\Gamma_{21}^4 = g_2\Gamma_{13}^4, \quad E_4(g_1) = -g_1\Gamma_{11}^4.
\end{cases}
\tag{6.6}
$$

Letting $(i, j, k) = (1, 4, 4)$ in (6.1), taking account of (6.2), (6.3) and (6.6), we get

$$
\begin{aligned}
E_1(f_4) = &-\frac{f_4(g_1\Gamma_{11}^2 + g_2\Gamma_{21}^2 + g_3\Gamma_{21}^3)}{g_2} \\
&- \frac{(f_4^2 + 4(1 + g_1^2 + g_2^2 + g_3^2))\Gamma_{13}^4}{g_3}.
\end{aligned}
\tag{6.7}
$$

Letting $(i, j, k) = (2, 3, 2)$ in (6.1), taking account of (6.3) and (6.4), we get

$$
\begin{cases}
\Gamma_{22}^4 = (g_2 g_3)^{-1} \left[(1 + g_1^2 + g_2^2)\Gamma_{32}^4 - g_1 g_2 \Gamma_{13}^4\right], \quad E_3(g_2) = -g_2 \Gamma_{22}^3 - 2g_1 \Gamma_{12}^3, \\
E_2(g_2) = -(g_2 g_3)^{-1} \left[g_1(1 + g_1^2 + g_2^2 - g_3^2)\Gamma_{21}^3 + g_2(1 + g_2^2 - g_3^2)\Gamma_{22}^3\right] \\
\quad - (g_2 g_3)^{-1} \left[g_3(1 + g_2^2)\Gamma_{32}^3 + g_1^2(g_2 \Gamma_{22}^3 + g_3(-\Gamma_{11}^2 + \Gamma_{32}^3))\right].
\end{cases} \tag{6.8}
$$

Letting $(i, j, k) = (2, 3, 3)$ in (6.1), taking account of (6.2) and (6.8), we get

$$
\Gamma_{32}^4 = \Gamma_{23}^4, \quad E_2(g_3) = -2g_1 \Gamma_{21}^3 - 2g_2 \Gamma_{22}^3 - g_3 \Gamma_{32}^3. \tag{6.9}
$$

Letting $(i, j, k) = (2, 4, 2)$ in (6.1), taking account of (6.6), we get

$$
\begin{cases}
\Gamma_{42}^4 = g_3^{-1}(g_1 \Gamma_{21}^3 + g_2 \Gamma_{22}^3 + g_3 \Gamma_{32}^3 + f_4 \Gamma_{23}^4), \quad \Gamma_{42}^3 = \Gamma_{23}^4, \\
E_4(g_2) = -g_3^{-1} \left[(1 + g_1^2 + g_2^2 - g_3^2)\Gamma_{42}^3 + g_1 g_2 \Gamma_{13}^4 + g_3^2 \Gamma_{23}^4\right].
\end{cases} \tag{6.10}
$$

Letting $(i, j, k) = (2, 4, 4)$ in (6.1), taking account of (6.10), we get

$$
\begin{aligned}
E_2(f_4) = -g_3^{-1} \big[&f_4(g_1 \Gamma_{21}^3 + g_2 \Gamma_{22}^3 + g_3 \Gamma_{32}^3) \\
&+ (f_4^2 + 4(1 + g_1^2 + g_2^2 + g_3^2))\Gamma_{23}^4\big].
\end{aligned} \tag{6.11}
$$

Letting $(i, j, k) = (3, 4, 3)$ in (6.1), taking account of (6.6) and (6.10), we get

$$
\begin{cases}
E_3(g_3) = \dfrac{g_1(1 + g_1^2 + 2g_2^2)\Gamma_{21}^3 + g_2^3 \Gamma_{22}^3 + f_4(g_1 g_2 \Gamma_{13}^4 + g_2^2 \Gamma_{23}^4 + g_2 g_3 \Gamma_{33}^4)}{g_2 g_3} \\
\qquad - g_3^{-1}(1 + g_1^2 + g_2^2 + g_3^2)\Gamma_{43}^4, \\
E_4(g_3) = -2g_1 \Gamma_{13}^4 - 2g_2 \Gamma_{23}^4 - g_3 \Gamma_{33}^4.
\end{cases} \tag{6.12}
$$

