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Results in Mathematics

Generalized q-Bernoulli Polynomials Generated by Jackson q-Bessel Functions

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Abstract. In this paper, we introduce the polynomials $B^{(k)}_{n,\alpha}(x;q)$ generated by a function including Jackson q-Bessel functions $J^{(k)}_{\alpha}(x;q)$ $(k = 1,2,3)$, $\alpha > -1$. The cases $\alpha = \pm\frac{1}{2}$ are the q-analogs of Bernoulli and Euler's polynomials introduced by Ismail and Mansour for $(k = 1,2)$, Mansour and Al-Towalib for $(k = 3)$. We study the main properties of these polynomials, their large n degree asymptotics and give their connection coefficients with the q-Laguerre polynomials and little q-Legendre polynomials.

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1. Introduction and Preliminaries

The Bernoulli polynomials $(B_n(x))_n$ are defined by the generating function

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad \mid t \mid < 2\pi.$$

In a series of papers, Frappier [9–11] studied the generalized Bernoulli polynomials $B_{n,\alpha}(x)$, defined by the generating function

$$\frac{e^{(x-\frac{1}{2})t}}{g_\alpha(\frac{it}{2})} = \sum_{n=0}^{\infty} B_{n,\alpha}(x)\frac{t^n}{n!}, \quad \mid t \mid < 2j_{1,\alpha}, \tag{1.1}$$

where

$$g_\alpha(t) = 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(t)}{t^\alpha},$$

$J_\alpha(t)$ is the Bessel function of the first kind of order α, and $j_{1,\alpha}$ is the smallest positive zero of $J_\alpha(t)$. Ismail and Mansour, see [19], introduced a pair of q-analogs of the Bernoulli polynomials by the generating functions

$$\begin{aligned} \frac{te_q(xt)}{e_q(\frac{t}{2})E_q(\frac{t}{2})-1} &= \sum_{n=0}^{\infty} b_n(x;q)\frac{t^n}{[n]_q!}, \\ \frac{tE_q(xt)}{e_q(\frac{t}{2})E_q(\frac{t}{2})-1} &= \sum_{n=0}^{\infty} B_n(x;q)\frac{t^n}{[n]_q!}. \end{aligned} \tag{1.2}$$

They also defined a pair of q-analogs of the Euler polynomials by the generating functions

$$\begin{aligned} \frac{2e_q(xt)}{E_q(\frac{t}{2})e_q(\frac{t}{2})+1} &= \sum_{n=0}^{\infty} e_n(x;q)\frac{t^n}{[n]_q!}, \\ \frac{2E_q(xt)}{E_q(\frac{t}{2})e_q(\frac{t}{2})+1} &= \sum_{n=0}^{\infty} E_n(x;q)\frac{t^n}{[n]_q!}, \end{aligned} \tag{1.3}$$

where

$$[n]_q! = \frac{(q;q)_n}{(1-q)^n} \quad (n \in \mathbb{N}), \quad (a;q)_n = \begin{cases} 1, & n = 0; \\ \prod_{k=0}^{n-1}(1-aq^k), & n \in \mathbb{N}, \end{cases}$$

and $a \in \mathbb{C}$, see [12]. The functions $E_q(x)$ and $e_q(x)$ are the q-analogs of the exponential functions defined by

$$\begin{aligned} E_q(x) &:= (-x(1-q);q)_\infty = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}(1-q)^n x^n}{(q;q)_n}, \ x \in \mathbb{C}, \\ e_q(x) &:= \frac{1}{(x(1-q);q)_\infty} = \sum_{n=0}^{\infty} \frac{(1-q)^n x^n}{(q;q)_n}, \ |x| < \frac{1}{1-q}, \end{aligned} \tag{1.4}$$

see [12].

In [22], Mansour and Al-Towalib introduced q-analogs of Bernoulli and Euler polynomials by the generating functions

$$\begin{aligned} \frac{t\exp_q(xt)\exp_q(\frac{-t}{2})}{\exp_q(\frac{t}{2})-\exp_q(\frac{-t}{2})} &= \sum_{n=0}^{\infty} \tilde{B}_n(x;q)\frac{t^n}{[n]_q!}, \\ \frac{2\exp_q(xt)\exp_q(\frac{-t}{2})}{\exp_q(\frac{t}{2})+\exp_q(\frac{-t}{2})} &= \sum_{n=0}^{\infty} \tilde{E}_n(x;q)\frac{t^n}{[n]_q!}, \end{aligned} \tag{1.5}$$

where

$$\exp_q(x) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{4}}\frac{x^n}{[n]_q!}, \quad x \in \mathbb{C},$$

is a q-analog of the exponential function. This q-exponential function has the property $\lim_{q\to 1}\exp_q(x) = e^x$ for $x \in \mathbb{C}$. It is an entire function of x of order zero, see [12, Eq. (1.3.27), p. 12].

In this paper, we use $\mathbb{N}$ to denote the set of positive integers and $\mathbb{N}_0$ to denote the set of non-negative integers. Throughout this paper, unless otherwise is stated, q is a positive number that is less than one. We follow Gasper and Rahman [12] to define the q-shifted factorial, the q-binomial coefficients, and the q-gamma function. The q-integer number $[n]_q$ is defined by

$$[n]_q = \frac{1-q^n}{1-q}, \quad n \in \mathbb{N}_0.$$

Jackson in [20] defined the q-difference operator by

$$D_q f(z) = \frac{f(qz)-f(z)}{z(q-1)}, \quad z \neq 0.$$

The symmetric q-difference operator is defined by, see [8,12],

$$\delta_{q,z} f(z) = \frac{f(q^{\frac{1}{2}}z)-f(q^{\frac{-1}{2}}z)}{(q^{\frac{1}{2}}-q^{\frac{-1}{2}})z}, \quad z \neq 0.$$

The q-trigonometric functions $\sin_q z$, $\cos_q z$, $Sin_q z$ and $Cos_q z$ are defined by

$$\begin{aligned} \sin_q z &= \frac{e_q(iz)-e_q(-iz)}{2i}, \quad \cos_q z = \frac{e_q(iz)+e_q(-iz)}{2}, \ \mid z \mid < 1,\\ Sin_q z &= \frac{E_q(iz)-E_q(-iz)}{2i}, \quad Cos_q z = \frac{E_q(iz)+E_q(-iz)}{2}, \ z \in \mathbb{C}, \end{aligned}$$

see [5,12]. The q-sine and cosine functions $S_q(z)$, $C_q(z)$ are defined by the q-Euler formula

$$\exp_q(iz) := C_q(z) + iS_q(z),$$

where

$$C_q(z) = \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n-\frac{1}{2})}}{[2n]_q!} z^{2n}, \quad S_q(z) = \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+\frac{1}{2})}}{[2n+1]_q!} z^{2n+1},$$

cf. [8, p. 2]. The hyperbolic functions $Sh_q(z)$ and $Ch_q(z)$ are defined for $z \in \mathbb{C}$ by

$$\begin{aligned} Sh_q(z) &:= -iS_q(iz) = \frac{\exp_q(z)-\exp_q(-z)}{2},\\ Ch_q(z) &:= C_q(iz) = \frac{\exp_q(z)+\exp_q(-z)}{2}. \end{aligned} \tag{1.6}$$

There are three known q-analogs of the Bessel function that are due to Jackson [20]. These are denoted by $J_\alpha^{(k)}(t;q)\,(k=1,2,3)$ and defined by

$$J_\alpha^{(1)}(t;q)=\frac{(q^{\alpha+1};q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n\frac{(\frac{t}{2})^{2n+\alpha}}{(q;q)_n(q^{\alpha+1};q)_n}\quad(\mid t\mid<2),$$
$$J_\alpha^{(2)}(t;q)=\frac{(q^{\alpha+1};q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n\frac{q^{n(\alpha+n)}(\frac{t}{2})^{2n+\alpha}}{(q;q)_n(q^{\alpha+1};q)_n}\quad(t\in\mathbb{C}),$$
$$J_\alpha^{(3)}(t;q)=\frac{(q^{\alpha+1};q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n\frac{q^{\frac{n(n+1)}{2}}t^{2n+\alpha}}{(q;q)_n(q^{\alpha+1};q)_n}\quad(t\in\mathbb{C}).$$

For convenience, we set

$$\begin{aligned}\mathcal{J}_\alpha^{(k)}(t;q)&:=\frac{(q;q)_\infty}{(q^{\alpha+1};q)_\infty}(\frac{t}{2})^{-\alpha}J_\alpha^{(k)}(t;q)\quad(k=1,2),\\ \mathcal{J}_\alpha^{(3)}(t;q)&:=\frac{(q;q)_\infty}{(q^{\alpha+1};q)_\infty}t^{-\alpha}J_\alpha^{(3)}(t;q).\end{aligned}\tag{1.7}$$

The functions $\mathcal{J}_\alpha^{(k)}(t;q)\,(k=1,2,3)$ are called the modified Jackson q-Bessel functions. From now on, we use $(j_{m,\alpha}^{(k)})_{m=1}^\infty$ to denote the positive zeros of $J_\alpha^{(k)}(\cdot;q^2)$ arranged in increasing order of magnitude. Consequently, $j_{1,\alpha}^{(k)}$ is the smallest positive zero of $J_\alpha^{(k)}(\cdot;q^2)\,(k=1,2,3)$.

