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Results in Mathematics

Compact Almost Hermitian Manifolds with Quasi-negative Curvature and the Almost Hermitian Curvature Flow

Masaya Kawamura

Abstract. We show that along the almost Hermitian curvature flow, the non-positivity of the first Chern–Ricci curvature can be preserved if the initial almost Hermitian metric has the Griffiths non-positive Chern curvature. If additionally, the first Chern–Ricci curvature of the initial metric is negative at some point, then we show that the almost complex structure of a compact non-quasi-Kähler almost Hermitian manifold equipped with such a metric cannot be integrable.

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1. Introduction

It is known that the Ricci flow can be used to give an alternative proof of the uniformization theorem of Riemann surfaces. Likewise, it can be questioned whether a geometric flow is able to be applied for classifying non-Kähler complex surfaces. These flows should preserve Hermitianness and some additional structures such as pluriclosedness, and should be close to the Ricci flow as much as possible. From this point of view, J. Streets and G. Tian introduced a parabolic evolution equation of pluriclosed metrics with a pluriclosed initial metric ω_0 on a compact Hermitian manifold,

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$$\begin{cases} \dfrac{\partial}{\partial t}\omega(t) = \partial\partial^*_{g(t)}\omega(t) + \bar{\partial}\bar{\partial}^*_{g(t)}\omega(t) - P(g(t)), \\ \omega(0) = \omega_0, \end{cases}$$

which is called the pluriclosed flow, where $\partial^*_{g(t)}$ and $\bar{\partial}^*_{g(t)}$ are decompositions of the L^2-adjoint operator of the exterior differential operator and $P(g(t))$ is the Ricci-type curvature of the Chern connection with respect to metrics $g(t)$ (cf. [9]). Their studies motivated us to generalize their results to almost Hermitian geometry. In [4], the author defined two parabolic flows; the almost Hermitian flow (AHF) and the almost Hermitian curvature flow (AHCF) on almost complex manifolds, which coincide with the pluriclosed flow and the Hermitian curvature flow respectively on complex manifolds, and studied the relationship between these parabolic evolution equations on a compact almost Hermitian manifold. In [5], we derived higher order derivative etimates in the presence of a curvature bound. And moreover, we exhibited a long-time existence obstruction for solutions to the almost Hermitian curvature flow by showing smoothing estimates for the curvature and torsion.

In the present paper, we investigate the AHCF, which coincides with the AHF (cf. [4, Theorem 1.2]), on a compact almost Hermitian manifold and we show that if the initial metric has the non-positive first Chern–Ricci curvature, then this positivity can be preserved along the AHCF. We mimic the argument in the proof of the positivity preservation properties of the Hermitian curvature flow (cf. [7]). Such an argument was initiated by G. Liu in order to prove that the Kähler–Ricci flow preserves the non-positivity of Ricci curvature if the initial metric has non-positive bisectional curvature (cf. [8]).

Let (M, J) be a compact almost complex manifold and let g be an almost Hermitian metric on M. Let $\{Z_r\}$ be an arbitrary local $(1,0)$-frame around a fixed point $p \in M$ and let $\{\zeta^r\}$ be the associated coframe. Then the associated real $(1,1)$-form ω with respect to g takes the local expression $\omega = \sqrt{-1}g_{r\bar{k}}\zeta^r \wedge \zeta^{\bar{k}}$. We will also refer to ω as to an almost Hermitian metric. Let Ω be the curvature of the Chern connection and Ω splits in $\Omega = H + R + \bar{H}$, where $R \in \Gamma(\Lambda^{1,1}M \otimes \Lambda^{1,1}M)$, $H \in \Gamma(\Lambda^{2,0}M \otimes \Lambda^{1,1}M)$ (see Sect. 2 in detail). The almost Hermitian flow (AHF) with an almost Hermitian initial metric ω_0 on (M, J) is as follows:

$$\begin{cases} \dfrac{\partial}{\partial t}\omega(t) = \partial\partial^*_{g(t)}\omega(t) + \bar{\partial}\bar{\partial}^*_{g(t)}\omega(t) - P(\omega(t)), \\ \omega(0) = \omega_0, \end{cases}$$

where $\partial^*_{g(t)}$ and $\bar{\partial}^*_{g(t)}$ are the L^2-adjoint operators with respect to metrics $g(t)$, and $P(\omega)$ is one of the Ricci-type curvatures of the Chern curvature, which is callaed the first Chern–Ricci curvature and locally given by $P_{i\bar{j}} = g^{k\bar{l}}R_{i\bar{j}k\bar{l}}$.

We have proven the following short-time existence result for the AHF.

Proposition 1.1. (cf. [4, Theorem 1.1]) *Given a compact almost Hermitian manifold* (M, ω_0, J)*, there exists a unique solution to the AHF with initial condition* ω_0 *on* $[0, \varepsilon)$ *for some* $\varepsilon > 0$.

We denote by S one of the Ricci-type curvatures of the Chern curvature, which is called the second Chern–Ricci curvature and is locally given by $S_{i\bar{j}} = g^{k\bar{l}} R_{k\bar{l}i\bar{j}}$.

It has been proved that a solution of the almost Hermitian flow with initial condition g_0 is equivalent to a solution of the following parabolic flow on a compact almost complex manifold with an almost Hermitian metric, we call it the almost Hermitian curvature flow (AHCF):

$$\begin{cases} \dfrac{\partial}{\partial t} g(t) = -S(g(t)) - Q^7(g(t)) - Q^8(g(t)) + BT'(g(t)) + \bar{Z}(T')(g(t)), \\ \\ g(0) = g_0, \end{cases}$$

where Q^1, Q^7, Q^8 are quadratics in the torsion of the Chern connection (cf. [11, pg. 712])

$$Q^1_{i\bar{j}} := T_{ik\bar{r}} T_{\bar{j}\bar{k}r}, \quad Q^7_{i\bar{j}} := T_{irk} T_{\bar{r}\bar{k}\bar{j}}, \quad Q^8_{i\bar{j}} := T_{irk} T_{\bar{j}\bar{k}\bar{r}}, \quad w_i := T_{ir\bar{r}},$$
$$BT'_{i\bar{j}} := B^{j}_{\bar{r}p} T_{ir\bar{p}} + B^{r}_{\bar{p}i} T_{pr\bar{j}} + B^{p}_{\bar{r}r} T_{pi\bar{j}} + B^{r}_{\bar{j}i} w_r,$$

and

$$\bar{Z}(T')_{i\bar{j}} := -Z_{\bar{r}}(T^s_{ri}) g_{s\bar{j}} - Z_{\bar{j}}(w_i) - g^{p\bar{q}} T^r_{pi} Z_{\bar{j}}(g_{r\bar{q}}).$$

These components are defined using an arbitrary unitary frame. In all this paper, we assume the Einstein convention omitting the symbol of sum over repeated indexes.

Note that $P = S + \mathrm{div}^{\nabla} T' - \nabla \bar{w} + Q^7 + Q^8$ for any almost Hermitian metric g (cf. [11, Lemma 3.5]), where T' is the torsion of the Chern connection ∇ associated to g, $(\mathrm{div}^{\nabla} T')_{i\bar{j}} = g^{k\bar{l}} \nabla_{\bar{l}} T_{ki\bar{j}}$, $(\nabla \bar{w})_{i\bar{j}} = g^{k\bar{l}} \nabla_i T_{\bar{j}\bar{l}k}$.

We say that a Hermitian metric g is pluriclosed if its associated real $(1,1)$ form ω satisfies $\partial\bar{\partial}\omega = 0$. We often write ω as a metric and say that ω is pluriclosed as well. In this paper, we will call the following parabolic flow on a compact complex manifold the Hermitian curvature flow (HCF_{Q^1}):

$$\begin{cases} \dfrac{\partial}{\partial t} g(t) = -S(g(t)) + Q^1(g(t)), \\ \\ g(0) = g_0, \end{cases}$$

where g_0 is a pluriclosed metric. We can obtain the following equivalence between the AHCF and the HCF_{Q^1} when the almost complex structure J is integrable.

Proposition 1.2. (cf. [4, Proposition 1.1]) *The AHCF coincides with the HCF_{Q^1} starting at a pluriclosed metric if J is integrable.*

It is known that the HCF_{Q^1} coincides with the pluriclosed flow starting at the same initial metric, which is called Streets–Tian identifiability theorem (cf. [9, Proposition 3.3]). The following result is the generalized version of Streets–Tian identifiability theorem.

Proposition 1.3. (cf. [4, Theorem 1.2]) *Let (M, g_0, J) be a compact almost Hermitian manifold with the associated real $(1,1)$-form ω_0. Then a solution to the AHCF with initial condition g_0 is equivalent to a solution to the AHF starting at the initial condition ω_0.*

As in [10, Theorem 1.1], we have developed some regurarity results for the AHCF. And also, we obtained the long-time existence obstruction for the AHCF in [5]. The long-time existence obstruction will be used for estimating the evolution equation of the Chern curvature R along the AHCF in order to prove our main theorem.

Proposition 1.4. (cf. [5, Theorem 1.1]) *Let $(M^{2n}, J, g(t))$ be a solution to the AHCF for a maximal time interval $[0, \tau_{max})$ on a compact almost Hermitian manifold which starts at the initial almost Hermitian metric g_0. The following statements* (i), (ii) *and* (iii) *hold.*

(i) We choose arbitrary $0 < \tau < \tau_{max}$. Assume that, for a positive constants α with $\alpha/\tau > 1$, the following inequalities hold:

$$\sup_{M\times[0,\tau)} |R|_{g(t)} \leq \frac{\alpha}{\tau}, \qquad \sup_{M\times[0,\tau)} |T'|^2_{g(t)} \leq \frac{\alpha}{\tau}, \qquad \sup_{M\times[0,\tau)} |\nabla T'|_{g(t)} \leq \frac{\alpha}{\tau}.$$

Then, for any $m \in \mathbb{N}$, the following inequalities hold:

$$|\nabla^m R|_{g(t)} \leq \frac{C_{m,n,\alpha}}{\tau \cdot t^{\frac{m}{2}}}, \quad |\nabla^{m+1} T'|_{g(t)} \leq \frac{C_{m,n,\alpha}}{\tau \cdot t^{\frac{m}{2}}}$$

for any $t \in (0, \tau]$, where $C_{m,n,\alpha}$ is some positive constant depending only on m, n and α.

(ii) If $\tau_{max} < \infty$, then

$$\limsup_{t\to\tau_{max}} \max\left\{ \max_M |R|_{g(t)}, \max_M |T'|^2_{g(t)}, \max_M |\nabla T'|_{g(t)} \right\} = \infty.$$

(iii) If J is integrable and g_0 is pluriclosed, then $g(t)$ is pluriclosed for all time in the existence interval and $g(t)$ is a solution to the HCF_{Q^1}. If furthermore g_0 is Kähler, then $g(t)$ is Kähler for all time and $g(t)$ solves the Kähler–Ricci flow.

Remark 1.1. Notice the fact that for any C^∞-family of C^∞-functions $\{\rho_t\}_{t\in[0,\tau_{\max})}$ such that $\limsup_{t\to\tau_{\max}} \max_M |\rho_t| = \infty$ and for any $0 < \tau < \tau_{\max} < \infty$,

$$\sup_{t\in[0,\tau)} \max_M |\rho_t| \leq \max_{t\in[0,\tau]} \max_M |\rho_t| < \infty$$

holds because $M \times [0, \tau]$ is compact. We may apply this for $\max_M |T'|^2_{g(t)}$, $\max_M |\nabla T'|_{g(t)}$ and $\max_M |R|_{g(t)}$ since we have the long-time existence obstruction in Proposition 1.4 (ii). Hence, we may assume that the quantities $|T'|^2_{C^0(g(t))}$, $|\nabla T'|_{C^0(g(t))}$ and $|R|_{C^0(g(t))}$ are uniformly bounded on $[0, \tau)$, where $|\cdot|_{C^0(g(t))} = \max_M |\cdot|_{g(t)}$.

Note that we have shown that the uniform equivalence between almost Hermitian metrics and the solution to the AHF (equivalently, the AHCF) in [6]. By applying the strong maximum principle (see Sect. 5), we have the following main result.

Theorem 1.1. *Suppose (M, J, g_0) is a compact almost Hermitian manifold with the Griffiths non-positive Chern curvature (see Definition 2.1). Let $g(t)$, $t \in [0, \tau_{\max})$ be a solution to the AHCF starting from the initial metric $g(0) = g_0$, where $\tau_{\max}$ is the finite explosion time of the AHCF. Then there exists $0 < \tau < \tau_{\max}$ such that the first Chern–Ricci curvature $P(g(t))$ is non-positive on $[0, \tau]$. Moreover, if the metric g_0 has the first Chern–Ricci curvature which is negative at some point, then there exists $0 < \tau < \tau_{\max}$ such that for any $t \in (0, \tau]$, $P(g(t)) < 0$ everywhere on M.*

Note that if the initial metric g_0 has the Griffiths non-positive Chern curvature and the first Chern–Ricci curvature which is negative at some point, then g_0 has the quasi-negative first Chern–Ricci curvature (see Definition 2.2).

A quasi-Kähler structure is an almost Hermitian structure whose real $(1,1)$-form ω satisfies $(d\omega)^{(1,2)} = \bar{\partial}\omega = 0$, which is equivalent to the original definition of quasi-Kählerianity: $D_X J(Y) + D_{JX} J(JY) = 0$ for all vector fields X, Y, where D is the Levi-Civita connection with respect to the metric ω. By letting $\mathcal{K}$, $\mathcal{QK}$, and $\mathcal{H}$ denote the class of Kähler manifolds, the class of quasi-Kähler manifolds, and the class of Hermitian manifolds respectively, we have that $\mathcal{K} = \mathcal{H} \cap \mathcal{QK}$ (cf. [2]).

Notice that in [7], M.-C. Lee has proven that the canonical line bundle of a compact Hermitian manifold with nonpositive curvature in the sense of Griffiths and quasi-negative Ricci curvature must be ample. The following result can be easily given by applying this Lee's result. We introduce another proof by applying Theorem 1.1. The condition "non-quasi-Kähler" means that the almost complex structure J admits no quasi-Kähler metric.

Corollary 1.1. *Suppose (M, J, g_0) is a compact non-quasi-Kähler almost Hermitian manifold with Griffiths non-positive Chern curvature. Moreover, if the metric g_0 has the first Chern–Ricci curvature which is negative at some point, then J cannot be integrable.*

Proof. By the short-time existence result in [4], there exists a shot-time solution $g(t)$ to the AHCF starting from the metric g_0. By Theorem 1.1, there exists $\tau > 0$ such that $P(g(\tau)) < 0$ on M. Now, let us assume that the almost complex structure J is integrable. Then the manifold becomes Hermitian and the assumption “non-quasi-Kähler” implies that the complex structure J admits no Kähler metric from the fact that $\mathcal{K} = \mathcal{H} \cap \mathcal{QK}$. Since we obtain that $c_1(K_M) > 0$ from that $P(g(\tau)) < 0$ on M and the assumption that J is integrable, then the manifold M must be Kähler, which is contradictory to that J does not admit any Kähler metrics. Therefore, the almost complex structure J cannot be integrable under these assumptions. □

Note that if $M \subset \mathbb{R}^7$, M is almost-Kähler if and only if M is Kähler, i.e., M is non-almost-Kähler if and only if M is non-Kähler (cf. [3]). This tells us that we may change the condition “non-quasi-Kähler” to “non-almost-Kähler”, which means that the almost complex structure J admits no almost-Kähler metric, for a compact almost Hermitian manifold $M \subset \mathbb{R}^7$ in Corollary 1.1.

This paper is organized as follows: in Sect. 2, we recall some basic definitions and computations. In Sect. 3, we show some estimates for torsions and the term $\bar{Z}(T')$ and the curvature H. And also we compute the evolution equation of the curvature R and the first Chern–Ricci curvature P. In Sect. 4, we show the preservation of non-positivity of the first Chern–Ricci curvature on a compact almost Hermitian manifold with non-positive bisectional curvature. In the last section, we prove if moreover, the initial metric has the first Chern–Ricci curvature which is negative at some point, then there exists $\tau > 0$ such that the first Chern–Ricci curvature is negative on $(0, \tau]$ by applying the strong maximal principle. Notice that we assume the Einstein convention omitting the symbol of sum over repeated indexes in all this paper.

2. Preliminaries

2.1. The Nijenhuis Tensor of the Almost Complex Structure

Let M be a $2n$-dimensional smooth differentiable manifold. An almost complex structure on M is an endomorphism J of TM, $J \in \Gamma(\mathrm{End}(TM))$, satisfying $J^2 = -Id_{TM}$, where TM is the real tangent vector bundle of M. The pair (M, J) is called an almost complex manifold. Let (M, J) be an almost complex manifold. We define a bilinear map on $C^\infty(M)$ for $X, Y \in \Gamma(TM)$ by

$$4N(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y],$$

which is the Nijenhuis tensor of J. The Nijenhuis tensor N satisfies $N(X, Y) = -N(Y, X)$, $N(JX, Y) = -JN(X, Y)$, $N(X, JY) = -JN(X, Y)$, $N(JX, JY) = -N(X, Y)$. For any $(1, 0)$-vector fields W and V, $N(V, W) = -[V, W]^{(0,1)}$, $N(V, \bar{W}) = N(\bar{V}, W) = 0$ and $N(\bar{V}, \bar{W}) = -[\bar{V}, \bar{W}]^{(1,0)}$ since we have $4N(V, W) = -2([V, W] + \sqrt{-1}J[V, W])$, $4N(\bar{V}, \bar{W}) = -2([\bar{V}, \bar{W}] - \sqrt{-1}J[\bar{V}, \bar{W}])$. An almost complex structure J is called integrable if $N = 0$ on M. There are several equivalent conditions for integrability: as we just mentioned that the Nijenhuis tensor N^i_{jk} of J vanishes, which is equivalent to that there exist holomorphic coordinates compatible with J, and also equivalent to that the space of $(1, 0)$-vector fields related to J is closed under Lie bracket. Giving a complex structure to a differentiable manifold M is equivalent to giving an integrable almost complex structure to M. Let (M, J) be an almost complex manifold. A Riemannian metric g on M is called J-invariant if J is compatible with g, i.e., for any $X, Y \in \Gamma(TM)$, $g(X,Y) = g(JX, JY)$. In this case, the pair (g, J) is called an almost Hermitian structure. The fundamental 2-form ω associated to a J-invariant Riemannian metric g, i.e., an almost Hermitian metric, is determined by, for $X, Y \in \Gamma(TM)$, $\omega(X,Y) = g(JX, Y)$. Indeed we have, for any $X, Y \in \Gamma(TM)$,

$$\omega(Y,X) = g(JY,X) = g(J^2Y, JX) = -g(JX,Y) = -\omega(X,Y)$$

and $\omega \in \Gamma(\bigwedge^2 T^*M)$. We will also refer to the associated real fundamental $(1,1)$-form ω as an almost Hermitian metric. The form ω is related to the volume form dV_g by $n!dV_g = \omega^n$.

The complexified tangent vector bundle is given by $T^{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C}$ for the real tangent vector bundle TM. By extending J $\mathbb{C}$-linearly and g, ω $\mathbb{C}$-bilinearly to $T^{\mathbb{C}}M$, they are also defined on $T^{\mathbb{C}}M$ and we observe that the complexified tangent vector bundle $T^{\mathbb{C}}M$ can be decomposed as

$$T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M,$$

where $T^{1,0}M$, $T^{0,1}M$ are the eigenspaces of J corresponding to eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively:

$$T^{1,0}M = \{X - \sqrt{-1}JX \,\big|\, X \in TM\}, \quad T^{0,1}M = \{X + \sqrt{-1}JX \,\big|\, X \in TM\}.$$

Let $\Lambda^r M = \bigoplus_{p+q=r} \Lambda^{p,q}M$ for $0 \le r \le 2n$ denote the decomposition of complex differential r-forms into (p,q)-forms, where $\Lambda^{p,q}M = \Lambda^p(\Lambda^{1,0}M) \otimes \Lambda^q(\Lambda^{0,1}M)$,

$$\Lambda^{1,0}M = \{\eta + \sqrt{-1}J\eta \,\big|\, \eta \in \Lambda^1 M\}, \quad \Lambda^{0,1}M = \{\eta - \sqrt{-1}J\eta \,\big|\, \eta \in \Lambda^1 M\}$$

and $\Lambda^1 M$ denotes the dual of $T^{\mathbb{C}}M$.

Let $\{Z_r\}$ be a local $(1,0)$-frame on (M, J) with an almost Hermitian metric g and let $\{\zeta^r\}$ be a local associated coframe with respect to $\{Z_r\}$, i.e.,

$$\zeta^i(Z_j) = \delta^i_j, \quad i, j = 1, \ldots, n.$$

Since g is almost Hermitian, its components satsfy $g_{ij} = g_{\bar{i}\bar{j}} = 0$ and $g_{i\bar{j}} = g_{\bar{j}i} = \bar{g}_{\bar{i}j}$. By using these local frame $\{Z_r\}$ and coframe $\{\zeta^r\}$, we have

$$N(Z_{\bar{i}}, Z_{\bar{j}}) = -[Z_{\bar{i}}, Z_{\bar{j}}]^{(1,0)} =: N^k_{\bar{i}\bar{j}} Z_k, \quad N(Z_i, Z_j) = -[Z_i, Z_j]^{(0,1)} = \overline{N^k_{\bar{i}\bar{j}}} Z_{\bar{k}},$$

and

$$N = \frac{1}{2}\overline{N^k_{\bar{i}\bar{j}}} Z_{\bar{k}} \otimes (\zeta^i \wedge \zeta^j) + \frac{1}{2} N^k_{\bar{i}\bar{j}} Z_k \otimes (\zeta^{\bar{i}} \wedge \zeta^{\bar{j}}).$$

Let (M, g, J) be an almost Hermitian manifold with $\dim_{\mathbb{R}} M = 2n$. An affine connection D on $T^{\mathbb{C}}M$ is called almost Hermitian connection if $Dg = DJ = 0$. For the almost Hermitian connection, we have the following Lemma (cf. [1]).

Lemma 2.1. *Let* (M, J, g) *be an almost Hermitian manifold with* $\dim_{\mathbb{R}} M = 2n$. *Then for any given vector valued* $(1,1)$*-form* $\Theta = (\Theta^i)_{1\le i\le n}$*, there exists a unique almost Hermitian connection* ∇ *on* (M, J, g) *such that the* $(1,1)$*-part of the torsion is equal to the given* Θ.

If the $(1,1)$-part of the torsion of an almost Hermitian connection vanishes everywhere, then the connction is called the second canonical connection or the Chern connection. We will refer the connection as the Chern connection and denote it by ∇.

Now let ∇ be the Chern connection on M. We can write

$$[Z_i, Z_j] = B^r_{ij} Z_r + B^{\bar r}_{ij} Z_{\bar r}, \quad [Z_i, Z_{\bar j}] = B^r_{i\bar j} Z_r + B^{\bar r}_{i\bar j} Z_{\bar r},$$
$$[Z_{\bar i}, Z_{\bar j}] = B^r_{\bar i\bar j} Z_r + B^{\bar r}_{\bar i\bar j} Z_{\bar r}$$

and also we here note that for instance, $[Z_i, Z_{\bar j}] = [Z_i, Z_{\bar j}]^{(1,0)} + [Z_i, Z_{\bar j}]^{(0,1)}$, where

$$[Z_i, Z_{\bar j}]^{(1,0)} = \frac{1}{2}([Z_i, Z_{\bar j}] - \sqrt{-1} J [Z_i, Z_{\bar j}]),$$
$$[Z_i, Z_{\bar j}]^{(0,1)} = \frac{1}{2}([Z_i, Z_{\bar j}] + \sqrt{-1} J [Z_i, Z_{\bar j}]).$$

For any p-form ψ, there holds that

$$d\psi(X_1, \dots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} X_i(\psi(X_1, \dots, \widehat{X_i}, \dots, X_{p+1})) + \sum_{i<j} (-1)^{i+j} \psi([X_i, X_j], X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{p+1})$$

for any vector fields $X_1, \dots, X_{p+1}$ on M (cf. [13]). We directly compute that

$$d\zeta^s = -\frac{1}{2} B^s_{kl} \zeta^k \wedge \zeta^l - B^s_{k\bar l} \zeta^k \wedge \zeta^{\bar l} - \frac{1}{2} B^s_{\bar k\bar l} \zeta^{\bar k} \wedge \zeta^{\bar l}.$$

For any real $(1,1)$-form $\eta = \sqrt{-1}\eta_{i\bar j}\zeta^i \wedge \zeta^{\bar j}$, we have

$$\partial\eta = \frac{\sqrt{-1}}{2}\Big(Z_i(\eta_{j\bar k}) - Z_j(\eta_{i\bar k}) - B^s_{ij}\eta_{s\bar k} - B^{\bar s}_{i\bar k}\eta_{j\bar s} + B^{\bar s}_{j\bar k}\eta_{i\bar s}\Big)\zeta^i \wedge \zeta^j \wedge \zeta^{\bar k},$$
$$\bar\partial\eta = \frac{\sqrt{-1}}{2}\Big(Z_{\bar j}(\eta_{k\bar i}) - Z_{\bar i}(\eta_{k\bar j}) - B^s_{k\bar i}\eta_{s\bar j} + B^s_{k\bar j}\eta_{s\bar i} + B^{\bar s}_{\bar i\bar j}\eta_{k\bar s}\Big)\zeta^k \wedge \zeta^{\bar i} \wedge \zeta^{\bar j}.$$

From these computations above, we have

$$\begin{aligned}\partial\omega &= \frac{\sqrt{-1}}{2}\Big(Z_i(g_{j\bar k}) - Z_j(g_{i\bar k}) - B^s_{ij} g_{s\bar k} - B^{\bar s}_{i\bar k} g_{j\bar s} + B^{\bar s}_{j\bar k} g_{i\bar s}\Big)\zeta^i \wedge \zeta^j \wedge \zeta^{\bar k} \\ &= \frac{\sqrt{-1}}{2} T_{ij\bar k}\zeta^i \wedge \zeta^j \wedge \zeta^{\bar k}\end{aligned}$$

and

$$\begin{aligned}\bar{\partial}\omega &= \frac{\sqrt{-1}}{2}\Big(Z_{\bar{j}}(g_{k\bar{i}}) - Z_{\bar{i}}(g_{k\bar{j}}) - B^s_{k\bar{i}}g_{s\bar{j}} + B^s_{k\bar{j}}g_{s\bar{i}} + B^{\bar{s}}_{\bar{i}\bar{j}}g_{k\bar{s}}\Big)\zeta^k\wedge\zeta^{\bar{i}}\wedge\zeta^{\bar{j}}\\ &= \frac{\sqrt{-1}}{2}T_{\bar{j}\bar{i}k}\zeta^k\wedge\zeta^{\bar{i}}\wedge\zeta^{\bar{j}}.\end{aligned}$$

And in this sense, we also obtain for any $(0,1)$-form β,

$$(\partial\beta)_{k\bar{j}} = Z_k(\beta_{\bar{j}}) - B^{\bar{m}}_{k\bar{j}}\beta_{\bar{m}} = \nabla_k\beta_{\bar{j}}.$$

2.2. The Torsion and the Curvature on Almost Complex Manifolds

Since the Chern connection ∇ preserves J, we have

$$\nabla_i Z_j = \Gamma^r_{ij}Z_r, \quad \nabla_i Z_{\bar{j}} = \Gamma^{\bar{r}}_{i\bar{j}}Z_{\bar{r}},$$

where

$$\Gamma^r_{ij} = g^{r\bar{s}}Z_i(g_{j\bar{s}}) - g^{r\bar{s}}g_{j\bar{l}}B^{\bar{l}}_{i\bar{s}}, \quad \Gamma^p_{ip} = Z_i(\log\det g) - B^{\bar{s}}_{i\bar{s}}.$$

We can obtain that

$$\Gamma^{\bar{r}}_{i\bar{j}} = B^{\bar{r}}_{i\bar{j}}$$

since the $(1,1)$-part of the torsion of the Chern connection vanishes everywhere (cf. [4]). For any $(0,1)$-form β, we have $\beta = \beta_{\bar{j}}\zeta^{\bar{j}}$,

$$\nabla_k\beta_{\bar{j}} = Z_k(\beta_{\bar{j}}) - \Gamma^{\bar{l}}_{k\bar{j}}\beta_{\bar{l}} = Z_k(\beta_{\bar{j}}) - B^{\bar{l}}_{k\bar{j}}\beta_{\bar{l}}.$$

Note that the mixed derivatives $\nabla_i Z_{\bar{j}}$ do not depend on g (cf. [11]). Let $\{\gamma^i_j\}$ be the connection form, which is defined by $\gamma^i_j = \Gamma^i_{sj}\zeta^s + \Gamma^i_{\bar{s}j}\zeta^{\bar{s}}$. The torsion T of the Chern connection ∇ is given by $T^i = d\zeta^i - \zeta^p\wedge\gamma^i_p$, $T^{\bar{i}} = d\zeta^{\bar{i}} - \zeta^{\bar{p}}\wedge\gamma^{\bar{i}}_{\bar{p}}$, which has no $(1,1)$-part and the only non-vanishing components are as follows:

$$T^s_{ij} = \Gamma^s_{ij} - \Gamma^s_{ji} - B^s_{ij} \quad T^{\bar{s}}_{ij} = -B^{\bar{s}}_{ij}.$$

These tell us that $T = (T^i)$ splits into $T = T' + T''$, where $T' \in \Gamma(\Lambda^{2,0}M\otimes T^{1,0}M)$, $T'' \in \Gamma(\Lambda^{0,2}M\otimes T^{1,0}M)$. We also lower the index of torsion and denote it by

$$T_{ij\bar{k}} = T^s_{ij}g_{s\bar{k}} = Z_i(g_{j\bar{k}}) - Z_j(g_{i\bar{k}}) + B^{\bar{q}}_{\bar{k}i}g_{j\bar{q}} - B^{\bar{q}}_{\bar{k}j}g_{i\bar{q}} - B^s_{ij}g_{s\bar{k}}.$$

Note that T'' depends only on J and it can be regarded as the Nijenhuis tensor of J, that is, J is integrable if and only if T'' vanishes.