Finally, letting $(i, j, k) = (3, 4, 4)$ in (6.1), taking account of (6.6), (6.10) and (6.12), we get

$$
E_3(f_4) = -4(g_1 \Gamma_{13}^4 + g_2 \Gamma_{23}^4 + g_3 \Gamma_{33}^4) - f_4 \Gamma_{43}^4. \tag{6.13}
$$

7. Determining the Connection Coefficients

For the purpose of proving Theorem 1.2, we now assume that $x : M^4 \to \mathbb{R}^5$ is a flat hyperbolic centroaffine Tchebychev hypersurface.

In this section, applying for the Tchebychev condition we shall solve the Eqs. (6.2)–(6.13) to express $\{\Gamma_{ij}^k\}$ and $E_4(f_4)$ by the coefficients $\{f_i, g_j\}$ of the difference tensor in (5.3). Similar as in the last section, the computations below are complicated but straightforward.

First of all, by $T = \frac{1}{4} \sum_{i=1}^4 K(E_i, E_i)$ we express the Tchebychev condition as

$$
\sum_{i=1}^4 \hat{\nabla}_{E_j} K(E_i, E_i) - 4H E_j = 0, \quad 1 \le j \le 4. \tag{7.1}
$$

Then, taking $j = 1$ in (7.1), by calculations with applying (5.3), we can derive that

$$
\begin{aligned}
0 = & \left[E_1(f_1 + 3g_1) + (f_2 + 2g_2)\Gamma_{12}^1 + (f_3 + g_3)\Gamma_{13}^1 + f_4\Gamma_{14}^1 - 4H \right] E_1 \\
& + \left[E_1(f_2 + 2g_2) + (f_1 + 3g_1)\Gamma_{11}^2 + (f_3 + g_3)\Gamma_{13}^2 + f_4\Gamma_{14}^2 \right] E_2 \\
& + \left[E_1(f_3 + g_3) + (f_1 + 3g_1)\Gamma_{11}^3 + (f_2 + 2g_2)\Gamma_{12}^3 + f_4\Gamma_{14}^3 \right] E_3 \\
& + \left[E_1(f_4) + (f_1 + 3g_1)\Gamma_{11}^4 + (f_2 + 2g_2)\Gamma_{12}^4 + (f_3 + g_3)\Gamma_{13}^4 \right] E_4.
\end{aligned}
\tag{7.2}
$$

From (5.4) we have

$$
\begin{cases}
f_1 + 3g_1 = g_1^{-1}(4g_1^2 - 1), \quad f_2 + 2g_2 = g_2^{-1}\left[3g_2^2 - (1 + g_1^2) \right], \\
f_3 + g_3 = g_3^{-1}\left[2g_3^2 - (1 + g_1^2 + g_2^2) \right].
\end{cases}
\tag{7.3}
$$

From (6.2), (6.3) and (6.6), we get

$$
\begin{aligned}
& \Gamma_{11}^3 = (g_1 g_2)^{-1}(1 + g_1^2)\Gamma_{12}^3, \quad \Gamma_{11}^4 = (g_1 g_3)^{-1}(1 + g_1^2)\Gamma_{13}^4, \\
& \Gamma_{14}^2 = -g_3^{-1}g_2\Gamma_{13}^4.
\end{aligned}
\tag{7.4}
$$