This paper is organized as follows. In Sect. 2, we introduce three q-analogs of the generalized Bernoulli polynomials defined in (1.1). The generating functions of these q-analogs include the three q-analogs of Jackson q-Bessel functions mentioned above. We also include the main properties of these q-analogs. Section 3 introduces a q-Fourier expansion for the generalized Bernoulli numbers related to the first and second Jackson q-Bessel functions. Also, their large n degree asymptotic is derived. Finally, in Sect. 4 as an application, we introduce the connection coefficients between q-analogs and certain q-orthogonal polynomials.

2. Generalized q-Bernoulli Polynomials Generated by Jackson q-Bessel Functions

This section introduces three q-analogs of the generalized Bernoulli polynomials introduced by Frappier in [9–11].

Definition 2.1. The generalized q-Bernoulli polynomials $B^{(k)}_{n,\alpha}(x;q)\,(k=1,2,3)$ are defined by the generating functions

$$\frac{e_q(xt)e_q(\frac{-t}{2})}{g^{(1)}_\alpha(it;q)}=\sum_{n=0}^{\infty}B^{(1)}_{n,\alpha}(x;q)\frac{t^n}{[n]_q!},\quad |\,t\,|<\frac{j^{(1)}_{1,\alpha}}{1-q},\tag{2.1}$$

$$\frac{E_q(xt)E_q(\frac{-t}{2})}{g^{(2)}_\alpha(it;q)}=\sum_{n=0}^{\infty}B^{(2)}_{n,\alpha}(x;q)\frac{t^n}{[n]_q!},\quad |\,t\,|<\frac{j^{(2)}_{1,\alpha}}{1-q},\tag{2.2}$$

$$\frac{\exp_q(xt)\exp_q(\frac{-t}{2})}{g^{(3)}_\alpha(it;q)}=\sum_{n=0}^{\infty}B^{(3)}_{n,\alpha}(x;q)\frac{t^n}{[n]_q!},\quad |\,t\,|<\frac{2q^{\frac{1}{4}}j^{(3)}_{1,\alpha}}{1-q},\tag{2.3}$$

where $g^{(k)}_\alpha(t;q)\,(k=1,2,3)$ are the functions defined for $(k=1,2)$ by

$$g^{(k)}_\alpha(t;q):=(1+q)^\alpha\Gamma_{q^2}(\alpha+1)\left(\frac{t}{2}\right)^{-\alpha}J^{(k)}_\alpha(t(1-q);q^2)=\mathcal{J}^{(k)}_\alpha(t(1-q);q^2),$$

and

$$\begin{aligned}g^{(3)}_\alpha(t;q)&:=(1+q)^\alpha\Gamma_{q^2}(\alpha+1)\left(\frac{q^{\frac{-1}{4}}t}{2}\right)^{-\alpha}J^{(3)}_\alpha\left(\frac{t}{2}(1-q)q^{\frac{-1}{4}};q^2\right)\\&=\mathcal{J}^{(3)}_\alpha\left(\frac{t}{2}(1-q)q^{\frac{-1}{4}};q^2\right).\end{aligned}$$

Since the generating functions in (1.2), (1.3), and (1.5) can be written as

$$\frac{te_q(xt)e_q(\frac{-t}{2})}{2\sinh_q\frac{t}{2}}=\sum_{n=0}^{\infty}b_n(x;q)\frac{t^n}{[n]_q!},$$

$$\frac{tE_q(xt)E_q(\frac{-t}{2})}{2Sinh_q\frac{t}{2}}=\sum_{n=0}^{\infty}B_n(x;q)\frac{t^n}{[n]_q!},\tag{2.4}$$

$$\frac{e_q(xt)e_q(\frac{-t}{2})}{\cosh_q\frac{t}{2}}=\sum_{n=0}^{\infty}e_n(x;q)\frac{t^n}{[n]_q!},$$

$$\frac{E_q(xt)E_q(\frac{-t}{2})}{Cosh_q\frac{t}{2}}=\sum_{n=0}^{\infty}E_n(x;q)\frac{t^n}{[n]_q!},\tag{2.5}$$

and

$$\begin{aligned}\frac{t\exp_q(xt)\exp_q(\frac{-t}{2})}{2Sh_q(\frac{t}{2})}&=\sum_{n=0}^{\infty}\tilde{B}_n(x;q)\frac{t^n}{[n]_q!},\\\frac{\exp_q(xt)\exp_q(\frac{-t}{2})}{Ch_q(\frac{t}{2})}&=\sum_{n=0}^{\infty}\tilde{E}_n(x;q)\frac{t^n}{[n]_q!},\end{aligned}\tag{2.6}$$

then, if we substitute with $\alpha=\pm\frac{1}{2}$ in (2.1), (2.2), and (2.3), we obtain the q-Bernoulli and Euler polynomials defined in (2.4), (2.5) and (2.6), respectively.

Lemma 2.2. *For* $n \in \mathbb{N}_0$ *and* $Re\,\alpha > -1$,

$$\frac{e_q(\frac{-t}{2})}{g_\alpha^{(1)}(it;q)} = \frac{E_q(\frac{-t}{2})}{g_\alpha^{(2)}(it;q)}, \quad |\, t \,|< \frac{1}{1-q}\min\{j_{1,\alpha}^{(1)}, j_{1,\alpha}^{(2)}, 2\}.$$

Proof. Hahn in [14] proved the identity

$$J_\alpha^{(2)}(t;q) = \left(\frac{-t^2}{4};q\right)_\infty J_\alpha^{(1)}(t;q), \quad |\, t \,|< 2. \tag{2.7}$$

Since

$$g_\alpha^{(k)}(it;q) = (1+q)^\alpha \Gamma_{q^2}(\alpha+1)\left(\frac{it}{2}\right)^{-\alpha} J_\alpha^{(k)}(it(1-q);q^2)\ (k=1,2), \tag{2.8}$$

then, substituting from (2.8) into (2.7), we conclude that

$$g_\alpha^{(2)}(it;q) = \left(\frac{t^2}{4}(1-q)^2;q^2\right)_\infty g_\alpha^{(1)}(it;q) = E_q\left(\frac{t}{2}\right)E_q\left(\frac{-t}{2}\right) g_\alpha^{(1)}(it;q). \tag{2.9}$$

Hence

$$\frac{E_q(\frac{-t}{2})}{g_\alpha^{(2)}(it;q)} = \frac{E_q(\frac{-t}{2})}{E_q(\frac{t}{2})E_q(\frac{-t}{2})\,g_\alpha^{(1)}(it;q)} = \frac{e_q(\frac{-t}{2})}{g_\alpha^{(1)}(it;q)},$$

which completes the proof. □

Definition 2.3. The generalized q-Bernoulli numbers $\beta_{n,\alpha}(q)$, $\beta_{n,\alpha}^{(3)}(q)$ are defined respectively in terms of the generating functions

$$\frac{e_q(\frac{-t}{2})}{g_\alpha^{(1)}(it;q)} = \frac{E_q(\frac{-t}{2})}{g_\alpha^{(2)}(it;q)} = \sum_{n=0}^\infty \beta_{n,\alpha}(q)\frac{t^n}{[n]_q!}, \tag{2.10}$$

$$\frac{\exp_q(\frac{-t}{2})}{g_\alpha^{(3)}(it;q)} = \sum_{n=0}^\infty \beta_{n,\alpha}^{(3)}(q)\frac{t^n}{[n]_q!}. \tag{2.11}$$

Proposition 2.4. *For* $n \in \mathbb{N}$, *we have*

$$B_{2n+1,\alpha}^{(k)}\left(\frac{1}{2};q\right) = 0 \quad (k=1,2,3).$$

Proof. If we substitute with $x = \frac{1}{2}$ in Eqs. (2.1)–(2.3), we find that their left hand side are even functions. Therefore, the coefficients of the odd powers of t^n on the right hand sides of Eqs. (2.1)–(2.3) vanish. This proves the proposition. □

Proposition 2.5. *For $k \in \{1, 2, 3\}$ and $n \in \mathbb{N}$, the polynomials $B^{(k)}_{n,\alpha}(x; q)$ have the representation* $B^{(k)}_{0,\alpha}(x; q) = 1$,

$$B^{(1)}_{n,\alpha}(x; q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \beta_{n-k,\alpha}(q)x^k, \tag{2.12}$$

$$B^{(2)}_{n,\alpha}(x; q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} \beta_{n-k,\alpha}(q)x^k, \tag{2.13}$$

$$B^{(3)}_{n,\alpha}(x; q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{4}} \beta^{(3)}_{n-k,\alpha}(q)x^k. \tag{2.14}$$

