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Results in Mathematics

Ricci-like Solitons with Arbitrary Potential and Gradient Almost Ricci-like Solitons on Sasaki-like Almost Contact B-metric Manifolds

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Abstract. Ricci-like solitons with arbitrary potential are introduced and studied on Sasaki-like almost contact B-metric manifolds. A manifold of this type can be considered as an almost contact complex Riemannian manifold which complex cone is a holomorphic complex Riemannian manifold. The soliton under study is characterized and proved that its Ricci tensor is equal to the vertical component of both B-metrics multiplied by a constant. Thus, the scalar curvatures with respect to both B-metrics are equal and constant. In the 3-dimensional case, it is found that the special sectional curvatures with respect to the structure are constant. Gradient almost Ricci-like solitons on Sasaki-like almost contact B-metric manifolds have been proved to have constant soliton coefficients. Explicit examples are provided of Lie groups as manifolds of dimensions 3 and 5 equipped with the structures under study.

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Introduction

Ricci soliton is a special self-similar solution of the Hamilton's Ricci flow and it is a natural generalization of the notion of Einstein metric. According to [19], a (pseudo-)Riemannian manifold admits a *Ricci soliton* if the metric, its Ricci tensor and the Lie derivative of the metric along a vector field (called potential) are linear dependent. If the coefficients of this dependence are functions then the soliton is called an *almost Ricci soliton* [32]. If a function exists so that the potential is its gradient, then the (almost) Ricci soliton is called a *gradient (almost) Ricci soliton* (see, e.g., [14,15,35]).

The topic became more popular after Perelman's proof of the Poincaré conjecture, following Hamilton's program to use the Ricci flow (see [31]). Ricci solitons have been explored by a number of authors (see, e.g., [2,3,9–11,17,21, 30,33]).

Ricci solitons are also of interest to physicists, and in physical literature are called *quasi-Einstein* (see, e.g., [12,16]).

The presence of the structure 1-form η on manifolds with almost contact or almost paracontact structure motivates the need to introduce so-called η-Ricci solitons. Then, $\eta \otimes \eta$ is the restriction of the metric on the orthogonal complement to the (para)contact distribution, determined by the structure vector field ξ. By adding a term proportional to $\eta\otimes\eta$ into the defining equality of a Ricci soliton, it is defined the notion of *η-Ricci soliton*, introduced in [13]. Later, it has been studied on almost contact and almost paracontact manifolds by many authors (e.g., [1,4,5,8,34]). For almost η-Ricci solitons, see, for example [6,7].

Our global goal is to study the differential geometry of almost contact B-metric manifolds investigated since 1993 [18,29].

Unlike almost contact metric manifolds, almost contact B-metric manifolds have two metrics that are mutually associated with structural endomorphism. The restrictions of both B-metrics on the orthogonal distribution to the contact distribution is $\eta \otimes \eta$. This is the reason for introducing in [27] a further generalization of the notions of a Ricci soliton and an η-Ricci soliton, the so-called *Ricci-like soliton*, using both B-metrics and $\eta\otimes\eta$. There, we have explored these objects with potential Reeb vector field on some important kinds of manifolds under consideration: Einstein-like, Sasaki-like and having a torse-forming Reeb vector field. In [28], we continue to study Ricci-like solitons, whose potential is the Reeb vector field or pointwise collinear to it.

In the present paper, our goal is to investigate Ricci-like solitons with arbitrary potential on almost contact B-metric manifolds of Sasaki-like type, as well as gradient almost Ricci-like solitons on these manifolds.

The paper is organized as follows. In Sect. 1, we recall basic definitions and properties of almost contact B-metric manifolds of Sasaki-like type and obtain several immediate consequences. Section 2 includes some necessary results and a 5-dimensional example for a Ricci-like soliton with a potential Reeb vector field. In Sect. 3, we study Ricci-like solitons with an arbitrary potential. Then, we prove an identity for the soliton constants and a property of the potential, as well as that the Ricci tensor is a constant multiple of $\eta\otimes\eta$. For the 3-dimensional case, we find the values of the sectional curvatures of the special 2-planes with respect to the structure and construct an explicit example. In

Sect. 4, we introduce gradient almost Ricci-like solitons on Sasaki-like manifolds and prove that their Ricci tensor has the same form as in the previous section. For the example in Sect. 3, we find a potential function to illustrate the obtained results.

1. Sasaki-like Almost Contact B-metric Manifolds

A differentiable manifold M of dimension $(2n+1)$, equipped with an almost contact structure (φ, ξ, η) and a B-metric g is called an *almost contact B-metric manifold* and it is denoted by $(M, \varphi, \xi, \eta, g)$. More concretely, φ is an endomorphism of the tangent bundle TM, ξ is a Reeb vector field, η is its dual contact 1-form and g is a pseudo-Riemannian metric of signature $(n+1, n)$ satisfying the following conditions [18]

$$\begin{aligned} &\varphi\xi = 0, \qquad \varphi^2 = -\iota + \eta \otimes \xi, \qquad \eta \circ \varphi = 0, \qquad \eta(\xi) = 1, \\ &g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y), \end{aligned} \tag{1.1}$$

where ι stands for the identity transformation on $\Gamma(TM)$.

In the latter equality and further, x, y, z, w will stand for arbitrary elements of $\Gamma(TM)$ or vectors in the tangent space T_pM of M at an arbitrary point p in M.

The following equations are immediate consequences of (1.1)

$$\begin{aligned} &g(\varphi x, y) = g(x, \varphi y), \; g(x, \xi) = \eta(x), \\ &g(\xi, \xi) = 1, \qquad\qquad \eta(\nabla_x \xi) = 0, \end{aligned} \tag{1.2}$$

where ∇ denotes the Levi-Civita connection of g.

The associated metric $\tilde{g}$ of g on M is also a B-metric and it is defined by

$$\tilde{g}(x, y) = g(x, \varphi y) + \eta(x)\eta(y).$$

In [18], almost contact B-metric manifolds (also known as almost contact complex Riemannian manifolds) are classified with respect to the (0,3)-tensor F defined by

$$F(x, y, z) = g\big((\nabla_x \varphi)\, y, z\big).$$

It has the following basic properties:

$$\begin{aligned} &F(x, y, z) = F(x, z, y) = F(x, \varphi y, \varphi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi), \\ &F(x, \varphi y, \xi) = (\nabla_x \eta)y = g(\nabla_x \xi, y). \end{aligned}$$

This classification consists of eleven basic classes $\mathcal{F}_i$, $i \in \{1, 2, \dots, 11\}$.

In [20], it is introduced the type of a *Sasaki-like* manifold among almost contact B-metric manifolds. The definition condition is its complex cone to be a Kähler-Norden manifold, i.e. with a parallel complex structure. A Sasaki-like manifold with almost contact B-metric structure is determined by the condition

$$(\nabla_x \varphi)\, y = -g(x, y)\xi - \eta(y)x + 2\eta(x)\eta(y)\xi, \tag{1.3}$$

which is equivalent to the following $(\nabla_x\varphi)\,y = g(\varphi x, \varphi y)\xi + \eta(y)\varphi^2 x$.