Then, from (7.2), taking account of (7.3) and (7.4), $\Gamma_{ij}^k = -\Gamma_{ik}^j$, and the expressions of $E_1(g_1)$, $E_1(g_2)$, $E_1(g_3)$ and $E_1(f_4)$ in Sect. 6, we can derive a system of four equations with unknowns Γ_{11}^2, Γ_{21}^2, Γ_{12}^3 and Γ_{13}^4. Solving these equations, we then obtain

$$
\begin{aligned}
\Gamma_{11}^2 = & \; 4H g_1^4 g_2 \left[\left(-g_1^2(1 + g_1^2 + g_2^2)^2 + (-1 - g_1^2 + g_2^2)g_3^2 \right) (\lambda\beta)^{-1} \right. \\
& \left. + \left(2 + 4g_1^4 + g_1^2(6 + f_4^2 + 4g_2^2 + 4g_3^2) \right) \gamma^{-1} \right], \\
\Gamma_{21}^2 = & -4H g_1^3 g_2^2 \left[g_1^2(1 + g_1^2 + g_2^2)^2(1 + 6g_1^6 + g_1^2(6 + f_4^2 + 2g_2^2) \right. \\
& + g_1^4(11 + 2f_4^2 + 6g_2^2)) + ((1 + g_1^2)^2(1 + (6 + f_4^2)g_1^2 \\
& + 2(7 + f_4^2)g_1^4 + 14g_1^6) + 2g_1^2(1 + 9g_1^2 + (22 + f_4^2)g_1^4 \\
& + 14g_1^6)g_2^2 + 2g_1^4(1 + 7g_1^2)g_2^4)g_3^2 + g_1^2(4 + g_1^2(f_4^2(1 + 2g_1^2) \\
& \left. + 4(5 + g_2^2 + 4g_1^2(2 + g_1^2 + g_2^2))))g_3^4 + 4g_1^4(1 + 2g_1^2)g_3^6 \right] (\lambda\beta\gamma)^{-1}, \\
\Gamma_{12}^3 = & \; 4H g_1^5 g_2 g_3 \left[-(1 + g_1^2 + g_2^2)(1 + 2g_1^4 + g_1^2(3 + f_4^2 + 2g_2^2)) \right. \\
& \left. + (2 + g_1^2(f_4^2 + 2(2 + g_1^2 + g_2^2)))g_3^2 + 4g_1^2 g_3^4 \right] (\beta\gamma)^{-1}, \\
\Gamma_{13}^4 = & \; 4H f_4 g_1^5 g_3 \gamma^{-1},
\end{aligned}
\tag{7.5}
$$

where

$$
\begin{aligned}
\lambda = & \; (1 + g_1^2)(g_1^2 + g_2^2), \\
\beta = & \; g_3^2 + g_1^2((1 + g_1^2 + g_2^2)^2 + (2 + g_1^2 + g_2^2)g_3^2), \\
\gamma = & \; 1 + g_1^2(f_4^2(1 + g_1^4 + g_1^2(2 + g_2^2 + g_3^2)) \\
& + (3 + 2g_1^2 + 2g_2^2 + 2g_3^2)(2 + 2g_1^4 + g_1^2(3 + 2g_2^2 + 2g_3^2))).
\end{aligned}
\tag{7.6}
$$

Similarly, we take $j = 2$ in (7.1). By straightforward calculations with applying (5.3), we can further use (7.5) to solve the unknowns Γ_{22}^3, Γ_{32}^3 and