Proof. We prove the case ($k = 1$). The proofs for ($k = 2, 3$) are similar and are omitted. Substituting with the series representation of $e_q(x)$ from (1.4) into (2.1) gives

$$\sum_{n=0}^{\infty} B^{(1)}_{n,\alpha}(x; q)\frac{t^n}{[n]_q!} = \frac{e_q(\frac{-t}{2})}{g^{(1)}_{\alpha}(it; q)} e_q(xt)$$
$$= \left(\sum_{n=0}^{\infty} \beta_{n,\alpha}(q)\frac{t^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty} \frac{(xt)^n}{[n]_q!}\right).$$

Hence

$$\sum_{n=0}^{\infty} B^{(1)}_{n,\alpha}(x; q)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \beta_{n-k,\alpha}\, x^k, \tag{2.15}$$

where we applied the Cauchy product formula. Equating the *nth* power of t in (2.15), we obtain (2.12). □

Proposition 2.6. *For $n \in \mathbb{N}$ and $k \in \{1, 2, 3\}$, the polynomials $B^{(k)}_{n,\alpha}(x; q)$ satisfy the q-difference equations*

$$D_{q,x}\, B^{(1)}_{n,\alpha}(x; q) = [n]_q\, B^{(1)}_{n-1,\alpha}(x; q), \tag{2.16}$$

$$D_{q^{-1},x}\, B^{(2)}_{n,\alpha}(x; q) = [n]_q\, B^{(2)}_{n-1,\alpha}(x; q), \tag{2.17}$$

$$\delta_{q,x}\, B^{(3)}_{n,\alpha}(x; q) = [n]_q B^{(3)}_{n-1,\alpha}(x; q). \tag{2.18}$$

Proof. We only prove the case ($k = 1$) and the proofs of ($k = 2, 3$) are similar. Calculating the q-derivative of both sides of (2.1) with respect to the variable x and taking into consideration that

$$D_{q,x}\, e_q(xt) = t\, e_q(xt),$$

we obtain

$$\frac{te_q(xt)e_q(\frac{-t}{2})}{g^{(1)}_{\alpha}(it; q)} = \sum_{n=1}^{\infty} D_{q,x}B^{(1)}_{n,\alpha}(x; q)\frac{t^n}{[n]_q!}.$$

Therefore,

$$\sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}(x;q)\frac{t^{n+1}}{[n]_q!} = \sum_{n=1}^{\infty} D_{q,x}B_{n,\alpha}^{(1)}(x;q)\frac{t^n}{[n]_q!}. \tag{2.19}$$

Equating the corresponding *nth* power of t in (2.19), we obtain (2.16). □

Corollary 2.7. *Let $n \in \mathbb{N}$ and k be a positive integer such that $k \leq n$. Then for $x \in \mathbb{C}$,*

$$D_{q,x}^k \frac{B_{n,\alpha}^{(1)}(x;q)}{[n]_q!} = \frac{B_{n-k,\alpha}^{(1)}(x;q)}{[n-k]_q!},$$

$$D_{q^{-1},x}^k \frac{B_{n,\alpha}^{(2)}(x;q)}{[n]_q!} = \frac{B_{n-k,\alpha}^{(2)}(x;q)}{[n-k]_q!},$$

$$\delta_{q,x}^k \frac{B_{n,\alpha}^{(3)}(x;q)}{[n]_q!} = \frac{B_{n-k,\alpha}^{(3)}(x;q)}{[n-k]_q!}.$$

Proof. The proofs follow from Proposition 2.6 and the mathematical induction. □

Proposition 2.8. *For $|\, t \,| < \frac{1}{1-q}\min\{j_{1,\alpha}^{(1)}, j_{1,\alpha}^{(2)}, 2\}$,*

$$\sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}\left(\frac{1}{2};q\right)\frac{t^n}{[n]_q!} = \frac{1}{g_\alpha^{(2)}(it;q)}. \tag{2.20}$$

$$\sum_{n=0}^{\infty} B_{n,\alpha}^{(2)}\left(\frac{1}{2};q\right)\frac{t^n}{[n]_q!} = \frac{1}{g_\alpha^{(1)}(it;q)}. \tag{2.21}$$

Proof. Set $x = \frac{1}{2}$ in (2.1), we obtain

$$\frac{e_q(\frac{t}{2})e_q(\frac{-t}{2})}{g_\alpha^{(1)}(it;q)} = \sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}\left(\frac{1}{2};q\right)\frac{t^n}{[n]_q!}. \tag{2.22}$$

Substituting from (2.9) into (2.22), we obtain (2.20). Similarly, we can prove (2.21). □

The following Lemma from [22] gives the reciprocal of $\exp_q(z)$ in a certain domain.

Lemma 2.9. *Let $z \in \Omega$, $\Omega := \{z \in \mathbb{C} : |\, 1 - \exp_q(-z) \,| < 1\}$. Then*

$$\frac{1}{\exp_q(z)} := \sum_{n=0}^{\infty} c_n\, z^n,$$

where

$$c_n = \sum_{k=1}^{n}(-1)^k \sum_{\substack{s_1+s_2+\ldots+s_k=n \\ s_i>0\,(i=1,\ldots,k)}} \frac{q^{\sum_{i=1}^k s_i(s_i-1)/2}}{[s_1]_q![s_2]_q!\ldots[s_k]_q!}. \tag{2.23}$$

Proposition 2.10. *For* $Re\,\alpha > -1$ *and* $t \in \Omega = \{t \in \mathbb{C} : |\, 1 - \exp_q(-t)\, | < 1\}$,

$$\frac{1}{g_\alpha^{(3)}(it;q)} = \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \frac{(-1)^k\, c_k}{2^k [n-k]_q!} \beta_{n-k,\alpha}^{(3)}(q), \tag{2.24}$$

where c_n *is defined in* (2.23).

Proof. Substitute with $x = 0$ in Eq. (2.3). This gives

$$\frac{\exp_q(\frac{-t}{2})}{g_\alpha^{(3)}(it;q)} = \sum_{n=0}^{\infty} \beta_{n,\alpha}^{(3)}(q) \frac{t^n}{[n]_q!}.$$

From Lemma 2.9,

$$\begin{aligned}\frac{1}{g_\alpha^{(3)}(it;q)} &= \frac{1}{\exp_q(\frac{-t}{2})} \sum_{n=0}^{\infty} \beta_{n,\alpha}^{(3)}(q) \frac{t^n}{[n]_q!} \\ &= \left(\sum_{n=0}^{\infty} c_n \frac{(-1)^n t^n}{2^n}\right) \left(\sum_{n=0}^{\infty} \beta_{n,\alpha}^{(3)}(q) \frac{t^n}{[n]_q!}\right).\end{aligned}$$

Applying the Cauchy product formula, we obtain (2.24) and completes the proof. □

Theorem 2.11. *For* $n \in \mathbb{N}_0$ *and* $x \in \mathbb{C}$,

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{k,\alpha}^{(1)}(-x;q) B_{n-k,\alpha}^{(2)}(x;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \beta_{k,\alpha}(q) \beta_{n-k,\alpha}(q).$$

Proof. If we replace x by $-x$ in (2.1), then

$$\frac{e_q(-xt) e_q(\frac{-t}{2})}{g_\alpha^{(1)}(it;q)} = \sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}(-x;q) \frac{t^n}{[n]_q!}. \tag{2.25}$$

Since $e_q(-xt) E_q(xt) = 1$, then multiplying (2.2) by (2.25) gives

$$\frac{E_q(\frac{-t}{2}) e_q(\frac{-t}{2})}{g_\alpha^{(2)}(it;q) g_\alpha^{(1)}(it;q)} = \left(\sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}(-x;q) \frac{t^n}{[n]_q!}\right) \left(\sum_{n=0}^{\infty} B_{n,\alpha}^{(2)}(x;q) \frac{t^n}{[n]_q!}\right).$$

From (2.10), we obtain

$$\left(\sum_{n=0}^{\infty} \beta_{n,\alpha}(q) \frac{t^n}{[n]_q!}\right)^2 = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{k,\alpha}^{(1)}(-x;q) B_{n-k,\alpha}^{(2)}(x;q).$$

Hence

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \beta_{k,\alpha}(q) \beta_{n-k,\alpha}(q) \\ &\quad = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{k,\alpha}^{(1)}(-x;q) B_{n-k,\alpha}^{(2)}(x;q).\end{aligned} \tag{2.26}$$