Obviously, Sasaki-like manifolds form a subclass of the class $\mathcal{F}_4$. Moreover, the following identities are valid for it [20]

$$\begin{array}{ll} \nabla_x\xi = -\varphi x, & (\nabla_x\eta)\,(y) = -g(x, \varphi y), \\ R(x,y)\xi = \eta(y)x - \eta(x)y, & \rho(x,\xi) = 2n\,\eta(x), \\ R(\xi,y)z = g(y,z)\xi - \eta(z)y, & \rho(\xi,\xi) = 2n, \end{array} \tag{1.4}$$

where R and ρ stand for the curvature tensor and the Ricci tensor of ∇.

The corresponding curvature tensor of type $(0,4)$ is determined as usually by $R(x,y,z,w) = g(R(x,y)z,w)$.

Further, we use an arbitrary basis $\{e_i\}$, $i \in \{1,2,\dots,2n+1\}$ of T_pM, $p \in M$.

On an arbitrary almost contact B-metric manifold, there exists a $(0,2)$-tensor ρ^*, which is associated with ρ regarding φ. It is defined by $\rho^*(y,z) = g^{ij}R(e_i,y,z,\varphi e_j)$ and due to the first equality in (1.2) ρ^* is symmetric.

The following relation between ρ^* and ρ is valid for a Sasaki-like manifold

$$\rho^*(y,z) = \rho(y,\varphi z) + (2n-1)g(y,\varphi z). \tag{1.5}$$

It follows by taking the trace for $x = e_i$ and $w = e_j$ the following property of a Sasaki-like manifold [20]

$$\begin{aligned} &R(x,y,\varphi z,w) - R(x,y,z,\varphi w) \\ &\quad = \{g(y,z) - 2\eta(y)\eta(z)\}\,g(x,\varphi w) + \{g(y,w) - 2\eta(y)\eta(w)\}\,g(x,\varphi z) \\ &\qquad - \{g(x,z) - 2\eta(x)\eta(z)\}\,g(y,\varphi w) - \{g(x,w) - 2\eta(x)\eta(w)\}\,g(y,\varphi z). \end{aligned} \tag{1.6}$$

As a corollary of (1.5) we have that $\rho(y,\varphi z) = \rho(\varphi y, z)$, i.e. $Q \circ \varphi = \varphi \circ Q$, where Q is the Ricci operator, i.e. $\rho(y,z) = g(Qy,z)$.

The scalar curvature $\tilde{\tau}$ of $\tilde{g}$ is defined by $\tilde{\tau} = \tilde{g}^{ij}\tilde{g}^{kl}\tilde{R}(e_i,e_k,e_l,e_j)$, where $\tilde{R}$ is the curvature tensor of $\tilde{g}$ and $\tilde{g}^{ij} = -\varphi^j_k g^{ik} + \xi^i\xi^j$ holds. In addition, the associated quantity τ^* of τ with respect to φ is determined by $\tau^* = g^{ij}\rho(e_i,\varphi e_j)$. For them, using (1.6) for a Sasaki-like manifold, we infer the following relation

$$\tilde{\tau} = -\tau^* + 2n. \tag{1.7}$$

In [26], it is given the following relations between τ and $\tilde{\tau}$

$$\mathrm{d}\tau \circ \varphi = \mathrm{d}\tilde{\tau} + 2(\tau - 2n)\eta, \qquad \mathrm{d}\tilde{\tau} \circ \varphi = -\mathrm{d}\tau + 2(\tilde{\tau} - 2n)\eta.$$

As corollaries we have

$$\mathrm{d}\tau \circ \varphi^2 = \mathrm{d}\tilde{\tau} \circ \varphi, \qquad \mathrm{d}\tilde{\tau} \circ \varphi^2 = -\mathrm{d}\tau \circ \varphi, \tag{1.8}$$

$$\mathrm{d}\tau(\xi) = 2(\tilde{\tau} - 2n), \qquad \mathrm{d}\tilde{\tau}(\xi) = -2(\tau - 2n). \tag{1.9}$$

Proposition 1.1. *On a Sasaki-like manifold* $(M, \varphi, \xi, \eta, g)$ *of dimension* $2n+1$, *the following formulae for the Ricci operator* Q *are valid*

$$(\nabla_x Q)\xi = Q\varphi x - 2n\,\varphi x, \tag{1.10}$$

$$(\nabla_\xi Q)y = 2Q\varphi y. \tag{1.11}$$

Proof. For a Sasaki-like manifold, according to (1.4), the equalities $Q\xi = 2n\,\xi$ and $\nabla_x \xi = -\varphi x$ holds. Using them, we obtain immediately the covariant derivative in (1.10).

Now, we apply ∇_z to the expression of $R(x,y)\xi$ in (1.4) and then, using the form of $\nabla\eta$ in (1.4), we get the following

$$(\nabla_z R)\,(x,y)\xi = R(x,y)\varphi z - g(y,\varphi z)x + g(x,\varphi z)y.$$

We take the trace of the above equality for $z = e_i$ and $x = e_j$ and use (1.5) to obtain

$$g^{ij}(\nabla_{e_i} R)(e_j, y)\xi = -Q\varphi y - 2n\varphi y.$$

By virtue of the following consequence the second Bianchi identity

$$g^{ij}(\nabla_{e_i} R)(\xi, y)e_j = (\nabla_y Q)\,\xi - (\nabla_\xi Q)\,y,$$

the symmetries of R and (1.10), we get (1.11). □

As consequences of (1.10) and (1.11) we obtain respectively

$$\eta\big((\nabla_x Q)\xi\big) = 0, \qquad \eta\big((\nabla_\xi Q)y\big) = 0. \tag{1.12}$$

Let us recall [27], an almost contact B-metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be *Einstein-like* if its Ricci tensor ρ satisfies

$$\rho = a\,g + b\,\tilde{g} + c\,\eta\otimes\eta \tag{1.13}$$

for some triplet of constants (a, b, c). In particular, when $b = 0$ and $b = c = 0$, the manifold is called an *η-Einstein manifold* and an *Einstein manifold*, respectively.

If a, b, c are functions on M, then the manifold is called *almost Einstein-like*, *almost η-Einstein* and *almost Einstein*, respectively.