We denote by Ω the curvature of the Chern connection ∇. We can regard Ω as a section of $\Lambda^2M\otimes\Lambda^{1,1}M$, $\Omega\in\Gamma(\Lambda^2M\otimes\Lambda^{1,1}M)$ and Ω splits in $\Omega = H + R + \bar{H}$, where $R\in\Gamma(\Lambda^{1,1}M\otimes\Lambda^{1,1}M)$, $H\in\Gamma(\Lambda^{2,0}M\otimes\Lambda^{1,1}M)$. The curvature form can be expressed by $\Omega^i_j = d\gamma^i_j + \gamma^i_s\wedge\gamma^s_j$.

In terms of Z_r's, we have

$$\begin{aligned}R_{i\bar{j}k}{}^r &= \Omega^r_k(Z_i, Z_{\bar{j}}) = Z_i(\Gamma^r_{\bar{j}k}) - Z_{\bar{j}}(\Gamma^r_{ik})\\ &\quad + \Gamma^r_{is}\Gamma^s_{\bar{j}k} - \Gamma^r_{\bar{j}s}\Gamma^s_{ik} - B^s_{i\bar{j}}\Gamma^r_{sk} + B^{\bar{s}}_{\bar{j}i}\Gamma^r_{\bar{s}k} = -R_{\bar{j}ik}{}^r,\end{aligned}$$

$$H_{ijk}{}^{r} = \Omega^r_k(Z_i, Z_j) = Z_i(\Gamma^r_{jk}) - Z_j(\Gamma^r_{ik}) + \Gamma^r_{is}\Gamma^s_{jk} - \Gamma^r_{js}\Gamma^s_{ik} - B^s_{ij}\Gamma^r_{sk}$$
$$-B^{\bar{s}}_{ij}\Gamma^r_{\bar{s}k} = -H_{jik}{}^{r},$$
$$H_{\bar{i}\bar{j}k}{}^{r} = \Omega^r_k(Z_{\bar{i}}, Z_{\bar{j}}) = Z_{\bar{i}}(\Gamma^r_{\bar{j}k}) - Z_{\bar{j}}(\Gamma^r_{\bar{i}k}) + \Gamma^r_{\bar{i}s}\Gamma^s_{\bar{j}k} - \Gamma^r_{\bar{j}s}\Gamma^s_{\bar{i}k} - B^s_{\bar{i}\bar{j}}\Gamma^r_{sk}$$
$$-B^{\bar{s}}_{\bar{i}\bar{j}}\Gamma^r_{\bar{s}k} = -H_{\bar{j}\bar{i}k}{}^{r}.$$

We can write $\Omega = (\Omega^i_j) = \Omega^{(2,0)} + \Omega^{(1,1)} + \Omega^{(0,2)} = H + R + \bar{H}$, with

$$\Omega^{(2,0)} = \left(\frac{1}{2}H_{ijk}{}^{r}\zeta^i \wedge \zeta^j\right), \quad \Omega^{(1,1)} = \left(R_{i\bar{j}k}{}^{r}\zeta^i \wedge \zeta^{\bar{j}}\right),$$
$$\Omega^{(0,2)} = \left(\frac{1}{2}H_{\bar{i}\bar{j}k}{}^{r}\zeta^{\bar{i}} \wedge \zeta^{\bar{j}}\right).$$

Then the Chern–Ricci form is $(\sqrt{-1}\Omega^i_i) \in c_1(M, J) \in H^2(M, \mathbb{R})$, where $c_1(M, J)$ is the first Chern class of (M, J).

We deduce that by using $\Gamma^p_{kp} = Z_k(\log\det g) - B^{\bar{p}}_{k\bar{p}}$,

$$\begin{aligned}
P_{i\bar{j}} &= R_{i\bar{j}r}{}^{r} \\
&= Z_i(\Gamma^r_{\bar{j}r}) - Z_{\bar{j}}(\Gamma^r_{ir}) - B^s_{i\bar{j}}\Gamma^r_{sr} - B^{\bar{s}}_{i\bar{j}}\Gamma^r_{\bar{s}r} \\
&= Z_i(\Gamma^r_{\bar{j}r}) - Z_{\bar{j}}Z_i(\log\det g) + Z_{\bar{j}}(B^{\bar{r}}_{i\bar{r}}) - B^s_{i\bar{j}}Z_s(\log\det g) + B^s_{i\bar{j}}B^{\bar{r}}_{s\bar{r}} - B^{\bar{s}}_{i\bar{j}}\Gamma^r_{\bar{s}r} \\
&= Z_i(\Gamma^r_{\bar{j}r}) + [Z_i, Z_{\bar{j}}](\log\det g) - Z_iZ_{\bar{j}}(\log\det g) + Z_{\bar{j}}(B^{\bar{r}}_{i\bar{r}}) \\
&\quad -[Z_i, Z_{\bar{j}}]^{(1,0)}(\log\det g) + B^s_{i\bar{j}}B^{\bar{r}}_{s\bar{r}} - B^{\bar{s}}_{i\bar{j}}\Gamma^r_{\bar{s}r} \\
&= -(Z_iZ_{\bar{j}} - [Z_i, Z_{\bar{j}}]^{(0,1)})(\log\det g) + Z_{\bar{j}}(B^{\bar{r}}_{i\bar{r}}) + Z_i(B^r_{\bar{j}r}) + B^s_{i\bar{j}}B^{\bar{r}}_{s\bar{r}} - B^{\bar{s}}_{i\bar{j}}B^r_{\bar{s}r} \\
R_{ij} &= H_{ijr}{}^{r} \\
&= Z_i(\Gamma^r_{jr}) - Z_j(\Gamma^r_{ir}) - B^s_{ij}\Gamma^r_{sr} - B^{\bar{s}}_{ij}\Gamma^r_{\bar{s}r} \\
&= Z_iZ_j(\log\det g) - Z_i(B^{\bar{r}}_{j\bar{r}}) - Z_jZ_i(\log\det g) + Z_j(B^{\bar{r}}_{i\bar{r}}) \\
&\quad -B^s_{ij}Z_s(\log\det g) + B^s_{ij}B^{\bar{r}}_{s\bar{r}} - B^{\bar{s}}_{ij}B^r_{\bar{s}r} \\
&= ([Z_i, Z_j] - [Z_i, Z_j]^{(1,0)})(\log\det g) - Z_i(B^{\bar{r}}_{j\bar{r}}) + Z_j(B^{\bar{r}}_{i\bar{r}}) + B^s_{ij}B^{\bar{r}}_{s\bar{r}} - B^{\bar{s}}_{ij}B^r_{\bar{s}r} \\
&= [Z_i, Z_j]^{(0,1)}(\log\det g) - Z_i(B^{\bar{r}}_{j\bar{r}}) + Z_j(B^{\bar{r}}_{i\bar{r}}) + B^s_{ij}B^{\bar{r}}_{s\bar{r}} - B^{\bar{s}}_{ij}B^r_{\bar{s}r}
\end{aligned}$$

and

$$\begin{aligned}
R_{\bar{i}\bar{j}} &= H_{\bar{i}\bar{j}r}{}^{r} \\
&= Z_{\bar{i}}(\Gamma^r_{\bar{j}r}) - Z_{\bar{j}}(\Gamma^r_{\bar{i}r}) - B^s_{\bar{i}\bar{j}}\Gamma^r_{sr} - B^{\bar{s}}_{\bar{i}\bar{j}}\Gamma^r_{\bar{s}r} \\
&= Z_{\bar{i}}(B^r_{\bar{j}r}) - Z_{\bar{j}}(B^r_{\bar{i}r}) - B^s_{\bar{i}\bar{j}}Z_s(\log\det g) + B^s_{\bar{i}\bar{j}}B^{\bar{r}}_{s\bar{r}} - B^{\bar{s}}_{\bar{i}\bar{j}}B^r_{\bar{s}r} \\
&= -[Z_{\bar{i}}, Z_{\bar{j}}]^{(1,0)}(\log\det g) + Z_{\bar{i}}(B^r_{\bar{j}r}) - Z_{\bar{j}}(B^r_{\bar{i}r}) + B^s_{\bar{i}\bar{j}}B^{\bar{r}}_{s\bar{r}} - B^{\bar{s}}_{\bar{i}\bar{j}}B^r_{\bar{s}r}.
\end{aligned}$$

The Chern–Ricci form $\mathrm{Ric}(\omega)$ is defined by

$$\mathrm{Ric}(\omega) := \frac{\sqrt{-1}}{2}R_{kl}\zeta^k \wedge \zeta^l + \sqrt{-1}P_{k\bar{l}}\zeta^k \wedge \zeta^{\bar{l}} + \frac{\sqrt{-1}}{2}R_{\bar{k}\bar{l}}\zeta^{\bar{k}} \wedge \zeta^{\bar{l}}.$$

It is a closed real 2-form. If J is integrable, it is a closed real $(1,1)$-form. If furthermore, J is integrable and $d\omega = 0$, then the Chern–Ricci form coincides with the Ricci form defined by the Levi-Civita connection of ω. Assume that $\tilde{\omega} = \sqrt{-1}\tilde{g}_{i\bar{j}}\zeta^i \wedge \zeta^{\bar{j}}$ is another almost Hermitian metric. Then we have

$$\mathrm{Ric}(\tilde{\omega}) - \mathrm{Ric}(\omega) = -\frac{1}{2}dJd\log\frac{\tilde{\omega}^n}{\omega^n},$$

with $\mathrm{Ric}(\omega) \in 2\pi c_1(M,J) \in H^2(M,\mathbb{R})$ (cf. [13]).

Lemma 2.2. (The first Bianchi identity for the Chern curvature) *For any* $X, Y, Z \in T^{\mathbb{C}}M$*,*

$$\sum \Omega(X,Y)Z = \sum \Big(T(T(X,Y),Z) + \nabla_X T(Y,Z)\Big),$$

where the sum is taken over all cyclic permutations.

This identity induces the following formulae:

$$R_{i\bar{j}k}{}^{l} = R_{k\bar{j}i}{}^{l} - T^{\bar{r}}_{ik}T^{l}_{\bar{r}\bar{j}} + \nabla_{\bar{j}}T^{l}_{ki} = R_{k\bar{j}i}{}^{l} - B^{\bar{r}}_{ik}B^{l}_{\bar{r}\bar{j}} + \nabla_{\bar{j}}T^{l}_{ki}, \tag{2.1}$$

$$H_{ijk}{}^{l} = T^{\bar{r}}_{ji}T^{\bar{k}}_{\bar{r}\bar{l}} + \nabla_{\bar{l}}T^{\bar{k}}_{ji} = -B^{\bar{r}}_{ji}T^{\bar{k}}_{\bar{r}\bar{l}} - \nabla_{\bar{l}}B^{\bar{k}}_{ji}, \tag{2.2}$$

where used that $R_{ij\bar{k}\bar{l}} = R_{\bar{i}\bar{j}kl} = H_{j\bar{l}ik} = H_{\bar{j}l\bar{i}\bar{k}} = H_{\bar{l}ijk} = H_{l\bar{i}\bar{j}\bar{k}} = 0$.

Lemma 2.3. (The second Bianchi identity for the Chern curvature) *For any* $X, Y, Z \in T^{\mathbb{C}}M$*,*

$$\sum \nabla_X \Omega(Y,Z) = -\sum \Omega(T(X,Y),Z),$$

where the sum is taken over all cyclic permutations.

This identity induces the following formulae:

$$\nabla_i R_{r\bar{s}k}{}^{l} = \nabla_r R_{i\bar{s}k}{}^{l} + \nabla_{\bar{s}}H_{rik}{}^{l} + T^{m}_{ri}R_{m\bar{s}k}{}^{l} + T^{\bar{m}}_{ri}H_{\bar{m}\bar{s}k}{}^{l}, \tag{2.3}$$

$$\nabla_{\bar{i}} H_{rsk}{}^{l} = \nabla_s R_{r\bar{i}k}{}^{l} + \nabla_r R_{\bar{i}sk}{}^{l} + T^{q}_{rs}R_{q\bar{i}k}{}^{l} + T^{\bar{q}}_{rs}H_{\bar{q}\bar{i}k}{}^{l}. \tag{2.4}$$

Taking into account the Bianchi identities, we have the following lemma:

Lemma 2.4. ([11, Lemma 3.5]) *The following formula holds*

$$P = S + div^{\nabla}T' - \nabla\bar{w} + Q^7 + Q^8 \tag{2.5}$$

for any almost Hermitian metric g *, where* T' *is the torsion of the Chern connection* ∇ *associated to* g*,*

$$(div^{\nabla}T')_{i\bar{j}} = g^{k\bar{l}}\nabla_{\bar{l}}T_{ki\bar{j}}, \quad (\nabla\bar{w})_{i\bar{j}} = g^{k\bar{l}}\nabla_i T_{\bar{j}\bar{l}k}.$$

We define the curvature condition as follows:

Definition 2.1. We say that an almost Hermitian manifold (M, J, g) has the Griffiths non-positive Chern curvature if there is non-positive function κ such that for any $p \in M$, $X, Y \in T^{1,0}_p M$,

$$R(X,\bar{X},Y,\bar{Y}) \le \kappa(p)B(X,\bar{X},Y,\bar{Y}),$$

where $B_{i\bar{j}k\bar{l}} := g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}$.

Definition 2.2. We say that an almost Hermitian manifold (M, J, g) has the first Chern–Ricci curvature bounded above by a function κ if for any $p \in M$, $X \in T_p^{1,0}M$,

$$P(X, \bar{X}) \le \kappa(p) g(X, \bar{X}).$$

If the function κ is non-positive (resp. negative), then we say that g has the non-positive (resp. negative) first Chern–Ricci curvature. If the function κ is non-positive and negative at some point, then we say that g has the quasi-negative first Chern–Ricci curvature.

3. Evolution Equations Along the Almost Hermitian Curvature Flow

Let (M^{2n}, g, J) be a compact almost Hermitian manifold. Let ∇ be the Chern connection on (M^{2n}, g, J). Let $\{Z_r\}$ be a local unitary $(1,0)$-frame with respect to g around a fixed point $p \in M$. Note that unitary frames always exist locally since we can take any frame and apply the Gram-Schmidt process. Then with respect to a local g-unitary frame, we have $g_{i\bar{j}} = \delta_{ij}$, $Z_k(g_{i\bar{j}}) = 0$ for any $i, j, k = 1, \ldots, n$, and the Christoffel symbols satisfy

$$\Gamma_{ij}^{k} = -\Gamma_{i\bar{k}}^{\bar{j}}, \quad \Gamma_{\bar{i}\bar{j}}^{\bar{k}} = -\Gamma_{\bar{i}k}^{j},$$

since we have

$$\begin{aligned}
\Gamma_{ij}^{k} &= g(\nabla_i Z_j, Z_{\bar{k}}) = Z_i(g_{j\bar{k}}) - g(Z_j, \nabla_i Z_{\bar{k}}) = -\Gamma_{i\bar{k}}^{\bar{j}}, \\
\Gamma_{\bar{i}\bar{j}}^{\bar{k}} &= g(Z_k, \nabla_{\bar{i}} Z_{\bar{j}}) = Z_{\bar{i}}(g_{k\bar{j}}) - g(\nabla_{\bar{i}} Z_k, Z_{\bar{j}}) = -\Gamma_{\bar{i}k}^{j}.
\end{aligned}$$

With respect to such a frame, the components of the torsion can be written as

$$T_{ij}^{k} = -B_{i\bar{k}}^{\bar{j}} + B_{j\bar{k}}^{\bar{i}} - B_{ij}^{k}$$

and the components of w can be written as

$$w_j = -B_{jr}^{r} - B_{j\bar{r}}^{\bar{r}} + B_{r\bar{r}}^{\bar{j}}.$$

And also we have

$$\begin{aligned}
R_{i\bar{j}k}{}^{r} &= Z_i(\Gamma_{\bar{j}k}^{r}) - Z_{\bar{j}}(\Gamma_{ik}^{r}) + \Gamma_{is}^{r}\Gamma_{\bar{j}k}^{s} - \Gamma_{\bar{j}s}^{r}\Gamma_{ik}^{s} - B_{i\bar{j}}^{s}\Gamma_{sk}^{r} + B_{\bar{j}i}^{\bar{s}}\Gamma_{\bar{s}k}^{r} \\
&= -Z_i(\Gamma_{\bar{j}\bar{r}}^{\bar{k}}) + Z_{\bar{j}}(\Gamma_{i\bar{r}}^{\bar{k}}) + \Gamma_{i\bar{r}}^{\bar{s}}\Gamma_{\bar{j}\bar{s}}^{\bar{k}} - \Gamma_{\bar{j}\bar{r}}^{\bar{s}}\Gamma_{i\bar{s}}^{\bar{k}} + B_{i\bar{j}}^{s}\Gamma_{s\bar{r}}^{\bar{k}} - B_{\bar{j}i}^{\bar{s}}\Gamma_{\bar{s}\bar{r}}^{\bar{k}} \\
&= -R_{i\bar{j}\bar{r}}{}^{\bar{k}}, \\
H_{ijk}{}^{r} &= Z_i(\Gamma_{jk}^{r}) - Z_j(\Gamma_{ik}^{r}) + \Gamma_{is}^{r}\Gamma_{jk}^{s} - \Gamma_{js}^{r}\Gamma_{ik}^{s} - B_{ij}^{s}\Gamma_{sk}^{r} - B_{ij}^{\bar{s}}\Gamma_{\bar{s}k}^{r} \\
&= -Z_i(\Gamma_{j\bar{r}}^{\bar{k}}) - Z_j(\Gamma_{i\bar{r}}^{\bar{k}}) + \Gamma_{i\bar{r}}^{\bar{s}}\Gamma_{j\bar{s}}^{\bar{k}} - \Gamma_{j\bar{r}}^{\bar{s}}\Gamma_{i\bar{s}}^{\bar{k}} + B_{ij}^{s}\Gamma_{s\bar{r}}^{\bar{k}} + B_{ij}^{\bar{s}}\Gamma_{\bar{s}\bar{r}}^{\bar{k}} \\
&= -H_{ij\bar{r}}{}^{\bar{k}}
\end{aligned}$$

and

$$\begin{aligned}
\overline{R_{i\bar{j}k}{}^{r}} &= Z_{\bar{i}}(\Gamma^{\bar{r}}_{j\bar{k}}) - Z_j(\Gamma^{\bar{r}}_{\bar{i}\bar{k}}) + \Gamma^{\bar{r}}_{\bar{i}\bar{s}}\Gamma^{\bar{s}}_{j\bar{k}} - \Gamma^{\bar{r}}_{j\bar{s}}\Gamma^{\bar{s}}_{\bar{i}\bar{k}} - B^{\bar{s}}_{\bar{i}j}\Gamma^{\bar{r}}_{\bar{s}\bar{k}} + B^{s}_{j\bar{i}}\Gamma^{\bar{r}}_{s\bar{k}} \\
&= Z_j(\Gamma^{k}_{\bar{i}r}) - Z_{\bar{i}}(\Gamma^{k}_{jr}) + \Gamma^{s}_{\bar{i}r}\Gamma^{k}_{js} - \Gamma^{s}_{jr}\Gamma^{k}_{\bar{i}s} - B^{s}_{j\bar{i}}\Gamma^{k}_{sr} + B^{\bar{s}}_{\bar{i}j}\Gamma^{k}_{\bar{s}r} \\
&= R_{j\bar{i}r}{}^{k}, \\
\overline{H_{ijk}{}^{r}} &= Z_{\bar{i}}(\Gamma^{\bar{r}}_{\bar{j}\bar{k}}) - Z_{\bar{j}}(\Gamma^{\bar{r}}_{\bar{i}\bar{k}}) + \Gamma^{\bar{r}}_{\bar{i}\bar{s}}\Gamma^{\bar{s}}_{\bar{j}\bar{k}} - \Gamma^{\bar{r}}_{\bar{j}\bar{s}}\Gamma^{\bar{s}}_{\bar{i}\bar{k}} - B^{\bar{s}}_{\bar{i}\bar{j}}\Gamma^{\bar{r}}_{\bar{s}\bar{k}} - B^{s}_{\bar{i}\bar{j}}\Gamma^{\bar{r}}_{s\bar{k}} \\
&= -Z_{\bar{i}}(\Gamma^{k}_{\bar{j}r}) + Z_{\bar{j}}(\Gamma^{k}_{\bar{i}r}) + \Gamma^{s}_{\bar{i}r}\Gamma^{k}_{\bar{j}s} - \Gamma^{s}_{\bar{j}r}\Gamma^{k}_{\bar{i}s} - B^{\bar{s}}_{\bar{j}\bar{i}}\Gamma^{k}_{\bar{s}r} - B^{s}_{\bar{j}\bar{i}}\Gamma^{k}_{sr} \\
&= H_{\bar{j}\bar{i}r}{}^{k}.
\end{aligned}$$

Hence we obtain $R_{i\bar{j}k\bar{r}} = -R_{i\bar{j}\bar{r}k}$, $H_{ijk\bar{r}} = -H_{ij\bar{r}k}$ and $\overline{R_{i\bar{j}k\bar{r}}} = R_{j\bar{i}r\bar{k}}$, $\overline{H_{ijk\bar{r}}} = H_{\bar{j}\bar{i}r\bar{k}}$ by using a local unitary frame with respect to g.

Let B° be the terms of B's depending only on J, which means that these terms do not depend on t along with the solution to the AHCF. Note that $B^{\bar{q}}_{j\bar{b}}$, $B^{q}_{\bar{j}b}$'s do not depend on g, which depend only on J since the mixed derivatives $\nabla_j Z_{\bar{b}}$, $\nabla_{\bar{j}} Z_b$ do not depend on g. Since we have $B^{q}_{b\bar{j}} = -B^{q}_{\bar{j}b}$, we have that $B^{q}_{b\bar{j}}$, $B^{\bar{q}}_{\bar{b}j}$'s also do not depend on g. Also note that $B^{\bar{s}}_{ri}$, $B^{s}_{\bar{r}\bar{i}}$ do not depend on g, depend only on J. These components $B^{\bar{q}}_{j\bar{b}}$, $B^{q}_{\bar{j}b}$, $B^{q}_{b\bar{j}}$, $B^{\bar{q}}_{\bar{b}j}$, $B^{\bar{s}}_{ri}$ and $B^{s}_{\bar{r}\bar{i}}$ are denoted by B°.

Lemma 3.1. (cf. [4, Lemma 3.1], [5, Lemma 3.1]) *Let $(M, \omega(t), J)$ be a solution to the AHCF for $t \in [0, \tau)$ starting at the initial almost Hermitian metric ω_0. Then one has for all $l = 0, 1, 2, \ldots$, for any fixed time $t_0 \in [0, \tau)$,*

$$|Z^l(\Gamma(g(t_0)))|_{g(t_0)} \le C_l$$

for some uniform constant $C_l > 0$.

Proof. Fix an arbitrary chosen time $t_0 \in [0, \tau)$. By using a local unitary frame with respect to $g(t_0)$, since we have $g(t_0)_{i\bar{j}} = \delta_{ij}$, and $\Gamma^k_{ij}(g(t_0)) = -\Gamma^{\bar{j}}_{i\bar{k}}(g(t_0)) = -B^{\bar{j}}_{i\bar{k}} = -B^{\circ}$, which means that these coefficients do not depend on t_0, we obtain on M,

$$\begin{aligned}
|\Gamma^k_{ij}(g(t_0))|^2_{g(t_0)} &= g(t_0)_{k\bar{l}}\, g(t_0)^{i\bar{r}} g(t_0)^{j\bar{s}} \Gamma^k_{ij}(g(t_0)) \Gamma^{\bar{l}}_{\bar{r}\bar{s}}(g(t_0)) \\
&= \Gamma^{\bar{j}}_{i\bar{k}}(g(t_0)) \Gamma^{j}_{\bar{i}k}(g(t_0)) \\
&= B^{\bar{j}}_{i\bar{k}} B^{j}_{\bar{i}k} \le C_0
\end{aligned}$$

for some uniform constant $C > 0$ since $B^{\bar{q}}_{j\bar{b}}$, $B^{q}_{\bar{j}b}$'s do not depend on g, which depend only on J because the mixed derivatives $\nabla_j Z_{\bar{b}}$ do not depend on g (cf. [11, Lemma 5.2]).

Likewise, using a local unitary frame with respect to $g(t_0)$, for all $l = 0, 1, 2, \ldots$, we have

$$|Z^l(\Gamma^k_{ij}(g(t_0))|^2_{g(t_0)} = |Z^l(\Gamma^{\bar{j}}_{i\bar{k}}(g(t_0))|^2_{g(t_0)} = |Z^l(B^{\bar{j}}_{i\bar{k}})|^2_{g(t_0)} = |Z^l(B^{\circ})|^2_{g(t_0)} \le C_l$$

for some uniform constants $C_l > 0$. □

We introduce some evolution equations in the following.

Lemma 3.2. (cf. [11, Lemma 5.1, 5.2 and 5.5]) *Let $g(t)$ be a smooth family of metric on M compatible with J. We denote by $h = \frac{\partial}{\partial t} g$ the variation of g. Then one has*

$$\frac{\partial}{\partial t}\Gamma^k_{ij} = g^{k\bar{l}}\nabla_i h_{j\bar{l}}, \quad \frac{\partial}{\partial t}\Gamma^{\bar{k}}_{ij} = 0, \quad \frac{\partial}{\partial t}R_{i\bar{j}k\bar{l}} = R^r_{i\bar{j}k} h_{r\bar{l}} - \nabla_{\bar{j}}\nabla_i h_{k\bar{l}}$$

and

$$\frac{\partial}{\partial t}T_{ij\bar{k}} = \nabla_i h_{j\bar{k}} - \nabla_j h_{i\bar{k}} + T^m_{ij} h_{m\bar{k}}.$$

The second formula follows from the fact that the $\Gamma^{\bar{k}}_{ij}$'s do not depend on t.

We need the following computation for estimating components $\bar{Z}(T')_{i\bar{j}}$.

In order to avoid a notational quagmire, we adopte the following $*$-convention $A_1 * A_2$ between two quantities A_1 and A_2 with respect to a metric g:

(i) Summation over pairs of maching upper and lower indices.
(ii) Contraction on upper indices with respect to the metric.
(iii) Contraction on lower indices with respect to the dual metrics.

Lemma 3.3. *Let (M^{2n}, g, J) be a compact almost Hermitian manifold. Let $\{Z_r\}$ be a local $(1,0)$-frame with respect to g around a fixed point $p \in M$. Then one has that*

$$\begin{aligned}\bar{Z}(T') = {} & \bar{Z}(\Gamma) + Z(B^\circ) + B^\circ * \Gamma \\ & + T' * \bar{\Gamma} + \Gamma * \bar{\Gamma} + B^\circ * T' + B^\circ * \bar{T}' + T' * \bar{T}' + \Gamma * \bar{T}' \\ & + B^\circ * B^\circ + \mathcal{O}(Z(g)) + \mathcal{O}(\bar{Z}(g)).\end{aligned} \tag{3.1}$$

Moreover, we then have the estimate

$$\bar{Z}(T') \leq C(|\nabla\Gamma|_g + |\Gamma|^2_g + |T'|^2_g + 1)\omega + \mathcal{O}(Z(g)) + \mathcal{O}(\bar{Z}(g)),$$

where ω is the associated real $(1,1)$-form with respect to g.