Γ_{23}^4. Then, we obtain

$$
\begin{aligned}
\Gamma_{22}^3 = \frac{4Hg_2^2 g_3}{(1+g_1^2)^2} \big[& g_2^2(2 + 2g_1^4 + 4g_2^4 + g_1^2(4 + 6g_2^2) \\
& + g_2^2(6 + f_4^2 + 4g_3^2))\varphi^{-1} - g_2^2\psi^{-1} - g_1^6(1 + g_1^2 + g_2^2)\beta^{-1} \\
& + g_1^6(2 + 4g_1^4 + g_1^2(6 + f_4^2 + 4g_2^2 + 4g_3^2))\gamma^{-1} \big],
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{32}^3 = \frac{4H}{(1+g_1^2)^2} \big[& g_2^5\psi^{-1} + g_1^6 g_2(1 + g_1^2 + g_2^2)^2\beta^{-1} \\
& - (g_2^3(1 + g_1^2 + g_2^2)((1 + g_1^2)^2 + (4 + f_4^2 + 4g_1^2)g_2^2 + 4g_2^4) \\
& + 2g_2^5(1 + g_1^2 + 2g_2^2)g_3^2)\varphi^{-1} \\
& + g_1^4 g_2(-1 + g_1^2(-f_4^2(1 + g_1^4 + g_1^2(2 + g_2^2)) \\
& - (3 + 2g_1^2 + 2g_2^2)(2 + 2g_1^4 + g_1^2(3 + 2g_2^2)) \\
& - 2(1 + 2g_1^4 + g_1^2(3 + 2g_2^2))g_3^2))\gamma^{-1} \big],
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{23}^4 = 4H f_4 g_2 g_3 \big[& g_1^{12} + g_2^4 + g_1^{10}(3 + 6g_2^2) + g_1^2 g_2^4(5 + f_4^2 + 4g_2^2 + 4g_3^2) \\
& + g_1^4 g_2^4(4(g_2^2 + g_3^2)^2 + 8(1 + g_2^2 + g_3^2) + f_4^2(1 + g_2^2 + g_3^2)) \\
& + g_1^8(3 + 13g_2^4 + g_2^2(12 + f_4^2 + 4g_3^2)) + g_1^6(1 + 12g_2^6 + g_2^2(6 + f_4^2 + 4g_3^2) \\
& + g_2^4(17 + 2f_4^2 + 12g_3^2)) \big] (\gamma\varphi)^{-1}, \tag{7.7}
\end{aligned}
$$

where

$$
\begin{aligned}
\varphi = & (1 + g_1^2 + g_2^2)^2((1 + g_1^2)^2 + (4 + f_4^2 + 4g_1^2)g_2^2 + 4g_2^4) \\
& + g_2^2(4(1 + g_1^2)^2 + (f_4^2 + 12(1 + g_1^2))g_2^2 + 8g_2^4)g_3^2 + 4g_2^4 g_3^4, \tag{7.8} \\
\psi = & (1 + g_1^2 + g_2^2)(g_2^2 + g_3^2),
\end{aligned}
$$

Finally, taking in (7.1) $j = 3$ and $j = 4$, respectively, by direct calculations with using (7.5) and (7.7), we can derive

$$
\Gamma_{33}^4 = 4H f_4 \big[g_3^4\theta^{-1} + g_2^6 g_3^2(\eta\varphi)^{-1} + g_1^6 g_3^2((1 + g_1^2)\gamma)^{-1} \big],
$$

$$
\begin{aligned}
\Gamma_{43}^4 = 4H g_3 \big[& -g_3^2(1 + g_1^2 + g_2^2 + 2g_3^2)\theta^{-1} - g_2^4(1 + g_1^4 + 2g_2^4 \\
& + g_1^2(2 + 3g_2^2) + g_2^2(3 + 2g_3^2))(\eta\varphi)^{-1} \\
& - g_1^4(1 + 2g_1^4 + g_1^2(3 + 2g_2^2 + 2g_3^2))((1 + g_1^2)\gamma)^{-1} \big],
\end{aligned}
$$

$$
\begin{aligned}
E_4(f_4) = 4H \big[& 2 - (g_2^2(1 + g_1^2 + g_2^2)^2(2(1 + g_1^2)^2 + (4 + f_4^2 + 4g_1^2)g_2^2) \tag{7.9} \\
& + g_2^4(4(1 + g_1^2)^2 + (4 + f_4^2 + 4g_1^2)g_2^2)g_3^2)(\eta\varphi)^{-1} \\
& - ((1 + g_1^2 + g_2^2)^3 + (1 + g_1^2 + g_2^2)(f_4^2 + 6(1 + g_1^2 + g_2^2))g_3^2 \\
& + (f_4^2 + 8(1 + g_1^2 + g_2^2))g_3^4)\theta^{-1} - ((2 + (4 + f_4^2)g_1^2)(g_1 + g_1^3)^2 \\
& + g_1^4(4 + (4 + f_4^2)g_1^2)g_2^2 + g_1^4(4 + (4 + f_4^2)g_1^2)g_3^2)((1 + g_1^2)\gamma)^{-1} \big],
\end{aligned}
$$