So, equating the nth power of t in (2.26), we obtain the required result. □

Proposition 2.12. *For $n \in \mathbb{N}_0$, $x \in \mathbb{C}$ and $q \neq 0$,*

$$B^{(2)}_{n,\alpha}(x;q) = q^{\frac{n(n-1)}{2}} B^{(1)}_{n,\alpha}\left(x;\frac{1}{q}\right). \tag{2.27}$$

In particular,

$$\beta_{n,\alpha}(q) = q^{\frac{n(n-1)}{2}} \beta_{n,\alpha}\left(\frac{1}{q}\right). \tag{2.28}$$

Proof. Replacing q by $\frac{1}{q}$ on the generating function in (2.1) and using $E_q(x) = e_{\frac{1}{q}}(x)$, we obtain

$$\frac{E_q(xt)E_q(\frac{-t}{2})}{g^{(1)}_\alpha(it;\frac{1}{q})} = \sum_{n=0}^{\infty} B^{(1)}_{n,\alpha}\left(x;\frac{1}{q}\right)\frac{t^n}{[n]_{\frac{1}{q}}!}. \tag{2.29}$$

Since

$$\begin{aligned} g^{(1)}_\alpha\left(it;\frac{1}{q}\right) &= \sum_{n=0}^{\infty} \frac{(1-q^{-1})^{2n}(\frac{t}{2})^{2n}}{(q^{-2};q^{-2})_n(q^{-2\alpha-2};q^{-2})_n} \\ &= \sum_{n=0}^{\infty} \frac{(1-q)^{2n}q^{2n(n+\alpha)}(\frac{t}{2})^{2n}}{(q^2,q^{2\alpha+2};q^2)_n} = g^{(2)}_\alpha(it;q), \end{aligned}$$

where we used the identity $(a;q^{-1})_n = (a^{-1};q)_n(-a)^n q^{-\frac{n(n-1)}{2}}$. Since $[n]_{1/q}! = q^{\frac{n(1-n)}{2}}[n]_q!$, then (2.29) takes the form

$$\frac{E_q(xt)E_q(\frac{-t}{2})}{g^{(2)}_\alpha(it;q)} = \sum_{n=0}^{\infty} B^{(1)}_{n,\alpha}\left(x;\frac{1}{q}\right) q^{\frac{n(n-1)}{2}}\frac{t^n}{[n]_q!}.$$

Therefore,

$$\sum_{n=0}^{\infty} B^{(2)}_{n,\alpha}(x;q)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} B^{(1)}_{n,\alpha}\left(x;\frac{1}{q}\right)\frac{q^{\frac{n(n-1)}{2}}t^n}{[n]_q!}. \tag{2.30}$$

Equating the coefficients of t^n in (2.30) gives (2.27) and substituting with $x = 0$ into (2.27) yields directly (2.28). □

Al-Salam in [3] introduced the polynomials

$$H_n(x) := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k, \quad G_n(x) := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2-nk}x^k. \tag{2.31}$$

He also proved that

$$E_q(x)E_q(-x) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} G_n(-1)\frac{x^n}{[n]_q!}, \quad x \in \mathbb{C}, \tag{2.32}$$

$$e_q(x)e_q(-x) = \sum_{n=0}^{\infty} H_n(-1)\frac{x^n}{[n]_q!}, \quad |\, x \,| < \frac{1}{1-q}. \tag{2.33}$$

The following theorem introduces connection relations between the polynomials $B^{(1)}_{n,\alpha}(x;q)$ and $B^{(2)}_{n,\alpha}(x;q)$.

Theorem 2.13. *For $n \in \mathbb{N}_0$,*

$$B^{(1)}_{n,\alpha}(x;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k H_k(-1) B^{(2)}_{n-k,\alpha}(x;q), \tag{2.34}$$

$$B^{(2)}_{n,\alpha}(x;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^k G_k(-1) B^{(1)}_{n-k,\alpha}(x;q). \tag{2.35}$$

Proof. Since $E_q(xt)e_q(-xt) = 1$, $\mid xt \mid < \frac{1}{1-q}$, then from (2.9), the generating function of $B^{(1)}_{n,\alpha}(x;q)$ can be represented as

$$\frac{e_q(xt)e_q(\frac{-t}{2})}{g^{(1)}_\alpha(it;q)} = \frac{E_q(xt)E_q(\frac{-t}{2})}{g^{(2)}_\alpha(it;q)} e_q(xt)e_q(-xt).$$

From (2.1), (2.2) and (2.33), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} B^{(1)}_{n,\alpha}(x;q) \frac{t^n}{[n]_q!} &= \left(\sum_{n=0}^{\infty} B^{(2)}_{n,\alpha}(x;q) \frac{t^n}{[n]_q!} \right) \left(\sum_{n=0}^{\infty} H_n(-1) \frac{(xt)^n}{[n]_q!} \right) \\ &= \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k H_k(-1) B^{(2)}_{n-k,\alpha}(x;q). \end{aligned} \tag{2.36}$$

Therefore, equating the coefficients of the *nth* power of t in the series of the outside parts of (2.36) gives (2.34). The proof for $B^{(2)}_{n,\alpha}(x;q)$ follows similarly from the generating function of $B^{(2)}_{n,\alpha}(x;q)$ and the identity (2.32), and is omitted. □

Theorem 2.14. *Let x and α be two complex numbers, with $Re\,\alpha > -1$, and n a positive integer. Then*

$$\begin{aligned} \sum_{k=0}^{[\frac{n}{2}]} \frac{(1-q)^{2k}\, B^{(1)}_{n-2k,\alpha}(-\frac{x}{2};q)}{2^{2k}\,[n-2k]_q!\,(q^2, q^{2n+\alpha}; q^2)_k} &= \frac{(-1/2)^n}{[n]_q!} H_n(x), \\ \sum_{k=0}^{[\frac{n}{2}]} \frac{(1-q)^{2k} q^{2k(k+\alpha)}\, B^{(2)}_{n-2k,\alpha}(-\frac{x}{2};q)}{2^{2k}\,[n-2k]_q!\,(q^2, q^{2n+\alpha}; q^2)_k} &= \frac{(-1/2)^n}{[n]_q!} q^{\frac{n(n-1)}{2}} G_n(x), \\ \sum_{k=0}^{[\frac{n}{2}]} \frac{(1-q)^{2k} q^{k^2+k/2}\, B^{(3)}_{n-2k,\alpha}(-\frac{x}{2};q)}{2^{2k}\,[n-2k]_q!\,(q^2, q^{2\alpha+2}; q^2)_k} &= \frac{(-1/2)^n}{[n]_q!} q^{\frac{n(n-1)}{4}} (-xq^{\frac{1-n}{2}};q)_n. \end{aligned} \tag{2.37}$$

Proof. We can write the generating function of the polynomials $B^{(1)}_{n,\alpha}(x;q)$ as

$$e_q(xt)e_q\left(\frac{-t}{2}\right) = g^{(1)}_\alpha(it;q)\sum_{n=0}^{\infty} B^{(1)}_{n,\alpha}(x;q)\frac{t^n}{[n]_q!}$$
$$= \left(\sum_{n=0}^{\infty}\frac{(1-q)^{2n}t^{2n}}{2^{2n}(q^2,q^{2\alpha+2};q^2)_n}\right)\left(\sum_{n=0}^{\infty}B^{(1)}_{n,\alpha}(x;q)\frac{t^n}{[n]_q!}\right). \tag{2.38}$$

On one hand, applying the Cauchy product formula in (2.38), we obtain

$$e_q(xt)e_q\left(\frac{-t}{2}\right) = \sum_{n=0}^{\infty} t^n \sum_{k=0}^{[\frac{n}{2}]}\frac{(1-q)^{2k}B^{(1)}_{n-2k,\alpha}(x;q)}{2^{2k}[n-2k]_q!(q^2,q^{2\alpha+2};q^2)_k}.$$

On the other hand, using the series representation of $e_q(x)$ in (1.4) followed by the Cauchy product formula, and using (2.31) yields

$$e_q(xt)e_q\left(\frac{-t}{2}\right) = \sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}\left(\frac{-1}{2}\right)^n H_n(-2x). \tag{2.39}$$

Hence

$$\sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}\left(\frac{-1}{2}\right)^n H_n(-2x) = \sum_{n=0}^{\infty} t^n \sum_{k=0}^{[\frac{n}{2}]}\frac{(1-q)^{2k}B^{(1)}_{n-2k,\alpha}(x;q)}{2^{2k}[n-2k]_q!(q^2,q^{2\alpha+2};q^2)_k}, \tag{2.40}$$

equating the coefficients of t^n in (2.40), we get

$$\frac{(\frac{-1}{2})^n H_n(-2x)}{[n]_q!} = \sum_{k=0}^{[\frac{n}{2}]}\frac{(1-q)^{2k}B^{(1)}_{n-2k,\alpha}(x;q)}{2^{2k}[n-2k]_q!(q^2,q^{2\alpha+2};q^2)_k}. \tag{2.41}$$