Proof. Let $\{Z_r\}$ be an arbitrary local $(1,0)$-frame around a fixed point $p \in M$. Now let D be the Levi-Civita connection with respect to g and let ∇ be the Chern connection with respect to g. The relation between D and ∇ is as follows (cf. [12, Lemma 3.1]):

$$g(D_Y X, Z) = g(\nabla_Y X, Z) + \frac{1}{2}(g(T(X,Y),Z) + g(T(Y,Z),X) - g(T(Z,X),Y))$$

for any tangent vector fields X, Y and Z. Here notice that the torsion T of the Chern connection ∇ is also defined as

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

and in this sense, we compute as follows with a local $(1,0)$-frame $\{Z_r\}$ with respect to g:

$$\begin{aligned} T_{ij} &:= T(Z_i, Z_j) = \nabla_{Z_i} Z_j - \nabla_{Z_j} Z_i - [Z_i, Z_j] \\ &= \Gamma_{ij}^k Z_k - \Gamma_{ji}^k Z_k - B_{ij}^k Z_k - B_{ij}^{\bar{k}} Z_{\bar{k}} = T_{ij}^k Z_k + T_{ij}^{\bar{k}} Z_{\bar{k}}, \\ T_{i\bar{j}} &= T(Z_i, Z_{\bar{j}}) = \nabla_{Z_i} Z_{\bar{j}} - \nabla_{Z_{\bar{j}}} Z_i - [Z_i, Z_{\bar{j}}] \\ &= (B_{\bar{j}i}^k - \Gamma_{\bar{j}i}^k) Z_k + (\Gamma_{i\bar{j}}^{\bar{k}} - B_{i\bar{j}}^{\bar{k}}) Z_{\bar{k}} = 0. \end{aligned}$$

Then we have the following:

$$\begin{aligned} g(D_r Z_i - D_i Z_r, Z_{\bar{s}}) &= g(\nabla_r Z_i, Z_{\bar{s}}) + \frac{1}{2}(g(T_{ir}, Z_{\bar{s}}) + g(T_{r\bar{s}}, Z_i) - g(T_{\bar{s}i}, Z_r)) \\ &\quad - g(\nabla_i Z_r, Z_{\bar{s}}) - \frac{1}{2}(g(T_{ri}, Z_{\bar{s}}) + g(T_{i\bar{s}}, Z_r) - g(T_{\bar{s}r}, Z_i)) \\ &= \Gamma_{ri}^k g_{k\bar{s}} - \Gamma_{ir}^k g_{k\bar{s}} + T_{ir}^k g_{k\bar{s}} \\ &= B_{ri}^k g_{k\bar{s}} \\ &= g([Z_r, Z_i], Z_{\bar{s}}), \\ g(D_{\bar{r}} Z_i - D_i Z_{\bar{r}}, Z_{\bar{s}}) &= g(\nabla_{\bar{r}} Z_i, Z_{\bar{s}}) + \frac{1}{2}(g(T_{i\bar{r}}, Z_{\bar{s}}) + g(T_{\bar{r}\bar{s}}, Z_i) - g(T_{\bar{s}i}, Z_{\bar{r}})) \\ &\quad - g(\nabla_i Z_{\bar{r}}, Z_{\bar{s}}) - \frac{1}{2}(g(T_{\bar{r}i}, Z_{\bar{s}}) + g(T_{i\bar{s}}, Z_{\bar{r}}) - g(T_{\bar{s}\bar{r}}, Z_i)) \\ &= \Gamma_{\bar{r}i}^k g_{k\bar{s}} \\ &= B_{\bar{r}i}^k g_{k\bar{s}} \\ &= g([Z_{\bar{r}}, Z_i], Z_{\bar{s}}). \end{aligned}$$

We compute

$$\begin{aligned} & Z_{\bar{r}}(g([Z_r, Z_i], Z_{\bar{j}})) = g(D_{\bar{r}}[Z_r, Z_i], Z_{\bar{j}}) + g([Z_r, Z_i], D_{\bar{r}} Z_{\bar{j}}), D_{\bar{r}}[Z_r, Z_i] \\ &\quad = D_{\bar{r}}(D_r Z_i - D_i Z_r) \\ &\quad = [Z_{\bar{r}}, Z_r] Z_i + D_r[Z_{\bar{r}}, Z_i] - [Z_{\bar{r}}, Z_i] Z_r - D_i[Z_{\bar{r}}, Z_r] + [Z_r, Z_i] Z_{\bar{r}} \end{aligned}$$

and

$$\begin{aligned} & Z_{\bar{r}}(g([Z_r, Z_i], Z_{\bar{j}})) \\ &\quad = g([Z_{\bar{r}}, Z_r] Z_i, Z_{\bar{j}}) + g(D_r[Z_{\bar{r}}, Z_i], Z_{\bar{j}}) \\ &\qquad - g([Z_{\bar{r}}, Z_i] Z_r, Z_{\bar{j}}) - g(D_i[Z_{\bar{r}}, Z_r], Z_{\bar{j}}) + g([Z_r, Z_i] Z_{\bar{r}}, Z_{\bar{j}}) + g([Z_r, Z_i], D_{\bar{r}} Z_{\bar{j}}) \\ &\quad = g([Z_{\bar{r}}, Z_r] Z_i, Z_{\bar{j}}) + Z_r(g([Z_{\bar{r}}, Z_i], Z_{\bar{j}})) - g([Z_{\bar{r}}, Z_i], D_r Z_{\bar{j}}) - g([Z_{\bar{r}}, Z_i] Z_r, Z_{\bar{j}}) \\ &\qquad - Z_i(g([Z_{\bar{r}}, Z_r], Z_{\bar{j}})) + g([Z_{\bar{r}}, Z_r], D_i Z_{\bar{j}}) + g([Z_r, Z_i] Z_{\bar{r}}, Z_{\bar{j}}) + g([Z_r, Z_i], D_{\bar{r}} Z_{\bar{j}}) \\ &\quad = g([Z_{\bar{r}}, Z_r] Z_i, Z_{\bar{j}}) - g([Z_{\bar{r}}, Z_i], D_r Z_{\bar{j}}) - g([Z_{\bar{r}}, Z_i] Z_r, Z_{\bar{j}}) + g([Z_{\bar{r}}, Z_r], D_i Z_{\bar{j}}) \\ &\qquad + g([Z_r, Z_i] Z_{\bar{r}}, Z_{\bar{j}}) + g([Z_r, Z_i], D_{\bar{r}} Z_{\bar{j}}) + Z_r(B_{\bar{r}i}^s) g_{s\bar{j}} - Z_i(B_{\bar{r}r}^s) g_{s\bar{j}} + \mathcal{O}(Z(g)), \end{aligned}$$

where we used that

$$\begin{aligned} & Z_r(g([Z_{\bar{r}}, Z_i], Z_{\bar{j}})) - Z_i(g([Z_{\bar{r}}, Z_r], Z_{\bar{j}})) \\ &\quad = Z_r(g(B_{\bar{r}i}^s Z_s + B_{\bar{r}i}^{\bar{s}} Z_{\bar{s}}, Z_{\bar{j}})) - Z_i(g(B_{\bar{r}r}^s Z_s + B_{\bar{r}r}^{\bar{s}} Z_{\bar{s}}, Z_{\bar{j}})) \end{aligned}$$

$$= Z_r(B^s_{\bar{r}i})g_{s\bar{j}} - Z_i(B^s_{\bar{r}r})g_{s\bar{j}} + \mathcal{O}(Z(g)).$$

We compute by using that $[Z_r, Z_i] = B^s_{ri}Z_s + B^{\bar{s}}_{ri}Z_{\bar{s}}$,

$$\begin{aligned}
Z_{\bar{r}}(B^s_{ri})g_{s\bar{j}} &= g(\nabla_{\bar{r}}(B^s_{ri})Z_s, Z_{\bar{j}}) \\
&= g(\nabla_{\bar{r}}(B^s_{ri}Z_s) - B^s_{ri}\nabla_{\bar{r}}Z_s, Z_{\bar{j}}) \\
&= Z_{\bar{r}}(g(B^s_{ri}Z_s, Z_{\bar{j}})) - g(B^s_{ri}Z_s, \nabla_{\bar{r}}Z_{\bar{j}}) - B^s_{ri}g(\nabla_{\bar{r}}Z_s, Z_{\bar{j}}) \\
&= Z_{\bar{r}}(g([Z_r, Z_i] - B^{\bar{s}}_{ri}Z_{\bar{s}}, Z_{\bar{j}})) - g(B^s_{ri}Z_s, \Gamma^{\bar{k}}_{\bar{r}\bar{j}}Z_{\bar{k}}) - B^s_{ri}g(\Gamma^k_{\bar{r}s}Z_k, Z_{\bar{j}}) \\
&= g([Z_{\bar{r}}, Z_r]Z_i, Z_{\bar{j}}) - g([Z_{\bar{r}}, Z_i], D_rZ_{\bar{j}}) - g([Z_{\bar{r}}, Z_i]Z_r, Z_{\bar{j}}) \\
&\quad + g([Z_{\bar{r}}, Z_r], D_iZ_{\bar{j}}) + g([Z_r, Z_i]Z_{\bar{r}}, Z_{\bar{j}}) + g([Z_r, Z_i], D_{\bar{r}}Z_{\bar{j}}) \\
&\quad + Z_r(B^s_{\bar{r}i})g_{s\bar{j}} - Z_i(B^s_{\bar{r}r})g_{s\bar{j}} - B^s_{ri}\Gamma^{\bar{k}}_{\bar{r}\bar{j}}g_{s\bar{k}} - B^s_{ri}\Gamma^k_{\bar{r}s}g_{k\bar{j}} + \mathcal{O}(Z(g)).
\end{aligned}$$

We also compute

$$\begin{aligned}
g([Z_{\bar{r}}, Z_r]Z_i, Z_{\bar{j}}) &= g(B^k_{\bar{r}r}D_kZ_i + B^{\bar{k}}_{\bar{r}r}D_{\bar{k}}Z_i, Z_{\bar{j}}) \\
&= B^k_{\bar{r}r}(g(\nabla_kZ_i, Z_{\bar{j}}) + \frac{1}{2}(g(T_{ik}, Z_{\bar{j}}) + g(T_{k\bar{j}}, Z_i) - g(T_{\bar{j}i}, Z_k))) \\
&\quad + B^{\bar{k}}_{\bar{r}r}(g(\nabla_{\bar{k}}Z_i, Z_{\bar{j}}) + \frac{1}{2}(g(T_{i\bar{k}}, Z_{\bar{j}}) + g(T_{\bar{k}\bar{j}}, Z_i) - g(T_{\bar{j}i}, Z_{\bar{k}}))) \\
&= B^k_{\bar{r}r}\Gamma^s_{ki}g_{s\bar{j}} + \frac{1}{2}B^k_{\bar{r}r}T^s_{ik}g_{s\bar{j}} + B^{\bar{k}}_{\bar{r}r}\Gamma^s_{\bar{k}i}g_{s\bar{j}} + \frac{1}{2}B^{\bar{k}}_{\bar{r}r}\overline{T^s_{kj}}g_{i\bar{s}} \\
&= B^\circ * \Gamma + B^\circ * T' + B^\circ * B^\circ + B^\circ * \bar{T}', \\
g([Z_{\bar{r}}, Z_i], D_rZ_{\bar{j}}) &= g(B^s_{\bar{r}i}Z_s + B^{\bar{s}}_{\bar{r}i}Z_{\bar{s}}, D_rZ_{\bar{j}}) \\
&= B^s_{\bar{r}i}(g(\nabla_rZ_{\bar{j}}, Z_s) + \frac{1}{2}(g(T_{\bar{j}r}, Z_s) + g(T_{rs}, Z_{\bar{j}}) - g(T_{s\bar{j}}, Z_r))) \\
&\quad + B^{\bar{s}}_{\bar{r}i}(g(\nabla_rZ_{\bar{j}}, Z_{\bar{s}}) + \frac{1}{2}(g(T_{\bar{j}r}, Z_{\bar{s}}) + g(T_{r\bar{s}}, Z_{\bar{j}}) - g(T_{\bar{s}\bar{j}}, Z_r))) \\
&= B^s_{\bar{r}i}\Gamma^{\bar{k}}_{r\bar{j}}g_{s\bar{k}} + \frac{1}{2}B^s_{\bar{r}i}T^k_{rs}g_{k\bar{j}} + \frac{1}{2}B^{\bar{s}}_{\bar{r}i}\overline{T^k_{js}}g_{r\bar{k}} \\
&= B^\circ * B^\circ + B^\circ * T' + B^\circ * \bar{T}', \\
g([Z_{\bar{r}}, Z_i]Z_r, Z_{\bar{j}}) &= g(B^s_{\bar{r}i}D_sZ_r + B^{\bar{s}}_{\bar{r}i}D_{\bar{s}}Z_r, Z_{\bar{j}}) \\
&= B^s_{\bar{r}i}(g(\nabla_sZ_r, Z_{\bar{j}}) + \frac{1}{2}(g(T_{rs}, Z_{\bar{j}}) + g(T_{s\bar{j}}, Z_r) - g(T_{\bar{j}r}, Z_s))) \\
&\quad + B^{\bar{s}}_{\bar{r}i}(g(\nabla_{\bar{s}}Z_r, Z_{\bar{j}}) + \frac{1}{2}(g(T_{r\bar{s}}, Z_{\bar{j}}) + g(T_{\bar{s}\bar{j}}, Z_r) - g(T_{\bar{j}r}, Z_{\bar{s}}))) \\
&= B^s_{\bar{r}i}\Gamma^k_{sr}g_{k\bar{j}} + \frac{1}{2}B^s_{\bar{r}i}T^k_{rs}g_{k\bar{j}} + B^{\bar{s}}_{\bar{r}i}\Gamma^k_{\bar{s}r}g_{k\bar{j}} + \frac{1}{2}B^{\bar{s}}_{\bar{r}i}\overline{T^k_{sj}}g_{r\bar{k}} \\
&= B^\circ * \Gamma + B^\circ * T' + B^\circ * B^\circ + B^\circ * \bar{T}', \\
g([Z_{\bar{r}}, Z_r], D_iZ_{\bar{j}}) &= g(B^s_{\bar{r}r}Z_s + B^{\bar{s}}_{\bar{r}r}Z_{\bar{s}}, D_iZ_{\bar{j}}) \\
&= B^s_{\bar{r}r}(g(\nabla_iZ_{\bar{j}}, Z_s) + \frac{1}{2}(g(T_{\bar{j}i}, Z_s) + g(T_{is}, Z_{\bar{j}}) - g(T_{s\bar{j}}, Z_i))) \\
&\quad + B^{\bar{s}}_{\bar{r}r}(g(\nabla_iZ_{\bar{j}}, Z_{\bar{s}}) + \frac{1}{2}(g(T_{\bar{j}i}, Z_{\bar{s}}) + g(T_{i\bar{s}}, Z_{\bar{j}}) - g(T_{\bar{s}\bar{j}}, Z_i))) \\
&= B^s_{\bar{r}r}\Gamma^{\bar{l}}_{i\bar{j}}g_{s\bar{l}} + \frac{1}{2}B^s_{\bar{r}r}T^l_{is}g_{l\bar{j}} + \frac{1}{2}B^{\bar{s}}_{\bar{r}r}\overline{T^l_{js}}g_{i\bar{l}}
\end{aligned}$$

$$
\begin{aligned}
&= B^\circ * B^\circ + B^\circ * T' + B^\circ * \bar{T}',\\
g([Z_r, Z_i]Z_{\bar r}, Z_{\bar j}) &= g(B^s_{ri} D_s Z_{\bar r} + B^{\bar s}_{ri} D_{\bar s} Z_{\bar r}, Z_{\bar j})\\
&= B^s_{ri}(g(\nabla_s Z_{\bar r}, Z_{\bar j}) + \frac{1}{2}(g(T_{\bar r s}, Z_{\bar j}) + g(T_{s\bar j}, Z_{\bar r}) - g(T_{\bar j \bar r}, Z_s)))\\
&\quad + B^{\bar s}_{\bar r i}(g(\nabla_{\bar s} Z_{\bar r}, Z_{\bar j}) + \frac{1}{2}(g(T_{\bar r \bar s}, Z_{\bar j}) + g(T_{\bar s \bar j}, Z_{\bar r}) - g(T_{\bar j \bar r}, Z_{\bar s})))\\
&= \frac{1}{2} B^s_{ri} \overline{T^l_{rj}} g_{s\bar l} + \frac{1}{2} B^{\bar s}_{ri} T^l_{\bar r \bar s} g_{l\bar j} + \frac{1}{2} B^{\bar s}_{ri} T^l_{\bar s \bar j} g_{l\bar r} + \frac{1}{2} B^{\bar s}_{ri} T^l_{\bar r \bar j} g_{l\bar s}\\
&= B * \bar{T}' + B^\circ * B^\circ,\\
g([Z_r, Z_i], D_{\bar r} Z_{\bar j}) &= g(B^s_{ri} Z_s + B^{\bar s}_{ri} Z_{\bar s}, D_{\bar r} Z_{\bar j})\\
&= B^s_{ri}(g(\nabla_{\bar r} Z_{\bar j}, Z_s) + \frac{1}{2}(g(T_{\bar j \bar r}, Z_s) + g(T_{\bar r s}, Z_{\bar j}) - g(T_{s\bar j}, Z_{\bar r})))\\
&\quad + B^{\bar s}_{ri}(g(\nabla_{\bar r} Z_{\bar j}, Z_{\bar s}) + \frac{1}{2}(g(T_{\bar j \bar r}, Z_{\bar s}) + g(T_{\bar r \bar s}, Z_{\bar j}) - g(T_{\bar s \bar j}, Z_{\bar r})))\\
&= B^s_{ri} \Gamma^{\bar l}_{\bar r \bar j} g_{s\bar l} + \frac{1}{2} B^s_{ri} \overline{T^l_{jr}} g_{s\bar l} + \frac{1}{2} B^{\bar s}_{ri} T^l_{\bar j \bar r} g_{l\bar s} + \frac{1}{2} B^{\bar s}_{ri} T^l_{\bar r \bar s} g_{l\bar j} + \frac{1}{2} B^{\bar s}_{ri} T^l_{\bar j \bar s} g_{l\bar r}\\
&= B * \bar\Gamma + B * \bar{T}' + B^\circ * B^\circ,
\end{aligned}
$$

where we used that $g_{\bar l \bar s} = 0$, $\Gamma^{\bar k}_{r\bar j} = B^{\bar k}_{r\bar j}$, $\Gamma^s_{\bar k i} = \overline{\Gamma^{\bar s}_{k\bar i}} = \overline{B^{\bar s}_{k\bar i}} = B^s_{\bar k i}$, $T^l_{\bar r \bar s} = \overline{T^{\bar l}_{rs}} = -\overline{B^{\bar l}_{rs}} = -B^l_{\bar r \bar s}$. Using these computations, we obtain

$$
\begin{aligned}
&-Z_{\bar r}(T^s_{ri}) g_{s\bar j} = -Z_{\bar r}(\Gamma^s_{ri} - \Gamma^s_{ir} - B^s_{ri}) g_{s\bar j}\\
&= -Z_{\bar r}(\Gamma^s_{ri}) g_{s\bar j} + Z_{\bar r}(\Gamma^s_{ir}) g_{s\bar j} + Z_{\bar r}(B^s_{ri}) g_{s\bar j}\\
&= -Z_{\bar r}(\Gamma^s_{ri}) g_{s\bar j} + Z_{\bar r}(\Gamma^s_{ir}) g_{s\bar j} + g([Z_{\bar r}, Z_r]Z_i, Z_{\bar j})\\
&\quad - g([Z_{\bar r}, Z_i], D_r Z_{\bar j})\\
&\quad - g([Z_{\bar r}, Z_i]Z_r, Z_{\bar j}) + g([Z_{\bar r}, Z_r], D_i Z_{\bar j})\\
&\quad + g([Z_r, Z_i]Z_{\bar r}, Z_{\bar j}) + g([Z_r, Z_i], D_{\bar r} Z_{\bar j})\\
&\quad + Z_r(B^s_{\bar r i}) g_{s\bar j} - Z_i(B^s_{\bar r r}) g_{s\bar j} - B^s_{ri} \Gamma^{\bar k}_{\bar r \bar j} g_{s\bar k}\\
&\quad - B^s_{ri} \Gamma^k_{\bar r s} g_{k\bar j} + \mathcal{O}(Z(g))\\
&= -Z_{\bar r}(\Gamma^s_{ri}) g_{s\bar j} + Z_{\bar r}(\Gamma^s_{ir}) g_{s\bar j} + Z_r(B^s_{\bar r i}) g_{s\bar j}\\
&\quad - Z_i(B^s_{\bar r r}) g_{s\bar j} - B^s_{ri} \Gamma^{\bar k}_{\bar r \bar j} g_{s\bar k} - B^s_{ri} \Gamma^k_{\bar r s} g_{k\bar j}\\
&\quad + B^\circ * \Gamma + B * \bar\Gamma + B^\circ * T' + B^\circ * \bar{T}' + B * \bar{T}' + B^\circ * B^\circ + \mathcal{O}(Z(g)).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
-Z_{\bar j}(w_i) &= -Z_{\bar j}(T_{ir\bar r})\\
&= -Z_{\bar j}(T^s_{ir} g_{s\bar r})\\
&= -Z_{\bar j}(T^s_{ir}) g_{s\bar r} - T^s_{ir} Z_{\bar j}(g_{s\bar r})\\
&= -Z_{\bar j}(\Gamma^s_{ir}) g_{s\bar r} + Z_{\bar j}(\Gamma^s_{ri}) g_{s\bar r}\\
&\quad + Z_{\bar j}(B^s_{ir}) g_{s\bar r} - T^s_{ir} Z_{\bar j}(g_{s\bar r})\\
&= -Z_{\bar j}(\Gamma^s_{ir}) g_{s\bar r} + Z_{\bar j}(\Gamma^s_{ri}) g_{s\bar r}
\end{aligned}
$$

$$
\begin{aligned}
&+g([Z_{\bar{j}}, Z_i]Z_r, Z_{\bar{r}}) + g(D_i[Z_{\bar{j}}, Z_r], Z_{\bar{r}}) \\
&-g([Z_{\bar{j}}, Z_r]Z_i, Z_{\bar{r}}) - g(D_r[Z_{\bar{j}}, Z_i], Z_{\bar{r}}) + g([Z_i, Z_r]Z_{\bar{j}}, Z_{\bar{r}}) \\
&+g([Z_i, Z_r], D_{\bar{j}} Z_{\bar{r}}) - B^s_{ir}\overline{\Gamma^k_{jr}} g_{s\bar{k}} - B^s_{ir}B^k_{\bar{j}s} g_{k\bar{r}} - T^s_{ir} Z_{\bar{j}}(g_{s\bar{r}}) \\
= &-Z_{\bar{j}}(\Gamma^s_{ir}) g_{s\bar{r}} + Z_{\bar{j}}(\Gamma^s_{ri}) g_{s\bar{r}} \\
&-B^s_{ir}\overline{\Gamma^k_{jr}} g_{s\bar{k}} - B^s_{ir}\Gamma^k_{\bar{j}s} g_{k\bar{r}} - T^s_{ir} Z_{\bar{j}}(g_{s\bar{r}}) \\
&+g([Z_{\bar{j}}, Z_i]Z_r, Z_{\bar{r}}) + Z_i(B^s_{\bar{j}r}) g_{s\bar{r}} + B^s_{\bar{j}r} Z_i(g_{s\bar{r}}) - g([Z_{\bar{j}}, Z_r], D_i Z_{\bar{r}}) \\
&-g([Z_{\bar{j}}, Z_r]Z_i, Z_{\bar{r}}) \\
&-Z_r(B^s_{\bar{j}i}) g_{s\bar{r}} - B^s_{\bar{j}i} Z_r(g_{s\bar{r}}) + g([Z_{\bar{j}}, Z_i], D_r Z_{\bar{r}}) + g([Z_i, Z_r]Z_{\bar{j}}, Z_{\bar{r}}) \\
&+g([Z_i, Z_r], D_{\bar{j}} Z_{\bar{r}}) \\
= &-Z_{\bar{j}}(\Gamma^s_{ir}) g_{s\bar{r}} + Z_{\bar{j}}(\Gamma^s_{ri}) g_{s\bar{r}} \\
&+Z_i(B^s_{\bar{j}r}) g_{s\bar{r}} - Z_r(B^s_{\bar{j}i}) g_{s\bar{r}} \\
&-B^s_{ir}\overline{\Gamma^k_{jr}} g_{s\bar{k}} - B^s_{ir}\Gamma^k_{\bar{j}s} g_{k\bar{r}} \\
&+B^\circ * \Gamma + B * \bar{\Gamma} + B^\circ * T' + B^\circ * \bar{T}' + B * \bar{T}' \\
&+B^\circ * B^\circ + \mathcal{O}(Z(g)) + \mathcal{O}(\bar{Z}(g)).
\end{aligned}
$$

Combining these with the term $-g^{p\bar{q}} T^r_{pi} Z_{\bar{j}}(g_{r\bar{q}}) = T^s_{ir} Z_{\bar{j}}(g_{s\bar{r}})$, we have

$$
\begin{aligned}
\bar{Z}(T')_{i\bar{j}} &= -Z_{\bar{r}}(T^s_{ri}) g_{s\bar{j}} - Z_{\bar{j}}(w_i) - g^{p\bar{q}} T^r_{pi} Z_{\bar{j}}(g_{r\bar{q}}) \\
&= -Z_{\bar{r}}(\Gamma^s_{ri}) g_{s\bar{j}} + Z_{\bar{r}}(\Gamma^s_{ir}) g_{s\bar{j}} + Z_r(B^s_{\bar{r}i}) g_{s\bar{j}} \\
&\quad -Z_i(B^s_{\bar{r}r}) g_{s\bar{j}} - B^s_{ri}\Gamma^{\bar{k}}_{\bar{r}\bar{j}} g_{s\bar{k}} - B^s_{ri}\Gamma^k_{\bar{r}s} g_{k\bar{j}} \\
&\quad -Z_{\bar{j}}(\Gamma^s_{ir}) g_{s\bar{r}} + Z_{\bar{j}}(\Gamma^s_{ri}) g_{s\bar{r}} + Z_i(B^s_{\bar{j}r}) g_{s\bar{r}} \\
&\quad -Z_r(B^s_{\bar{j}i}) g_{s\bar{r}} - B^s_{ir}\Gamma^{\bar{k}}_{\bar{j}\bar{r}} g_{s\bar{k}} - B^s_{ir}\Gamma^k_{\bar{j}s} g_{k\bar{r}} \\
&\quad +B^\circ * \Gamma + B * \bar{\Gamma} + B^\circ * T' + B^\circ * \bar{T}' \\
&\quad +B * \bar{T}' + B^\circ * B^\circ + \mathcal{O}(Z(g)) + \mathcal{O}(\bar{Z}(g)) \\
&= \bar{Z}(\Gamma) + Z(B^\circ) + B * B^\circ + B * \bar{\Gamma} \\
&\quad +B^\circ * \Gamma + B * \bar{\Gamma} + B^\circ * T' + B^\circ * \bar{T}' + B * \bar{T}' \\
&\quad +B^\circ * B^\circ + \mathcal{O}(Z(g)) + \mathcal{O}(\bar{Z}(g)) \\
&= \bar{Z}(\Gamma) + Z(B^\circ) + B^\circ * \Gamma + T' * \bar{\Gamma} + \Gamma * \bar{\Gamma} \\
&\quad +B^\circ * T' + B^\circ * \bar{T}' + T' * \bar{T}' + \Gamma * \bar{T}' \\
&\quad +B^\circ * B^\circ + \mathcal{O}(Z(g)) + \mathcal{O}(\bar{Z}(g)),
\end{aligned}
$$

where we used that $B = T' + \Gamma$ and then $B * B^\circ = T' * B^\circ + \Gamma * B^\circ$, $B * \bar{\Gamma} = T' * \bar{\Gamma} + \Gamma * \bar{\Gamma}$ $B * \bar{T}' = T' * \bar{T}' + \Gamma * \bar{T}'$ at the third equality. □

Lemma 3.4. *Let $(M, g(t), J)$ be a solution to the AHCF for $t \in [0, \tau)$ starting at the initial almost Hermitian metric ω_0. Suppose that there exist uniform bounds for $|T'(g(t))|^2_{C^0(g(t))}$ and $|\nabla T'(g(t))|_{C^0(g(t))}$. Then we have that for any fixed time $t_0 \in [0, \tau)$,*

$$|\bar\nabla\nabla T'(g(t_0))|_{C^0(g(t_0))} \leq C, \quad |\bar\nabla\nabla \bar T'(g(t_0))|_{C^0(g(t_0))} \leq C,$$
$$|\bar\nabla^2 T'(g(t_0))|_{C^0(g(t_0))} \leq C, \quad |\nabla^2 \bar T'(g(t_0))|_{C^0(g(t_0))} \leq C$$

for some uniform constant $C > 0$.