where

$$
\begin{aligned}
\eta = & (1 + g_1^2)(1 + g_1^2 + g_2^2), \\
\theta = & (1 + g_1^2 + g_2^2)(1 + g_1^2 + g_2^2 + g_3^2)((1 + g_1^2 + g_2^2)^2 \tag{7.10} \\
& + (f_4^2 + 4(1 + g_1^2 + g_2^2))g_3^2 + 4g_3^4).
\end{aligned}
$$

8. Completion of Theorem 1.2's Proof

Let $x : M^4 \to \mathbb{R}^5$ be a flat hyperbolic centroaffine Tchebychev hypersurface. In this section, with the materials we derived in preceding sections, we shall complete the proof of Theorem 1.2. We begin with the equation

$$E_3(E_4(g_1)) - E_4(E_3(g_1)) = (\hat{\nabla}_{E_3} E_4)(g_1) - (\hat{\nabla}_{E_4} E_3)(g_1). \tag{8.1}$$

By using the results in Sects. 6 and 7, we can calculate (8.1) to obtain that

$$H^2 f_4 g_1^9 g_3 \left[(1 + g_1^2 + g_2^2)^3 - (1 + g_1^2 + g_2^2)(f_4^2 + 3(1 + g_1^2 + g_2^2))g_3^2 \right.$$
$$\left. + f_4^2 g_3^4 + 4 g_3^6 \right] = 0.$$

From the fact $g_1 \neq 0$ and $g_3 \neq 0$, we then obtain

$$H^2 f_4 \mu = 0, \tag{8.2}$$

where

$$\mu := (1 + g_1^2 + g_2^2)^3 - (1 + g_1^2 + g_2^2)(f_4^2$$
$$+ 3(1 + g_1^2 + g_2^2))g_3^2 + f_4^2 g_3^4 + 4 g_3^6. \tag{8.3}$$

From (8.2), we shall divide the following discussions into three cases:

Case I: $H = 0$;

Case II: $H \neq 0$ and $f_4 = 0$;

Case III: $H \neq 0$, $f_4 \neq 0$ and $\mu = 0$.

In **Case I**, we can apply Theorem 1.1. Then, combining with the results in Sect. 3 and taking into account that $x : M^4 \to \mathbb{R}^5$ is hyperbolic, we conclude that x is centroaffinely equivalent to one of the hypersurfaces (cf. Remark 1.2):

$$x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4} x_5^{\alpha_5} = 1, \quad \alpha_i > 0 \text{ for } 1 \leq i \leq 5.$$

In **Case II**, substituting $f_4 = 0$ into the expression of $E_4(f_4)$ in (7.9), we obtain that

$$0 = 4H \left[2 - (g_2^2(1 + g_1^2 + g_2^2)^2(2(1 + g_1^2)^2 + (4 + 4g_1^2)g_2^2) + g_2^4(4(1 + g_1^2)^2 \right.$$
$$+ (4 + 4g_1^2)g_2^2)g_3^2)(\eta\varphi)^{-1} - ((1 + g_1^2 + g_2^2)^3 + (1 + g_1^2 + g_2^2)(6(1 + g_1^2 + g_2^2))g_3^2$$
$$+ (8(1 + g_1^2 + g_2^2))g_3^4)\theta^{-1} - ((2 + 4g_1^2)(g_1 + g_1^3)^2 + g_1^4(4 + 4g_1^2)g_2^2$$
$$\left. + g_1^4(4 + 4g_1^2)g_3^2)((1 + g_1^2)\gamma)^{-1} \right] = 4H\Omega\nu^{-1},$$