Tracing (1.13) and using (1.7), the scalar curvatures τ and $\tilde{\tau}$ of an Einstein-like almost contact B-metric manifold have the form

$$\tau = (2n+1)a + b + c, \qquad \tilde{\tau} = 2n(b+1). \tag{1.14}$$

For a Sasaki-like manifold $(M, \varphi, \xi, \eta, g)$ with $\dim M = 2n+1$ and a scalar curvature τ regarding g, which is Einstein-like with a triplet of constants (a, b, c), the following equalities are given in [27]:

$$a + b + c = 2n, \qquad \tau = 2n(a+1). \tag{1.15}$$

Then, for $\tilde{\tau}$ on an Einstein-like Sasaki-like manifold we obtain

$$\tilde{\tau} = 2n(b+1) \tag{1.16}$$

and because (1.14)–(1.16), the expression (1.13) becomes

$$\rho = \left(\frac{\tau}{2n} - 1\right) g + \left(\frac{\tilde{\tau}}{2n} - 1\right) \tilde{g} + \left(2(n+1) - \frac{\tau + \tilde{\tau}}{2n}\right) \eta \otimes \eta.$$

Proposition 1.2. *Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional Sasaki-like manifold. If it is almost Einstein-like with functions (a, b, c) then the scalar curvatures τ and $\tilde{\tau}$ of g and $\tilde{g}$, respectively, are constants*

$$\tau = const, \qquad \tilde{\tau} = 2n$$

and $(M, \varphi, \xi, \eta, g)$ is η-Einstein with constants

$$(a, b, c) = \left(\frac{\tau}{2n} - 1,\ 0,\ 2n + 1 - \frac{\tau}{2n}\right).$$

Proof. If $(M, \varphi, \xi, \eta, g)$ is almost Einstein-like then ρ has the form in (1.13), where (a, b, c) are a triad of functions. Then, according to (1.7), (1.13), (1.14) and the expression for $\rho(\xi, \xi)$ on a Sasaki-like manifold, given in (1.4), we have the following

$$a + b + c = 2n, \qquad \tau = 2n(a+1), \qquad \tilde{\tau} = 2n(b+1). \tag{1.17}$$

Using (1.4), we can express $R(x, y)\xi$ and $R(x, \xi)y$ as follows

$$\begin{aligned} R(x,y)\xi = \frac{1}{4n^2}\big\{&2n\left[\eta(x)Qy - \eta(y)Qx + \rho(x,\xi)y - \rho(y,\xi)x\right] \\ &+(\tau - 2n)\left[\eta(y)x - \eta(x)y\right] + (\tilde{\tau} - 2n)\left[\eta(y)\varphi x - \eta(x)\varphi y\right]\big\}, \\ R(x,\xi)y = \frac{1}{4n^2}\big\{&2n\left[\rho(x,y)\xi + g(x,y)Q\xi - \rho(y,\xi)x - \eta(y)Qx\right] \\ &+(\tau - 2n)\left[\eta(y)x - g(x,y)\xi\right] \\ &+(\tilde{\tau} - 2n)\left[\eta(y)\varphi x - g(x,\varphi y)\xi\right]\big\}. \end{aligned}$$

Then, for $y = \xi$ in either of the last two equalities, we have

$$R(x,\xi)\xi = \eta(x)\xi - \frac{1}{2n}Qx - \frac{1}{4n^2}\big\{[\tau - 2n(2n+1)]\varphi^2 x - [\tilde{\tau} - 2n]\varphi x\big\}.$$

After that, we compute the covariant derivative of $R(x, \xi)\xi$ with respect to ∇_z. Since (1.3) and (1.4), we obtain

$$\begin{aligned} \left(\nabla_z R\right)(x,\xi)\xi = -\frac{1}{2n}\left\{(\nabla_z Q)x - \eta(x)Q\varphi z\right\} - \frac{1}{4n^2}\big\{&\mathrm{d}\tau(z)\varphi^2 x - \mathrm{d}\tilde{\tau}(z)\varphi x \\ -[\tau - 2n(2n+1)]g(x,\varphi z)\xi + [\tilde{\tau} - 2n]g(\varphi x, \varphi z)\xi&\big\} - \eta(x)\varphi z, \end{aligned}$$

which by taking the trace for $z = e_i$ and $x = e_j$ and (1.8) gives the following

$$g^{ij} g\big(\left(\nabla_{e_i} R\right)(e_j,\xi)\xi, y\big) = -\frac{1}{4n}\mathrm{d}\tau(y) - \left\{\frac{\tilde{\tau}}{2n} - 1\right\}\eta(y). \tag{1.18}$$

By virtue of the following consequence of the second Bianchi identity

$$g^{ij} g\big((\nabla_{e_i} R)(y,\xi)\xi, e_j\big) = \eta\big((\nabla_y Q)\,\xi\big) - \eta\big((\nabla_\xi Q)\,y\big) \tag{1.19}$$

and (1.12), we have that the trace in the left hand side of (1.19) vanishes. Then, (1.18) and (1.19) imply

$$\mathrm{d}\tau(y) = -2\{\tilde{\tau} - 2n\}\eta(y),$$

which comparing with (1.9) implies

$$\mathrm{d}\tau(\xi) = 0, \qquad \tilde{\tau} = 2n.$$

The latter equalities together with (1.8) and (1.17) complete the proof. □

2. Ricci-like Solitons with Potential Reeb Vector Field on Sasaki-like Manifolds

In [27], by a condition for Ricci tensor, it is introduced the notion of a Ricci-like soliton with potential ξ on an almost contact B-metric manifold.

Now, we generalize this notion for a potential, which is an arbitrary vector field as follows. We say that $(M, \varphi, \xi, \eta, g)$ admits a *Ricci-like soliton with potential vector field* v if the following condition is satisfied for a triplet of constants (λ, μ, ν)

$$\frac{1}{2}\mathcal{L}_v g + \rho + \lambda\, g + \mu\, \tilde{g} + \nu\, \eta \otimes \eta = 0, \tag{2.1}$$

where $\mathcal{L}$ denotes the Lie derivative.

If $\mu = 0$ (respectively, $\mu = \nu = 0$), then (2.1) defines an *η-Ricci soliton* (respectively, a *Ricci soliton*) on $(M, \varphi, \xi, \eta, g)$.

If λ, μ, ν are functions on M, then the soliton is called *almost Ricci-like soliton*, *almost η-Ricci soliton* and *almost Ricci soliton*, respectively.

If $(M, \varphi, \xi, \eta, g)$ is Sasaki-like, we have

$$(\mathcal{L}_\xi g)(x, y) = g(\nabla_x \xi, y) + g(x, \nabla_y \xi) = -2g(x, \varphi y),$$

i.e. $\frac{1}{2}\mathcal{L}_\xi g = -\tilde{g} + \eta \otimes \eta$. Then, because of (2.1), ρ takes the form

$$\rho = -\lambda g + (1 - \mu)\tilde{g} - (1 + \nu)\eta \otimes \eta.$$

Theorem 2.1 ([27]). *Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional Sasaki-like manifold and let a, b, c, λ, μ, ν be constants that satisfy the following equalities:*

$$a + \lambda = 0, \qquad b + \mu - 1 = 0, \qquad c + \nu + 1 = 0.$$

Then, the manifold admits a Ricci-like soliton with potential ξ and constants (λ, μ, ν), where $\lambda + \mu + \nu = -2n$, if and only if it is Einstein-like with constants (a, b, c), where $a + b + c = 2n$.