Proof. We compute

$$\begin{aligned}
\nabla_{\bar i}\nabla_m(T_{jk\bar r}) &= \nabla_{\bar i}\nabla_m(T^l_{jk})g_{l\bar r}\\
&= \nabla_{\bar i}(Z_m(T^l_{jk}) - \Gamma^s_{mj}T^l_{sk} - \Gamma^s_{mk}T^l_{js} + \Gamma^l_{ms}T^s_{jk})g_{l\bar r}\\
&= \nabla_{\bar i}(Z_m(T^l_{jk})) + \bar\nabla\Gamma * T' + \Gamma * \bar\nabla T'
\end{aligned}$$

and

$$\begin{aligned}
\nabla_{\bar i}(Z_m(T^l_{jk})) &= Z_{\bar i}Z_m(T^l_{jk}) - \Gamma^s_{\bar i m}Z_s(T^l_{jk}) - \Gamma^s_{\bar i j}Z_m(T^l_{sk}) - \Gamma^s_{\bar i k}Z_m(T^l_{js}) + \Gamma^l_{\bar i s}Z_m(T^s_{jk})\\
&= Z_m Z_{\bar i}(T^l_{jk}) + [Z_{\bar i}, Z_m](T^l_{jk})\\
&\quad -\Gamma^s_{\bar i m}Z_s(T^l_{jk}) - \Gamma^s_{\bar i j}Z_m(T^l_{sk}) - \Gamma^s_{\bar i k}Z_m(T^l_{js}) + \Gamma^l_{\bar i s}Z_m(T^s_{jk}).\\
&= Z_m Z_{\bar i}(T^l_{jk}) + B^\circ * Z(T') + B^\circ * \bar Z(T')
\end{aligned}$$

Using that $[Z_r, Z_i] = B^s_{ri}Z_s + B^{\bar s}_{ri}Z_{\bar s}$, we compute

$$\begin{aligned}
Z_{\bar i}(B^l_{jk})g_{l\bar r} &= g(\nabla_{\bar i}(B^l_{jk})Z_l, Z_{\bar r})c\\
&= g(\nabla_{\bar i}(B^l_{jk}Z_l) - B^l_{jk}\nabla_{\bar i}Z_l, Z_{\bar r})\\
&= Z_{\bar i}(g(B^l_{jk}Z_l, Z_{\bar r})) - g(B^l_{jk}Z_l, \nabla_{\bar i}Z_{\bar r}) - B^l_{jk}g(\nabla_{\bar i}Z_l, Z_{\bar r})\\
&= Z_{\bar i}(g([Z_j, Z_k] - B^{\bar l}_{jk}Z_{\bar l}, Z_{\bar r})) - g(B^l_{jk}Z_l, \Gamma^{\bar s}_{\bar i\bar r}Z_{\bar s}) - B^l_{jk}g(\Gamma^s_{\bar i l}Z_s, Z_{\bar r})\\
&= g(D_{\bar i}[Z_j, Z_k], Z_{\bar r}) + g([Z_j, Z_k], D_{\bar i}Z_{\bar r}) - B^l_{jk}\Gamma^{\bar s}_{\bar i\bar r}g(Z_l, Z_{\bar s}) - B^l_{jk}\Gamma^s_{\bar i l}g(Z_s, Z_{\bar r})\\
&= g([Z_{\bar i}, Z_j]Z_k, Z_{\bar r}) + g([Z_k, Z_{\bar i}]Z_j, Z_{\bar r}) + g(D_j[Z_{\bar i}, Z_k], Z_{\bar r})\\
&\quad +g(D_k[Z_j, Z_{\bar i}], Z_{\bar r}) + g([Z_j, Z_k]Z_{\bar i}, Z_{\bar r}) + g([Z_j, Z_k], D_{\bar i}Z_{\bar r})\\
&\quad -B^l_{jk}\Gamma^{\bar s}_{\bar i\bar r}g_{l\bar s} - B^l_{jk}\Gamma^s_{\bar i l}g_{s\bar r}\\
&= g([Z_{\bar i}, Z_j]Z_k, Z_{\bar r}) + g([Z_k, Z_{\bar i}]Z_j, Z_{\bar r}) + Z_j(g([Z_{\bar i}, Z_k], Z_{\bar r}))\\
&\quad -g([Z_{\bar i}, Z_k], D_jZ_{\bar r})\\
&\quad +Z_k(g([Z_j, Z_{\bar i}], Z_{\bar r})) - g([Z_j, Z_{\bar i}], D_kZ_{\bar r}) + g([Z_j, Z_k]Z_{\bar i}, Z_{\bar r})\\
&\quad +g([Z_j, Z_k], D_{\bar i}Z_{\bar r})\\
&\quad -B^l_{jk}\Gamma^{\bar s}_{\bar i\bar r}g_{l\bar s} - B^l_{jk}\Gamma^s_{\bar i l}g_{s\bar r},
\end{aligned}$$

where we used that

$$D_{\bar i}[Z_j, Z_k] = [Z_{\bar i}, Z_j]Z_k + [Z_k, Z_{\bar i}]Z_j + D_j[Z_{\bar i}, Z_k] + D_k[Z_j, Z_{\bar i}] + [Z_j, Z_k]Z_{\bar i}.$$

We also compute

$$\begin{aligned}
g([Z_{\bar i}, Z_j]Z_k, Z_{\bar r}) &= g(B^s_{\bar i j}D_sZ_k + B^{\bar s}_{\bar i j}D_{\bar s}Z_k, Z_{\bar r})\\
&= B^s_{\bar i j}(g(\nabla_sZ_k, Z_{\bar r}) + \frac{1}{2}(g(T_{ks}, Z_{\bar r}) + g(T_{s\bar r}, Z_k) - g(T_{\bar r k}, Z_s)))\\
&\quad +B^{\bar s}_{\bar i j}(g(\nabla_{\bar s}Z_k, Z_{\bar r}) + \frac{1}{2}(g(T_{k\bar s}, Z_{\bar r}) + g(T_{\bar s\bar r}, Z_k) - g(T_{\bar r k}, Z_{\bar s})))\\
&= B^s_{\bar i j}\Gamma^l_{sk}g_{l\bar r} + \frac{1}{2}B^s_{\bar i j}T^l_{ks}g_{l\bar r} + B^{\bar s}_{\bar i j}\Gamma^l_{\bar s k}g_{l\bar r} + \frac{1}{2}B^{\bar s}_{\bar i j}\overline{T^l_{sr}}g_{k\bar l}\\
&= B^\circ * \Gamma + B^\circ * T' + B^\circ * B^\circ + B^\circ * \bar T',
\end{aligned}$$

$$\begin{aligned}
g([Z_k, Z_{\bar{i}}]Z_j, Z_{\bar{r}}) &= g(B^s_{k\bar{i}} D_s Z_j + B^{\bar{s}}_{k\bar{i}} D_{\bar{s}} Z_j, Z_{\bar{r}}) \\
&= B^s_{k\bar{i}}(g(\nabla_s Z_j, Z_{\bar{r}}) + \frac{1}{2}(g(T_{js}, Z_{\bar{r}}) + g(T_{s\bar{r}}, Z_j) - g(T_{\bar{r}j}, Z_s))) \\
&\quad + B^{\bar{s}}_{k\bar{i}}(g(\nabla_{\bar{s}} Z_j, Z_{\bar{r}}) + \frac{1}{2}(g(T_{j\bar{s}}, Z_{\bar{r}}) + g(T_{\bar{s}\bar{r}}, Z_j) - g(T_{\bar{r}j}, Z_{\bar{s}}))) \\
&= B^s_{k\bar{i}} \Gamma^l_{sj} g_{l\bar{r}} + \frac{1}{2} B^s_{k\bar{i}} T^l_{js} g_{l\bar{r}} + B^{\bar{s}}_{k\bar{i}} \Gamma^l_{\bar{s}j} g_{l\bar{r}} + \frac{1}{2} B^{\bar{s}}_{k\bar{i}} \overline{T^l_{sr}} g_{j\bar{l}} \\
&= B^\circ * \Gamma + B^\circ * T' + B^\circ * B^\circ + B^\circ * \bar{T}', \\
g([Z_{\bar{i}}, Z_k], D_j Z_{\bar{r}}) &= g(B^s_{\bar{i}k} Z_s + B^{\bar{s}}_{\bar{i}k} Z_{\bar{s}}, D_j Z_{\bar{r}}) \\
&= B^s_{\bar{i}k}(g(\nabla_j Z_{\bar{r}}, Z_s) + \frac{1}{2}(g(T_{\bar{r}j}, Z_s) + g(T_{js}, Z_{\bar{r}}) - g(T_{s\bar{r}}, Z_j))) \\
&\quad + B^{\bar{s}}_{\bar{i}k}(g(\nabla_j Z_{\bar{r}}, Z_{\bar{s}}) + \frac{1}{2}(g(T_{\bar{r}j}, Z_{\bar{s}}) + g(T_{j\bar{s}}, Z_{\bar{r}}) - g(T_{\bar{s}\bar{r}}, Z_j))) \\
&= B^s_{\bar{i}k} \Gamma^{\bar{l}}_{j\bar{r}} g_{s\bar{l}} + \frac{1}{2} B^s_{\bar{i}k} T^l_{js} g_{l\bar{r}} + \frac{1}{2} B^{\bar{s}}_{\bar{i}k} \overline{T^l_{rs}} g_{j\bar{l}} \\
&= B^\circ * B^\circ + B^\circ * T' + B^\circ * \bar{T}', \\
g([Z_j, Z_{\bar{i}}], D_k Z_{\bar{r}}) &= g(B^s_{j\bar{i}} Z_s + B^{\bar{s}}_{j\bar{i}} Z_{\bar{s}}, D_k Z_{\bar{r}}) \\
&= B^s_{j\bar{i}}(g(\nabla_k Z_{\bar{r}}, Z_s) + \frac{1}{2}(g(T_{\bar{r}k}, Z_s) + g(T_{ks}, Z_{\bar{r}}) - g(T_{s\bar{r}}, Z_k))) \\
&\quad + B^{\bar{s}}_{j\bar{i}}(g(\nabla_k Z_{\bar{r}}, Z_{\bar{s}}) + \frac{1}{2}(g(T_{\bar{r}k}, Z_{\bar{s}}) + g(T_{k\bar{s}}, Z_{\bar{r}}) - g(T_{\bar{s}\bar{r}}, Z_k))) \\
&= B^s_{j\bar{i}} \Gamma^{\bar{l}}_{k\bar{r}} g_{s\bar{l}} + \frac{1}{2} B^s_{j\bar{i}} T^l_{ks} g_{l\bar{r}} + \frac{1}{2} B^{\bar{s}}_{j\bar{i}} \overline{T^l_{rs}} g_{k\bar{l}} \\
&= B^\circ * B^\circ + B^\circ * T' + B^\circ * \bar{T}', \\
g([Z_j, Z_k]Z_{\bar{i}}, Z_{\bar{r}}) &= g(B^s_{jk} D_s Z_{\bar{i}} + B^{\bar{s}}_{jk} D_{\bar{s}} Z_{\bar{i}}, Z_{\bar{r}}) \\
&= B^s_{jk}(g(\nabla_s Z_{\bar{i}}, Z_{\bar{r}}) + \frac{1}{2}(g(T_{\bar{i}s}, Z_{\bar{r}}) + g(T_{s\bar{r}}, Z_{\bar{i}}) - g(T_{\bar{r}\bar{i}}, Z_s))) \\
&\quad + B^{\bar{s}}_{jk}(g(\nabla_{\bar{s}} Z_{\bar{i}}, Z_{\bar{r}}) + \frac{1}{2}(g(T_{\bar{i}\bar{s}}, Z_{\bar{r}}) + g(T_{\bar{s}\bar{r}}, Z_{\bar{i}}) - g(T_{\bar{r}\bar{i}}, Z_{\bar{s}}))) \\
&= \frac{1}{2} B^s_{jk} \overline{T^l_{ir}} g_{s\bar{l}} + \frac{1}{2} B^{\bar{s}}_{jk} T^l_{\bar{i}\bar{s}} g_{l\bar{r}} + \frac{1}{2} B^{\bar{s}}_{jk} T^l_{\bar{s}\bar{r}} g_{l\bar{i}} + \frac{1}{2} B^{\bar{s}}_{jk} T^l_{\bar{i}\bar{r}} g_{l\bar{s}} \\
&= B * \bar{T}' + B^\circ * B^\circ, \\
g([Z_j, Z_k], D_{\bar{i}} Z_{\bar{r}}) &= g(B^s_{jk} Z_s + B^{\bar{s}}_{jk} Z_{\bar{s}}, D_{\bar{i}} Z_{\bar{r}}) \\
&= B^s_{jk}(g(\nabla_{\bar{i}} Z_{\bar{r}}, Z_s) + \frac{1}{2}(g(T_{\bar{r}\bar{i}}, Z_s) + g(T_{\bar{i}s}, Z_{\bar{r}}) - g(T_{s\bar{r}}, Z_{\bar{i}}))) \\
&\quad + B^{\bar{s}}_{jk}(g(\nabla_{\bar{i}} Z_{\bar{r}}, Z_{\bar{s}}) + \frac{1}{2}(g(T_{\bar{r}\bar{i}}, Z_{\bar{s}}) + g(T_{\bar{i}\bar{s}}, Z_{\bar{r}}) - g(T_{\bar{s}\bar{r}}, Z_{\bar{i}}))) \\
&= B^s_{jk} \overline{\Gamma^l_{ir}} g_{s\bar{l}} + \frac{1}{2} B^s_{jk} \overline{T^l_{ri}} g_{s\bar{l}} + \frac{1}{2} B^{\bar{s}}_{jk} T^l_{\bar{r}\bar{i}} g_{l\bar{s}} + \frac{1}{2} B^{\bar{s}}_{jk} T^l_{\bar{i}\bar{s}} g_{l\bar{r}} + \frac{1}{2} B^{\bar{s}}_{jk} T^l_{\bar{r}\bar{s}} g_{l\bar{i}} \\
&= B * \bar{\Gamma} + B * \bar{T}' + B^\circ * B^\circ.
\end{aligned}$$

And we compute that

$$Z_j(g([Z_{\bar{i}}, Z_k], Z_{\bar{r}}])) = Z_j(B^l_{\bar{i}k} g_{l\bar{r}}) = Z_j(B^l_{\bar{i}k}) g_{l\bar{r}} + \mathcal{O}(Z(g)),$$
$$Z_k(g([Z_j, Z_{\bar{i}}], Z_{\bar{r}}])) = Z_k(B^l_{j\bar{i}} g_{l\bar{r}}) = Z_k(B^l_{j\bar{i}}) g_{l\bar{r}} + \mathcal{O}(Z(g)).$$

Therefore we have

$$Z_{\bar{i}}(B^l_{jk})g_{l\bar{r}} = B^\circ * \Gamma + B^\circ * T' + B^\circ * B^\circ + B^\circ * \bar{T}' + B * \bar{T}' + B * \bar{\Gamma} \\ + Z_j(B^s_{\bar{i}k}g_{s\bar{r}}) + Z_k(B^s_{j\bar{i}}g_{s\bar{r}}) - B^l_{jk}\Gamma^{\bar{s}}_{\bar{i}\bar{r}}g_{l\bar{s}} - B^l_{jk}\Gamma^s_{\bar{i}l}g_{s\bar{r}}.$$

Using these computations, we obtain

$$\begin{aligned} Z_mZ_{\bar{i}}(T^l_{jk})g_{l\bar{r}} &= Z_m(Z_{\bar{i}}(\Gamma^l_{jk}) - Z_{\bar{i}}(\Gamma^l_{kj}) - Z_{\bar{i}}(B^l_{jk}))g_{l\bar{r}} \\ &= Z_mZ_{\bar{i}}(\Gamma^l_{jk})g_{l\bar{r}} - Z_mZ_{\bar{i}}(\Gamma^l_{kj})g_{l\bar{r}} - Z_m(Z_{\bar{i}}(B^l_{jk})g_{l\bar{r}}) + Z_{\bar{i}}(B^l_{jk})Z_m(g_{l\bar{r}}) \\ &= Z_mZ_{\bar{i}}(\Gamma^l_{jk})g_{l\bar{r}} - Z_mZ_{\bar{i}}(\Gamma^l_{kj})g_{l\bar{r}} \\ &\quad -Z_m(B^\circ * \Gamma + B^\circ * T' + B^\circ * B^\circ + B^\circ * \bar{T}' + B * \bar{T}' + B * \bar{\Gamma}) \\ &\quad -Z_mZ_j(B^l_{\bar{i}k})g_{l\bar{r}} - Z_mZ_k(B^l_{j\bar{i}})g_{l\bar{r}} + Z_m(B^l_{jk}\Gamma^{\bar{s}}_{\bar{i}\bar{r}})g_{l\bar{s}} + Z_m(B^l_{jk}\Gamma^s_{\bar{i}l})g_{s\bar{r}} + \mathcal{O}(Z(g)) \\ &= Z\bar{Z}(\Gamma) + ZZ(B^\circ) + Z(B^\circ * \Gamma + B^\circ * T' + B^\circ * B^\circ + B^\circ * \bar{T}' + T' * \bar{\Gamma} \\ &\quad +\Gamma * \bar{\Gamma} + \Gamma * \bar{T}' + T' * \bar{T}') + \mathcal{O}(Z(g)), \end{aligned}$$

where we used that $B = T' + \Gamma$. Combining these computations, we obtain

$$\nabla_{\bar{i}}\nabla_m(T_{jk\bar{r}}) = Z\bar{Z}(\Gamma) + ZZ(B^\circ) + Z(B^\circ * \Gamma + B^\circ * T' + B^\circ * B^\circ \\ + B^\circ * \bar{T}' + T' * \bar{\Gamma} + \Gamma * \bar{\Gamma} + \Gamma * \bar{T}' + T' * \bar{T}') + \mathcal{O}(Z(g)) \\ + B^\circ * Z(T') + B^\circ * \bar{Z}(T') + \bar{\nabla}\Gamma * T' + \Gamma * \bar{\nabla}T'.$$

Now fix an arbitrary chosen time $t_0 \in [0, \tau)$ and using a local $g(t_0)$-unitary frame, we have that at t_0, $|Z_iZ_{\bar{i}}(\Gamma^l_{jk})| \le C$ as in Lemma 3.1, and we also have that $\mathcal{O}(Z(g)) = 0$. By using Lemma 3.3, then we obtain that $|\bar{\nabla}\nabla T'(g(t_0))|_{C^0(g(t_0))} \le C$ for some uniform constant $C > 0$ under our assumptions that $|T'(g(t))|^2_{C^0(g(t))}$ and $|\nabla T'(g(t))|_{C^0(g(t))}$ have uniform bounds. Similarly, one can obtain the rest of uniform bounds, since we have

$$\begin{aligned} \nabla_{\bar{i}}\nabla_{\bar{m}}(T_{jk\bar{r}}) &= Z_{\bar{i}}Z_{\bar{m}}(T^l_{jk})g_{l\bar{r}} - \Gamma^{\bar{s}}_{\bar{i}\bar{m}}Z_{\bar{s}}(T^l_{jk}) - \Gamma^s_{\bar{i}j}Z_{\bar{m}}(T^l_{sk}) - \Gamma^s_{\bar{i}k}Z_{\bar{m}}(T^l_{js}) + \Gamma^l_{\bar{i}s}Z_{\bar{m}}(T^s_{jk}) \\ &\quad -\nabla_{\bar{i}}(B^s_{\bar{m}j}T^l_{sk} + B^s_{\bar{m}k}T^l_{js} - B^l_{\bar{m}s}T^s_{jk})g_{l\bar{r}} \\ &= Z_{\bar{i}}Z_{\bar{m}}(\Gamma^l_{jk})g_{l\bar{r}} - Z_{\bar{i}}Z_{\bar{m}}(\Gamma^l_{kj})g_{l\bar{r}} - Z_{\bar{i}}(Z_{\bar{m}}(B^l_{jk})g_{l\bar{r}}) + Z_{\bar{m}}(B^l_{jk})Z_{\bar{i}}(g_{l\bar{r}}) \\ &\quad +\bar{\Gamma} * \bar{Z}(T') + B^\circ * \bar{Z}(T') + \bar{\nabla}B^\circ * T' + B^\circ * \bar{\nabla}T' \\ &= \bar{Z}\bar{Z}(\Gamma) - Z_{\bar{i}}(B^\circ * \Gamma + B^\circ * T' + B^\circ * B^\circ + B^\circ * \bar{T}' + B * \bar{T}' + B * \bar{\Gamma}) \\ &\quad -Z_{\bar{i}}Z_j(B^l_{\bar{m}k}g_{l\bar{r}}) - Z_{\bar{i}}Z_k(B^l_{j\bar{m}}g_{l\bar{r}}) + Z_{\bar{i}}(B^l_{jk}\Gamma^{\bar{s}}_{\bar{m}\bar{r}}g_{l\bar{s}} + B^l_{jk}\Gamma^s_{\bar{m}l}g_{s\bar{r}}) \\ &\quad +Z_{\bar{m}}(B^l_{jk})Z_{\bar{i}}(g_{l\bar{r}}) + \bar{\Gamma} * \bar{Z}(T') + B^\circ * \bar{Z}(T') + \bar{\nabla}B^\circ * T' + B^\circ * \bar{\nabla}T' \\ &= \bar{Z}\bar{Z}(\Gamma) + \bar{Z}(B^\circ * \Gamma + B^\circ * T' + B^\circ * B^\circ + B^\circ * \bar{T}' + B * \bar{T}' + B * \bar{\Gamma}) \\ &\quad +\bar{Z}Z(B^\circ) + \bar{Z}(B) * \bar{\Gamma} + B * \bar{Z}(\bar{\Gamma}) + \bar{Z}(B) * B^\circ + B * \bar{Z}(B^\circ) \\ &\quad +\bar{\Gamma} * \bar{Z}(T') + B^\circ * \bar{Z}(T') + \bar{\nabla}B^\circ * T' + B^\circ * \bar{\nabla}T' + \mathcal{O}(Z(g)), \\ \nabla_m\nabla_i(T_{\bar{j}\bar{k}r}) &= ZZ(\bar{\Gamma}) + \Gamma * Z(\bar{T}') + B^\circ * Z(\bar{T}') + \nabla B^\circ * \bar{T}' + B^\circ * \nabla\bar{T}' \\ &\quad -Z_m(\overline{Z_{\bar{i}}(B^l_{jk})g_{l\bar{r}}}) + Z_i(\overline{B^l_{jk}})Z_m(g_{r\bar{l}}) \\ &= ZZ(\Gamma) + \Gamma * Z(\bar{T}') + B^\circ * Z(\bar{T}') + \nabla B^\circ * \bar{T}' + B^\circ * \nabla\bar{T}' \\ &\quad +Z(B^\circ * \bar{\Gamma} + B^\circ * \bar{T}' + B^\circ * B^\circ + B^\circ * T' + \bar{B} * T' + \bar{B} * \Gamma) \\ &\quad -Z_mZ_{\bar{j}}(B^{\bar{l}}_{i\bar{k}}g_{r\bar{l}}) - Z_mZ_{\bar{k}}(B^{\bar{l}}_{\bar{j}i}g_{r\bar{l}}) \\ &\quad +Z_m(\overline{B^l_{jk}}\Gamma^s_{ir}g_{s\bar{l}} + \overline{B^l_{jk}}\Gamma^{\bar{s}}_{i\bar{l}}g_{r\bar{s}}) + Z_i(\overline{B^l_{jk}})Z_m(g_{r\bar{l}}) \end{aligned}$$

$$\begin{aligned}&= ZZ(\Gamma) + \Gamma * Z(\bar{T}') + B^\circ * Z(\bar{T}') + \nabla B^\circ * \bar{T}' + B^\circ * \nabla \bar{T}'\\ &\quad + Z(B^\circ * \bar{\Gamma} + B^\circ * \bar{T}' + B^\circ * B^\circ + B^\circ * T' + \bar{B} * T' + \bar{B} * \Gamma)\\ &\quad + Z\bar{Z}(B^\circ) + Z(\bar{B}) * \Gamma + \bar{B} * Z(\Gamma) + Z(\bar{B}) * B^\circ + \bar{B} * Z(B^\circ) + \mathcal{O}(Z(g))\end{aligned}$$

and

$$\begin{aligned}\nabla_{\bar{i}} \nabla_m (T_{\bar{j}\bar{k}r}) &= \bar{Z}Z(\bar{\Gamma}) + B^\circ * Z(\bar{T}') + \bar{\Gamma} * Z(\bar{T}') + \bar{\nabla} B^\circ * \bar{T}' + B^\circ * \bar{\nabla}\bar{T}'\\ &\quad - Z_{\bar{i}}(\overline{Z_{\bar{m}}(B^l_{jk}) g_{l\bar{r}}}) + Z_m(\overline{B^l_{jk}}) Z_{\bar{i}}(g_{r\bar{l}})\\ &= \bar{Z}Z(\bar{\Gamma}) + B^\circ * Z(\bar{T}') + \bar{\Gamma} * Z(\bar{T}') + \bar{\nabla} B^\circ * \bar{T}' + B^\circ * \bar{\nabla}\bar{T}'\\ &\quad - Z_{\bar{i}}(B^\circ * \bar{\Gamma} + B^\circ * \bar{T}' + B^\circ * B^\circ + B^\circ * T' + \bar{B} * T' + \bar{B} * \Gamma)\\ &\quad - Z_{\bar{i}} Z_{\bar{j}}(B^{\bar{l}}_{m\bar{k}} g_{r\bar{l}}) - Z_{\bar{i}} Z_{\bar{k}}(B^{\bar{l}}_{\bar{j}m} g_{r\bar{l}}) + Z_{\bar{i}}(\overline{B^l_{jk}} \Gamma^s_{mr} g_{s\bar{l}} + \overline{B^l_{jk}} \Gamma^{\bar{s}}_{m\bar{l}} g_{r\bar{s}})\\ &\quad + Z_m(\overline{B^l_{jk}}) Z_{\bar{i}}(g_{r\bar{l}})\\ &= \bar{Z}Z(\bar{\Gamma}) + \bar{Z}\bar{Z}(B^\circ) + B^\circ * Z(\bar{T}') + \bar{\Gamma} * Z(\bar{T}') + \bar{\nabla} B^\circ * \bar{T}' + B^\circ * \bar{\nabla}\bar{T}'\\ &\quad + \bar{Z}(B^\circ * \bar{\Gamma} + B^\circ * \bar{T}' + B^\circ * B^\circ + B^\circ * T' + \bar{B} * T' + \bar{B} * \Gamma)\\ &\quad + \bar{Z}(\bar{B}) * \Gamma + \bar{B} * \bar{Z}(\Gamma) + \bar{Z}(\bar{B}) * B^\circ + \bar{B} * \bar{Z}(B^\circ) + \mathcal{O}(\bar{Z}(g))\end{aligned}$$

Then, by applying the estimates in Lemmas 3.1, 3.3, we obtain the uniform bound for $|\bar{\nabla}^2 T'(g(t_0))|^2_{C^0(g(t_0))}$, $|\bar{\nabla}\nabla\bar{T}'(g(t_0))|^2_{C^0(g(t_0))}$ for any fixed time $t_0 \in [0, \tau)$. □

Recall that we have

$$\begin{aligned}H &= H_{ijk\bar{r}}\\ &= H_{ijk}{}^l g_{l\bar{r}}\\ &= (Z_i(\Gamma^l_{jk}) - Z_j(\Gamma^l_{ik}) + \Gamma^l_{is}\Gamma^s_{jk} - \Gamma^l_{js}\Gamma^s_{ik} - B^s_{ij}\Gamma^l_{sk} - B^{\bar{s}}_{ij}\Gamma^l_{\bar{s}k}) g_{l\bar{r}}\\ &= Z(\Gamma) + \Gamma * \Gamma + B * \Gamma + B^\circ * B^\circ\\ &= Z(\Gamma) + \Gamma * \Gamma + T' * \Gamma + B^\circ * B^\circ.\end{aligned}$$

Note that we have for any $j = 0, 1, 2 \ldots,$

$$\begin{aligned}\bar{\nabla}^j H &= \bar{\nabla}^j (Z(\Gamma) + \Gamma * \Gamma + T' * \Gamma + B^\circ * B^\circ)\\ &= \bar{\nabla}^j (Z(\Gamma)) + \sum_{l=0}^{j} \bar{\nabla}^l \Gamma * \bar{\nabla}^{j-l}\Gamma + \sum_{l=0}^{j} \bar{\nabla}^l T' * \bar{\nabla}^{j-l}\Gamma + \sum_{l=0}^{j} \bar{\nabla}^l B^\circ * \bar{\nabla}^{j-l} B^\circ.\end{aligned}$$

From the equality above and Lemma 3.1, we have the following estimate.

Lemma 3.5. *One can obtain the following estimate for any $j = 0, 1, 2, \ldots$ and for a time $t_0 \in [0, \tau)$,*

$$|\bar{\nabla}^j H(g(t_0))|_{C^0(g(t_0))} \leq C \sum_{l=0}^{j} |\bar{\nabla}^{j-l} T'(g(t_0))|_{C^0(g(t_0))} + C.$$

Especially, from Lemmas 3.4 and 3.5, under the assumption that $|T'(g(t))|^2_{C^0(g(t))}$, $|\nabla T'(g(t))|_{C^0(g(t))}$ have uniform bounds, then we obtain for any fixed time $t_0 \in [0, \tau)$,

$$|\bar{\nabla}^2 H(g(t_0))|_{C^0(g(t_0))} \le C, \quad |\nabla^2 \bar{H}(g(t_0))|_{C^0(g(t_0))} \le C$$

for some uniform constant $C > 0$ independent t_0.