Replacing x *by* $\frac{-x}{2}$ in (2.41) gives

$$\frac{(\frac{-1}{2})^n H_n(x)}{[n]_q!} = \sum_{k=0}^{[\frac{n}{2}]}\frac{(1-q)^{2k}B^{(1)}_{n-2k,\alpha}(\frac{-x}{2};q)}{2^{2k}[n-2k]_q!(q^2,q^{2\alpha+2};q^2)_k},$$

which readily completes the proof for $B^{(1)}_{n,\alpha}(x;q)$. The proofs for $B^{(2)}_{n,\alpha}(x;q)$ and $B^{(3)}_{n,\alpha}(x;q)$ are similar and are omitted. □

If we set $x=0$ in (2.37), we obtain the following recurrence relations for $\beta_{n,\alpha}(q)$ and $\beta^{(3)}_{n,\alpha}(q)$,

$$\begin{aligned}\sum_{k=0}^{[\frac{n}{2}]}\frac{(1-q)^{2k}\,\beta_{n-2k,\alpha}(q)}{2^{2k}\,[n-2k]_q!\,(q^2,q^{2\alpha+2};q^2)_k} &= \frac{(\frac{-1}{2})^n}{[n]_q!} \quad (n\in\mathbb{N}),\\ \sum_{k=0}^{[\frac{n}{2}]}\frac{(1-q)^{2k}q^{k^2+k/2}\,\beta^{(3)}_{n-2k,\alpha}(q)}{2^{2k}\,[n-2k]_q!\,(q^2,q^{2\alpha+2};q^2)_k} &= \frac{q^{\frac{n(n-1)}{4}}(\frac{-1}{2})^n}{[n]_q!} \quad (n\in\mathbb{N}).\end{aligned} \tag{2.42}$$

As a consequence of the recursive relations in (2.42), and the fact that

$$\beta_{0,\alpha}(q) = \beta_{0,\alpha}^{(3)}(q) = 1,$$

we can prove that

$$\beta_{1,\alpha}(q) = -\frac{1}{2}, \quad \beta_{2,\alpha}(q) = \frac{q(1-q^{2\alpha+1})}{4(1-q^{2\alpha+2})}, \quad \beta_{3,\alpha}(q) = \frac{-q^3(1-q^{2\alpha-1})}{8(1-q^{2\alpha+2})},$$

$$\beta_{4,\alpha}(q) = \frac{1}{16} - \frac{(q+q^3)(1-q^3)(1-q^{2\alpha+1})}{16(1-q^{2\alpha+2})^2} - \frac{(1-q)(1-q^3)}{16(q^{2\alpha+2};q^2)_2},$$

$$\beta_{5,\alpha}(q) = \frac{(1+q^2)(1-q^5)(q^3-q^{2\alpha+2})}{32(1-q^{2\alpha+2})^2} + \frac{(1-q^3)(1-q^5)}{32(q^{2\alpha+2};q^2)_2} - \frac{1}{32},$$

and

$$\beta_{1,\alpha}^{(3)}(q) = \frac{-1}{2}, \qquad \beta_{2,\alpha}^{(3)}(q) = \frac{q^{1/2}(1-q^{2\alpha+2}) - q^{3/2}(1-q)}{4(1-q^{2\alpha+2})},$$

$$\beta_{3,\alpha}^{(3)}(q) = \frac{-q^{3/2}(q^3-q^{2\alpha+2})}{8(1-q^{2\alpha+2})},$$

$$\beta_{4,\alpha}^{(3)}(q) = \frac{q^3(q^{2\alpha+2};q^2)_2(1-q^{2\alpha+2})}{16(1-q^{2\alpha+2})^2(1-q^{2\alpha+4})} - \frac{[3]_q q^5(1-q)^2(1-q^{2\alpha+2})}{16(1-q^{2\alpha+2})^2(1-q^{2\alpha+4})} + \frac{[4]_q[3]_q q^{3/2}(1-q^{2\alpha+4})\left(q^{1/2}(1-q^{2\alpha+2}) - q^{3/2}(1-q)\right)}{16(1-q^{2\alpha+2})^2(1-q^{2\alpha+4})},$$

$$\beta_{5,\alpha}^{(3)}(q) = \frac{[5]_q q^3(1-q)(1+q^2)(q^3-q^{2\alpha+2})(1-q^{2\alpha+4})}{32(1-q^{2\alpha+2})^2(1-q^{2\alpha+4})} + \frac{[5]_q q^5(1-q)(1-q^3)(1-q^{2\alpha+2})}{32(1-q^{2\alpha+2})^2(1-q^{2\alpha+4})} - \frac{q^5(1-q^{2\alpha+2})^2(1-q^{2\alpha+4})}{32(1-q^{2\alpha+2})^2(1-q^{2\alpha+4})}.$$

Theorem 2.15. *For $n \in \mathbb{N}_0$ and complex numbers a and x,*

$$B_{n,\alpha}^{(1)}(x;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (a;q)_k\, x^k B_{n-k,\alpha}^{(1)}(ax;q), \tag{2.43}$$

$$B_{n,\alpha}^{(2)}(x;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-a)^k (1/a;q)_k\, x^k B_{n-k,\alpha}^{(2)}(ax;q). \tag{2.44}$$

Proof. The proof of (2.43) follows from the generating function (2.1) since

$$\frac{e_q(xt)e_q(\frac{-t}{2})}{g_\alpha^{(1)}(it;q)} = \frac{e_q(tax)e_q(\frac{-t}{2})}{g_\alpha^{(1)}(it;q)}\,\frac{e_q(tx)}{e_q(atx)}, \quad |\, tx \,| < \frac{1}{1-q}.$$

From the q-binomial theorem (see [12, Eq.(1.3.2), p. 8]), we can prove that

$$\frac{e_q(tx)}{e_q(atx)} = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n}((1-q)tx)^n, \quad |\, tx \,| < \frac{1}{1-q}.$$

Therefore,

$$\sum_{n=0}^{\infty} B^{(1)}_{n,\alpha}(x;q)\frac{t^n}{[n]_q!} = \left(\sum_{n=0}^{\infty} B^{(1)}_{n,\alpha}(ax;q)\frac{t^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty}\frac{(a;q)_n}{[n]_q!}(tx)^n\right)$$
$$= \sum_{n=0}^{\infty}\frac{t^n}{[n]_q!}\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q (a;q)_k x^k B^{(1)}_{n-k,\alpha}(ax;q), \tag{2.45}$$

where we used the Cauchy product formula. Equating the coefficients of t^n in (2.45), we obtain (2.43). The proof for $B^{(2)}_{n,\alpha}(x;q)$ is similar and is omitted. □

Lemma 2.16. *For $n \in \mathbb{N}_0$, $Re\,\alpha > -1$, and $\mid \frac{(1-q)t}{2} \mid < 1$,*

$$g^{(1)}_{\alpha}(it;q)E_q(\frac{t}{2}) = g^{(2)}_{\alpha}(it;q)e_q(\frac{t}{2}) = {}_2\phi_1\left(q^{\alpha+\frac{1}{2}}, -q^{\alpha+\frac{1}{2}}; q^{2\alpha+1}; q, \frac{(1-q)t}{2}\right). \tag{2.46}$$

Proof. From Lemma 2.2, we conclude that

$$g^{(2)}_{\alpha}(it;q)e_q\left(\frac{t}{2}\right) = g^{(1)}_{\alpha}(it;q)E_q\left(\frac{t}{2}\right).$$

From the series representations of $E_q(x)$ and $g^{(1)}_{\alpha}(it;q)$ in (1.4) and (2.38), respectively we obtain

$$g^{(1)}_{\alpha}(it;q)E_q(\frac{t}{2}) = \left(\sum_{n=0}^{\infty}\frac{(1-q)^{2n}t^{2n}}{2^{2n}(q^2,q^{2\alpha+2};q^2)_n}\right)\left(\sum_{n=0}^{\infty}\frac{q^{\frac{n(n-1)}{2}}(1-q)^n t^n}{2^n(q;q)_n}\right)$$
$$= \sum_{n=0}^{\infty}\frac{(1-q)^n q^{\frac{n(n-1)}{2}}t^n}{2^n}\sum_{k=0}^{[\frac{n}{2}]}\frac{q^{2k^2-2nk+k}}{(q;q)_{n-2k}(q^2,q^{2\alpha+2};q^2)_k}$$
$$= \sum_{n=0}^{\infty}\frac{q^{\frac{n(n-1)}{2}}(1-q)^n t^n}{2^n(q;q)_n}\sum_{k=0}^{[\frac{n}{2}]}\frac{q^{2k}(q^{-n};q)_{2k}}{(q^2,q^{2\alpha+2};q^2)_k},$$

where we used the identity, see [12, Eq. (1.2.32), p. 6],

$$(a;q)_{n-k} = \frac{(a;q)_n}{(a^{-1}q^{1-n};q)_k}(-qa^{-1})^k q^{\frac{k(k-1)}{2}-nk} \quad (k = 0,1,\ldots,n). \tag{2.47}$$