In particular, we get:

(i) *The manifold admits an η-Ricci soliton with potential ξ and constants $(\lambda, 0, -2n - \lambda)$ if and only if the manifold is Einstein-like with constants $(-\lambda, 1, \lambda + 2n - 1)$.*

(ii) *The manifold admits a shrinking Ricci soliton with potential* ξ *and constant* $-2n$ *if and only if the manifold is Einstein-like with constants* $(2n, 1, -1)$.
(iii) *The manifold is* η*-Einstein with constants* $(a, 0, 2n-a)$ *if and only if it admits a Ricci-like soliton with potential* ξ *and constants* $(-a, 1, a-2n-1)$.
(iv) *The manifold is Einstein with constant* $2n$ *if and only if it admits a Ricci-like soliton with potential* ξ *and constants* $(-2n, 1, -1)$.

2.1. Example 1

In Example 2 of [20], it is given a Lie group G of dimension 5 (i.e. $n = 2$) with a basis of left-invariant vector fields $\{e_0, \dots, e_4\}$ and the corresponding Lie algebra is defined as follows

$$\begin{aligned}&[e_0, e_1] = pe_2 + e_3 + qe_4, \quad [e_0, e_2] = -pe_1 - qe_3 + e_4,\\&[e_0, e_3] = -e_1 - qe_2 + pe_4, [e_0, e_4] = qe_1 - e_2 - pe_3, \qquad p, q \in \mathbb{R}.\end{aligned}$$

After that G is equipped with an almost contact B-metric structure defined by

$$\begin{aligned}&g(e_0, e_0) = g(e_1, e_1) = g(e_2, e_2) = -g(e_3, e_3) = -g(e_4, e_4) = 1,\\&g(e_i, e_j) = 0, \quad i, j \in \{0, 1, \dots, 4\},\ i \neq j,\\&\qquad \xi = e_0, \quad \varphi e_1 = e_3, \quad \varphi e_2 = e_4, \quad \varphi e_3 = -e_1, \quad \varphi e_4 = -e_2.\end{aligned}$$

It is verified that the constructed almost contact B-metric manifold $(G, \varphi, \xi, \eta, g)$ is Sasaki-like.

In [27], it is proved that $(G, \varphi, \xi, \eta, g)$ is η-Einstein with constants

$$(a, b, c) = (0, 0, 4). \tag{2.2}$$

Moreover, it is clear that $\tau = \tilde{\tau} = 4$.

It is also found there that $(G, \varphi, \xi, \eta, g)$ admits a Ricci-like soliton with potential ξ and constants

$$(\lambda, \mu, \nu) = (0, 1, -5). \tag{2.3}$$

Therefore, this example is in unison with Theorem 2.1 (iii) for $a = 0$.

3. Ricci-like Solitons with Arbitrary Potential on Sasaki-like Manifolds

Theorem 3.1. *Let* $(M, \varphi, \xi, \eta, g)$ *be a* $(2n+1)$*-dimensional Sasaki-like manifold. If it admits a Ricci-like soliton with arbitrary potential vector field* v *and constants* (λ, μ, ν) *then it is valid the following identities*

$$\begin{aligned}\lambda + \mu + \nu &= -2n,\\ \nabla_\xi v &= -\varphi v.\end{aligned} \tag{3.1}$$

Proof. According to (2.1), a Ricci-like soliton with arbitrary potential vector field v is defined by $\big(\mathcal{L}_v g\big)(y,z) = -2\rho(y,z) - 2\lambda\, g(y,z) - 2\mu\, \tilde{g}(y,z) - 2\nu\, \eta(y)\eta(z)$. Then, bearing in mind (1.4), the covariant derivative with respect to ∇_x has the form

$$\begin{aligned}\big(\nabla_x \mathcal{L}_v g\big)(y,z) = &-2\big(\nabla_x \rho\big)(y,z) - 2\mu\{g(\varphi x,\varphi y)\eta(z) + g(\varphi x,\varphi z)\eta(y)\}\\ &+2(\mu+\nu)\{g(x,\varphi y)\eta(z) + g(x,\varphi z)\eta(y)\}.\end{aligned} \tag{3.2}$$

We use of the following formula from [36] for a metric connection ∇

$$\big(\nabla_x \mathcal{L}_v g\big)(y,z) = g\left((\mathcal{L}_v \nabla)(x,y),z\right) + g\left((\mathcal{L}_v \nabla)(x,z),y\right),$$

which due to symmetry of $\mathcal{L}_v\nabla$ can read as

$$2g\big((\mathcal{L}_v\nabla)(x,y),z\big) = \big(\nabla_x \mathcal{L}_v g\big)(y,z) + \big(\nabla_y \mathcal{L}_v g\big)(z,x) - \big(\nabla_z \mathcal{L}_v g\big)(x,y). \tag{3.3}$$

Applying (3.3) to (3.2), we obtain

$$\begin{aligned}g\big((\mathcal{L}_v\nabla)(x,y),z\big) = &-\big(\nabla_x \rho\big)(y,z) - \big(\nabla_y \rho\big)(z,x) + \big(\nabla_z \rho\big)(x,y)\\ &-2\mu\, g(\varphi x,\varphi y)\eta(z) + 2(\mu+\nu) g(x,\varphi y)\eta(z).\end{aligned} \tag{3.4}$$

Setting $y=\xi$ in the equality above and using (1.10) and (1.11), we get

$$(\mathcal{L}_v\nabla)(x,\xi) = -2Q\varphi x. \tag{3.5}$$

The covariant derivative of the above equation by using of (1.4) has the form

$$\begin{aligned}\big(\nabla_y \mathcal{L}_v\nabla\big)(x,\xi) = &(\mathcal{L}_v\nabla)(x,\varphi y) - 2(\nabla_y Q)\varphi x + 2\eta(x)Qy\\ &-4n\, g(x,y) - 2(2n+1)\eta(x)\eta(y)\xi.\end{aligned} \tag{3.6}$$

We apply the latter equality to the following formula from [36]

$$(\mathcal{L}_v R)(x,y)z = \big(\nabla_x \mathcal{L}_v\nabla\big)(y,z) - \big(\nabla_y \mathcal{L}_v\nabla\big)(x,z) \tag{3.7}$$

and owing to symmetry of $\mathcal{L}_v\nabla$, we obtain the following consequence of (3.5)–(3.7)

$$\begin{aligned}g\big((\mathcal{L}_v R)(x,y)\xi,z\big) = &-(\nabla_x\rho)(\varphi y,z) + (\nabla_{\varphi y}\rho)(x,z) - (\nabla_z\rho)(x,\varphi y)\\ &+(\nabla_y\rho)(\varphi x,z) - (\nabla_{\varphi x}\rho)(y,z) + (\nabla_z\rho)(\varphi x,y)\\ &-2\,\eta(x)\rho(y,z) + 2\,\eta(y)\rho(x,z).\end{aligned} \tag{3.8}$$