From now on, we consider the solution $g = g(t)$ of the AHCF starting at the initial almost Hermitian metric g_0 on a compact almost Hermitian manifold M satisfying

$$\begin{cases} \dfrac{\partial}{\partial t} g(t) = -S(g(t)) - Q^7(g(t)) - Q^8(g(t)) + BT'(g(t)) + \bar{Z}(T')(g(t)), \\ g(0) = g_0, \end{cases}$$

Lemma 3.6. *For a solution to the AHCF, one has that*

$$\begin{aligned} \frac{\partial}{\partial t} R_{i\bar{j}k\bar{l}} &= \Delta R_{i\bar{j}k\bar{l}} + g^{r\bar{s}}(T^{\bar{q}}_{\bar{s}\bar{j}} \nabla_r R_{i\bar{q}k\bar{l}} + T^p_{ri} \nabla_{\bar{s}} R_{p\bar{j}k\bar{l}} + T^p_{ri} T^{\bar{q}}_{\bar{s}\bar{j}} R_{p\bar{q}k\bar{l}} + R_{i\bar{j}r}{}^p R_{p\bar{s}k\bar{l}} \\ &\quad + R_{r\bar{j}k}{}^p R_{i\bar{s}p\bar{l}} - R_{r\bar{j}p\bar{l}} R_{i\bar{s}k}{}^p) \\ &\quad - g^{m\bar{n}} R_{m\bar{j}}{}^{\bar{s}}{}_{\bar{n}} R_{i\bar{s}k\bar{l}} + g^{n\bar{m}} T^s_{mi} (\nabla_s H_{\bar{j}\bar{n}k\bar{l}} + T^p_{\bar{n}\bar{j}} H_{spk\bar{l}}) \\ &\quad + g^{m\bar{n}} (\nabla_m \nabla_i H_{\bar{j}\bar{n}k\bar{l}} + \nabla_m T^{\bar{s}}_{\bar{n}\bar{j}} R_{i\bar{s}k\bar{l}} + \nabla_m T^s_{\bar{n}\bar{j}} H_{isk\bar{l}} + T^s_{\bar{n}\bar{j}} \nabla_m H_{isk\bar{l}} \\ &\quad + \Gamma^s_{\bar{j}i} \Gamma^p_{ms} R_{p\bar{n}k\bar{l}} + \Gamma^s_{\bar{j}i} \Gamma^p_{mk} R_{s\bar{n}p\bar{l}} + \Gamma^s_{\bar{j}i} \Gamma^{\bar{p}}_{m\bar{l}} R_{s\bar{n}k\bar{p}} + \Gamma^{\bar{s}}_{\bar{j}\bar{n}} \Gamma^p_{mk} R_{i\bar{s}p\bar{l}} + \Gamma^{\bar{s}}_{\bar{j}\bar{n}} \Gamma^{\bar{p}}_{m\bar{l}} R_{i\bar{s}k\bar{p}} \\ &\quad + \Gamma^s_{\bar{j}k} \Gamma^p_{mi} R_{p\bar{n}s\bar{l}} + \Gamma^s_{\bar{j}k} \Gamma^{\bar{p}}_{m\bar{n}} R_{i\bar{p}s\bar{l}} + \Gamma^s_{\bar{j}k} \Gamma^p_{ms} R_{i\bar{n}p\bar{l}} + \Gamma^s_{\bar{j}k} \Gamma^{\bar{p}}_{m\bar{l}} R_{i\bar{n}s\bar{p}} + \Gamma^{\bar{s}}_{\bar{j}\bar{l}} \Gamma^p_{mi} R_{p\bar{n}k\bar{s}} \\ &\quad + \Gamma^{\bar{s}}_{\bar{j}\bar{l}} \Gamma^{\bar{p}}_{m\bar{n}} R_{i\bar{p}k\bar{s}} + \Gamma^{\bar{s}}_{\bar{j}\bar{l}} \Gamma^p_{mk} R_{i\bar{n}p\bar{s}} + \Gamma^{\bar{s}}_{\bar{j}\bar{l}} \Gamma^{\bar{p}}_{m\bar{s}} R_{i\bar{n}k\bar{p}} + B^s_{\bar{j}m} \Gamma^p_{si} R_{p\bar{n}k\bar{l}} + B^s_{\bar{j}m} \Gamma^p_{sk} R_{i\bar{n}p\bar{l}} \\ &\quad + B^s_{\bar{j}m} \Gamma^{\bar{p}}_{s\bar{l}} R_{i\bar{n}k\bar{p}} + B^{\bar{s}}_{\bar{j}m} \Gamma^p_{\bar{s}i} R_{p\bar{n}k\bar{l}} + B^{\bar{s}}_{\bar{j}m} \Gamma^{\bar{p}}_{\bar{s}\bar{n}} R_{i\bar{p}k\bar{l}} + B^{\bar{s}}_{\bar{j}m} \Gamma^p_{\bar{s}k} R_{i\bar{n}p\bar{l}} + B^{\bar{s}}_{\bar{j}m} \Gamma^{\bar{p}}_{\bar{s}\bar{l}} R_{i\bar{n}k\bar{p}} \\ &\quad - \Gamma^s_{mi} \Gamma^p_{\bar{j}s} R_{p\bar{n}k\bar{l}} - \Gamma^s_{mi} \Gamma^p_{\bar{j}k} R_{s\bar{n}p\bar{l}} - \Gamma^s_{mi} \Gamma^{\bar{p}}_{\bar{j}\bar{l}} R_{s\bar{n}k\bar{p}} - \Gamma^{\bar{s}}_{m\bar{n}} \Gamma^{\bar{p}}_{\bar{j}\bar{s}} R_{i\bar{p}k\bar{l}} - \Gamma^{\bar{s}}_{m\bar{n}} \Gamma^p_{\bar{j}k} R_{i\bar{s}p\bar{l}} \\ &\quad - \Gamma^{\bar{s}}_{m\bar{n}} \Gamma^{\bar{p}}_{\bar{j}\bar{l}} R_{i\bar{s}k\bar{p}} - \Gamma^s_{mk} \Gamma^p_{\bar{j}i} R_{p\bar{n}s\bar{l}} - \Gamma^s_{mk} \Gamma^{\bar{p}}_{\bar{j}\bar{n}} R_{i\bar{p}s\bar{l}} - \Gamma^s_{mk} \Gamma^p_{\bar{j}s} R_{i\bar{n}p\bar{l}} - \Gamma^s_{mk} \Gamma^{\bar{p}}_{\bar{j}\bar{l}} R_{i\bar{n}s\bar{p}} \\ &\quad - \Gamma^{\bar{s}}_{m\bar{l}} \Gamma^p_{\bar{j}i} R_{p\bar{n}k\bar{s}} - \Gamma^{\bar{s}}_{m\bar{l}} \Gamma^{\bar{p}}_{\bar{j}\bar{n}} R_{i\bar{p}k\bar{s}} - \Gamma^{\bar{s}}_{m\bar{l}} \Gamma^p_{\bar{j}k} R_{i\bar{n}p\bar{s}} - \Gamma^{\bar{s}}_{m\bar{l}} \Gamma^{\bar{p}}_{\bar{j}\bar{s}} R_{i\bar{n}k\bar{p}} + \nabla_{\bar{j}} \nabla_{\bar{n}} H_{mik\bar{l}} \\ &\quad + \nabla_{\bar{j}} T^s_{mi} R_{s\bar{n}k\bar{l}} - \nabla_{\bar{j}} B^{\bar{s}}_{mi} H_{\bar{s}\bar{n}k\bar{l}} - B^{\bar{s}}_{mi} \nabla_{\bar{j}} H_{\bar{s}\bar{n}k\bar{l}}) \\ &\quad - g^{m\bar{n}} (B^{\bar{q}}_{mi} B^s_{\bar{q}\bar{j}} - \nabla_{\bar{j}} T^s_{im} + \Gamma^s_{mp} \Gamma^p_{\bar{j}i} - \Gamma^s_{\bar{j}p} \Gamma^p_{mi} - B^p_{m\bar{j}} \Gamma^s_{pi} + B^{\bar{p}}_{\bar{j}m} \Gamma^s_{\bar{p}i}) R_{s\bar{n}k\bar{l}} \\ &\quad - g^{m\bar{n}} (\Gamma^s_{mp} \Gamma^p_{\bar{j}k} - \Gamma^s_{\bar{j}p} \Gamma^p_{mk} - B^p_{m\bar{j}} \Gamma^s_{pk} + B^{\bar{p}}_{\bar{j}m} \Gamma^s_{\bar{p}k}) R_{i\bar{n}s\bar{l}} \\ &\quad + g^{m\bar{n}} (\Gamma^{\bar{s}}_{\bar{j}\bar{p}} \Gamma^{\bar{p}}_{m\bar{n}} - \Gamma^{\bar{s}}_{m\bar{p}} \Gamma^{\bar{p}}_{\bar{j}\bar{n}} - B^{\bar{p}}_{\bar{j}m} \Gamma^{\bar{s}}_{\bar{p}\bar{n}} + B^p_{m\bar{j}} \Gamma^{\bar{s}}_{p\bar{n}}) R_{i\bar{s}k\bar{l}} \\ &\quad + g^{m\bar{n}} (\Gamma^{\bar{s}}_{\bar{j}\bar{p}} \Gamma^{\bar{p}}_{m\bar{l}} - \Gamma^{\bar{s}}_{m\bar{p}} \Gamma^{\bar{p}}_{\bar{j}\bar{l}} - B^{\bar{p}}_{\bar{j}m} \Gamma^{\bar{s}}_{\bar{p}\bar{l}} + B^p_{m\bar{j}} \Gamma^{\bar{s}}_{p\bar{l}}) R_{i\bar{n}k\bar{s}} \\ &\quad - R_{i\bar{j}k}{}^r (S + Q^7 + Q^8 - BT' - \bar{Z}(T'))_{r\bar{l}} + \nabla_{\bar{j}} \nabla_i (Q^7 + Q^8 - BT' - \bar{Z}(T'))_{k\bar{l}} \\ &=: \Delta R_{i\bar{j}k\bar{l}} + g^{r\bar{s}}(T^{\bar{q}}_{\bar{s}\bar{j}} \nabla_r R_{i\bar{q}k\bar{l}} + T^p_{ri} \nabla_{\bar{s}} R_{p\bar{j}k\bar{l}} + T^p_{ri} T^{\bar{q}}_{\bar{s}\bar{j}} R_{p\bar{q}k\bar{l}} \\ &\quad + R_{i\bar{j}r}{}^p R_{p\bar{s}k\bar{l}} + R_{r\bar{j}k}{}^p R_{i\bar{s}p\bar{l}} - R_{r\bar{j}p\bar{l}} R_{i\bar{s}k}{}^p) \\ &\quad - R_{i\bar{j}k}{}^r (S + Q^7 + Q^8 - BT' - \bar{Z}(T'))_{r\bar{l}} + E_{i\bar{j}k\bar{l}}. \end{aligned} \tag{3.2}$$

Proof. We consider the term $-S$ in the evolution equation of g. Using Lemma 3.2, the evolution $\frac{\partial}{\partial t} g = -S$ yields

$$\frac{\partial}{\partial t} R_{i\bar{j}k\bar{l}} = -R_{i\bar{j}k}{}^r S_{r\bar{l}} + \nabla_{\bar{j}} \nabla_i S_{k\bar{l}}.$$

Applying the second Bianchi identity in Lemma 2.2, we have

$$\begin{aligned}\nabla_{\bar{j}}\nabla_i S_{k\bar{l}} &= g^{m\bar{n}}\nabla_{\bar{j}}\nabla_i R_{m\bar{n}k\bar{l}}\\ &= g^{m\bar{n}}\nabla_{\bar{j}}(\nabla_m R_{i\bar{n}k\bar{l}} + \nabla_{\bar{n}} H_{mik\bar{l}} + T^s_{mi}R_{s\bar{n}k\bar{l}} - B^{\bar{s}}_{mi}H_{\bar{s}\bar{n}k\bar{l}})\\ &= g^{m\bar{n}}(\nabla_{\bar{j}}\nabla_m R_{i\bar{n}k\bar{l}} + \nabla_{\bar{j}}\nabla_{\bar{n}} H_{mik\bar{l}} + \nabla_{\bar{j}}T^s_{mi}R_{s\bar{n}k\bar{l}} + T^s_{mi}\nabla_{\bar{j}}R_{s\bar{n}k\bar{l}}\\ &\quad -\nabla_{\bar{j}}B^{\bar{s}}_{mi}H_{\bar{s}\bar{n}k\bar{l}} - B^{\bar{s}}_{mi}\nabla_{\bar{j}}H_{\bar{s}\bar{n}k\bar{l}}).\end{aligned}$$

We compute

$$\begin{aligned}\nabla_{\bar{j}}\nabla_m R_{i\bar{n}k\bar{l}} &= \nabla_m\nabla_{\bar{j}}R_{i\bar{n}k\bar{l}} + \Gamma^{\bar{s}}_{m\bar{j}}\nabla_{\bar{s}}R_{i\bar{n}k\bar{l}} + \Gamma^s_{mi}\nabla_{\bar{j}}R_{s\bar{n}k\bar{l}} + \Gamma^{\bar{s}}_{m\bar{n}}\nabla_{\bar{j}}R_{i\bar{s}k\bar{l}}\\ &\quad +\Gamma^s_{mk}\nabla_{\bar{j}}R_{i\bar{n}s\bar{l}} + \Gamma^{\bar{s}}_{m\bar{l}}\nabla_{\bar{j}}R_{i\bar{n}k\bar{s}} + Z_m(\Gamma^s_{\bar{j}i}R_{s\bar{n}k\bar{l}} + \Gamma^{\bar{s}}_{\bar{j}\bar{n}}R_{i\bar{s}k\bar{l}} + \Gamma^s_{\bar{j}k}R_{i\bar{n}s\bar{l}} + \Gamma^{\bar{s}}_{\bar{j}\bar{l}}R_{i\bar{n}k\bar{s}})\\ &\quad +[Z_{\bar{j}}, Z_m]R_{i\bar{n}k\bar{l}} - Z_{\bar{j}}(\Gamma^s_{mi}R_{s\bar{n}k\bar{l}} + \Gamma^{\bar{s}}_{m\bar{n}}R_{i\bar{s}k\bar{l}} + \Gamma^s_{mk}R_{i\bar{n}s\bar{l}} + \Gamma^{\bar{s}}_{m\bar{l}}R_{i\bar{n}k\bar{s}})\\ &\quad -\Gamma^s_{\bar{j}i}\nabla_m R_{s\bar{n}k\bar{l}} - \Gamma^{\bar{s}}_{\bar{j}\bar{n}}\nabla_m R_{i\bar{s}k\bar{l}} - \Gamma^s_{\bar{j}k}\nabla_m R_{i\bar{n}s\bar{l}} - \Gamma^{\bar{s}}_{\bar{j}\bar{l}}\nabla_m R_{i\bar{n}k\bar{s}} - \Gamma^s_{\bar{j}m}\nabla_s R_{i\bar{n}k\bar{l}}\end{aligned}$$

and

$$\begin{aligned}\nabla_m\nabla_{\bar{j}}R_{i\bar{n}k\bar{l}} &= \nabla_m\overline{\nabla_j R_{n\bar{i}l\bar{k}}}\\ &= \nabla_m(\overline{\nabla_n R_{j\bar{i}l\bar{k}} + \nabla_{\bar{i}}H_{njl\bar{k}} + T^s_{nj}R_{s\bar{i}l\bar{k}} + T^{\bar{s}}_{nj}H_{\bar{s}\bar{i}l\bar{k}}})\\ &= \nabla_m\nabla_{\bar{n}}R_{i\bar{j}k\bar{l}} + \nabla_m\nabla_i H_{\bar{j}\bar{n}k\bar{l}} + \nabla_m(T^{\bar{s}}_{\bar{n}\bar{j}}R_{i\bar{s}k\bar{l}} + T^s_{\bar{n}\bar{j}}H_{isk\bar{l}}).\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\nabla_{\bar{j}}\nabla_i S_{k\bar{l}} &= \Delta R_{i\bar{j}k\bar{l}} + g^{m\bar{n}}(\nabla_m\nabla_i H_{\bar{j}\bar{n}k\bar{l}} + \nabla_m(T^{\bar{s}}_{\bar{n}\bar{j}}R_{i\bar{s}k\bar{l}} + T^s_{\bar{n}\bar{j}}H_{isk\bar{l}}))\\ &\quad +g^{m\bar{n}}(\Gamma^{\bar{s}}_{m\bar{j}}\nabla_{\bar{s}}R_{i\bar{n}k\bar{l}} + \Gamma^s_{mi}\nabla_{\bar{j}}R_{s\bar{n}k\bar{l}} + \Gamma^{\bar{s}}_{m\bar{n}}\nabla_{\bar{j}}R_{i\bar{s}k\bar{l}} + \Gamma^s_{mk}\nabla_{\bar{j}}R_{i\bar{n}s\bar{l}}\\ &\quad +\Gamma^{\bar{s}}_{m\bar{l}}\nabla_{\bar{j}}R_{i\bar{n}k\bar{s}}) + g^{m\bar{n}}Z_m(\Gamma^s_{\bar{j}i}R_{s\bar{n}k\bar{l}} + \Gamma^{\bar{s}}_{\bar{j}\bar{n}}R_{i\bar{s}k\bar{l}} + \Gamma^s_{\bar{j}k}R_{i\bar{n}s\bar{l}} + \Gamma^{\bar{s}}_{\bar{j}\bar{l}}R_{i\bar{n}k\bar{s}})\\ &\quad +g^{m\bar{n}}[Z_{\bar{j}}, Z_m]R_{i\bar{n}k\bar{l}} - g^{m\bar{n}}Z_{\bar{j}}(\Gamma^s_{mi}R_{s\bar{n}k\bar{l}} + \Gamma^{\bar{s}}_{m\bar{n}}R_{i\bar{s}k\bar{l}} + \Gamma^s_{mk}R_{i\bar{n}s\bar{l}} + \Gamma^{\bar{s}}_{m\bar{l}}R_{i\bar{n}k\bar{s}})\\ &\quad -g^{m\bar{n}}(\Gamma^s_{\bar{j}i}\nabla_m R_{s\bar{n}k\bar{l}} + \Gamma^{\bar{s}}_{\bar{j}\bar{n}}\nabla_m R_{i\bar{s}k\bar{l}} + \Gamma^s_{\bar{j}k}\nabla_m R_{i\bar{n}s\bar{l}} + \Gamma^{\bar{s}}_{\bar{j}\bar{l}}\nabla_m R_{i\bar{n}k\bar{s}}\\ &\quad +\Gamma^s_{\bar{j}m}\nabla_s R_{i\bar{n}k\bar{l}}) + g^{m\bar{n}}(\nabla_{\bar{j}}\nabla_{\bar{n}}H_{mik\bar{l}} + \nabla_{\bar{j}}(T^s_{mi}R_{s\bar{n}k\bar{l}} - B^{\bar{s}}_{mi}H_{\bar{s}\bar{n}k\bar{l}}))\\ &= \Delta R_{i\bar{j}kl} + g^{m\bar{n}}(\nabla_m\nabla_i H_{\bar{j}\bar{n}k\bar{l}} + \nabla_m T^{\bar{s}}_{\bar{n}\bar{j}}R_{i\bar{s}k\bar{l}} + T^{\bar{s}}_{\bar{n}\bar{j}}\nabla_m R_{i\bar{s}k\bar{l}} + \nabla_m T^s_{\bar{n}\bar{j}}H_{isk\bar{l}}\\ &\quad +T^s_{\bar{n}\bar{j}}\nabla_m H_{isk\bar{l}} + \Gamma^{\bar{s}}_{m\bar{j}}\nabla_{\bar{s}}R_{i\bar{n}k\bar{l}} + \Gamma^s_{mi}\nabla_{\bar{j}}R_{s\bar{n}k\bar{l}} + \Gamma^{\bar{s}}_{m\bar{n}}\nabla_{\bar{j}}R_{i\bar{s}k\bar{l}} + \Gamma^s_{mk}\nabla_{\bar{j}}R_{i\bar{n}s\bar{l}}\\ &\quad +\Gamma^{\bar{s}}_{m\bar{l}}\nabla_{\bar{j}}R_{i\bar{n}k\bar{s}} + Z_m(\Gamma^s_{\bar{j}i})R_{s\bar{n}k\bar{l}} + \Gamma^s_{\bar{j}i}\nabla_m R_{s\bar{n}k\bar{l}} + \Gamma^s_{\bar{j}i}\Gamma^p_{ms}R_{p\bar{n}k\bar{l}} + \Gamma^s_{\bar{j}i}\Gamma^{\bar{p}}_{m\bar{n}}R_{s\bar{p}k\bar{l}}\\ &\quad +\Gamma^s_{\bar{j}i}\Gamma^p_{mk}R_{s\bar{n}p\bar{l}} + \Gamma^s_{\bar{j}i}\Gamma^{\bar{p}}_{m\bar{l}}R_{s\bar{n}k\bar{p}} + Z_m(\Gamma^{\bar{s}}_{\bar{j}\bar{n}})R_{i\bar{s}k\bar{l}} + \Gamma^{\bar{s}}_{\bar{j}\bar{n}}\nabla_m R_{i\bar{s}k\bar{l}} + \Gamma^{\bar{s}}_{\bar{j}\bar{n}}\Gamma^p_{mi}R_{p\bar{s}k\bar{l}}\\ &\quad +\Gamma^{\bar{s}}_{\bar{j}\bar{n}}\Gamma^{\bar{p}}_{m\bar{s}}R_{i\bar{p}k\bar{l}} + \Gamma^{\bar{s}}_{\bar{j}\bar{n}}\Gamma^p_{mk}R_{i\bar{s}p\bar{l}} + \Gamma^{\bar{s}}_{\bar{j}\bar{n}}\Gamma^{\bar{p}}_{m\bar{l}}R_{i\bar{s}k\bar{p}} + Z_m(\Gamma^s_{\bar{j}k})R_{i\bar{n}s\bar{l}} + \Gamma^s_{\bar{j}k}\nabla_m R_{i\bar{n}s\bar{l}}\\ &\quad +\Gamma^s_{\bar{j}k}\Gamma^p_{mi}R_{p\bar{n}s\bar{l}} + \Gamma^s_{\bar{j}k}\Gamma^{\bar{p}}_{m\bar{n}}R_{i\bar{p}s\bar{l}} + \Gamma^s_{\bar{j}k}\Gamma^p_{ms}R_{i\bar{n}p\bar{l}} + \Gamma^s_{\bar{j}k}\Gamma^{\bar{p}}_{m\bar{l}}R_{i\bar{n}s\bar{p}} + Z_m(\Gamma^{\bar{s}}_{\bar{j}\bar{l}})R_{i\bar{n}k\bar{s}}\\ &\quad +\Gamma^{\bar{s}}_{\bar{j}\bar{l}}\nabla_m R_{i\bar{n}k\bar{s}} + \Gamma^{\bar{s}}_{\bar{j}\bar{l}}\Gamma^p_{mi}R_{p\bar{n}k\bar{s}} + \Gamma^{\bar{s}}_{\bar{j}\bar{l}}\Gamma^{\bar{p}}_{m\bar{n}}R_{i\bar{p}k\bar{s}} + \Gamma^{\bar{s}}_{\bar{j}\bar{l}}\Gamma^p_{mk}R_{i\bar{n}p\bar{s}} + \Gamma^{\bar{s}}_{\bar{j}\bar{l}}\Gamma^{\bar{p}}_{m\bar{s}}R_{i\bar{n}k\bar{p}}\\ &\quad +B^s_{\bar{j}m}\nabla_s R_{i\bar{n}k\bar{l}} + B^s_{\bar{j}m}\Gamma^p_{si}R_{p\bar{n}k\bar{l}} + B^s_{\bar{j}m}\Gamma^{\bar{p}}_{s\bar{n}}R_{i\bar{p}k\bar{l}} + B^s_{\bar{j}m}\Gamma^p_{sk}R_{i\bar{n}p\bar{l}} + B^s_{\bar{j}m}\Gamma^{\bar{p}}_{s\bar{l}}R_{i\bar{n}k\bar{p}}\\ &\quad +B^{\bar{s}}_{\bar{j}m}\nabla_{\bar{s}}R_{i\bar{n}k\bar{l}} + B^{\bar{s}}_{\bar{j}m}\Gamma^p_{\bar{s}i}R_{p\bar{n}k\bar{l}} + B^{\bar{s}}_{\bar{j}m}\Gamma^{\bar{p}}_{\bar{s}\bar{n}}R_{i\bar{p}k\bar{l}} + B^{\bar{s}}_{\bar{j}m}\Gamma^p_{\bar{s}k}R_{i\bar{n}p\bar{l}} + B^{\bar{s}}_{\bar{j}m}\Gamma^{\bar{p}}_{\bar{s}\bar{l}}R_{i\bar{n}k\bar{p}}\\ &\quad -Z_{\bar{j}}(\Gamma^s_{mi})R_{s\bar{n}k\bar{l}} - \Gamma^s_{mi}\nabla_{\bar{j}}R_{s\bar{n}k\bar{l}} - \Gamma^s_{mi}\Gamma^p_{\bar{j}s}R_{p\bar{n}k\bar{l}} - \Gamma^s_{mi}\Gamma^{\bar{p}}_{\bar{j}\bar{n}}R_{s\bar{p}k\bar{l}} - \Gamma^s_{mi}\Gamma^p_{\bar{j}k}R_{s\bar{n}p\bar{l}}\\ &\quad -\Gamma^s_{mi}\Gamma^{\bar{p}}_{\bar{j}\bar{l}}R_{s\bar{n}k\bar{p}} - Z_{\bar{j}}(\Gamma^{\bar{s}}_{m\bar{n}})R_{i\bar{s}k\bar{l}} - \Gamma^{\bar{s}}_{m\bar{n}}\nabla_{\bar{j}}R_{i\bar{s}k\bar{l}} - \Gamma^{\bar{s}}_{m\bar{n}}\Gamma^p_{\bar{j}i}R_{p\bar{s}k\bar{l}} - \Gamma^{\bar{s}}_{m\bar{n}}\Gamma^{\bar{p}}_{\bar{j}\bar{s}}R_{i\bar{p}k\bar{l}}\\ &\quad -\Gamma^{\bar{s}}_{m\bar{n}}\Gamma^p_{\bar{j}k}R_{i\bar{s}p\bar{l}} - \Gamma^{\bar{s}}_{m\bar{n}}\Gamma^{\bar{p}}_{\bar{j}\bar{l}}R_{i\bar{s}k\bar{p}} - Z_{\bar{j}}(\Gamma^s_{mk})R_{i\bar{n}s\bar{l}} - \Gamma^s_{mk}\nabla_{\bar{j}}R_{i\bar{n}s\bar{l}} - \Gamma^s_{mk}\Gamma^p_{\bar{j}i}R_{p\bar{n}s\bar{l}}\\ &\quad -\Gamma^s_{mk}\Gamma^{\bar{p}}_{\bar{j}\bar{n}}R_{i\bar{p}s\bar{l}} - \Gamma^s_{mk}\Gamma^p_{\bar{j}s}R_{i\bar{n}p\bar{l}} - \Gamma^s_{mk}\Gamma^{\bar{p}}_{\bar{j}\bar{l}}R_{i\bar{n}s\bar{p}} - Z_{\bar{j}}(\Gamma^{\bar{s}}_{m\bar{l}})R_{i\bar{n}k\bar{s}} - \Gamma^{\bar{s}}_{m\bar{l}}\nabla_{\bar{j}}R_{i\bar{n}k\bar{s}}\end{aligned}$$

$$-\Gamma^{\bar{s}}_{m\bar{l}}\Gamma^{p}_{\bar{j}i}R_{p\bar{n}k\bar{s}} - \Gamma^{\bar{s}}_{m\bar{l}}\Gamma^{\bar{p}}_{\bar{j}\bar{n}}R_{i\bar{p}k\bar{s}} - \Gamma^{\bar{s}}_{m\bar{l}}\Gamma^{p}_{\bar{j}k}R_{i\bar{n}p\bar{s}} - \Gamma^{\bar{s}}_{m\bar{l}}\Gamma^{\bar{p}}_{\bar{j}\bar{s}}R_{i\bar{n}k\bar{p}}$$
$$-\Gamma^{s}_{\bar{j}i}\nabla_m R_{s\bar{n}k\bar{l}} - \Gamma^{\bar{s}}_{\bar{j}\bar{n}}\nabla_m R_{i\bar{s}k\bar{l}} - \Gamma^{s}_{\bar{j}k}\nabla_m R_{i\bar{n}s\bar{l}} - \Gamma^{\bar{s}}_{\bar{j}\bar{l}}\nabla_m R_{i\bar{n}k\bar{s}} - \Gamma^{s}_{\bar{j}m}\nabla_s R_{i\bar{n}k\bar{l}}$$
$$+\nabla_{\bar{j}}\nabla_{\bar{n}}H_{mik\bar{l}} + \nabla_{\bar{j}}T^{s}_{mi}R_{s\bar{n}k\bar{l}} + T^{s}_{mi}\nabla_{\bar{j}}R_{s\bar{n}k\bar{l}} - \nabla_{\bar{j}}B^{\bar{s}}_{mi}H_{\bar{s}\bar{n}k\bar{l}} - B^{\bar{s}}_{mi}\nabla_{\bar{j}}H_{\bar{s}\bar{n}k\bar{l}}),$$

where we used that $R_{\bar{p}\bar{n}k\bar{l}} = R_{ipk\bar{l}} = 0$,

$$[Z_{\bar{j}}, Z_m](R_{i\bar{n}k\bar{l}}) = B^{s}_{\bar{j}m}Z_s(R_{i\bar{n}k\bar{l}}) + B^{\bar{s}}_{\bar{j}m}Z_{\bar{s}}(R_{i\bar{n}k\bar{l}}).$$

Note that we have that $\Gamma^{\bar{s}}_{m\bar{j}}\nabla_{\bar{s}}R_{i\bar{n}k\bar{l}} + B^{\bar{s}}_{\bar{j}m}\nabla_{\bar{s}}R_{i\bar{n}k\bar{l}} = 0$ since $B^{\bar{s}}_{\bar{j}m} = -B^{\bar{s}}_{m\bar{j}} = -\Gamma^{\bar{s}}_{m\bar{j}}$.