Therefore, using the identity $(a;q)_{2n} = (a;q^2)_n(aq;q^2)_n$ yields

$$g^{(1)}_{\alpha}(it;q)E_q\left(\frac{t}{2}\right) = \sum_{n=0}^{\infty}\frac{q^{\frac{n(n-1)}{2}}(1-q)^n(\frac{t}{2})^n}{(q;q)_n}\sum_{k=0}^{[\frac{n}{2}]}\frac{q^{2k}(q^{-n};q^2)_k(q^{-n+1};q^2)_k}{(q^2;q^{2\alpha+2};q^2)_k}$$
$$= \sum_{n=0}^{\infty}\frac{q^{\frac{n(n-1)}{2}}(1-q)^n(\frac{t}{2})^n}{(q;q)_n}\ {}_2\phi_1(q^{-n},q^{-n+1};q^{2\alpha+2};q^2,q^2).$$

Since

$$ {}_2\phi_1\left(q^{-n}, q^{1-n}; qb^2; q^2, q^2\right) = \frac{(b^2;q^2)_n}{(b^2;q)_n} q^{\frac{-n(n-1)}{2}} \ (n \in \mathbb{N}), $$

see [12, p. 26], then

$$ \begin{aligned} g_\alpha^{(1)}(it;q)E_q\left(\frac{t}{2}\right) &= \sum_{n=0}^{\infty} \frac{(q^{\alpha+\frac{1}{2}};q)_n(-q^{\alpha+\frac{1}{2}};q)_n(\frac{(1-q)t}{2})^n}{(q;q)_n(q^{2\alpha+1};q)_n} \\ &= {}_2\phi_1\left(q^{\alpha+\frac{1}{2}}, -q^{\alpha+\frac{1}{2}}; q^{2\alpha+1}; q, \frac{(1-q)t}{2}\right). \end{aligned} \tag{2.48} $$

Hence from Lemma 2.2 and (2.48), we obtain (2.46) and completes the proof. □

Theorem 2.17. *Let α be a complex number such that $Re\,\alpha > -1$. Then*

$$ \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(q^{2\alpha+1};q^2)_m B_{n-m,\alpha}^{(1)}(x;q)}{2^m (q^{2\alpha+1};q)_m} = x^n, $$

$$ \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(q^{2\alpha+1};q^2)_m B_{n-m,\alpha}^{(2)}(x;q)}{2^m (q^{2\alpha+1};q)_m} = q^{\frac{n(n-1)}{2}} x^n, $$

$$ \sum_{m=0}^{n} (-\frac{1}{2})^m \left(\sum_{k=0}^{[\frac{m}{2}]} \frac{q^{k^2+k/2}(1-q)^{2k} c_{m-2k}}{(q^2, q^{2\alpha+2}; q^2)_k} \right) \frac{B_{n-m,\alpha}^{(3)}(x;q)}{[n-m]_q!} = \frac{q^{\frac{n(n-1)}{4}} x^n}{[n]_q!}, $$

where $(c_k)_k$ are the coefficients defined in (2.23).

Proof. We can write Eq. (2.1) in the form

$$ \begin{aligned} e_q(xt) &= E_q\left(\frac{t}{2}\right) g_\alpha^{(1)}(it;q) \sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}(x;q) \frac{t^n}{[n]_q!} \\ &= \left(\sum_{n=0}^{\infty} d_n t^n\right) \left(\sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}(x;q) \frac{t^n}{[n]_q!}\right). \end{aligned} \tag{2.49} $$

From Lemma 2.16, we obtain

$$ g_\alpha^{(1)}(it;q)E_q\left(\frac{t}{2}\right) = \sum_{n=0}^{\infty} d_n t^n, \tag{2.50} $$

where

$$ d_n = \frac{(1-q)^n (q^{2\alpha+1};q^2)_n}{2^n (q;q)_n (q^{2\alpha+1};q)_n}. \tag{2.51} $$

Now, applying the Cauchy product formula in (2.49) gives

$$e_q(xt) = \sum_{n=0}^{\infty} t^n \sum_{m=0}^{n} \frac{d_m B^{(1)}_{n-m,\alpha}(x;q)}{[n-m]_q!} = \sum_{n=0}^{\infty} \frac{(xt)^n}{[n]_q!}. \tag{2.52}$$

Equating the coefficients of the nth power of t in (2.52) gives

$$\sum_{m=0}^{n} \frac{d_m B^{(1)}_{n-m,\alpha}(x;q)}{[n-m]_q!} = \frac{x^n}{[n]_q!}. \tag{2.53}$$

Substituting from (2.51) into (2.53), we get the result for $B^{(1)}_{n,\alpha}(x;q)$. Similarly, we can prove the result for $B^{(k)}_{n,\alpha}(x;q)$ $(k=2,3)$. □

Theorem 2.18. *Let n be a positive integer and x be a complex number. If $Re\,\alpha > -1$, then*

$$\begin{aligned} &B^{(1)}_{n,\alpha}(x;q) - (-1)^n B^{(1)}_{n,\alpha}(-x;q) \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\left(\frac{-1}{2}\right)^k H_k(-2x) - \left(\frac{1}{2}\right)^k H_k(2x) \right) B^{(2)}_{n-k,\alpha}\left(\frac{1}{2};q\right), \end{aligned} \tag{2.54}$$

$$\begin{aligned} &B^{(2)}_{n,\alpha}(x;q) - (-1)^n B^{(2)}_{n,\alpha}(-x;q) \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} \left((\frac{-1}{2})^k G_k(-2x) - (\frac{1}{2})^k G_k(2x) \right) B^{(1)}_{n-k,\alpha}(\frac{1}{2};q). \end{aligned} \tag{2.55}$$

Proof. We give only the proof of (2.54) since the proof of (2.55) is similar. From (2.1),

$$\frac{e_q(xt)e_q(\frac{-t}{2})}{g^{(1)}_\alpha(it;q)} - \frac{e_q(xt)e_q(\frac{t}{2})}{g^{(1)}_\alpha(-it;q)} = \sum_{n=0}^{\infty} B^{(1)}_{n,\alpha}(x;q)\frac{t^n}{[n]_q!} - \sum_{n=0}^{\infty} B^{(1)}_{n,\alpha}(-x;q)\frac{(-t)^n}{[n]_q!}. \tag{2.56}$$

Since

$$g^{(1)}_\alpha(-it;q) = g^{(1)}_\alpha(it;q),$$

then Eq. (2.56) can be written as

$$\frac{e_q(xt)e_q(\frac{-t}{2}) - e_q(xt)e_q(\frac{t}{2})}{g^{(1)}_\alpha(it;q)} = \sum_{n=0}^{\infty} \left[B^{(1)}_{n,\alpha}(x;q) - (-1)^n B^{(1)}_{n,\alpha}(-x;q) \right] \frac{t^n}{[n]_q!}. \tag{2.57}$$

Replacing x, t by $-x, -t$, respectively in (2.39) gives

$$e_q(xt)e_q(\frac{t}{2}) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!}(\frac{1}{2})^n H_n(2x). \tag{2.58}$$

From (2.39) and (2.58), the left hand side of (2.57) can be written as

$$\frac{e_q(xt)e_q(\frac{-t}{2}) - e_q(xt)e_q(\frac{t}{2})}{g_\alpha^{(1)}(it;q)} = \frac{1}{g_\alpha^{(1)}(it;q)} \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \times \left(\left(\frac{-1}{2}\right)^n H_n(-2x) - \left(\frac{1}{2}\right)^n H_n(2x) \right).$$

Therefore, by (2.21) and the Cauchy product formula, we get

$$\frac{e_q(xt)e_q(\frac{-t}{2}) - e_q(xt)e_q(\frac{t}{2})}{g_\alpha^{(1)}(it;q)} = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\left(\frac{-1}{2}\right)^k H_k(-2x) - \left(\frac{1}{2}\right)^k H_k(2x) \right) B_{n-k,\alpha}^{(2)}\left(\frac{1}{2};q\right). \tag{2.59}$$

Since the left hand side of (2.57) and (2.59) are equal, then equating the coefficients of t^n on the right hand sides of (2.57) and (2.59) yields (2.54) and completes the proof. □

Proposition 2.19. *If* $\alpha_0 > -1$ *satisfies the condition*

$$q^{2(\alpha_0+1)}(1-q)^2 < (1-q^2)(1-q^{2\alpha_0+2}), \tag{2.60}$$

then $(t/2)^{-\alpha} J_\alpha^{(2)}(t(1-q);q^2)$ *has no zeros in* $|\, t \,| \leq 1$ *for all* $\alpha \geq \alpha_0$.