Plugging $y=z=\xi$ in (3.8) and using (3.5), we obtain

$$(\mathcal{L}_v R)(x,\xi)\xi = 0. \tag{3.9}$$

On the other hand, applying $\mathcal{L}_v$ to the expression of $R(x,\xi)\xi$ from (1.4) and using (2.1), as well as the formulae for $R(x,y)\xi$ and $R(\xi,y)z$ from the same referent equalities, we get

$$(\mathcal{L}_v R)(x,\xi)\xi = (\mathcal{L}_v\eta)(x)\xi + g(x,\mathcal{L}_v\xi)\xi - 2\eta(\mathcal{L}_v\xi)x$$

or in an equivalent form

$$(\mathcal{L}_v R)(x,\xi)\xi = \{(\mathcal{L}_v\eta)(x) + g(x,\mathcal{L}_v\xi) - 2\eta(\mathcal{L}_v\xi)\eta(x)\}\xi + 2\eta(\mathcal{L}_v\xi)\varphi^2 x. \tag{3.10}$$

Comparing (3.9) and (3.10), we obtain the following system of equations

$$(\mathcal{L}_v\eta)(x) + g(x, \mathcal{L}_v\xi) - 2\eta(\mathcal{L}_v\xi)\eta(x) = 0, \qquad \eta(\mathcal{L}_v\xi) = 0,$$

i.e.

$$(\mathcal{L}_v\eta)(x) + g(x, \mathcal{L}_v\xi) = 0, \qquad \eta(\mathcal{L}_v\xi) = 0. \tag{3.11}$$

According to (2.1) and $\rho(x, \xi) = 2n\eta(x)$ from (1.4), we have for a Sasaki-like manifold

$$(\mathcal{L}_v g)(x, \xi) = -2(\lambda + \mu + \nu + 2n)\eta(x) \tag{3.12}$$

and as a consequence for $x = \xi$ the following

$$(\mathcal{L}_v g)(\xi, \xi) = -2(\lambda + \mu + \nu + 2n). \tag{3.13}$$

The Lie derivative of $g(x, \xi) = \eta(x)$ with respect to v gives

$$(\mathcal{L}_v g)(x, \xi) = (\mathcal{L}_v\eta)(x) - g\left(x, \mathcal{L}_v\xi\right), \tag{3.14}$$

which for $x = \xi$ leads to

$$(\mathcal{L}_v g)(\xi, \xi) = -2\eta\left(\mathcal{L}_v\xi\right). \tag{3.15}$$

From (3.13) and (3.15) we obtain

$$\eta(\mathcal{L}_v\xi) = \lambda + \mu + \nu + 2n.$$

The latter equality implies (3.1), by virtue of the second equality in (3.11).

Substituting (3.1) in (3.12) gives the vanishing of $(\mathcal{L}_v g)(x, \xi)$ and because of (3.14) we have $(\mathcal{L}_v\eta)(x) = g\left(x, \mathcal{L}_v\xi\right)$. Hence, bearing in mind the first equality in (3.11), we get

$$\mathcal{L}_v\xi = 0,$$

which together with $\nabla\xi = -\varphi$ from (1.4) completes the proof. □

Proposition 3.2. *Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional Sasaki-like manifold. If it admits a Ricci-like soliton with arbitrary potential v then the Ricci tensor ρ of g and the scalar curvatures τ and $\tilde{\tau}$ of g and $\tilde{g}$, respectively, satisfy the following equalities*

$$\left(\mathcal{L}_v\rho\right)(x, \xi) = 0, \qquad \tau = 2n, \qquad \tilde{\tau} = const.$$

Proof. By (3.7) we find the following

$$\begin{aligned} g\big(\left(\mathcal{L}_v R\right)(x, y)\xi, z\big) = & - g\big(\left(\mathcal{L}_v\nabla\right)(x, \varphi y), z\big) + g\big(\left(\mathcal{L}_v\nabla\right)(\varphi x, y), z\big) \\ & -2\left(\nabla_x\rho\right)(\varphi y, z) + 2\left(\nabla_y\rho\right)(\varphi x, z) \\ & -2\eta(x)\rho(y, z) + 2\eta(y)\rho(x, z). \end{aligned}$$

Taking the trace of the last equality for $x = e_i$ and $z = e_j$ and using (3.4) and (1.7), $\mathrm{d}\tau = 2\,\mathrm{div}\,\rho$, we obtain successively

$$\begin{aligned} g^{ij}g\big(\left(\mathcal{L}_v\nabla\right)(e_i, \varphi y), e_j\big) &= -\mathrm{d}\tau(\varphi y), \\ g^{ij}g\big(\left(\mathcal{L}_v\nabla\right)(\varphi e_i, y), e_j\big) &= \mathrm{d}\tilde{\tau}(y), \\ g^{ij}g\big(\left(\mathcal{L}_v R\right)(e_i, y)\xi, e_j\big) &= \left(\mathcal{L}_v\rho\right)(y, \xi) \end{aligned} \tag{3.16}$$

and therefore the following formula is valid

$$(\mathcal{L}_v\rho)(y,\xi) = -\,\mathrm{d}\tilde{\tau}(y) + 2(\tau - 2n)\eta(y), \tag{3.17}$$

which for $y = \xi$ implies

$$(\mathcal{L}_v\rho)(\xi,\xi) = -\,\mathrm{d}\tilde{\tau}(\xi) + 2(\tau - 2n). \tag{3.18}$$

On the other hand, according to (3.9) and (3.16), $(\mathcal{L}_v\rho)(\xi,\xi)$ vanishes and therefore (3.17) and (3.18) imply

$$(\mathcal{L}_v\rho)(x,\xi) = \mathrm{d}\tilde{\tau}(\varphi^2 x), \qquad \mathrm{d}\tilde{\tau}(\xi) = 2\tau - 4n. \tag{3.19}$$

The latter equalities, due to (1.8) and (1.9), imply consequently $\mathrm{d}\tilde{\tau}(\xi) = 0$ and

$$\tau = 2n, \qquad \tilde{\tau} = const.$$

In conclusion, because of (3.19), we infer the assertion. □

Theorem 3.3. *Let (M,φ,ξ,η,g) be a $(2n+1)$-dimensional Einstein-like Sasaki-like manifold. If it admits a Ricci-like soliton with potential v then the Ricci tensor is $\rho = 2n\,\eta\otimes\eta$ and the scalar curvatures are $\tau = \tilde{\tau} = 2n$.*

Proof. The assertion follows from Theorem 3.1, Propositions 1.2 and 3.2. □

Corollary 3.4. *Let (M,φ,ξ,η,g), $\dim M = 2n+1$, be an Einstein-like Sasaki-like manifold. Then it is η-Einstein with constants $(0,0,2n)$, which is equivalent to the existence on M of a Ricci-like soliton with potential ξ and constants $(0,1,-2n-1)$.*

Proof. Using Theorem 3.3, we obtain the following expression $\mathcal{L}_v g = -2\lambda\, g - 2\mu\,\tilde{g} + 2(\lambda+\mu)\eta\otimes\eta$, which holds for $\lambda = 0$, $\mu = 1$ in the case $v = \xi$. Therefore Theorem 2.1 is restricted to its case (iii) and $a = 0$. □

Let us recall, every non-degenerate 2-plane (or section) β with a basis $\{x,y\}$ with respect to g in T_pM, $p\in M$, has the following sectional curvature

$$k(\beta;p) = \frac{R(x,y,y,x)}{g(x,x)g(y,y) - [g(x,y)]^2}. \tag{3.20}$$

A section β is said to be φ-*holomorphic* if the condition $\beta = \varphi\beta$ holds. Every φ-holomorphic section has a basis of the form $\{\varphi x, \varphi^2 x\}$. A section β is called a ξ-*section* if it has a basis of the form $\{x,\xi\}$.