Then we obtain the following equality:

$$\nabla_{\bar{j}}\nabla_i S_{k\bar{l}} = \Delta R_{i\bar{j}k\bar{l}} + T^{\bar{s}}_{\bar{m}\bar{j}}\nabla_m R_{i\bar{s}k\bar{l}} + T^{s}_{mi}(\nabla_{\bar{j}}R_{s\bar{m}k\bar{l}} + \nabla_s H_{\bar{j}\bar{m}k\bar{l}} + T^{\bar{p}}_{\bar{m}\bar{j}}R_{s\bar{p}k\bar{l}}$$
$$+T^{p}_{\bar{m}\bar{j}}H_{spk\bar{l}}) + g^{m\bar{n}}(\nabla_m\nabla_i H_{\bar{j}\bar{n}k\bar{l}} + \nabla_m T^{\bar{s}}_{\bar{n}\bar{j}}R_{i\bar{s}k\bar{l}} + \nabla_m T^{s}_{\bar{n}\bar{j}}H_{isk\bar{l}} + T^{s}_{\bar{n}\bar{j}}\nabla_m H_{isk\bar{l}}$$
$$+Z_m(\Gamma^{s}_{\bar{j}i})R_{s\bar{n}k\bar{l}} + \Gamma^{s}_{\bar{j}i}\Gamma^{p}_{ms}R_{p\bar{n}k\bar{l}} + \Gamma^{s}_{\bar{j}i}\Gamma^{\bar{p}}_{m\bar{n}}R_{s\bar{p}k\bar{l}} + \Gamma^{s}_{\bar{j}i}\Gamma^{p}_{mk}R_{s\bar{n}p\bar{l}} + \Gamma^{s}_{\bar{j}i}\Gamma^{\bar{p}}_{m\bar{l}}R_{s\bar{n}k\bar{p}}$$
$$+Z_m(\Gamma^{\bar{s}}_{\bar{j}\bar{n}})R_{i\bar{s}k\bar{l}} + \Gamma^{\bar{s}}_{\bar{j}\bar{n}}\Gamma^{p}_{mi}R_{p\bar{s}k\bar{l}} + \Gamma^{\bar{s}}_{\bar{j}\bar{n}}\Gamma^{\bar{p}}_{m\bar{s}}R_{i\bar{p}k\bar{l}} + \Gamma^{\bar{s}}_{\bar{j}\bar{n}}\Gamma^{p}_{mk}R_{i\bar{s}p\bar{l}} + \Gamma^{\bar{s}}_{\bar{j}\bar{n}}\Gamma^{\bar{p}}_{m\bar{l}}R_{i\bar{s}k\bar{p}}$$
$$+Z_m(\Gamma^{s}_{\bar{j}k})R_{i\bar{n}s\bar{l}} + \Gamma^{s}_{\bar{j}k}\Gamma^{p}_{mi}R_{p\bar{n}s\bar{l}} + \Gamma^{s}_{\bar{j}k}\Gamma^{\bar{p}}_{m\bar{n}}R_{i\bar{p}s\bar{l}} + \Gamma^{s}_{\bar{j}k}\Gamma^{p}_{ms}R_{i\bar{n}p\bar{l}} + \Gamma^{s}_{\bar{j}k}\Gamma^{\bar{p}}_{m\bar{l}}R_{i\bar{n}s\bar{p}}$$
$$+Z_m(\Gamma^{\bar{s}}_{\bar{j}\bar{l}})R_{i\bar{n}k\bar{s}} + \Gamma^{\bar{s}}_{\bar{j}\bar{l}}\Gamma^{p}_{mi}R_{p\bar{n}k\bar{s}} + \Gamma^{\bar{s}}_{\bar{j}\bar{l}}\Gamma^{\bar{p}}_{m\bar{n}}R_{i\bar{p}k\bar{s}} + \Gamma^{\bar{s}}_{\bar{j}\bar{l}}\Gamma^{p}_{mk}R_{i\bar{n}p\bar{s}} + \Gamma^{\bar{s}}_{\bar{j}\bar{l}}\Gamma^{\bar{p}}_{m\bar{s}}R_{i\bar{n}k\bar{p}}$$
$$+B^{s}_{\bar{j}m}\Gamma^{p}_{si}R_{p\bar{n}k\bar{l}} + B^{s}_{\bar{j}m}\Gamma^{\bar{p}}_{s\bar{n}}R_{i\bar{p}k\bar{l}} + B^{s}_{\bar{j}m}\Gamma^{p}_{sk}R_{i\bar{n}p\bar{l}} + B^{s}_{\bar{j}m}\Gamma^{\bar{p}}_{s\bar{l}}R_{i\bar{n}k\bar{p}} + B^{\bar{s}}_{\bar{j}m}\Gamma^{p}_{\bar{s}i}R_{p\bar{n}k\bar{l}}$$
$$+B^{\bar{s}}_{\bar{j}m}\Gamma^{\bar{p}}_{\bar{s}\bar{n}}R_{i\bar{p}k\bar{l}} + B^{\bar{s}}_{\bar{j}m}\Gamma^{p}_{\bar{s}k}R_{i\bar{n}p\bar{l}} + B^{\bar{s}}_{\bar{j}m}\Gamma^{\bar{p}}_{\bar{s}\bar{l}}R_{i\bar{n}k\bar{p}} - Z_{\bar{j}}(\Gamma^{s}_{mi})R_{s\bar{n}k\bar{l}} - \Gamma^{s}_{mi}\Gamma^{p}_{\bar{j}s}R_{p\bar{n}k\bar{l}}$$
$$-\Gamma^{s}_{mi}\Gamma^{\bar{p}}_{\bar{j}\bar{n}}R_{s\bar{p}k\bar{l}} - \Gamma^{s}_{mi}\Gamma^{p}_{\bar{j}k}R_{s\bar{n}p\bar{l}} - \Gamma^{s}_{mi}\Gamma^{\bar{p}}_{\bar{j}\bar{l}}R_{s\bar{n}k\bar{p}} - Z_{\bar{j}}(\Gamma^{\bar{s}}_{m\bar{n}})R_{i\bar{s}k\bar{l}} - \Gamma^{\bar{s}}_{m\bar{n}}\Gamma^{p}_{\bar{j}i}R_{p\bar{s}k\bar{l}}$$
$$-\Gamma^{\bar{s}}_{m\bar{n}}\Gamma^{\bar{p}}_{\bar{j}\bar{s}}R_{i\bar{p}k\bar{l}} - \Gamma^{\bar{s}}_{m\bar{n}}\Gamma^{p}_{\bar{j}k}R_{i\bar{s}p\bar{l}} - \Gamma^{\bar{s}}_{m\bar{n}}\Gamma^{\bar{p}}_{\bar{j}\bar{l}}R_{i\bar{s}k\bar{p}} - Z_{\bar{j}}(\Gamma^{s}_{mk})R_{i\bar{n}s\bar{l}} - \Gamma^{s}_{mk}\Gamma^{p}_{\bar{j}i}R_{p\bar{n}s\bar{l}}$$
$$-\Gamma^{s}_{mk}\Gamma^{\bar{p}}_{\bar{j}\bar{n}}R_{i\bar{p}s\bar{l}} - \Gamma^{s}_{mk}\Gamma^{p}_{\bar{j}s}R_{i\bar{n}p\bar{l}} - \Gamma^{s}_{mk}\Gamma^{\bar{p}}_{\bar{j}\bar{l}}R_{i\bar{n}s\bar{p}} - Z_{\bar{j}}(\Gamma^{\bar{s}}_{m\bar{l}})R_{i\bar{n}k\bar{s}} - \Gamma^{\bar{s}}_{m\bar{l}}\Gamma^{p}_{\bar{j}i}R_{p\bar{n}k\bar{s}}$$
$$-\Gamma^{\bar{s}}_{m\bar{l}}\Gamma^{\bar{p}}_{\bar{j}\bar{n}}R_{i\bar{p}k\bar{s}} - \Gamma^{\bar{s}}_{m\bar{l}}\Gamma^{p}_{\bar{j}k}R_{i\bar{n}p\bar{s}} - \Gamma^{\bar{s}}_{m\bar{l}}\Gamma^{\bar{p}}_{\bar{j}\bar{s}}R_{i\bar{n}k\bar{p}} + \nabla_{\bar{j}}\nabla_{\bar{n}}H_{mik\bar{l}}$$
$$+\nabla_{\bar{j}}T^{s}_{mi}R_{s\bar{n}k\bar{l}} - \nabla_{\bar{j}}B^{\bar{s}}_{mi}H_{\bar{s}\bar{n}k\bar{l}} - B^{\bar{s}}_{mi}\nabla_{\bar{j}}H_{\bar{s}\bar{n}k\bar{l}}),$$

where we have used that by applying (2.3),

$$\begin{aligned}\nabla_{\bar{j}}R_{s\bar{m}k\bar{l}} &= \overline{\nabla_j R_{m\bar{s}l\bar{k}}}\\ &= \overline{\nabla_m R_{j\bar{s}l\bar{k}} + \nabla_{\bar{s}}H_{mjl\bar{k}} + T^{p}_{mj}R_{p\bar{s}l\bar{k}} + T^{\bar{p}}_{mj}H_{\bar{p}\bar{s}l\bar{k}}}\\ &= \nabla_{\bar{m}}R_{s\bar{j}k\bar{l}} + \nabla_s H_{\bar{j}\bar{m}k\bar{l}} + T^{\bar{p}}_{\bar{m}\bar{j}}R_{s\bar{p}k\bar{l}} + T^{p}_{\bar{m}\bar{j}}H_{spk\bar{l}}.\end{aligned}$$

We have the following:

$$\begin{aligned}-(Z_{\bar{j}}(\Gamma^{\bar{s}}_{m\bar{n}}) - Z_m(\Gamma^{\bar{s}}_{\bar{j}\bar{n}}))R_{i\bar{s}k\bar{l}} &= -\overline{(Z_j(\Gamma^{s}_{\bar{m}n}) - Z_{\bar{m}}(\Gamma^{s}_{jn}))}R_{i\bar{s}k\bar{l}}\\ &= -\overline{(R_{j\bar{m}n}{}^{s} - \Gamma^{s}_{jp}\Gamma^{p}_{\bar{m}n} + \Gamma^{s}_{\bar{m}p}\Gamma^{p}_{jn} + B^{p}_{j\bar{m}}\Gamma^{s}_{pn} - B^{\bar{p}}_{\bar{m}j}\Gamma^{s}_{\bar{p}n})}R_{i\bar{s}k\bar{l}}\\ &= (-R_{m\bar{j}}{}^{\bar{s}}{}_{\bar{n}} + \Gamma^{\bar{s}}_{\bar{j}\bar{p}}\Gamma^{\bar{p}}_{m\bar{n}} - \Gamma^{\bar{s}}_{m\bar{p}}\Gamma^{\bar{p}}_{\bar{j}\bar{n}} - B^{\bar{p}}_{\bar{j}m}\Gamma^{\bar{s}}_{\bar{p}\bar{n}} + B^{p}_{m\bar{j}}\Gamma^{\bar{s}}_{p\bar{n}})R_{i\bar{s}k\bar{l}},\end{aligned}$$

and similarly,

$$\begin{aligned}&-(Z_{\bar{j}}(\Gamma^{\bar{s}}_{m\bar{l}}) - Z_m(\Gamma^{\bar{s}}_{\bar{j}\bar{l}}))R_{i\bar{n}k\bar{s}}\\ &\quad= (-R_{m\bar{j}}{}^{\bar{s}}{}_{\bar{l}} + \Gamma^{\bar{s}}_{\bar{j}\bar{p}}\Gamma^{\bar{p}}_{m\bar{l}} - \Gamma^{\bar{s}}_{m\bar{p}}\Gamma^{\bar{p}}_{\bar{j}\bar{l}} - B^{\bar{p}}_{\bar{j}m}\Gamma^{\bar{s}}_{\bar{p}\bar{l}} + B^{p}_{m\bar{j}}\Gamma^{\bar{s}}_{p\bar{l}})R_{i\bar{n}k\bar{s}},\\ &(Z_m(\Gamma^{s}_{\bar{j}k}) - Z_{\bar{j}}(\Gamma^{s}_{mk}))R_{i\bar{n}s\bar{l}}\end{aligned}$$

$$= (R_{m\bar{j}k}{}^{s} - \Gamma^{s}_{mp}\Gamma^{p}_{\bar{j}k} + \Gamma^{s}_{\bar{j}p}\Gamma^{p}_{mk} + B^{p}_{m\bar{j}}\Gamma^{s}_{pk} - B^{\bar{p}}_{\bar{j}m}\Gamma^{s}_{\bar{p}k})R_{i\bar{n}s\bar{l}},$$

$$(Z_m(\Gamma^{s}_{\bar{j}i}) - Z_{\bar{j}}(\Gamma^{s}_{mi}))R_{s\bar{n}k\bar{l}}$$
$$= (R_{m\bar{j}i}{}^{s} - \Gamma^{s}_{mp}\Gamma^{p}_{\bar{j}i} + \Gamma^{s}_{\bar{j}p}\Gamma^{p}_{mi} + B^{p}_{m\bar{j}}\Gamma^{s}_{pi} - B^{\bar{p}}_{\bar{j}m}\Gamma^{s}_{\bar{p}i})R_{s\bar{n}k\bar{l}}$$
$$= (R_{i\bar{j}m}{}^{s} - B^{\bar{q}}_{mi}B^{s}_{\bar{q}\bar{j}} + \nabla_{\bar{j}}T^{s}_{im} - \Gamma^{s}_{mp}\Gamma^{p}_{\bar{j}i} + \Gamma^{s}_{\bar{j}p}\Gamma^{p}_{mi} + B^{p}_{m\bar{j}}\Gamma^{s}_{pi} - B^{\bar{p}}_{\bar{j}m}\Gamma^{s}_{\bar{p}i})R_{s\bar{n}k\bar{l}},$$

where we used that $R_{m\bar{j}i}{}^{s} = R_{i\bar{j}m}{}^{s} - B^{\bar{q}}_{mi}B^{s}_{\bar{q}\bar{j}} + \nabla_{\bar{j}}T^{s}_{im}$.

Choosing $t = t_0$ and a local unitary $(1,0)$-frame $\{Z_r\}$ around a fixed point $p \in M$ with respect to $g(t_0)$. Then using the local $g(t_0)$-unitary frame, $g(t_0)^{m\bar{n}} = \delta_{mn}$,

$$\Gamma^{\bar{p}}_{m\bar{j}} = -\Gamma^{j}_{mp}, \quad \Gamma^{\bar{s}}_{i\bar{j}} = -\Gamma^{j}_{\bar{i}s}.$$

By combining these calculations, we compute for the evolution $\frac{\partial}{\partial t}g = -S$, with a local unitary frame,

$$\begin{aligned}
\frac{\partial}{\partial t}R_{i\bar{j}k\bar{l}} &= -R_{i\bar{j}k}{}^{r}S_{r\bar{l}} + \nabla_{\bar{j}}\nabla_{i}S_{k\bar{l}}\\
&= -R_{i\bar{j}k}{}^{r}S_{r\bar{l}} + \Delta R_{i\bar{j}k\bar{l}} + g^{r\bar{s}}(T^{\bar{q}}_{\bar{s}\bar{j}}\nabla_{r}R_{i\bar{q}k\bar{l}} + T^{p}_{ri}\nabla_{\bar{s}}R_{p\bar{j}k\bar{l}})\\
&\quad + g^{n\bar{m}}T^{s}_{mi}(\nabla_{s}H_{\bar{j}\bar{n}k\bar{l}} + T^{\bar{p}}_{\bar{n}\bar{j}}R_{s\bar{p}k\bar{l}} + T^{p}_{\bar{n}\bar{j}}H_{spk\bar{l}})\\
&\quad + g^{m\bar{n}}(R_{i\bar{j}m}{}^{r}R_{r\bar{n}k\bar{l}} + R_{m\bar{j}k}{}^{s}R_{i\bar{n}s\bar{l}} - R_{m\bar{j}r\bar{l}}R_{i\bar{n}k}{}^{r}) - g^{m\bar{n}}R_{m\bar{j}}{}^{\bar{s}}{}_{\bar{n}}R_{i\bar{s}k\bar{l}}\\
&\quad + g^{m\bar{n}}(\nabla_{m}\nabla_{i}H_{\bar{j}\bar{n}k\bar{l}} + \nabla_{m}T^{\bar{s}}_{\bar{n}\bar{j}}R_{i\bar{s}k\bar{l}} + \nabla_{m}T^{s}_{\bar{n}\bar{j}}H_{isk\bar{l}} + T^{s}_{\bar{n}\bar{j}}\nabla_{m}H_{isk\bar{l}}\\
&\quad + \Gamma^{s}_{\bar{j}i}\Gamma^{p}_{ms}R_{p\bar{n}k\bar{l}} + \Gamma^{s}_{\bar{j}i}\Gamma^{p}_{mk}R_{s\bar{n}p\bar{l}} + \Gamma^{s}_{\bar{j}i}\Gamma^{\bar{p}}_{m\bar{l}}R_{s\bar{n}k\bar{p}} + \Gamma^{\bar{s}}_{\bar{j}\bar{n}}\Gamma^{p}_{mk}R_{i\bar{s}p\bar{l}}\\
&\quad + \Gamma^{\bar{s}}_{\bar{j}\bar{n}}\Gamma^{\bar{p}}_{m\bar{l}}R_{i\bar{s}k\bar{p}} + \Gamma^{s}_{\bar{j}k}\Gamma^{p}_{mi}R_{p\bar{n}s\bar{l}} + \Gamma^{s}_{\bar{j}k}\Gamma^{\bar{p}}_{m\bar{n}}R_{i\bar{p}s\bar{l}}\\
&\quad + \Gamma^{s}_{\bar{j}k}\Gamma^{p}_{ms}R_{i\bar{n}p\bar{l}} + \Gamma^{s}_{\bar{j}k}\Gamma^{\bar{p}}_{m\bar{l}}R_{i\bar{n}s\bar{p}} + \Gamma^{\bar{s}}_{\bar{j}\bar{l}}\Gamma^{p}_{mi}R_{p\bar{n}k\bar{s}} + \Gamma^{\bar{s}}_{\bar{j}\bar{l}}\Gamma^{\bar{p}}_{m\bar{n}}R_{i\bar{p}k\bar{s}}\\
&\quad + \Gamma^{\bar{s}}_{\bar{j}\bar{l}}\Gamma^{p}_{mk}R_{i\bar{n}p\bar{s}} + \Gamma^{\bar{s}}_{\bar{j}\bar{l}}\Gamma^{\bar{p}}_{m\bar{s}}R_{i\bar{n}k\bar{p}} + B^{s}_{\bar{j}m}\Gamma^{p}_{si}R_{p\bar{n}k\bar{l}} + B^{s}_{\bar{j}m}\Gamma^{p}_{sk}R_{i\bar{n}p\bar{l}}\\
&\quad + B^{s}_{\bar{j}m}\Gamma^{\bar{p}}_{s\bar{l}}R_{i\bar{n}k\bar{p}} + B^{\bar{s}}_{\bar{j}m}\Gamma^{p}_{\bar{s}i}R_{p\bar{n}k\bar{l}} + B^{\bar{s}}_{\bar{j}m}\Gamma^{\bar{p}}_{\bar{s}\bar{n}}R_{i\bar{p}k\bar{l}}\\
&\quad + B^{\bar{s}}_{\bar{j}m}\Gamma^{p}_{\bar{s}k}R_{i\bar{n}p\bar{l}} + B^{\bar{s}}_{\bar{j}m}\Gamma^{\bar{p}}_{\bar{s}\bar{l}}R_{i\bar{n}k\bar{p}}\\
&\quad - \Gamma^{s}_{mi}\Gamma^{p}_{\bar{j}s}R_{p\bar{n}k\bar{l}} - \Gamma^{s}_{mi}\Gamma^{p}_{\bar{j}k}R_{s\bar{n}p\bar{l}} - \Gamma^{s}_{mi}\Gamma^{\bar{p}}_{\bar{j}\bar{l}}R_{s\bar{n}k\bar{p}} - \Gamma^{\bar{s}}_{m\bar{n}}\Gamma^{\bar{p}}_{\bar{j}\bar{s}}R_{i\bar{p}k\bar{l}}\\
&\quad - \Gamma^{\bar{s}}_{m\bar{n}}\Gamma^{p}_{\bar{j}k}R_{i\bar{s}p\bar{l}} - \Gamma^{\bar{s}}_{m\bar{n}}\Gamma^{\bar{p}}_{\bar{j}\bar{l}}R_{i\bar{s}k\bar{p}} - \Gamma^{s}_{mk}\Gamma^{p}_{\bar{j}i}R_{p\bar{n}s\bar{l}} - \Gamma^{s}_{mk}\Gamma^{\bar{p}}_{\bar{j}\bar{n}}R_{i\bar{p}s\bar{l}}\\
&\quad - \Gamma^{s}_{mk}\Gamma^{p}_{\bar{j}s}R_{i\bar{n}p\bar{l}} - \Gamma^{s}_{mk}\Gamma^{\bar{p}}_{\bar{j}\bar{l}}R_{i\bar{n}s\bar{p}} - \Gamma^{\bar{s}}_{m\bar{l}}\Gamma^{p}_{\bar{j}i}R_{p\bar{n}k\bar{s}} - \Gamma^{\bar{s}}_{m\bar{l}}\Gamma^{\bar{p}}_{\bar{j}\bar{n}}R_{i\bar{p}k\bar{s}}\\
&\quad - \Gamma^{\bar{s}}_{m\bar{l}}\Gamma^{p}_{\bar{j}k}R_{i\bar{n}p\bar{s}} - \Gamma^{\bar{s}}_{m\bar{l}}\Gamma^{\bar{p}}_{\bar{j}\bar{s}}R_{i\bar{n}k\bar{p}}\\
&\quad + \nabla_{\bar{j}}\nabla_{\bar{n}}H_{mik\bar{l}} + \nabla_{\bar{j}}T^{s}_{mi}R_{s\bar{n}k\bar{l}} - \nabla_{\bar{j}}B^{\bar{s}}_{mi}H_{\bar{s}\bar{n}k\bar{l}} - B^{\bar{s}}_{mi}\nabla_{\bar{j}}H_{\bar{s}\bar{n}k\bar{l}})\\
&\quad - g^{m\bar{n}}(B^{\bar{q}}_{mi}B^{s}_{\bar{q}\bar{j}} - \nabla_{\bar{j}}T^{s}_{im} + \Gamma^{s}_{mp}\Gamma^{p}_{\bar{j}i} - \Gamma^{s}_{\bar{j}p}\Gamma^{p}_{mi} - B^{p}_{m\bar{j}}\Gamma^{s}_{pi} + B^{\bar{p}}_{\bar{j}m}\Gamma^{s}_{\bar{p}i})R_{s\bar{n}k\bar{l}}\\
&\quad - g^{m\bar{n}}(\Gamma^{s}_{mp}\Gamma^{p}_{\bar{j}k} - \Gamma^{s}_{\bar{j}p}\Gamma^{p}_{mk} - B^{p}_{m\bar{j}}\Gamma^{s}_{pk} + B^{\bar{p}}_{\bar{j}m}\Gamma^{s}_{\bar{p}k})R_{i\bar{n}s\bar{l}}\\
&\quad + g^{m\bar{n}}(\Gamma^{\bar{s}}_{\bar{j}\bar{p}}\Gamma^{\bar{p}}_{m\bar{n}} - \Gamma^{\bar{s}}_{m\bar{p}}\Gamma^{\bar{p}}_{\bar{j}\bar{n}} - B^{\bar{p}}_{\bar{j}m}\Gamma^{\bar{s}}_{\bar{p}\bar{n}} + B^{p}_{m\bar{j}}\Gamma^{\bar{s}}_{p\bar{n}})R_{i\bar{s}k\bar{l}}\\
&\quad + g^{m\bar{n}}(\Gamma^{\bar{s}}_{\bar{j}\bar{p}}\Gamma^{\bar{p}}_{m\bar{l}} - \Gamma^{\bar{s}}_{m\bar{p}}\Gamma^{\bar{p}}_{\bar{j}\bar{l}} - B^{\bar{p}}_{\bar{j}m}\Gamma^{\bar{s}}_{\bar{p}\bar{l}} + B^{p}_{m\bar{j}}\Gamma^{\bar{s}}_{p\bar{l}})R_{i\bar{n}k\bar{s}},
\end{aligned}$$

where we have used that

$$\Gamma^{\bar{s}}_{\bar{j}\bar{n}}\Gamma^{p}_{mi}R_{p\bar{s}k\bar{l}} - \Gamma^{s}_{mi}\Gamma^{\bar{p}}_{\bar{j}\bar{n}}R_{s\bar{p}k\bar{l}} = 0,$$
$$\Gamma^{\bar{s}}_{\bar{j}i}\Gamma^{\bar{p}}_{m\bar{n}}R_{s\bar{p}k\bar{l}} - \Gamma^{\bar{s}}_{m\bar{n}}\Gamma^{p}_{\bar{j}i}R_{p\bar{s}k\bar{l}} = 0,$$

and

$$\begin{aligned} & g^{m\bar{n}}(\Gamma^{\bar{s}}_{\bar{j}\bar{n}}\Gamma^{\bar{p}}_{m\bar{s}}R_{i\bar{p}k\bar{l}} + B^{s}_{\bar{j}m}\Gamma^{\bar{p}}_{s\bar{n}}R_{i\bar{p}k\bar{l}}) \\ &= \Gamma^{\bar{s}}_{\bar{j}\bar{m}}\Gamma^{\bar{p}}_{m\bar{s}}R_{i\bar{p}k\bar{l}} + B^{s}_{\bar{j}m}\Gamma^{\bar{p}}_{s\bar{m}}R_{i\bar{p}k\bar{l}} \\ &= -\Gamma^{m}_{\bar{j}s}\Gamma^{\bar{p}}_{m\bar{s}}R_{i\bar{p}k\bar{l}} + B^{s}_{\bar{j}m}\Gamma^{\bar{p}}_{s\bar{m}}R_{i\bar{p}k\bar{l}} = 0. \end{aligned}$$

Combining this formula and the terms from $-Q^7 - Q^8 + BT' + \bar{Z}(T')$ gives the result. □

Lemma 3.7. *Along the AHCF, we have the following evolution equation for the first Chern–Ricci curvature.*

$$\begin{aligned} \frac{\partial}{\partial t}P_{i\bar{j}} &= \Delta P_{i\bar{j}} + g^{r\bar{s}}(T^{\bar{q}}_{\bar{s}\bar{j}}\nabla_r P_{i\bar{q}} + T^{p}_{ri}\nabla_{\bar{s}}P_{p\bar{j}} + T^{p}_{ri}T^{\bar{q}}_{\bar{s}\bar{j}}P_{p\bar{q}} + R_{i\bar{j}r}{}^{p}P_{p\bar{s}}) \\ &\quad -g^{r\bar{s}}g^{m\bar{n}}R_{m\bar{j}r\bar{n}}P_{i\bar{s}} + g^{m\bar{n}}T^{s}_{mi}(\nabla_s R_{\bar{j}\bar{n}} + T^{p}_{\bar{n}\bar{j}}R_{sp}) \\ &\quad +g^{m\bar{n}}(\nabla_m\nabla_i R_{\bar{j}\bar{n}} + \nabla_m T^{\bar{s}}_{\bar{n}\bar{j}}\cdot P_{i\bar{s}} + \nabla_m T^{s}_{\bar{n}\bar{j}}\cdot R_{is} + T^{s}_{\bar{n}\bar{j}}\nabla_m R_{is} \\ &\quad +\Gamma^{s}_{\bar{j}i}\Gamma^{p}_{ms}P_{p\bar{n}} + B^{s}_{\bar{j}m}\Gamma^{p}_{si}P_{p\bar{n}} + B^{\bar{s}}_{\bar{j}m}\Gamma^{p}_{\bar{s}i}P_{p\bar{n}} + B^{\bar{s}}_{\bar{j}m}\Gamma^{\bar{p}}_{\bar{s}\bar{n}}P_{i\bar{p}} - \Gamma^{s}_{mi}\Gamma^{p}_{\bar{j}s}P_{p\bar{n}} \\ &\quad -\Gamma^{\bar{s}}_{m\bar{n}}\Gamma^{\bar{p}}_{\bar{j}\bar{s}}P_{i\bar{p}} + \nabla_{\bar{j}}\nabla_{\bar{n}}R_{mi} + \nabla_{\bar{j}}T^{s}_{mi}\cdot P_{s\bar{n}} - \nabla_{\bar{j}}B^{\bar{s}}_{mi}\cdot R_{\bar{s}\bar{n}} - B^{\bar{s}}_{mi}\nabla_{\bar{j}}R_{\bar{s}\bar{n}} \\ &\quad -g^{m\bar{n}}(B^{\bar{q}}_{mi}B^{s}_{\bar{q}\bar{j}} - \nabla_{\bar{j}}T^{s}_{im} + \Gamma^{s}_{mp}\Gamma^{p}_{\bar{j}i} - \Gamma^{s}_{\bar{j}p}\Gamma^{p}_{mi} - B^{p}_{m\bar{j}}\Gamma^{s}_{pi} + B^{\bar{p}}_{\bar{j}m}\Gamma^{s}_{\bar{p}i})P_{s\bar{n}} \\ &\quad -g^{m\bar{n}}g^{k\bar{l}}(\Gamma^{s}_{mp}\Gamma^{p}_{\bar{j}k} - \Gamma^{s}_{\bar{j}p}\Gamma^{p}_{mk} - B^{p}_{m\bar{j}}\Gamma^{s}_{pk} + B^{\bar{p}}_{\bar{j}m}\Gamma^{s}_{\bar{p}k})R_{i\bar{n}s\bar{l}} \\ &\quad +g^{m\bar{n}}(\Gamma^{\bar{s}}_{\bar{j}\bar{p}}\Gamma^{\bar{p}}_{m\bar{n}} - \Gamma^{\bar{s}}_{m\bar{p}}\Gamma^{\bar{p}}_{\bar{j}\bar{n}} - B^{\bar{p}}_{\bar{j}m}\Gamma^{\bar{s}}_{\bar{p}\bar{n}} + B^{p}_{m\bar{j}}\Gamma^{\bar{s}}_{p\bar{n}})P_{i\bar{s}} \\ &\quad +g^{m\bar{n}}g^{k\bar{l}}(\Gamma^{\bar{s}}_{\bar{j}\bar{p}}\Gamma^{\bar{p}}_{m\bar{l}} - \Gamma^{\bar{s}}_{m\bar{p}}\Gamma^{\bar{p}}_{\bar{j}\bar{l}} - B^{\bar{p}}_{\bar{j}m}\Gamma^{\bar{s}}_{\bar{p}\bar{l}} + B^{p}_{m\bar{j}}\Gamma^{\bar{s}}_{p\bar{l}})R_{i\bar{n}k\bar{s}} \\ &\quad +g^{k\bar{l}}\nabla_{\bar{j}}\nabla_i(Q^7 + Q^8 - BT' - \bar{Z}(T'))_{k\bar{l}} \\ &= \Delta P_{i\bar{j}} + g^{r\bar{s}}(T^{\bar{q}}_{\bar{s}\bar{j}}\nabla_r P_{i\bar{q}} + T^{p}_{ri}\nabla_{\bar{s}}P_{p\bar{j}} + T^{p}_{ri}T^{\bar{q}}_{\bar{s}\bar{j}}P_{p\bar{q}} + R_{i\bar{j}r}{}^{p}P_{p\bar{s}}) + E_{i\bar{j}}, \end{aligned} \tag{3.3}$$

where we put

$$E_{i\bar{j}} := g^{k\bar{l}}E_{i\bar{j}k\bar{l}}. \tag{3.4}$$

Proof. We compute that

$$\begin{aligned} \frac{\partial}{\partial t}P_{i\bar{j}}(g(t)) &= \frac{\partial}{\partial t}(g^{k\bar{l}}R_{i\bar{j}k\bar{l}}) \\ &= (S + Q^7 + Q^8 - BT' - \bar{Z}(T'))^{k\bar{l}}R_{i\bar{j}k\bar{l}} + g^{k\bar{l}}\frac{\partial}{\partial t}R_{i\bar{j}k\bar{l}}. \end{aligned}$$