Proof. Set

$$F(t) := \frac{(q;q)_\infty}{(q^{\alpha+1};q)_\infty} (t/2)^{-\alpha} J_\alpha^{(2)}(t(1-q);q^2) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{2k(k+\alpha)}(1-q)^{2k}}{2^{2k}(q^2,q^{2\alpha+2};q^2)_k} t^{2k},$$

and

$$a_k := \frac{q^{2k(k+\alpha)}(1-q)^{2k}}{2^{2k}(q^2,q^{2\alpha+2};q^2)_k}.$$

Then, under hypothesis (2.60) and since $0 < q < 1$,

$$\begin{aligned} q^{2(\alpha+1)}(1-q^2) &\leq q^{2(\alpha_0+1)}(1-q^2) < (1-q^2)(1-q^{2\alpha_0+2}) \\ &\leq (1-q^2)(1-q^{2\alpha+2}), \end{aligned}$$

holds whenever $\alpha \geq \alpha_0$. Hence

$$\frac{a_{k+1}}{a_k} = \frac{q^{4k+2(\alpha+1)}(1-q)^2}{4(1-q^{2k+2})(1-q^{2k+2\alpha+2})} \leq \frac{q^{2(\alpha+1)}(1-q)^2}{4(1-q^2)(1-q^{2\alpha+2})} < 1,$$

for $t \in \mathbb{R}$, $|\, t \,| \leq 1$

$$F(t) = \sum_{k=0}^{\infty} t^{2k}(a_{2k} - a_{2k+1}t^2) \geq (a_0 - a_1 t^2) \geq (a_0 - a_1) > 0.$$

This proves that $F(t)$ has no zeros on $[-1,1]$, since $F(t)$ has only real zeros, then $F(t)$ has no zeros in the unit disk. i.e $|\, F(t) \,| > 0,\ \textit{for}\ \ |\, t \,| \leq 1$. □

Corollary 2.20. *There exists* $\alpha_0 > -1$ *such that* $J_\alpha^{(2)}(t(1-q);q^2)$ *has no zeros in the unit disk for all* $\alpha \geq \alpha_0$.

Proof. Since for a fixed $q \in (0,1)$,

$$\lim_{\alpha\to\infty} q^{2\alpha+2} = 0, \ \lim_{\alpha\to\infty}(1-q^2)(1-q^{2\alpha+2}) = (1-q^2),$$

then there exists $\alpha_0 > -1$ such that the condition (2.60) holds for all $\alpha \geq \alpha_0$. Consequently from Proposition 2.19, $J_\alpha^{(2)}(t(1-q);q^2)$ has no zeros in the unit disk for all $\alpha \geq \alpha_0$. □

Theorem 2.21. *For* $n \in \mathbb{N}$,

$$\lim_{\alpha\to\infty} B_{n,\alpha}^{(2)}(x;q) = \left(-\frac{1}{2}\right)^n q^{\frac{n(n-1)}{2}} G_n(-2x), \tag{2.61}$$

$$\lim_{\alpha\to\infty} B_{n,\alpha}^{(1)}(x;q) = x^n \left(\frac{1}{2x};q\right)_n. \tag{2.62}$$

Proof. Taking the limit on both sides of Eq. (2.2) as $\alpha \to \infty$ we get

$$\lim_{\alpha\to\infty} \frac{E_q(xt)E_q(\frac{-t}{2})}{g_\alpha^{(2)}(it;q)} = \lim_{\alpha\to\infty} \sum_{n=0}^{\infty} B_{n,\alpha}^{(2)}(x;q)\frac{t^n}{[n]_q!}. \tag{2.63}$$

From Corollary 2.20, there exists $\alpha_0 > -1$ such that $g_\alpha^{(2)}(it;q)$ has no zeros in $\mid t \mid \leq 1$ for all $\alpha \geq \alpha_0$. This means that $\dfrac{E_q(xt)E_q(\frac{-t}{2})}{g_\alpha^{(2)}(it;q)}$ is analytic in $\mid t \mid \leq 1$ for all $\alpha \geq \alpha_0$. Therefore, we can interchange the limit with the summation in (2.63) when $\mid t \mid \leq 1$ to obtain

$$\frac{E_q(xt)E_q(\frac{-t}{2})}{\lim_{\alpha\to\infty} g_\alpha^{(2)}(it;q)} = \sum_{n=0}^{\infty} \lim_{\alpha\to\infty} B_{n,\alpha}^{(2)}(x;q)\frac{t^n}{[n]_q!}.$$

Since

$$\lim_{\alpha\to\infty} g_\alpha^{(2)}(it;q) = 1, \quad E_q(xt)E_q(yt) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}(ty)^n}{[n]_q!} G_n(\frac{x}{y}),$$

then from (2.32)

$$\sum_{n=0}^{\infty} \lim_{\alpha\to\infty} B_{n,\alpha}^{(2)}(x;q)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}t^n}{[n]_q!}\left(\frac{-1}{2}\right)^n G_n(-2x). \tag{2.64}$$

Equating the coefficients of t^n in (2.64) gives (2.61). The proof of (2.62) follows directly from the relation (2.9) since

$$1 = \lim_{\alpha\to\infty} g_\alpha^{(2)}(it;q) = E_q\left(\frac{t}{2}\right) E_q\left(\frac{-t}{2}\right) \lim_{\alpha\to\infty} g_\alpha^{(1)}(it;q).$$

Hence

$$\lim_{\alpha\to\infty} g_\alpha^{(1)}(it;q) = e_q\left(\frac{t}{2}\right) e_q\left(\frac{-t}{2}\right), \ \mid t(1-q) \mid < 2.$$

Therefore, computing the limit in both sides of (2.1) gives

$$\frac{e_q(xt)}{e_q(\frac{t}{2})} = \sum_{n=0}^{\infty} \lim_{\alpha\to\infty} B_{n,\alpha}^{(1)}(x;q)\frac{t^n}{[n]_q!}.$$

From the q-binomial theorem (see [12, Eq.(1.3.2), p. 8]), we have

$$\frac{e_q(xt)}{e_q(\frac{t}{2})} = \frac{(\frac{t}{2}(1-q);q)_\infty}{(xt(1-q);q)_\infty} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2x};q)_n}{(q;q)_n}(xt(1-q))^n, \ \mid xt(1-q) \mid < 1.$$

Hence

$$\sum_{n=0}^{\infty} \lim_{\alpha\to\infty} B_{n,\alpha}^{(1)}(x;q)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \frac{(xt)^n}{[n]_q!}(\frac{1}{2x};q)_n, \tag{2.65}$$

equating the coefficients of t^n in (2.65) yields the required result. □

Corollary 2.22. *For $n \in \mathbb{N}$,*

$$\lim_{\alpha\to\infty} \beta_{n,\alpha}(q) = (-1)^n 2^{-n} q^{\frac{n(n-1)}{2}}. \tag{2.66}$$

Proof. Since

$$\lim_{x\to 0} x^n \left(\frac{1}{2x};q\right)_n = \lim_{x\to 0} x^n \prod_{k=0}^{n-1}\left(1-\frac{q^k}{2x}\right) = \lim_{x\to 0}\prod_{k=0}^{n-1}\left(x-\frac{q^k}{2}\right)$$
$$= (-1)^n 2^{-n} q^{\frac{n(n-1)}{2}},$$

then substituting with $x = 0$ into (2.62) yields (2.66). □

Lemma 2.23. *Let $\alpha_0 > -1$. If $q^{3/2}(1-q)^2 < (1-q^2)(1-q^{2\alpha_0+2})$, then $(q^{\frac{1}{4}}t/2)^{-\alpha} J_\alpha^{(3)}(\frac{t}{2}(1-q)q^{\frac{-1}{4}};q^2)$ has no zeros in $\mid t \mid \leq 1$ for all $\alpha \geq \alpha_0$.*

Proof. The proof is similar to the proof of Proposition 2.19 and is omitted. □

Theorem 2.24. *For $n \in \mathbb{N}$,*

$$\lim_{\alpha\to\infty} B_{n,\alpha}^{(3)}(x;q) = q^{\frac{n(n-1)}{4}}\left(\frac{-1}{2}\right)^n \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k q^{k(n-k+3)}(q^{-n};q)_{2k}}{(q^2;q^2)_k}(2xq^{\frac{1-n}{2}};q)_{n-2k}, \tag{2.67}$$

$$\lim_{\alpha\to\infty} \beta_{n,\alpha}^{(3)}(q) = q^{\frac{n(n-1)}{4}}\left(\frac{-1}{2}\right)^n \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k q^{k(n-k+3)}(q^{-n};q)_{2k}}{(q^2;q^2)_k}. \tag{2.68}$$