Theorem 3.5. *Let (M,φ,ξ,η,g) be a 3-dimensional Sasaki-like manifold. If it admits a Ricci-like soliton with potential v then:*

(i) *the sectional curvatures of its φ-holomorphic sections are equal to -1;*
(ii) *the sectional curvatures of its ξ-sections are equal to 1.*

Proof. It is well known that the curvature tensor of a 3-dimensional manifold has the form

$$R(x,y)z = g(y,z)Qx - g(x,z)Qy + \rho(y,z)x - \rho(x,z)y \\ -\frac{\tau}{2}\{g(y,z)x - g(x,z)y\}. \tag{3.21}$$

Then, substituting $y = z = \xi$ and recalling (1.4), we have

$$\rho = \frac{1}{2}\{(\tau-2)g - (\tau-6)\eta\otimes\eta\},$$

which means that the manifold is η-Einstein. Therefore, because of Theorem 3.3, we have

$$\rho = 2\,\eta\otimes\eta, \qquad \tau = \tilde{\tau} = 2.$$

Substituting the latter two equalities for τ and ρ in (3.21), we get

$$R(x,y)z = -\,[g(y,z) - 2\,\eta(y)\eta(z)]x + 2\,g(y,z)\eta(x)\xi \\ +[g(x,z) - 2\,\eta(x)\eta(z)]y - 2\,g(x,z)\eta(y)\xi.$$

Using a basis $\{\varphi x, \varphi^2 x\}$ of an arbitrary φ-holomorphic section, we calculate its sectional curvature by (3.20), replacing x and y by φx and $\varphi^2 x$, respectively. Then, bearing in mind (1.1) and (1.2), we obtain $k(\varphi x, \varphi^2 x) = -1$.

Similarly, for a ξ-section with a basis $\{x, \xi\}$, we get $k(x,\xi) = 1$, which completes the proof. □

Remark 3.6. Examples of 3-dimensional Sasaki-like manifolds as a Lie group from type $Bia(VII_0)(1)$, a matrix Lie group, an S^1-solvable extension on a Kähler-Norden 2-manifold, and their geometrical properties are studied in [22–25], respectively.

Remark 3.7. The constructed 5-dimensional example in Sect. 3.1 of a Sasaki-like manifold with the results in (2.2) and (2.3) supports also Theorem 3.1, Proposition 3.2, Theorem 3.3 and Corollary 3.4 for the case of $v = \xi$ and $n = 2$.

3.1. Example 2

Let us consider M as a set of points in $\mathbb{R}^3$ with coordinates (x^1, x^2, x^3) and let M be equipped with an almost contact B-metric structure defined by

$$g(\partial_1,\partial_1) = -g(\partial_2,\partial_2) = \cos 2x^3, \qquad g(\partial_1,\partial_2) = \sin 2x^3, \\ g(\partial_1,\partial_3) = g(\partial_2,\partial_3) = 0, \qquad g(\partial_3,\partial_3) = 1, \\ \varphi\partial_1 = \partial_2, \qquad \varphi\partial_2 = -\partial_1, \qquad \xi = \partial_3,$$

where ∂_1, ∂_2, ∂_3 denote briefly $\frac{\partial}{\partial x^1}$, $\frac{\partial}{\partial x^2}$, $\frac{\partial}{\partial x^3}$, respectively. Then, the vectors determined by

$$e_1 = \cos x^3\partial_1 + \sin x^3\partial_2, \qquad e_2 = -\sin x^3\partial_1 + \cos x^3\partial_2, \qquad e_3 = \partial_3 \tag{3.22}$$

form an orthonormal φ-basis of T_pM, $p \in M$, i.e.

$$\begin{aligned} g(e_1,e_1) = -g(e_2,e_2) = g(e_3,e_3) = 1 \\ g(e_i,e_j) = 0, \quad i,j \in \{1,2,3\},\ i \neq j, \\ \varphi e_1 = e_2, \qquad \varphi e_2 = -e_1, \qquad \xi = e_3. \end{aligned} \tag{3.23}$$

Immediately from (3.22) we obtain the commutators of e_i as follows

$$[e_0,e_1] = e_2, \qquad [e_0,e_2] = -e_1, \qquad [e_1,e_2] = 0. \tag{3.24}$$

Then, according to Example 1 in [20] for $n = 1$, the solvable Lie group of dimension 3 with a basis of left-invariant vector fields $\{e_1, e_2, e_3\}$ defined by (3.24) and equipped with the (φ, ξ, η, g)-structure from (3.23) is a Sasaki-like almost contact B-metric manifold.

In the well-known way, we calculate the components of the Levi-Civita connection ∇ for g and from there the corresponding components $R_{ijkl} = R(e_i, e_j, e_k, e_l)$ and $\rho_{ij} = \rho(e_i, e_j)$ of the curvature tensor R and the Ricci tensor ρ, respectively. The non-zero ones of them are the following (keep in mind the symmetries of R)

$$\nabla_{e_1} e_2 = \nabla_{e_2} e_1 = -e_3, \qquad \nabla_{e_1} e_3 = -e_2, \qquad \nabla_{e_2} e_3 = e_1; \tag{3.25}$$

$$R_{1221} = R_{1331} = -R_{2332} = 1, \qquad \rho_{33} = 2. \tag{3.26}$$

The latter equality means that the Ricci tensor has the following form

$$\rho = 2\eta \otimes \eta, \tag{3.27}$$

i.e. the manifold is Einstein-like with constants $(a, b, c) = (0, 0, 2)$. Therefore, the scalar curvatures with respect to g and $\tilde{g}$ are $\tau = \tilde{\tau} = 2$.

The values of R_{ijkl} in (3.26) imply the sectional curvatures

$$k_{12} = -k_{13} = -k_{23} = -1,$$

which supports Theorem 3.5.

Let us consider a vector field, determined by the following

$$\begin{aligned} v &= v^1 e_1 + v^2 e_2 + v^3 e_3, \\ v^1 &= -\{c_1 \cos x^3 + c_2 \sin x^3\} x^1 + \{c_2 \cos x^3 - c_1 \sin x^3\} x^2 + \sin x^3, \\ v^2 &= -\{c_2 \cos x^3 - c_1 \sin x^3\} x^1 - \{c_1 \cos x^3 + c_2 \sin x^3\} x^2 + \cos x^3, \\ v^3 &= c_3, \end{aligned} \tag{3.28}$$

and c_1, c_2, c_3 are arbitrary constants.