Choosing $t = t_0$ and a local unitary $(1,0)$-frame $\{Z_r\}$ around a fixed point $p \in M$ with respect to $g(t_0)$. Then using the local $g(t_0)$-unitary frame, $g(t_0)^{m\bar{n}} =$

δ_{mn},

$$\Gamma^{\bar p}_{m\bar j} = -\Gamma^{j}_{mp}, \quad \Gamma^{\bar s}_{\bar i\bar j} = -\Gamma^{j}_{\bar i s}.$$

In the evolution equation in Lemma 3.6 (3.2), we see that

$$\begin{aligned}
&g^{k\bar l}g^{r\bar s}(R_{r\bar j k}{}^{p}R_{i\bar s p\bar l} - R_{r\bar j p\bar l}R_{i\bar s k}{}^{p}) = 0,\\
&g^{k\bar l}R_{i\bar j k}{}^{r}(S + Q^7 + Q^8 - BT' - \bar Z(T'))_{r\bar l}\\
&\quad = g^{k\bar l}g^{r\bar s}R_{i\bar j k\bar s}(S + Q^7 + Q^8 - BT' - \bar Z(T'))_{r\bar l}\\
&\quad = (S + Q^7 + Q^8 - BT' - \bar Z(T'))^{k\bar l}R_{i\bar j k\bar l},\\
&\quad \Gamma^{s}_{\bar j i}\Gamma^{p}_{mk}R_{s\bar m p\bar k} + \Gamma^{s}_{\bar j i}\Gamma^{\bar p}_{m\bar k}R_{s\bar m k\bar p} = \Gamma^{s}_{\bar j i}\Gamma^{p}_{mk}R_{s\bar m p\bar k} - \Gamma^{s}_{\bar j i}\Gamma^{k}_{mp}R_{s\bar m k\bar p} = 0,\\
&\quad \Gamma^{\bar s}_{\bar j\bar m}\Gamma^{p}_{mk}R_{i\bar s p\bar k} + \Gamma^{\bar s}_{\bar j\bar m}\Gamma^{\bar p}_{m\bar k}R_{i\bar s k\bar p} = \Gamma^{\bar s}_{\bar j\bar m}\Gamma^{p}_{mk}R_{i\bar s p\bar k} - \Gamma^{\bar s}_{\bar j\bar m}\Gamma^{k}_{mp}R_{i\bar s k\bar p} = 0,\\
&\quad \Gamma^{s}_{\bar j k}\Gamma^{\bar p}_{m\bar k}R_{i\bar m s\bar p} + \Gamma^{\bar s}_{\bar j\bar k}\Gamma^{\bar p}_{m\bar s}R_{i\bar m k\bar p} = \Gamma^{s}_{\bar j k}\Gamma^{\bar p}_{m\bar k}R_{i\bar m s\bar p} - \Gamma^{k}_{\bar j s}\Gamma^{\bar p}_{m\bar s}R_{i\bar m k\bar p} = 0,\\
&\quad \Gamma^{s}_{\bar j k}\Gamma^{p}_{mi}R_{p\bar m s\bar k} + \Gamma^{\bar s}_{\bar j\bar k}\Gamma^{p}_{mi}R_{p\bar m k\bar s} = \Gamma^{s}_{\bar j k}\Gamma^{p}_{mi}R_{p\bar m s\bar k} - \Gamma^{k}_{\bar j s}\Gamma^{p}_{mi}R_{p\bar m k\bar s} = 0,\\
&\quad \Gamma^{s}_{\bar j k}\Gamma^{\bar p}_{m\bar n}R_{i\bar p s\bar k} + \Gamma^{\bar s}_{\bar j\bar k}\Gamma^{\bar p}_{m\bar n}R_{i\bar p k\bar s} = \Gamma^{s}_{\bar j k}\Gamma^{\bar p}_{m\bar n}R_{i\bar p s\bar k} - \Gamma^{k}_{\bar j s}\Gamma^{\bar p}_{m\bar n}R_{i\bar p k\bar s} = 0,\\
&\quad \Gamma^{s}_{\bar j k}\Gamma^{p}_{ms}R_{i\bar m p\bar k} + \Gamma^{\bar s}_{\bar j\bar k}\Gamma^{p}_{mk}R_{i\bar m p\bar s} = \Gamma^{s}_{\bar j k}\Gamma^{p}_{ms}R_{i\bar m p\bar k} - \Gamma^{k}_{\bar j s}\Gamma^{p}_{mk}R_{i\bar m p\bar s} = 0,\\
&\quad \Gamma^{s}_{\bar j k}\Gamma^{\bar p}_{m\bar k}R_{i\bar m s\bar p} + \Gamma^{\bar s}_{\bar j\bar k}\Gamma^{\bar p}_{m\bar s}R_{i\bar m k\bar p} = \Gamma^{s}_{\bar j k}\Gamma^{\bar p}_{m\bar k}R_{i\bar m s\bar p} - \Gamma^{k}_{\bar j s}\Gamma^{\bar p}_{m\bar s}R_{i\bar m k\bar p} = 0,\\
&\quad B^{s}_{\bar j m}\Gamma^{p}_{sk}R_{i\bar m p\bar k} + B^{s}_{\bar j m}\Gamma^{\bar p}_{s\bar k}R_{i\bar m k\bar p} = B^{s}_{\bar j m}\Gamma^{p}_{sk}R_{i\bar m p\bar k} - B^{s}_{\bar j m}\Gamma^{k}_{sp}R_{i\bar m k\bar p} = 0,\\
&\quad B^{\bar s}_{\bar j m}\Gamma^{p}_{\bar s k}R_{i\bar m p\bar k} + B^{\bar s}_{\bar j m}\Gamma^{\bar p}_{\bar s\bar k}R_{i\bar m k\bar p} = B^{\bar s}_{\bar j m}\Gamma^{p}_{\bar s k}R_{i\bar m p\bar k} - B^{\bar s}_{\bar j m}\Gamma^{k}_{\bar s p}R_{i\bar m k\bar p} = 0,\\
&\quad \Gamma^{s}_{mi}\Gamma^{p}_{\bar j k}R_{s\bar m p\bar k} + \Gamma^{s}_{mi}\Gamma^{\bar p}_{\bar j\bar k}R_{s\bar m k\bar p} = \Gamma^{s}_{mi}\Gamma^{p}_{\bar j k}R_{s\bar m p\bar k} - \Gamma^{s}_{mi}\Gamma^{k}_{\bar j p}R_{s\bar m k\bar p} = 0,\\
&\quad \Gamma^{\bar s}_{m\bar m}\Gamma^{p}_{\bar j k}R_{i\bar s p\bar k} + \Gamma^{\bar s}_{m\bar m}\Gamma^{\bar p}_{\bar j\bar k}R_{i\bar s k\bar p} = \Gamma^{\bar s}_{m\bar m}\Gamma^{p}_{\bar j k}R_{i\bar s p\bar k} - \Gamma^{\bar s}_{m\bar m}\Gamma^{k}_{\bar j p}R_{i\bar s k\bar p} = 0,\\
&\quad \Gamma^{s}_{mk}\Gamma^{p}_{\bar j i}R_{p\bar n s\bar k} + \Gamma^{\bar s}_{m\bar k}\Gamma^{p}_{\bar j i}R_{p\bar m k\bar s} = \Gamma^{s}_{mk}\Gamma^{p}_{\bar j i}R_{p\bar n s\bar k} - \Gamma^{k}_{ms}\Gamma^{p}_{\bar j i}R_{p\bar m k\bar s} = 0,\\
&\quad \Gamma^{s}_{mk}\Gamma^{\bar p}_{\bar j\bar k}R_{i\bar m s\bar p} + \Gamma^{\bar s}_{m\bar k}\Gamma^{\bar p}_{\bar j\bar s}R_{i\bar n k\bar p} = \Gamma^{s}_{mk}\Gamma^{\bar p}_{\bar j\bar k}R_{i\bar m s\bar p} - \Gamma^{k}_{ms}\Gamma^{\bar p}_{\bar j\bar s}R_{i\bar n k\bar p} = 0,\\
&\quad \Gamma^{s}_{mk}\Gamma^{p}_{\bar j s}R_{i\bar m p\bar k} + \Gamma^{\bar s}_{m\bar k}\Gamma^{p}_{\bar j k}R_{i\bar m p\bar s} = \Gamma^{s}_{mk}\Gamma^{p}_{\bar j s}R_{i\bar m p\bar k} - \Gamma^{k}_{ms}\Gamma^{p}_{\bar j k}R_{i\bar m p\bar s} = 0,\\
&\quad \Gamma^{s}_{mk}\Gamma^{\bar p}_{\bar j\bar m}R_{i\bar p s\bar k} + \Gamma^{\bar s}_{m\bar k}\Gamma^{\bar p}_{\bar j\bar m}R_{i\bar p k\bar s} = \Gamma^{s}_{mk}\Gamma^{\bar p}_{\bar j\bar m}R_{i\bar p s\bar k} - \Gamma^{k}_{ms}\Gamma^{\bar p}_{\bar j\bar m}R_{i\bar p k\bar s} = 0,
\end{aligned}$$

which leads the desired result. □

We need the estimate of the term $\nabla_{\bar j}\nabla_i(Q^7+Q^8-BT'-\bar Z(T'))_{k\bar l}$ included in the terms $E_{i\bar j k\bar l}$ and $E_{i\bar j}$ for giving a proof of Theorem 1.1. From Lemma 3.3, we have that

$$\begin{aligned}
\bar Z(T') = \bar Z(\Gamma) + Z(B^\circ) + B^\circ * \Gamma + T' * \bar\Gamma + \Gamma * \bar\Gamma + B^\circ * T' + B^\circ * \bar T'\\
+T' * \bar T' + \Gamma * \bar T' + B^\circ * B^\circ + \mathcal{O}(Z(g)) + \mathcal{O}(\bar Z(g))
\end{aligned}$$

From the definitions of BT', $\bar Z(T')$ and the equality above, we have

$$\begin{aligned}
&(-Q^7 - Q^8 + BT' + \bar Z(T'))_{i\bar j}\\
&\quad = -B^{\bar s}_{ir}B^{l}_{\bar r\bar k}g_{k\bar s}g_{l\bar j} - B^{\bar s}_{ir}B^{l}_{\bar j\bar k}g_{k\bar s}g_{l\bar r} + B^{j}_{\bar r p}T_{ir\bar p} + B^{r}_{\bar p i}T_{pr\bar j} + B^{p}_{\bar r r}T_{pi\bar j} + B^{r}_{\bar j i}w_r
\end{aligned}$$

$$
\begin{aligned}
&-Z_{\bar r}(T^s_{ri})g_{s\bar j} - Z_{\bar j}(w_i) - g^{p\bar q}T^r_{pi}Z_{\bar j}(g_{r\bar q})\\
&= B^\circ * B^\circ + B^\circ * T' + B^\circ * \bar T' + T' * \bar\Gamma + \bar T' * \Gamma + T' * \bar T' + \Gamma * \bar\Gamma\\
&\quad +\Gamma * B^\circ + \bar Z(\Gamma) + Z(B^\circ) + \mathcal{O}(Z(g)) + \mathcal{O}(\bar Z(g))
\end{aligned}
$$

and then we obtain

$$
\begin{aligned}
&-\nabla_{\bar j}\nabla_i(-Q^7 - Q^8 + BT' + \bar Z(T'))_{k\bar l}\\
&= \bar\nabla\nabla B^\circ * B^\circ + \nabla B^\circ * \bar\nabla B^\circ + \bar\nabla\nabla B^\circ * T' + \nabla B^\circ * \bar\nabla T' + B^\circ * \bar\nabla\nabla T'\\
&\quad +\bar\nabla\nabla B^\circ * \bar T' + \nabla B^\circ * \bar\nabla \bar T' + \bar\nabla B^\circ * \nabla\bar T' + B^\circ * \bar\nabla\nabla\bar T' + \bar\nabla\nabla T' * \bar\Gamma\\
&\quad +\nabla T' * \bar\nabla\bar\Gamma + \bar\nabla T' * \nabla\bar\Gamma + T' * \bar\nabla\nabla\bar\Gamma + \bar\nabla\nabla\bar T' * \Gamma + \nabla\bar T' * \bar\nabla\Gamma\\
&\quad +\bar\nabla\bar T' * \nabla\Gamma + \bar T' * \bar\nabla\nabla\Gamma + \bar\nabla\nabla T' * \bar T' + \nabla T' * \bar\nabla\bar T' + \bar\nabla T' * \nabla\bar T'\\
&\quad +T' * \bar\nabla\nabla\bar T' + \bar\nabla\nabla\Gamma * \bar\Gamma + \nabla\Gamma * \bar\nabla\bar\Gamma + \bar\nabla\Gamma * \nabla\bar\Gamma + \Gamma * \bar\nabla\nabla\bar\Gamma + \bar\nabla\nabla\Gamma * B^\circ\\
&\quad +\nabla\Gamma * \bar\nabla B^\circ + \bar\nabla\Gamma * \nabla B^\circ + \Gamma * \bar\nabla\nabla B^\circ\\
&\quad +\bar\nabla\nabla\bar Z(\Gamma) + \bar\nabla\nabla\bar Z(B^\circ) + \bar\nabla\nabla\mathcal{O}(Z(g)) + \bar\nabla\nabla\mathcal{O}(\bar Z(g)).
\end{aligned}
$$

Hence, by applying Lemmas 3.1, 3.4, we have the following estimate since we may assume that the quantities $|T'|^2_{C^0(g(t))}$, $|\nabla T'|_{C^0(g(t))}$ and $|R|_{C^0(g(t))}$ are uniformly bounded on $[0,\tau)$ for any $0<\tau<\tau_{\max}<\infty$ (see Remark 1.1),

$$
\begin{aligned}
&\left|\nabla_{\bar j}\nabla_i(-Q^7 - Q^8 + BT' + \bar Z(T'))_{k\bar l}\right|_{g(t)}\\
&= \Big|\bar\nabla\nabla B^\circ * B^\circ + \nabla B^\circ * \bar\nabla B^\circ + \bar\nabla\nabla B^\circ * T' + \nabla B^\circ * \bar\nabla T'\\
&\quad +B^\circ * \bar\nabla\nabla T' + \bar\nabla\nabla B^\circ * \bar T' + \nabla B^\circ * \bar\nabla\bar T' + \bar\nabla B^\circ * \nabla\bar T' + B^\circ * \bar\nabla\nabla\bar T'\\
&\quad +\bar\nabla\nabla T' * \bar\Gamma + \nabla T' * \bar\nabla\bar\Gamma + \bar\nabla T' * \nabla\bar\Gamma + T' * \bar\nabla\nabla\bar\Gamma + \bar\nabla\nabla\bar T' * \Gamma\\
&\quad +\nabla\bar T' * \bar\nabla\Gamma + \bar\nabla\bar T' * \nabla\Gamma + \bar T' * \bar\nabla\nabla\Gamma + \bar\nabla\nabla T' * \bar T' + \nabla T' * \bar\nabla\bar T'\\
&\quad +\bar\nabla T' * \nabla\bar T' + T' * \bar\nabla\nabla\bar T' + \bar\nabla\nabla\Gamma * \bar\Gamma + \nabla\Gamma * \bar\nabla\bar\Gamma + \bar\nabla\Gamma * \nabla\bar\Gamma + \Gamma * \bar\nabla\nabla\bar\Gamma\\
&\quad +\bar\nabla\nabla\Gamma * B^\circ + \nabla\Gamma * \bar\nabla B^\circ + \bar\nabla\Gamma * \nabla B^\circ + \Gamma * \bar\nabla\nabla B^\circ + \bar\nabla\nabla\bar Z(\Gamma)\\
&\quad +\bar\nabla\nabla\bar Z(B^\circ) + \bar\nabla\nabla\mathcal{O}(Z(g)) + \bar\nabla\nabla\mathcal{O}(\bar Z(g))\Big|_{g(t)}\\
&\le C'R' + \left|\bar\nabla\nabla\mathcal{O}(Z(g)) + \bar\nabla\nabla\mathcal{O}(\bar Z(g))\right|_{g(t)} \qquad (3.5)
\end{aligned}
$$

for some uniform constant $C'>0$, where R' is the time dependent tensor field defined by $R'_{i\bar j k\bar l} := g(t)_{i\bar j}g(t)_{k\bar l}$.

4. Preservation of Curvature Conditions

Remark 4.1. As we confirm in Remark 1.1, we may assume that the quantities $|T'|^2_{C^0(g(t))}$, $|\nabla T'|_{C^0(g(t))}$ and $|R|_{C^0(g(t))}$ are uniformly bounded on $[0,\tau)$ for any $0<\tau<\tau_{\max}<\infty$. We define

$$
K_0 := \sup_{M\times\{0\}}\left(|R|_{g(0)} + |T'|^2_{g(0)} + |\nabla T'|_{g(0)}\right)
$$

and choose sufficiently large $K \gg K_0$ so that $\tau = K^{-1} < \tau_{\max} < \infty$ and

$$\sup_{M\times[0,\tau]} (|R|_{g(t)} + |T'|^2_{g(t)} + |\nabla T'|_{g(t)}) \le 2K_0. \tag{4.1}$$

Proposition 4.1. *Suppose that $(M, J, g(t))$ is a solution to the almost Hermitian curvature flow such that the initial metric $g(0) = g_0$ has the Griffiths non-positive Chern curvature. There exist $\tau > 0$ and $K > 0$ such that for all $t \in [0, \tau]$, $R = R(g(t))$ satisfies the following conditions:*

(i) $P \le 0$;
(ii) $|R_{u\bar{v}x\bar{x}}|^2 \le (1 + Kt)P_{u\bar{u}}P_{v\bar{v}}$ *for all* x, u, $v \in T^{1,0}M$, $|x| = 1$.

We consider $R^\varepsilon_{i\bar{j}k\bar{l}} := R_{i\bar{j}k\bar{l}} - \varepsilon B_{i\bar{j}k\bar{l}}$, where $B_{i\bar{j}k\bar{l}} = g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}$ and ε is a sufficiently small real number. The result of Proposition 4.1 follows directly as a consequence of the following Lemma by letting $\varepsilon \to 0$.

Lemma 4.1. *Under the assumption of Proposition 4.1, there exist $\tau > 0$ and $K > 0$ such that for any sufficiently small $\varepsilon > 0$, and for any $t \in [0, \tau]$, the following hold.*

(i) $P^\varepsilon_{i\bar{j}} < -\varepsilon e^{-Kt} g_{i\bar{j}}$, *where we put* $P^\varepsilon_{i\bar{j}} := g^{k\bar{l}} R^\varepsilon_{i\bar{j}k\bar{l}}$;
(ii) $|R^\varepsilon_{u\bar{v}x\bar{x}}|^2 < (1 + Kt)P^\varepsilon_{u\bar{u}}P^\varepsilon_{v\bar{v}}$ *for all* x, u, $v \in T^{1,0}M$, $|x| = 1$.

We firstly show that $g(0) = g_0$ satisfies the assumptions in Lemma 4.1.

Lemma 4.2. *Under the assumption of Proposition 4.1, $R^\varepsilon(g_0)$ satisfies for any sufficiently small $\varepsilon > 0$,*

(i) $P^\varepsilon(g_0)_{i\bar{j}} < -\varepsilon(g_0)_{i\bar{j}}$;
(ii) $|R^\varepsilon(g_0)_{u\bar{v}x\bar{x}}|^2_{g_0} < P(g_0)^\varepsilon_{u\bar{u}}P(g_0)^\varepsilon_{v\bar{v}}$ *for all* x, u, $v \in T^{1,0}M$, $|x| = 1$.

Proof. (i) follows from that $P(g_0) \le 0$ and $g_0^{k\bar{l}} B(g_0)_{i\bar{j}k\bar{l}} = (n + 1)(g_0)_{i\bar{j}}$. Next we show (ii). For a fixed $x \in T^{1,0}M$, since $R^\varepsilon(g_0)_{i\bar{j}x\bar{x}}$ is almost Hermitian form, we may choose eigenvectors $\{e_i\}_{i=1}^n$ such that $R^\varepsilon(g_0)_{i\bar{j}x\bar{x}} = \lambda_i\delta_{ij}$, where $\lambda_i < 0$ since we have assumed that the initial metric g_0 has the Griffiths non-positive Chern curvature. Hence, for $u = \sum_{i=1}^n u^i e_i$ and $v = \sum_{i=1}^n v^i e_i$,

$$\begin{aligned} |R^\varepsilon(g_0)_{u\bar{v}x\bar{x}}|^2_{g_0} &= \left|\sum_{i=1}^n \lambda_i u^i v^{\bar{i}}\right|^2_{g_0} \\ &\le \left(\sum_{i=1}^n \lambda_i |u^i|^2_{g_0}\right)\left(\sum_{i=1}^n \lambda_i |v^i|^2_{g_0}\right) \\ &= R^\varepsilon(g_0)_{u\bar{u}x\bar{x}} R^\varepsilon(g_0)_{v\bar{v}x\bar{x}} \\ &< P^\varepsilon(g_0)_{u\bar{u}} P^\varepsilon(g_0)_{v\bar{v}}. \end{aligned}$$

□

Proof of Lemma 4.1. As in Remark 4.1, we choose sufficiently large $K \gg K_0$ so that $\tau = K^{-1} < \tau_{\max}$ and satisfying (4.1). If conditions (i), (ii) are true on $[0,\tau]$, then we are done. Let $t_0 \in (0,\tau]$ be the first time such that one of conditions (i), (ii) fails. First, we assume that (i) is true on $[0,t_0)$ and fails at $t=t_0$. Then there exist $p_0 \in M$ and $X_0 \in T^{1,0}_{p_0}M$ with $|X_0|_{g(t_0)} = 1$ such that $P^\varepsilon(g(t_0))_{X_0\bar{X}_0} = -\varepsilon e^{-Kt_0} g(t_0)_{X_0\bar{X}_0} = -\varepsilon e^{-Kt_0}$. Moreover, for all $p \in M$, $t \in [0,t_0]$, $Y,U,V \in T^{1,0}_pM$ with $|Y|=1$,

$$P^\varepsilon_{Y\bar{Y}} \le -\varepsilon e^{-Kt} g_{Y\bar{Y}} = -\varepsilon e^{-Kt}, \quad |R^\varepsilon_{U\bar{V}Y\bar{Y}}|^2 < (1+Kt) P^\varepsilon_{U\bar{U}} P^\varepsilon_{V\bar{V}}.$$

Using polarization as in [7,8] and (4.1) to infer that for sufficiently small $\varepsilon > 0$, any $e_k, e_l \in T^{1,0}_pM$ with unit 1 and $e_i, e_j \in T^{1,0}_pM$ for all $p \in M$,

$$|R^\varepsilon_{i\bar{j}k\bar{l}}|^2 \le C_n P^\varepsilon_{i\bar{i}} P^\varepsilon_{j\bar{j}}, \quad |R^\varepsilon_{i\bar{j}k\bar{l}}|^2 \le C_n K_0 |P^\varepsilon_{i\bar{i}}|.$$

We consider

$$A_{i\bar{j}} := P_{i\bar{j}} + \varepsilon(e^{-Kt} - (n+1)) g_{i\bar{j}} = P^\varepsilon_{i\bar{j}} + \varepsilon e^{-Kt} g_{i\bar{j}},$$

which satisfies $A(X_0,\bar{X}_0) = 0$ and $A(Y,\bar{Y}) \le 0$ for all $Y \in T^{1,0}_pM$, $p \in M$. We may assume that $|X_0|_{g(t_0)} = 1$ by rescaling. We extend X_0 locally to a vector field around (p_0,t_0) such that at (p_0,t_0),

$$\nabla_{\bar{q}} X^p = 0, \quad \nabla_p X^q = T^q_{pl} X^l.$$

Then $A(X,\bar{X})$ locally defines a function and satisfies

$$\Box A(X,\bar{X}) \ge 0,$$

where $\Box := (\frac{\partial}{\partial t} - \Delta)$. At (p_0,t_0), we have

$$\begin{aligned}
\frac{\partial}{\partial t} A(X,\bar{X}) &= \Big(\frac{\partial}{\partial t} A_{i\bar{j}}\Big) X^i X^{\bar{j}} + A_{i\bar{j}} \Big(\frac{\partial}{\partial t} X^i \cdot X^{\bar{j}} + X^i \frac{\partial}{\partial t} X^{\bar{j}}\Big) \\
&= \Big(\frac{\partial}{\partial t} P_{i\bar{j}} - \varepsilon(e^{-Kt} - (n+1))(S + Q^7 + Q^8 - BT' - \bar{Z}(T'))_{i\bar{j}} - \varepsilon K e^{-Kt} g_{i\bar{j}}\Big) X^i X^{\bar{j}} \\
&\quad + A_{i\bar{j}} \Big(\frac{\partial}{\partial t} X^i X^{\bar{j}} + X^i \frac{\partial}{\partial t} X^{\bar{j}}\Big) \\
&\le \Big(\frac{\partial}{\partial t} P_{i\bar{j}}\Big) X^i X^{\bar{j}} - \frac{1}{2} \varepsilon K e^{-Kt},
\end{aligned} \tag{4.2}$$

where we used (4.1) and the fact that for any $Y \in T^{1,0}_{p_0}M$, $A_{X_0\bar{Y}} = 0$. Choosing a local unitary $(1,0)$-frame with respect to $g(t_0)$ around a point p_0, we have $g_{i\bar{j}}(t_0) = \delta_{i\bar{j}}$, $Z_r(g_{i\bar{j}}(t_0)) = 0$ at p_0. Then we obtain

$$\begin{aligned}
\Delta A(X,\bar{X}) &= \frac{1}{2} g^{r\bar{s}} (\nabla_r \nabla_{\bar{s}} + \nabla_{\bar{s}} \nabla_r)(A_{i\bar{j}} X^i X^{\bar{j}}) \\
&= \Delta A_{i\bar{j}} \cdot X^i X^{\bar{j}} + A_{i\bar{j}} X^i \Delta X^{\bar{j}} + A_{i\bar{j}} X^{\bar{j}} \Delta X^i \\
&\quad + \nabla_r A_{i\bar{j}} \nabla_{\bar{r}} X^i \cdot X^{\bar{j}} + \nabla_{\bar{r}} A_{i\bar{j}} \nabla_r X^i \cdot X^{\bar{j}} \\
&\quad + \nabla_r A_{i\bar{j}} \cdot X^i \nabla_{\bar{r}} X^{\bar{j}} + \nabla_{\bar{r}} A_{i\bar{j}} \cdot X^i \nabla_r X^{\bar{j}} \\
&\quad + A_{i\bar{j}} \nabla_r X^i \nabla_{\bar{r}} X^{\bar{j}} + A_{i\bar{j}} \nabla_{\bar{r}} X^i \nabla_r X^{\bar{j}} \\
&= \Delta P_{i\bar{j}} \cdot X^i X^{\bar{j}} + \nabla_r P_{i\bar{j}} \nabla_{\bar{r}} X^i \cdot X^{\bar{j}} + \nabla_{\bar{r}} P_{i\bar{j}} \nabla_r X^i \cdot X^{\bar{j}}
\end{aligned}$$

$$+\nabla_r P_{i\bar{j}} \cdot X^i \nabla_{\bar{r}} X^{\bar{j}} + \nabla_{\bar{r}} P_{i\bar{j}} \cdot X^i \nabla_r X^{\bar{j}}$$
$$+A_{i\bar{j}} \nabla_r X^i \nabla_{\bar{r}} X^{\bar{j}} + A_{i\bar{j}} \nabla_{\bar{r}} X^i \nabla_r X^{\bar{j}}, \tag{4.3}$$

where we used $\nabla g = 0$ and the fact that for any $Y \in T^{1,0}_{p_0} M$, $A_{X_0 \bar{Y}} = 0$ for the terms involve $A(\Delta X, \bar{X})$ or its conjugate.