where
$$\Omega = (1+g_1^2+g_2^2)^2(1+4g_1^8+2g_2^2+4g_2^4+2g_1^6(5+8g_2^2)$$
$$+ 4g_1^2(1+g_2^2)(1+g_2^2+2g_2^4)+g_1^4(9+22g_2^2+20g_2^4))$$
$$+ 2(1+g_1^2+g_2^2)((1+g_1^2)^2(1+3g_1^2+8g_1^4)+(1+g_1^2)(3+11g_1^2+36g_1^4)g_2^2$$
$$+ 4(2+7G_1^2+12g_1^4)g_2^4+20g_1^2g_2^6)g_3^2+4((1+g_1^2)^2(1+3g_1^2+5g_1^4)$$
$$+ (1+g_1^2)(3+12g_1^2+28g_1^4)g_2^2+(5+27g_1^2+41g_1^4)g_2^4+18g_1^2g_2^6)g_3^4$$
$$+ 8((g_1+g_1^3)^2+(1+5g_1^2+8g_1^4)g_2^2+7g_1^2g_2^4)g_3^6+16g_1^2g_2^2g_3^8 > 0,$$
$$\nu = (1+g_1^2+g_2^2+g_3^2)(1+g_1^2+g_2^2+2g_3^2)(1+g_1^4+2g_2^4+g_1^2(2+3g_2^2)$$
$$+ g_2^2(3+2g_3^2))(1+2g_1^4+g_1^2(3+2g_2^2+2g_3^2)) > 0.$$

This is a contradiction to $H \neq 0$. Hence, **Case II** does not occur.

In **Case III**, we first see from $g_3 \neq 0$ and (5.4) that $1+g_1^2+g_2^2-g_3^2 > 0$. Moreover, replacing E_4 by $-E_4$ if necessary, we can assume $f_4 < 0$. Then, from (8.3), we get

$$f_4 = -\sqrt{\frac{(1+g_1^2+g_2^2-2g_3^2)^2(1+g_1^2+g_2^2+g_3^2)}{g_3^2(1+g_1^2+g_2^2-g_3^2)}}, \tag{8.4}$$

and that $1+g_1^2+g_2^2-2g_3^2 \neq 0$.

Since $x : M^4 \to \mathbb{R}^5$ is a Tchebychev hypersurface, from (2.15) we have

$$E_3(\|T\|^2) = 2Hh(T, E_3). \tag{8.5}$$

However, by Lemma 5.2, it holds that

$$T = \tfrac{1}{4}\left[(f_1+3g_1)E_1 + (f_2+2g_2)E_2 + (f_3+g_3)E_3 + f_4E_4\right], \tag{8.6}$$
$$\|T\|^2 = \tfrac{1}{16}\left[(f_1+3g_1)^2 + (f_2+2g_2)^2 + (f_3+g_3)^2 + f_4^2\right]. \tag{8.7}$$

Moreover, combining with (7.5), (7.7) and (7.9), we can rewrite $E_3(g_1)$, $E_3(g_2)$, $E_3(g_3)$ and $E_3(f_4)$ in Sect. 6 as

$$E_3(g_1) = -4Hg_1^5(1+g_1^2)g_3\left[(-1-g_1^2-g_2^2)(1+2g_1^4+g_1^2(3+f_4^2+2g_2^2))\right.$$
$$\left. +(2+g_1^2(f_4^2+2(2+g_1^2+g_2^2)))g_3^2+4g_1^2g_3^4\right](\beta\gamma)^{-1},$$
$$E_3(g_2) = \frac{4Hg_2g_3}{(1+g_1^2)^2}\left[g_2^4\psi^{-1} - (g_2^4(2+2g_1^4+4g_2^4+g_1^2(4+6g_2^2)\right.$$
$$+ g_2^2(6+f_4^2+4g_3^2)))\varphi^{-1} + g_1^6(1+g_1^2+g_2^2)(2+2g_1^2+g_2^2)\beta^{-1}$$
$$\left. -g_1^6(2+2g_1^2+g_2^2)(2+4g_1^4+g_1^2(6+f_4^2+4g_2^2+4g_3^2))\gamma^{-1}\right], \tag{8.8}$$