Using (3.22), (3.23), (3.25) and (3.28), we obtain the following

$$\begin{aligned} \nabla_{e_1} v &= -c_1 e_1 - (c_2 + c_3) e_2 - v^2 e_3, \\ \nabla_{e_2} v &= (c_2 + c_3) e_1 - c_1 e_2 - v^1 e_3, \\ \nabla_{e_3} v &= v^2 e_1 - v^1 e_2, \end{aligned} \tag{3.29}$$

that allow us to calculate the components $(\mathcal{L}_v g)_{ij} = (\mathcal{L}_v g)(e_i, e_j)$ of the Lie derivative $\mathcal{L}_v g$. Then, we get the following nonzero ones

$$(\mathcal{L}_v g)_{11} = -(\mathcal{L}_v g)_{22} = -2c_1, \qquad (\mathcal{L}_v g)_{12} = 2(c_2 + c_3),$$

which implies that this tensor has the following expression

$$\mathcal{L}_v g = -2\,c_1\,g - 2(c_2 + c_3)\tilde{g} + 2(c_1 + c_2 + c_3)\eta \otimes \eta. \tag{3.30}$$

Substituting the latter equality and (3.27) in (2.1), we obtain that $(M, \varphi, \xi, \eta, g)$ admits a Ricci-like soliton with potential v determined by (3.28) and the potential constants are

$$\lambda = c_1, \qquad \mu = c_2 + c_3, \qquad \nu = -c_1 - c_2 - c_3 - 2.$$

These results are in accordance with Theorems 3.1 and 3.3. The conclusion in Proposition 3.2 follows from (3.27) and the subsequent formula $(\mathcal{L}_v\rho)(x,\xi) = -2g(\varphi x, v) + 2\eta(\nabla_x v)$, together with the equalities in (3.28) and (3.29).

4. Gradient Almost Ricci-like Solitons

Let us consider a Ricci-like soliton, defined by (2.1) with the condition λ, μ, ν to be functions on M. If its potential v is a gradient of a differentiable function f, i.e. $v = \operatorname{grad} f$, then the soliton is called a *gradient almost Ricci-like soliton* of $(M, \varphi, \xi, \eta, g)$. In this case (2.1) is reduced to the following condition

$$\operatorname{Hess} f + \rho + \lambda g + \mu\tilde{g} + \nu\eta \otimes \eta = 0, \tag{4.1}$$

where Hess denotes the Hessian operator with respect to g, i.e. Hess f is defined by

$$(\operatorname{Hess} f)(x, y) := (\nabla_x \mathrm{d}f)(y) = g(\nabla_x \operatorname{grad} f, y). \tag{4.2}$$

Taking the trace of (4.1), we obtain

$$\Delta f + \tau + (2n + 1)\lambda + \mu + \nu = 0,$$

where $\Delta := \operatorname{tr} \circ \operatorname{Hess}$ is the Laplacian operator of g. Also for the Laplacian of f, the formula $\Delta f = \operatorname{div}(\operatorname{grad} f)$ is valid, where div stands for the divergence operator.

The gradient Ricci-like soliton is said to be *trivial* when f is constant. Further, we consider only non-trivial gradient Ricci-like solitons.

Equality (4.1) with the recall of (4.2) provides the following

$$\nabla_x v = -Qx - \lambda x - \mu\varphi x - (\mu + \nu)\eta(x)\xi, \tag{4.3}$$

where Q is the Ricci operator and $v = \operatorname{grad} f$.

Theorem 4.1. *Let $(M, \varphi, \xi, \eta, g)$ be a Sasaki-like almost contact B-metric manifold of dimension $2n+1$. If it admits a gradient almost Ricci-like soliton with functions (λ, μ, ν) and a potential function f, then $(M, \varphi, \xi, \eta, g)$ has constant scalar curvatures $\tau = \tilde{\tau} = 2n$ for both B-metrics g and $\tilde{g}$, respectively, and its Ricci tensor is $\rho = 2n\,\eta \otimes \eta$.*

Proof. Using (4.3), we compute the following curvature tensor

$$\begin{aligned} R(x,y)v = &-(\nabla_x Q)\,y + (\nabla_y Q)\,x \\ &+\{\mathrm{d}\lambda(y)+\mu\eta(y)\}x - \{\mathrm{d}\lambda(x)+\mu\eta(x)\}y \\ &+\{\mathrm{d}\mu(y)+(\mu+\nu)\eta(y)\}\varphi x - \{\mathrm{d}\mu(x)+(\mu+\nu)\eta(x)\}\varphi y \\ &+\mathrm{d}(\mu+\nu)(y)\eta(x)\xi - \mathrm{d}(\mu+\nu)(x)\eta(y)\xi. \end{aligned} \tag{4.4}$$

The latter expression implies the following equality

$$\begin{aligned} R(\xi,y)v = &-(\nabla_\xi Q)\,y + (\nabla_y Q)\,\xi + \{\mathrm{d}\lambda(\xi)+\mu\}\varphi^2 y - \{\mathrm{d}\mu(\xi)+\mu+\nu\}\varphi y \\ &+\mathrm{d}(\lambda+\mu+\nu)(y)\xi - \mathrm{d}(\lambda+\mu+\nu)(\xi)\eta(y)\xi, \end{aligned}$$

where we apply (1.10) and (1.11) and get

$$\begin{aligned} R(\xi,y)v = &-Q\varphi y + \{\mathrm{d}\lambda(\xi)+\mu\}\varphi^2 y - \{\mathrm{d}\mu(\xi)+\mu+\nu+2n\}\varphi y \\ &+\mathrm{d}(\lambda+\mu+\nu)(y)\xi - \mathrm{d}(\lambda+\mu+\nu)(\xi)\eta(y)\xi. \end{aligned} \tag{4.5}$$

We put $z=v$ in the equality for $R(\xi,y)z$ in (1.4) and obtain the following expression

$$R(\xi,y)v = \mathrm{d}f(y)\xi - \mathrm{d}f(\xi)y. \tag{4.6}$$

Combining (4.5) and (4.6), we find the following formula

$$\begin{aligned} Q\varphi y = &\{\mathrm{d}(\lambda-f)(\xi)+\mu\}\varphi^2 y - \{\mathrm{d}\mu(\xi)+\mu+\nu+2n\}\varphi y \\ &+\mathrm{d}(\lambda+\mu+\nu-f)(y)\xi - \mathrm{d}(\lambda+\mu+\nu-f)(\xi)\eta(y)\xi. \end{aligned} \tag{4.7}$$