By combining them, we obtain by using the formula (3.3), and applying (4.2), (4.3),

$$\begin{aligned}
\Box A(X,\bar{X}) &\le X^i X^{\bar{j}} \Box P_{i\bar{j}} - \frac{1}{2}\varepsilon K e^{-Kt} \\
&\quad - g^{r\bar{s}}(\nabla_r P_{i\bar{j}} \nabla_{\bar{s}} X^i \cdot X^{\bar{j}} + \nabla_{\bar{s}} P_{i\bar{j}} \nabla_r X^i \cdot X^{\bar{j}} + \nabla_r P_{i\bar{j}} X^i \nabla_{\bar{s}} X^{\bar{j}} + \nabla_{\bar{s}} P_{i\bar{j}} X^i \nabla_r X^{\bar{j}}) \\
&\quad - g^{r\bar{s}}(P_{i\bar{j}} \nabla_r X^i \nabla_{\bar{s}} X^{\bar{j}} + P_{i\bar{j}} \nabla_{\bar{s}} X^i \nabla_r X^{\bar{j}}) \\
&\quad - \varepsilon(e^{-Kt} - (n+1)) g^{r\bar{s}}(\nabla_r X^i \nabla_{\bar{s}} X^{\bar{j}} + \nabla_{\bar{s}} X^i \nabla_r X^{\bar{j}}) \\
&= X^i X^{\bar{j}} g^{r\bar{s}}(T^{\bar{q}}_{\bar{s}\bar{j}} \nabla_r P_{i\bar{q}} + T^p_{ri} \nabla_{\bar{s}} P_{p\bar{j}} + T^p_{ri} T^{\bar{q}}_{\bar{s}\bar{j}} P_{p\bar{q}} + R_{i\bar{j}r}{}^p P_{p\bar{s}}) + E_{i\bar{j}} X^i X^{\bar{j}} - \frac{1}{2}\varepsilon K e^{-Kt} \\
&\quad - g^{r\bar{s}}(\nabla_{\bar{s}} P_{i\bar{j}} \cdot T^i_{rk} X^k X^{\bar{j}} + \nabla_r P_{i\bar{j}} \cdot X^i T^{\bar{j}}_{\bar{s}\bar{l}} X^{\bar{l}} + P_{i\bar{j}} T^i_{rk} X^k T^{\bar{j}}_{\bar{s}\bar{l}} X^{\bar{l}}) \\
&\quad - \varepsilon(e^{-Kt} - (n+1)) g^{r\bar{s}} T^i_{rk} X^k T^{\bar{j}}_{\bar{s}\bar{l}} X^{\bar{l}} \\
&= g^{r\bar{s}} R_{i\bar{j}r}{}^p X^i X^{\bar{j}} P_{p\bar{s}} + E_{i\bar{j}} X^i X^{\bar{j}} - \frac{1}{2}\varepsilon K e^{-Kt} - \varepsilon(e^{-Kt} - (n+1)) g^{r\bar{s}} T^i_{rk} X^k T^{\bar{j}}_{\bar{s}\bar{l}} X^{\bar{l}}.
\end{aligned}$$

We have the following estimate from the estiamte (3.5),

$$\left| \nabla_{\bar{j}} \nabla_i (-Q^7 - Q^8 + BT' + \bar{Z}(T'))_{k\bar{l}} \right|_{g(t)} \le C' R' \tag{4.4}$$

for a uniform constant $C' > 0$, where we used that $Z(g) = 0$ with a local unitary frame.

As $K \gg K_0$ sufficiently large, since we may assume that the quantities $|T'|^2_{C^0(g(t))}$, $|\nabla T'|_{C^0(g(t))}$ and $|R|_{C^0(g(t))}$ are uniformly bounded on $[0,\tau]$, we have the estimates in Lemma 3.1, 3.4, 3.5, and the estimate (4.4),

$$0 \le \Box A(X,\bar{X}) \le R_{X\bar{X}k}{}^p P_p{}^k - \frac{1}{4}\varepsilon K \le -\frac{1}{8}\varepsilon K,$$

which is a contrsdiction.

Next, we suppose that (ii) is not true at $t = t_0$. Then, there exist $p_0 \in M$, $X_0, U_0, V_0 \in T^{1,0}_{p_0} M$ with $|X_0|_{g(t_0)} = 1$ such that $|R^{\varepsilon}_{U_0 \bar{V}_0 X_0 \bar{X}_0}|^2_{g(t_0)} = (1 + Kt_0) P^{\varepsilon}_{U_0 \bar{U}_0} P^{\varepsilon}_{V_0 \bar{V}_0}$. By rescaling, we may assume that $|U_0|_{g(t_0)} = |V_0|_{g(t_0)} = 1$. For all $(p,t) \in M \times [0,t_0]$, $X, U, V \in T^{1,0}_p M$ with $|X| = 1$,

$$P^{\varepsilon}_{X\bar{X}} < -\varepsilon e^{-Kt} g_{X\bar{X}} = -\varepsilon e^{-Kt}, \quad |R^{\varepsilon}_{U\bar{V}X\bar{X}}|^2 \le (1+Kt) P^{\varepsilon}_{U\bar{U}} P^{\varepsilon}_{V\bar{V}}. \tag{4.5}$$

For sufficiently small $\varepsilon > 0$, and for any $e_k, e_l \in T^{1,0}_p M$ for all $p \in M$ with unit 1,

$$|R^{\varepsilon}_{i\bar{j}k\bar{l}}|^2 \le C_n P^{\varepsilon}_{i\bar{i}} P^{\varepsilon}_{j\bar{j}}, \quad |R^{\varepsilon}_{i\bar{j}k\bar{l}}|^2 \le C_n K_0 |P^{\varepsilon}_{i\bar{i}}|. \tag{4.6}$$

As in the previous case, we extend X_0, U_0 to a local vector field X, U around (p_0, t_0). We extend X_0, U_0, V_0 to X, U and V around (p_0, t_0) such that at

(p_0, t_0),

$$\nabla_{\bar{s}} U^r = 0, \quad \nabla_p U^r = T^r_{pq} U^q,$$
$$\nabla_{\bar{s}} V^r = 0, \quad \nabla_p V^r = T^r_{pq} V^q,$$
$$\nabla_{\bar{s}} X^r = 0, \quad \nabla_p X^r = 0, \quad \Box X^r = 0.$$

Notice that we have at (p_0, t_0),

$$\begin{aligned}\Box U^r &= \frac{\partial}{\partial t} U^r - \frac{1}{2} g^{k\bar{l}} (\nabla_k \nabla_{\bar{l}} + \nabla_{\bar{l}} \nabla_k) U^r \\ &= \frac{\partial}{\partial t} U^r - \frac{1}{2} g^{k\bar{l}} \nabla_{\bar{l}} (T^r_{ks} U^s) \\ &= \frac{\partial}{\partial t} U^r - \frac{1}{2} g^{k\bar{l}} (\nabla_{\bar{l}} T^r_{ks} \cdot U^s + T^r_{ks} \nabla_{\bar{l}} U^s) \\ &= \frac{\partial}{\partial t} U^r - \frac{1}{2} \nabla_{\bar{k}} T^r_{ks} \cdot U^s,\end{aligned}$$

and note that we get

$$|\Box U^r|_{g(t_0)} \leq \left| \frac{\partial}{\partial t} U^r \right|_{g(t_0)} + \frac{1}{2} |\nabla_{\bar{k}} T^r_{ks}|_{g(t_0)} |U^s|_{g(t_0)} \leq C$$

for some uniform constant $C > 0$, where we have used that (4.1) and Lemma 3.4.

In particular, $|X|_{g(t_0)} = 1$. Then the function

$$F(x,t) = g^{-2}_{X\bar{X}} |R^{\varepsilon}_{U\bar{V}X\bar{X}}|^2 - (1 + Kt) P^{\varepsilon}_{U\bar{U}} P^{\varepsilon}_{V\bar{V}}$$

attains its local maximum at (p_0, t_0) and therefore satisfies

$$\Box F|_{(p_0,t_0)} \geq 0.$$

We calculate by making use of (3.3),

$$\begin{aligned}\Box P^{\varepsilon}_{U\bar{U}} &= \Box P^{\varepsilon}_{i\bar{j}} \cdot U^i U^{\bar{j}} + P^{\varepsilon}_{i\bar{j}} U^i \Box U^{\bar{j}} + P^{\varepsilon}_{i\bar{j}} \Box U^i \cdot U^{\bar{j}} \\ &\quad - g^{r\bar{s}} P^{\varepsilon}_{i\bar{j}} \nabla_r U^i \nabla_{\bar{s}} U^{\bar{j}} - g^{r\bar{s}} \nabla_r P_{i\bar{j}} \cdot U^i \nabla_{\bar{s}} U^{\bar{j}} - g^{r\bar{s}} \nabla_{\bar{s}} P_{i\bar{j}} \nabla_r U^i \cdot U^{\bar{j}} \\ &= \Box P_{i\bar{j}} \cdot U^i U^{\bar{j}} + \varepsilon(n+1)(S + Q^7 + Q^8 - BT' - \bar{Z}(T'))_{i\bar{j}} U^i U^{\bar{j}} \\ &\quad + P^{\varepsilon}_{i\bar{j}} U^i \Box U^{\bar{j}} + P^{\varepsilon}_{i\bar{j}} U^{\bar{j}} \Box U^i + \varepsilon(n+1) g^{r\bar{s}} g_{i\bar{j}} T^i_{rp} T^{\bar{j}}_{\bar{s}\bar{q}} U^p U^{\bar{q}} \\ &\quad - g^{r\bar{s}} P_{i\bar{j}} T^i_{rp} T^{\bar{j}}_{\bar{s}\bar{q}} U^p U^{\bar{q}} - g^{r\bar{s}} \nabla_r P_{i\bar{j}} \cdot T^{\bar{j}}_{\bar{s}\bar{q}} U^i U^{\bar{q}} - g^{r\bar{s}} \nabla_{\bar{s}} P_{i\bar{j}} \cdot T^i_{rp} U^p U^{\bar{j}} \\ &= R_{i\bar{j}k}{}^p U^i U^{\bar{j}} P^k_p + P_{i\bar{j}} U^{\bar{j}} \Box U^i + P_{i\bar{j}} U^i \Box U^{\bar{j}} - 2\varepsilon(n+1) U^i \Box U^{\bar{i}} \\ &\quad + E_{i\bar{j}} U^i U^{\bar{j}} + \varepsilon(n+1) g^{r\bar{s}} g_{i\bar{j}} T^i_{rk} U^k T^{\bar{j}}_{\bar{s}\bar{l}} U^{\bar{l}} \\ &\quad + \varepsilon(n+1)(S + Q^7 + Q^8 - BT' - \bar{Z}(T'))_{i\bar{j}} U^i U^{\bar{j}}.\end{aligned} \tag{4.7}$$

Similarly we have

$$\begin{aligned}\Box P^{\varepsilon}_{V\bar{V}} &= R_{i\bar{j}k}{}^p V^i V^{\bar{j}} P^k_p + P_{i\bar{j}} V^{\bar{j}} \Box V^i + P_{i\bar{j}} V^i \Box V^{\bar{j}} - 2\varepsilon(n+1) V^i \Box V^{\bar{i}} \\ &\quad + E_{i\bar{j}} V^i V^{\bar{j}} + \varepsilon(n+1) g^{r\bar{s}} g_{i\bar{j}} T^i_{rk} V^k T^{\bar{j}}_{\bar{s}\bar{l}} V^{\bar{l}} \\ &\quad + \varepsilon(n+1)(S + Q^7 + Q^8 - BT' - \bar{Z}(T'))_{i\bar{j}} V^i V^{\bar{j}}.\end{aligned} \tag{4.8}$$

By combining these and chossing $K \gg K_0$ sufficiently large and $\tau = K^{-1}$, we obtain by applying (4.1), (4.5)–(4.8),

$$\begin{aligned}\Box[(1+Kt)P^{\varepsilon}_{U\bar{U}}P^{\varepsilon}_{V\bar{V}}] &\geq (1+Kt)P^{\varepsilon}_{U\bar{U}}(R^{\varepsilon}_{V\bar{V}k}{}^{p}P_{p}{}^{k}) + (1+Kt)P^{\varepsilon}_{V\bar{V}}(R^{\varepsilon}_{U\bar{U}k}{}^{p}P_{p}{}^{k}) \\ &\quad -2(1+Kt)\mathrm{Re}(g^{r\bar{s}}\nabla_r P^{\varepsilon}_{U\bar{U}}\nabla_{\bar{s}}P^{\varepsilon}_{V\bar{V}}) + \frac{1}{2}KP^{\varepsilon}_{U\bar{U}}P^{\varepsilon}_{V\bar{V}} \\ &\geq -2(1+Kt)\mathrm{Re}(g^{r\bar{s}}\nabla_r P^{\varepsilon}_{U\bar{U}}\nabla_{\bar{s}}P^{\varepsilon}_{V\bar{V}}) + \frac{1}{4}KP^{\varepsilon}_{U\bar{U}}P^{\varepsilon}_{V\bar{V}},\end{aligned} \tag{4.9}$$

where we used that the quantities $|T'|^2_{C^0(g(t))}$, $|\nabla T'|_{C^0(g(t))}$ and $|R|_{C^0(g(t))}$ are uniformly bounded on $[0,\tau]$, and we applied Lemmas 3.1, 3.4, 3.5, the estiamte (4.4).

We have by using (3.2),

$$\begin{aligned}\Box R^{\varepsilon}_{U\bar{V}X\bar{X}} &= \Box R_{i\bar{j}k\bar{l}}\cdot U^iV^{\bar{j}}X^kX^{\bar{l}} - \varepsilon\Box B_{i\bar{j}k\bar{l}}\cdot U^iV^{\bar{j}}X^kX^{\bar{l}} \\ &\quad + R^{\varepsilon}_{i\bar{j}k\bar{l}}\Box U^i\cdot V^{\bar{j}}X^kX^{\bar{l}} + R^{\varepsilon}_{i\bar{j}k\bar{l}}U^i\Box V^{\bar{j}}\cdot X^kX^{\bar{l}} \\ &\quad + R^{\varepsilon}_{i\bar{j}k\bar{l}}U^iV^{\bar{j}}(\Box X^k\cdot X^{\bar{l}} + X^k\Box X^{\bar{l}}) \\ &\quad - g^{r\bar{s}}\nabla_r R_{i\bar{j}k\bar{l}}(U^i\nabla_{\bar{s}}V^{\bar{j}}\cdot X^kX^{\bar{l}} + U^iV^{\bar{j}}X^k\nabla_{\bar{s}}X^{\bar{l}}) \\ &\quad - g^{r\bar{s}}\nabla_{\bar{s}} R_{i\bar{j}k\bar{l}}(\nabla_r U^i\cdot V^{\bar{j}}X^kX^{\bar{l}} + U^iV^{\bar{j}}\nabla_r X^k\cdot X^{\bar{l}}) \\ &\quad - R^{\varepsilon}_{i\bar{j}k\bar{l}}(\nabla_r U^i\nabla_{\bar{s}}V^{\bar{j}}\cdot X^kX^{\bar{l}} + U^i\nabla_{\bar{s}}V^{\bar{j}}\nabla_r X^k\cdot X^{\bar{l}} \\ &\quad + \nabla_r U^i\cdot V^{\bar{j}}X^k\nabla_{\bar{s}}X^{\bar{l}} + U^iV^{\bar{j}}\nabla_r X^k\nabla_{\bar{s}}X^{\bar{l}}) \\ &= g^{r\bar{s}}(R_{U\bar{V}r}{}^{p}R_{p\bar{s}X\bar{X}} + R_{r\bar{V}X}{}^{p}R_{U\bar{s}p\bar{X}} - R_{r\bar{V}p\bar{X}}R_{U\bar{s}X}{}^{p}) \\ &\quad - R_{U\bar{V}X}{}^{r}(S+Q^7+Q^8-BT'-\bar{Z}(T'))_{r\bar{X}} + R_{i\bar{V}X\bar{X}}\Box U^i \\ &\quad + R_{U\bar{j}X\bar{X}}\Box V^{\bar{j}} + \varepsilon((S+Q^7+Q^8-BT'-\bar{Z}(T'))_{U\bar{V}}g_{X\bar{X}} \\ &\quad + g_{U\bar{V}}(S+Q^7+Q^8-BT'-\bar{Z}(T'))_{X\bar{X}} \\ &\quad + (S+Q^7+Q^8-BT'-\bar{Z}(T'))_{U\bar{X}}g_{X\bar{V}} \\ &\quad + g_{U\bar{X}}(S+Q^7+Q^8-BT'-\bar{Z}(T'))_{X\bar{V}}) \\ &\quad + \varepsilon B_{i\bar{j}X\bar{X}}T^i_{rU}T^{\bar{j}}_{\bar{s}\bar{V}} - \varepsilon(B_{i\bar{V}X\bar{X}}\Box U^i + B_{U\bar{j}X\bar{X}}\Box V^{\bar{j}}) + E_{U\bar{V}X\bar{X}},\end{aligned} \tag{4.10}$$

where we have used that

$$R^{\varepsilon}_{i\bar{j}k\bar{l}}\nabla_r U^i\nabla_{\bar{s}}V^{\bar{j}}\cdot X^kX^{\bar{l}} = T^p_{rU}T^{\bar{q}}_{\bar{s}\bar{V}}R_{p\bar{q}X\bar{X}} - \varepsilon B_{i\bar{j}X\bar{X}}T^i_{rU}T^{\bar{j}}_{\bar{s}\bar{V}}.$$

Similarly, we have

$$\begin{aligned}\Box R^{\varepsilon}_{V\bar{U}X\bar{X}} &= g^{r\bar{s}}(R_{V\bar{U}r}{}^{p}R_{p\bar{s}X\bar{X}} + R_{r\bar{U}X}{}^{p}R_{V\bar{s}p\bar{X}} - R_{r\bar{U}p\bar{X}}R_{V\bar{s}X}{}^{p}) \\ &\quad - R_{V\bar{U}X}{}^{r}(S+Q^7+Q^8-BT'-\bar{Z}(T'))_{r\bar{X}} + R_{i\bar{U}X\bar{X}}\Box V^i + R_{V\bar{j}X\bar{X}}\Box U^{\bar{j}} \\ &\quad + \varepsilon((S+Q^7+Q^8-BT'-\bar{Z}(T'))_{V\bar{U}}g_{X\bar{X}} + g_{V\bar{U}}(S+Q^7+Q^8-BT' \\ &\quad - \bar{Z}(T'))_{X\bar{X}} + (S+Q^7+Q^8-BT'-\bar{Z}(T'))_{V\bar{X}}g_{X\bar{U}}\end{aligned}$$

$$+g_{V\bar{X}}(S+Q^7+Q^8-BT'-\bar{Z}(T'))_{X\bar{U}})+\varepsilon B_{i\bar{j}X\bar{X}}T^i_{rV}T^{\bar{j}}_{\bar{s}\bar{U}}$$
$$-\varepsilon(B_{i\bar{U}X\bar{X}}\Box V^i+B_{V\bar{j}X\bar{X}}\Box U^{\bar{j}})+E_{V\bar{U}X\bar{X}}. \tag{4.11}$$

By combining these, we obtain by applying (4.1), (4.4), (4.5), (4.9) and (4.10),

$$\Box(R^\varepsilon_{U\bar{V}X\bar{X}}R^\varepsilon_{V\bar{U}X\bar{X}})\leq -|\nabla R^\varepsilon_{U\bar{V}X\bar{X}}|^2-|\bar{\nabla}R^\varepsilon_{U\bar{V}X\bar{X}}|^2+C_nK_0P^\varepsilon_{U\bar{U}}P^\varepsilon_{V\bar{V}},$$

where we used that the quantities $|T'|^2_{C^0(g(t))}$, $|\nabla T'|_{C^0(g(t))}$ and $|R|_{C^0(g(t))}$ are uniformly bounded on $[0,\tau]$, and applied the estimates in Lemmas 3.1, 3.4, 3.5, the estimate (4.4).

Therefore, at (p_0,t_0),

$$\begin{aligned}\Box F\leq\ & 2(1+Kt)\mathrm{Re}(g^{r\bar{s}}\nabla_rP^\varepsilon_{U\bar{U}}\nabla_{\bar{s}}P^\varepsilon_{V\bar{V}})\\ &-|\nabla R^\varepsilon_{U\bar{V}X\bar{X}}|^2-|\bar{\nabla}R^\varepsilon_{U\bar{V}X\bar{X}}|^2\\ &-\frac{1}{8}KP^\varepsilon_{U\bar{U}}P^\varepsilon_{V\bar{V}}+2(S+Q^7+Q^8-BT'-\bar{Z}(T'))_{X\bar{X}}|R^\varepsilon_{U\bar{V}X\bar{X}}|^2\end{aligned}$$

By using that $\nabla F=0$ and $F=0$ at (p_0,t_0), one may conclude that

$$2(1+Kt)\mathrm{Re}(g^{r\bar{s}}\nabla_rP^\varepsilon_{U\bar{U}}\nabla_{\bar{s}}P^\varepsilon_{V\bar{V}})\leq|\nabla R^\varepsilon_{U\bar{V}X\bar{X}}|^2+|\bar{\nabla}R^\varepsilon_{U\bar{V}X\bar{X}}|^2.$$

Using (4.1) and $F(p_0,t_0)=0$, we deduce that

$$2(S+Q^7+Q^8-BT'-\bar{Z}(T'))_{X\bar{X}}|R^\varepsilon_{U\bar{V}X\bar{X}}|^2\leq C_nK_0P^\varepsilon_{U\bar{U}}P^\varepsilon_{V\bar{V}}$$

and hence for sufficiently large $K>\tilde{C}_nK_0$ for some $\tilde{C}_n\gg 1$, at (p,t_0),

$$0\leq\Box F<-\frac{1}{16}KP^\varepsilon_{U\bar{U}}P^\varepsilon_{V\bar{V}},$$

which is a contradiction. □

5. Strong Maximum Principle

Theorem 5.1. *Suppose $(M,J,g(t))$ is a solution to the AHCF on $t\in[0,\tau_{\max})$ be a solution to the AHCF starting from the initial metric $g(0)=g_0$, where $\tau_{\max}$ is the finite explosion time of the AHCF. If the metric g_0 has the Griffiths non-positive Chern curvature and its first Chern–Ricci curvature is negative at some $p\in M$, then there exists $0<\tau<\tau_{\max}<\infty$ such that $P(g(t))<0$ on $(0,\tau]$. Note that under these assumptions, the metric g_0 has the quasi-negative first Chern–Ricci curvature.*

Proof. Let τ be the constant obtained in Proposition 4.1. Let $y\in M$ be a point at which the first Chern–Ricci curvature is negative. Let ϕ_0 be a smooth non-negative function such that $\phi_0(y)>0$, $\phi_0=0$ outside a neighbour of y and

$$P(g_0)+\phi_0g_0\leq 0\quad\text{on } M.$$

Let $\phi(x,t)$ be the solution to the heat equation

$$\Big(\frac{\partial}{\partial t}-\Delta_{g(t)}\Big)\phi(x,t)=0,\qquad\text{on } M\times[0,\tau];$$

$$\phi(x,0)=\phi_0.$$

By the strong maximum principle, it follows that $\phi(x,t)>0$ on $M\times(0,\tau]$. By rescaling, one may assume that $\phi(x,t)\le 1$. Let $k:=c_nK_0$ with $c_n\gg 1$. For any $\varepsilon>0$, we consider

$$A^\varepsilon=A^\varepsilon(g(t)):=P(g(t))+e^{-kt}\phi^2g(t)-\varepsilon e^{\Lambda t}g(t),$$

where Λ is some sufficiently large constant which will be determined later. We claim that $A^\varepsilon\le 0$ on $M\times[0,\tau]$. Then the result follows by letting $\varepsilon\to 0$. Note that $A^\varepsilon(g(0))<0$ on M. Suppose not, there is $t_0\in(0,\tau]$ such that for all $(p,t)\in M\times[0,t_0]$, $U\in T^{1,0}_pM$, $A^\varepsilon_{U\bar U}(p,t)\le 0$. And there exists $p_0\in M$, $V\in T^{1,0}_{p_0}M$ so that $A^\varepsilon_{V\bar V}(p_0,t_0)=0$. By rescaling, we may assume that $|V|_{g(t_0)}=1$. We extend V around (p_0,t_0) locally such that

$$\nabla_{\bar j}V^i=0,\quad \nabla_rV^i=T^i_{rk}V^k.$$

Then the function $A^\varepsilon_{V\bar V}$ attains local maximum at (p_0,t_0) and obeys

$$\Box A^\varepsilon_{V\bar V}|_{(p_0,t_0)}\ge 0. \tag{5.1}$$

Choosing a local unitary $(1,0)$-frame with respect to $g(t_0)$ around p_0 such that $g_{i\bar j}(t_0)=\delta_{i\bar j}$, then we compute

$$\begin{aligned}\Box A^\varepsilon_{V\bar V}&=\Box A^\varepsilon_{i\bar j}\cdot V^iV^{\bar j}+A^\varepsilon_{i\bar j}(V^i\Box V^{\bar j}+V^{\bar j}\Box V^i)\\&\quad-g^{r\bar s}(A^\varepsilon_{i\bar j}\nabla_rV^i\nabla_{\bar s}V^{\bar j}+\nabla_rA^\varepsilon_{i\bar j}\cdot V^i\nabla_{\bar s}V^{\bar j}+\nabla_{\bar s}A^\varepsilon_{i\bar j}\nabla_rV^i\cdot V^{\bar j})\\&=\Box A^\varepsilon_{i\bar j}\cdot V^iV^{\bar j}-A^\varepsilon_{i\bar j}T^i_{rk}V^kT^{\bar j}_{\bar r\bar l}V^{\bar l}-\nabla_rA^\varepsilon_{i\bar j}\cdot T^{\bar j}_{\bar r\bar l}V^{\bar l}V^i-\nabla_{\bar r}A^\varepsilon_{i\bar j}\cdot T^i_{rk}V^kV^{\bar j},\end{aligned} \tag{5.2}$$

where we have used that $A^\varepsilon_{V\bar U}=0$ for all $U\in T^{1,0}_{p_0}M$, hence we get that

$$A^\varepsilon(V,\Box\bar V)=A^\varepsilon(\Box V,\bar V)=0,$$

which can be seen by considering the first derivation of functions: $A^\varepsilon(V+tU,\bar V+t\bar U)$ and $A^\varepsilon(V+t\sqrt{-1}U,\bar V-t\sqrt{-1}\bar U)$ at $t=0$. From the formula (3.3), we obtain at (p_0,t_0),

$$\begin{aligned}\Box A^\varepsilon_{i\bar j}\cdot V^iV^{\bar j}&=P_{i\bar j}T^i_{rs}V^sT^{\bar j}_{\bar r\bar l}V^{\bar l}+\nabla_rP_{i\bar j}\cdot T^{\bar j}_{\bar r\bar l}V^{\bar l}V^i+\nabla_{\bar r}P_{i\bar j}\cdot T^i_{rs}V^sV^{\bar j}\\&\quad+R_{i\bar js}{}^pV^iV^{\bar j}P_p{}^s+E_{i\bar j}V^iV^{\bar j}\\&\quad-ke^{-kt}\phi^2-2|\nabla\phi|^2e^{-kt}-\phi^2e^{-kt}(S+Q^7+Q^8-BT'\\&\quad-\bar Z(T'))_{i\bar j}V^iV^{\bar j}-\varepsilon\Lambda e^{\Lambda t}+\varepsilon e^{\Lambda t}(S+Q^7+Q^8-BT'\\&\quad-\bar Z(T'))_{i\bar j}V^iV^{\bar j}.\end{aligned}$$

Combining them with Proposition 4.1 and the fact that $A^\varepsilon_{V\bar V}(p_0,t_0)=0$, which gives us that $P_{V\bar V}|_{(p_0,t_0)}=-e^{-kt}\phi^2+\varepsilon e^{\Lambda t}$, we have at (p_0,t_0), using (4.1), (5.1) and (5.2),

$$0\le\Box A^\varepsilon_{V\bar V}=(-\phi^2e^{-kt}+\varepsilon e^{\Lambda t})g_{i\bar j}T^i_{rs}V^sT^{\bar j}_{\bar r\bar l}V^{\bar l}-4\phi e^{-kt}\mathrm{Re}(\phi_{\bar r}T_{rp\bar q}V^pV^{\bar q})$$

$$
\begin{aligned}
&+R_{i\bar{j}p\bar{q}}V^iV^{\bar{j}}P^{\bar{q}p}+E_{i\bar{j}}V^iV^{\bar{j}}-ke^{-kt}\phi^2-2|\nabla\phi|^2e^{-kt}\\
&-\varepsilon\Lambda e^{\Lambda t}+P_{s\bar{l}}V^sV^{\bar{l}}(S+Q^7+Q^8-BT'-\bar{Z}(T'))_{i\bar{j}}V^iV^{\bar{j}}\\
&\leq(-k+C_nK_0)\phi^2e^{-kt}+\varepsilon e^{\Lambda t}(-\Lambda+C_nK_0),
\end{aligned}
$$

which leads a contradiction by choosing k and Λ sufficiently large, since the quantities $|T'|^2_{C^0(g(t))}$, $|\nabla T'|_{C^0(g(t))}$ and $|R|_{C^0(g(t))}$ are uniformly bounded on $[0,\tau]$, where we applied Lemmas 3.1, 3.4, 3.5, the estimate (4.4). As a result, we have shown that there exist sufficiently large constants $k>0$, $\Lambda>0$ such that for all $\varepsilon>0$, $(x,t)\in M\times[0,\tau]$,

$$
P(g(t))\leq(-e^{-kt}\phi(x,t)+\varepsilon e^{\Lambda t})g(t).
$$

In particular, by letting $\varepsilon\to 0$, we obtain $P(g(t))<0$ for $t\in(0,\tau]$. □

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Conflict of interest The authors declare that they have no conflict of interest.

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Masaya Kawamura
Department of General Education
National Institute of Technology, Kagawa College
355, Chokushi-cho
Takamatsu Kagawa761-8058
Japan
e-mail: kawamura-m@t.kagawa-nct.ac.jp

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