and

$$E_3(g_3) = \frac{2H}{(1+g_1^2)^2}\left[-2g_2^6\psi^{-1} + g_2^2((1+g_1^2+g_2^2)^2(-1-g_1^2+2g_2^2)((1+g_1^2)^2\right.$$
$$+(4+f_4^2+4g_1^2)g_2^2+4g_2^4)+g_2^2(-2(1+g_1^2)^3+(1+g_1^2)(f_4^2$$
$$-2(1+g_1^2))g_2^2+8(1+g_1^2)g_2^4+8g_2^6)g_3^2)((1+g_1^2+g_1^2)\varphi)^{-1}$$
$$+(1+g_1^2)^2(-(1+g_1^2+g_2^2)^3-(1+g_1^2+g_2^2)(f_4^2+3(1+g_1^2+g_2^2))g_3^2$$
$$+(f_4^2-2(1+g_1^2+g_2^2))g_3^4)\theta^{-1} - (g_1^2(1+g_1^2+g_2^2)^2(-1-2g_1^2+2g_1^6$$

$$\begin{aligned}
&+g_1^4(1+2g_2^2)) - (1+g_1^2)^2(1+g_1^4+g_1^2(2+g_2^2))g_3^2)\beta^{-1}\\
&+g_1^2((-1+2g_1^4+g_1^2(1+2g_2^2))(1+g_1^2(f_4^2(1+g_1^4+g_1^2(2+g_2^2))\\
&+(3+2g_1^2+2g_2^2)(2+2g_1^4+g_1^2(3+2g_2^2)))) + g_1^2(-2+g_1^2(-4+6g_1^2\\
&+f_4^2(1+g_1^2)+8g_1^2(g_1^2+g_2^2)(2+g_1^2+g_2^2)))g_3^2)\gamma^{-1}\big]\,,
\end{aligned}$$

$$\begin{aligned}
E_3(f_4) = 4Hf_4g_3\big[&g_3^2(1+g_1^2+g_2^2-2g_3^2)\theta^{-1}\\
&-(g_2^4(-1-g_1^4+g_2^2+g_1^2(-2+g_2^2)+2g_2^2(g_2^2+g_3^2)))(\varphi\eta)^{-1}\\
&+(g_1^4-2g_1^8-g_1^6(1+2g_2^2+2g_3^2))((1+g_1^2)\gamma)^{-1}\big]\,,
\end{aligned} \tag{8.9}$$

where $\{\beta,\gamma,\varphi,\psi,\eta,\theta\}$ are defined by (7.6), (7.8) and (7.10), respectively.

Now, combining (7.3), (8.4), (8.6), (8.7), (8.8) and (8.9), we can carry a straightforward calculation of (8.5) to obtain the following relation:

$$\frac{1}{2g_3}(1+g_1^2+g_2^2-2g_3^2)H = 0. \tag{8.10}$$

But this is impossible. Hence, **Case III** does not occur.

In conclusion, we have completed the proof of Theorem 1.2.

Acknowledgements

Declarations

Competing Interests The authors hereby state that there are no conflicts of interest regarding the presented results.

References

[1] Binder, Th.: Relative Tchebychev hypersurfaces which are also translation hypersurfaces. Hokkaido Math. J. **38**, 97–110 (2009)

[2] Cheng, X., Hu, Z.: On the isolation phenomena of locally conformally flat manifolds with constant scalar curvature-submanifolds versions. J. Math. Anal. Appl. **464**, 1147–1157 (2018)

[3] Cheng, X., Hu, Z., Moruz, M.: Classification of the locally strongly convex centroaffine hypersurfaces with parallel cubic form. Results Math. **72**, 419–469 (2017)

[4] Cheng, X., Hu, Z., Moruz, M., Vrancken, L.: On product affine hyperspheres in $\mathbb{R}^{n+1}$. Sci China Math **63**, 2055–2078 (2020)