We apply η of equality (4.7) and since $Q\circ\varphi = \varphi\circ Q$ for a Sasaki-like manifold, we obtain the following

$$\mathrm{d}(\lambda+\mu+\nu-f)(y) = \mathrm{d}(\lambda+\mu+\nu-f)(\xi)\eta(y), \tag{4.8}$$

which changes (4.7) and (4.5) as follows

$$\begin{aligned} Q\varphi y &= \{\mathrm{d}(\lambda-f)(\xi)+\mu\}\varphi^2 y - \{\mathrm{d}\mu(\xi)+\mu+\nu+2n\}\varphi y, \\ R(\xi,y)v &= -Q\varphi y + \{\mathrm{d}\lambda(\xi)+\mu\}\varphi^2 y - \{\mathrm{d}\mu(\xi)+\mu+\nu+2n\}\varphi y \\ &\quad +\{\mathrm{d}f(y)-\mathrm{d}f(\xi)\eta(y)\}\xi \end{aligned}$$

and therefore we have

$$\begin{aligned} R(\xi,y,v,z) = &-\rho(y,\varphi z) + \{\mathrm{d}\lambda(\xi)+\mu\}g(\varphi y,\varphi z) \\ &-\{\mathrm{d}\mu(\xi)+\mu+\nu+2n\}g(y,\varphi z) \\ &+\{\mathrm{d}f(y)-\mathrm{d}f(\xi)\eta(y)\}\eta(z). \end{aligned} \tag{4.9}$$

On the other hand, the expression of $R(x,y)\xi$ from (1.4) and equality (4.4) imply respectively the following two equalities

$$\begin{aligned} R(x,y,\xi,v) &= \mathrm{d}f(x)\eta(y) - \mathrm{d}f(y)\eta(x), \\ R(x,y,v,\xi) &= -\eta\left((\nabla_x Q)y - (\nabla_y Q)x\right) + \mathrm{d}(\lambda+\mu+\nu)(y)\eta(x) \\ &\quad -\mathrm{d}(\lambda+\mu+\nu)(x)\eta(y). \end{aligned}$$

By summation of the latter two equalities, we find the following formula

$$\eta\big((\nabla_x Q)y - (\nabla_y Q)x\big) = -\,\mathrm{d}(\lambda+\mu+\nu-f)(x)\eta(y) \\ +\mathrm{d}(\lambda+\mu+\nu-f)(y)\eta(x),$$

which because of (4.8) is simplified to the following form

$$\eta\big((\nabla_x Q)y - (\nabla_y Q)x\big) = 0.$$

On the other hand, the expression of $R(\xi, y)z$ from (1.4) yield

$$R(\xi, y, z, v) = -\,\mathrm{d}f(\xi)g(\varphi y, \varphi z) - \{\mathrm{d}f(y) - \mathrm{d}f(\xi)\eta(y)\}\eta(z),$$

which together with (4.9) and the form of $\rho(x, \xi)$ from (1.4) implies

$$\rho(y,z) = \{\mathrm{d}\mu(\xi)+\mu+\nu+2n\}g(\varphi y, \varphi z) + \{\mathrm{d}(\lambda-f)(\xi)+\mu\}g(y,\varphi z) \\ +2n\eta(y)\eta(z).$$

The latter equality can be rewritten in the form

$$\rho = -\,\{\mathrm{d}\mu(\xi)+\mu+\nu+2n\}g + \{\mathrm{d}(\lambda-f)(\xi)+\mu\}\tilde{g} \\ +\{4n+\nu-\mathrm{d}(\lambda-\mu-f)(\xi)\}\eta\otimes\eta,$$

which means that the manifold is almst Einstein-like with coefficient functions

$$a = -\mathrm{d}\mu(\xi)-\mu-\nu-2n, \qquad b = \mathrm{d}(\lambda-f)(\xi)+\mu, \\ c = -\mathrm{d}(\lambda-\mu-f)(\xi)+\nu+4n. \tag{4.10}$$

Then, using (1.14), we obtain

$$\tau = -\,2n\{\mathrm{d}\mu(\xi)+\mu+\nu+2n-1\}, \qquad \tilde{\tau} = 2n\{\mathrm{d}(\lambda-f)(\xi)+\mu+1\}. \tag{4.11}$$

Contracting (4.4) with respect to x, we obtain

$$\rho(y,v) = \frac{1}{2}\mathrm{d}\tau(y) + 2n\,\mathrm{d}\lambda(y) + \mathrm{d}(\mu+\nu)(y) - \mathrm{d}\mu(\varphi y) - \{\mathrm{d}(\mu+\nu)(\xi) - 2n\,\mu\}\eta(y)$$

and consequently for $y = \xi$ we have

$$\rho(\xi, v) = \frac{1}{2}\mathrm{d}\tau(\xi) + 2n\,\mathrm{d}\lambda(\xi) + 2n\,\mu. \tag{4.12}$$

We compute the left side of (4.12) by the formula $\rho(x,\xi) = 2n\,\eta(x)$ from (1.4) and then from (4.12) and (1.9), we obtain

$$\tilde{\tau} = -2n\{\mathrm{d}(\lambda-f)(\xi)+\mu-1\}.$$

Comparing the latter equality with (4.11), we have

$$\mathrm{d}(\lambda-f)(\xi) = -\mu, \\ \tilde{\tau} = 2n.$$

The former equality implies $b = 0$ in (4.10) and therefore the manifold is almost η-Einstein and the latter one means that $\mathrm{d}\tilde{\tau} = 0$ and using (1.9), we obtain for τ the following

$$\tau = 2n.$$

Then, substituting the value of τ in (4.11), we obtain

$$\mathrm{d}\mu(\xi) = -\mu - \nu - 2n,$$

which implies $a = 0$ in (4.10) and finally we get $(a, b, c) = (0, 0, 2n)$. □

4.1. Example 3

Let $(M, \varphi, \xi, \eta, g)$ be the 3-dimensional Sasaki-like manifold, given in Example 2 of sect. 3.1. Now, let f be a differentiable function on M, defined by

$$f = -\frac{1}{2} s \{(x^1)^2 + (x^2)^2\} + x^2 + t\, x^3$$

for arbitrary constants s and t. Then, the gradient of f with respect to the B-metric g is the following

$$\begin{aligned} \operatorname{grad} f = & -\{s\, x^1 \cos x^3 + (s\, x^2 - 1) \sin x^3\} e_1 \\ & + \{s\, x^1 \sin x^3 - (s\, x^2 - 1) \cos x^3\} e_2 + t\, e_3. \end{aligned} \tag{4.13}$$

Using (3.22), we compute the components of $\mathcal{L}_{\operatorname{grad} f} g$ as follows

$$(\mathcal{L}_{\operatorname{grad} f} g)_{11} = -(\mathcal{L}_{\operatorname{grad} f} g)_{22} = -2s, \qquad (\mathcal{L}_{\operatorname{grad} f} g)_{12} = 2t,$$

which give us the following expression

$$(\mathcal{L}_{\operatorname{grad} f} g) = -2s\, g - 2t\tilde{g} + 2(s + t)\, \eta \otimes \eta.$$

The latter equality coincides with (3.30) for $s = c_1$, $t = c_2 + c_3$. Therefore, $(M, \varphi, \xi, \eta, g)$ admits a Ricci-like soliton with potential $v = \operatorname{grad} f$ determined by (4.13) and the potential constants are

$$\lambda = s, \qquad \mu = t, \qquad \nu = -s - t - 2.$$

In conclusion, the constructed 3-dimensional example of a Sasaki-like manifold with $\tau = \tilde{\tau} = 2$ and gradient Ricci-like soliton supports also Theorem 4.1.

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