[5] Cheng, X., Hu, Z., Vrancken, L.: Every centroaffine Tchebychev hyperovaloid is ellipsoid. Pacific J. Math. **315**, 27–44 (2021)

[6] Cheng, X., Hu, Z., Yao, Z.: A rigidity theorem for centroaffine Chebyshev hyperovaloids. Colloq. Math. **157**, 133–141 (2019)

[7] Hu, Z., Li, H., Vrancken, L.: Locally strongly convex affine hypersurfaces with parallel cubic form. J. Differ. Geom. **87**, 239–307 (2011)

[8] Hu, Z., Xing, C.: A new centroaffine characterization of the ellipsoids. Proc. Amer. Math. Soc. **149**, 3531–3540 (2021)

[9] Li, M.: Rigidity theorems for relative Tchebychev hypersurfaces. Results Math. **70**, 283–298 (2016)

[10] Li, A.-M., Li, H., Simon, U.: Centroaffine Bernstein problems. Differ. Geom. Appl. **20**, 331–356 (2004)

[11] Li, A.-M., Liu, H., Schwenk-Schellschmidt, A., Simon, U., Wang, C.P.: Cubic form methods and relative Tchebychev hypersurfaces. Geom. Dedicata. **66**, 203–221 (1997)

[12] Li, A.-M., Penn, G.: Uniqueness theorems in affine differential geometry part II. Results Math. **13**, 308–317 (1988)

[13] Lalléchère, S., Ramifidisoa, L., Ravelo, B.: Flat hyperbolic centro-affine Tchebychev hypersurfaces of $\mathbb{R}^4$. Results Math. **76**: 71, 18 pp. (2021)

[14] Li, A.-M., Simon, U., Zhao, G., Hu, Z.: Global Affine Differential Geometry of Hypersurfaces, 2nd edn. de Gruyter Expositions in Mathematics 11, Walter de Gruyter, Berlin/Boston (2015)

[15] Liu, H., Simon, U., Wang, C.P.: Conformal structures in affine geometry: complete Tchebychev hypersurfaces. Abh. Math. Sem. Hamburg **66**, 249–262 (1996)

[16] Li, A.-M., Wang, C.P.: Canonical centroaffine hypersurfaces in $\mathbb{R}^{n+1}$. Results Math. **20**, 660–681 (1991)

[17] Liu, H., Wang, C.P.: The centroaffine Tchebychev operator. Results Math. **27**, 77–92 (1995)

[18] Liu, H., Wang, C.P.: Relative Tchebychev surfaces in $\mathbb{R}^3$. Kyushu J. Math. **50**, 533–540 (1996)

[19] Nomizu, K., Sasaki, T.: Affine Differential Geometry. Cambridge University Press, Cambridge (1994)

[20] Simon, U., Schwenk-Schellschmidt, A., Viesel, H.: Introduction to the Affine Differential Geometry of Hypersurfaces. Lecture Notes of the Science University of Tokyo, Japan (1991)

[21] Simon, U., Trabelsi, H., Vrancken, L.: Complete hyperbolic Tchebychev hypersurfaces. J. Geom. **89**, 148–159 (2008)

[22] Vrancken, L.: The Magid-Ryan conjecture for equiaffine hyperspheres with constant sectional curvature. J. Differ. Geom. **54**, 99–138 (2000)

[23] Vrancken, L., Li, A.-M., Simon, U.: Affine spheres with constant affine sectional curvature. Math. Z. **206**, 651–658 (1991)

[24] Wang, C.P.: Centroaffine minimal hypersurfaces in $\mathbb{R}^{n+1}$. Geom. Dedicata. **51**, 63–74 (1994)

Xiuxiu Cheng, Zejun Hu and Cheng Xing
School of Mathematics and Statistics
Zhengzhou University
450001 Zhengzhou
People's Republic of China
e-mail: `chengxiuxiu1988@163.com`;
  `huzj@zzu.edu.cn`;
  `xingchengchn@yeah.net`

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.