Proof. Taking the limit as $\alpha \to \infty$ on both sides of (2.3), we obtain

$$\lim_{\alpha\to\infty} \frac{\exp_q(xt)\exp_q(\frac{-t}{2})}{g_\alpha^{(3)}(it;q)} = \lim_{\alpha\to\infty}\sum_{n=0}^{\infty} B_{n,\alpha}^{(3)}(x;q)\frac{t^n}{[n]_q!}. \tag{2.69}$$

We can choose $\alpha_0 > -1$ such that

$$q^{3/2} \leq \frac{(1-q^2)}{1-q}\frac{(1-q^{2\alpha_0+2})}{1-q} \leq \frac{(1-q^2)}{1-q}\frac{(1-q^{2\alpha+2})}{1-q},$$

for all $\alpha \geq \alpha_0$. Hence from Lemma 2.23, the function $g_\alpha^{(3)}(it;q)$ does not vanish on the unit disk, and the left hand side of (2.69) is analytic for $|\, t \,| \leq 1$. Therefore, we can interchange the limit as $\alpha \to \infty$ with the summation in (2.69) to obtain

$$\frac{\exp_q(xt)\exp_q(\frac{-t}{2})}{\lim_{\alpha\to\infty} g_\alpha^{(3)}(it;q)} = \sum_{n=0}^{\infty} \lim_{\alpha\to\infty} B_{n,\alpha}^{(3)}(x;q)\frac{t^n}{[n]_q!}.$$

Since

$$\begin{aligned}\lim_{\alpha\to\infty} g_\alpha^{(3)}(it;q) &= \sum_{n=0}^{\infty} \lim_{\alpha\to\infty} \frac{q^{n^2+\frac{n}{2}}(1-q)^{2n}}{(q^2, q^{2\alpha+2};q^2)_n}(\frac{t}{2})^{2n} \\ &= \sum_{n=0}^{\infty} \frac{q^{n^2+\frac{n}{2}}(1-q)^{2n}(\frac{t}{2})^{2n}}{(q^2;q^2)_n} = \left(-\frac{q^{\frac{3}{2}}(1-q)^2t^2}{4};q^2\right)_\infty.\end{aligned}$$

Hence

$$\frac{\exp_q(xt)\exp_q(\frac{-t}{2})}{\left(-\frac{q^{\frac{3}{2}}(1-q)^2t^2}{4};q^2\right)_\infty} = \sum_{n=0}^{\infty} \lim_{\alpha\to\infty} B_{n,\alpha}^{(3)}(x;q)\frac{t^n}{[n]_q!}. \tag{2.70}$$

But

$$\exp_q(xt)\exp_q\left(\frac{-t}{2}\right) = \sum_{n=0}^{\infty} \frac{(\frac{-1}{2})^n q^{\frac{n(n-1)}{4}} t^n}{[n]_q!}(2xq^{\frac{1-n}{2}};q)_n.$$

Therefore,

$$\begin{aligned}&\left(\frac{1}{(-q^{\frac{3}{2}}(1-q)^2t^2/4;q^2)_\infty}\right)\left(\exp_q(xt)\exp_q(\frac{-t}{2})\right) \\ &= \left(\sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{3}{2}n}(\frac{(1-q)t}{2})^{2n}}{(q^2;q^2)_n}\right)\left(\sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{4}}(\frac{-t}{2})^n}{[n]_q!}(2xq^{\frac{1-n}{2}};q)_n\right) \\ &= \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{4}}\left(\frac{-t(1-q)}{2}\right)^n \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k q^{\frac{3}{2}k} q^{k^2-nk+k/2}}{(q^2;q^2)_k(q;q)_{n-2k}}(2xq^{\frac{1-n}{2}};q)_{n-2k} \\ &= \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{4}}(\frac{-t}{2})^n}{[n]_q!} \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k q^{k(n-k+3)}(q^{-n};q)_{2k}}{(q^2;q^2)_k}(2xq^{\frac{1-n}{2}};q)_{n-2k}.\end{aligned} \tag{2.71}$$

Substituting from (2.71) into (2.70) and equating the coefficients of t^n yields (2.67). The proof of (2.68) follows directly by setting $x = 0$ in (2.67). □

Theorem 2.25. *Let α be a complex number such that $Re\,\alpha > -1$. Then for $n \in \mathbb{N}$, $n \geq 2$,*

$$\beta_{n,\alpha}(q) = -\frac{[n]_q!(1-q)^2}{4} \sum_{k=0}^{[\frac{n}{2}]-1} \frac{((1-q)/2)^{2k}\,\beta_{n-2k-2,\alpha}(q)}{[n-2k-2]_q!(q^2, q^{2\alpha+2}; q^2)_{k+1}} + \frac{(-1)^n}{2^n},$$
$$\beta^{(3)}_{n,\alpha}(q) = -\frac{[n]_q! q^{3/2}(1-q)^2}{4} \sum_{k=0}^{[\frac{n}{2}]-1} \frac{q^{k^2+5k/2}\,((1-q)/2)^{2k}\,\beta^{(3)}_{n-2k-2,\alpha}(q)}{[n-2k-2]_q!(q^2, q^{2\alpha+2}; q^2)_{k+1}}$$
$$+ \frac{(-1)^n q^{\frac{n(n-1)}{4}}}{2^n}. \tag{2.72}$$

Proof. We give in detail the proof of $\beta_{n,\alpha}(q)$ in (2.72). The proof for $\beta^{(3)}_{n,\alpha}(q)$ is similar. Since

$$\frac{e_q(\frac{-t}{2})}{g^{(1)}_\alpha(it;q)} = \sum_{n=0}^{\infty} \beta_{n,\alpha}(q) \frac{t^n}{[n]_q!}, \tag{2.73}$$

then

$$\frac{e_q(\frac{-t}{2})}{g^{(1)}_\alpha(it;q)} - e_q\left(\frac{-t}{2}\right) = \sum_{n=0}^{\infty} \beta_{n,\alpha}(q) \frac{t^n}{[n]_q!} - e_q\left(\frac{-t}{2}\right).$$

Consequently, from the series representation of $e_q(t)$ in (1.4), we get

$$\frac{e_q(\frac{-t}{2})}{g^{(1)}_\alpha(it;q)} \left(1 - g^{(1)}_\alpha(it;q)\right) = \sum_{n=0}^{\infty} \left(\beta_{n,\alpha}(q) - \frac{(-1)^n}{2^n}\right) \frac{t^n}{[n]_q!}. \tag{2.74}$$

Since

$$\left(g^{(1)}_\alpha(it;q) - 1\right) = t^2 \sum_{m=0}^{\infty} \frac{(1-q)^{2m+2} t^{2m}}{2^{2m+2}(q^2, q^{2\alpha+2}; q^2)_{m+1}}, \tag{2.75}$$

then substituting from (2.75) into (2.74) and using (2.73), we obtain

$$\left(\sum_{n=0}^{\infty} \beta_{n,\alpha}(q) \frac{t^n}{[n]_q!}\right) \left(-t^2 \sum_{n=0}^{\infty} \frac{(1-q)^{2n+2} t^{2n}}{2^{2n+2}(q^2, q^{2\alpha+2}; q^2)_{n+1}}\right)$$
$$= \sum_{n=0}^{\infty} \left(\beta_{n,\alpha}(q) - \frac{(-1)^n}{2^n}\right) \frac{t^n}{[n]_q!}.$$

Therefore, by the Cauchy product formula

$$-\frac{(1-q)^2}{4} \sum_{n=2}^{\infty} t^n \sum_{k=0}^{[\frac{n}{2}]-1} \frac{(1-q)^{2k} \beta_{n-2k-2,\alpha}(q)}{2^{2k}[n-2k-2]_q!(q^2, q^{2\alpha+2}; q^2)_{k+1}}$$
$$= \sum_{n=0}^{\infty} \left(\beta_{n,\alpha}(q) - \frac{(-1)^n}{2^n}\right) \frac{t^n}{[n]_q!}. \tag{2.76}$$

Equating the coefficient of t^n in (2.76), we get the required result and the theorem follows. □

The following theorem gives a recursive relations between the polynomials $B^{(k)}_{n,\alpha}(x;q)$ and $B^{(k)}_{n,\alpha+1}(x;q)$ $(k=2,3)$.

Theorem 2.26. *If* $Re\,\alpha > -1$, $x \in \mathbb{C}$, *and* $k \in \mathbb{N}$, *then*

$$\frac{B^{(2)}_{n,\alpha}(x;q)}{[n]_q!} = 2(1-q^{2\alpha+2}) \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{(1-q)^{2k}\, h^{(2)}_{k+1}(q^2)}{[n-2k]_q!} B^{(2)}_{n-2k,\alpha+1}(x;q), \tag{2.77}$$

$$\frac{B^{(3)}_{n,\alpha}(x;q)}{[n]_q!} = (1-q^{2\alpha+2}) \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{(1-q)^{2k} q^{\frac{-k}{2}}\, h^{(3)}_{k+1}(q^2)}{2^{2k}[n-2k]_q!} B^{(3)}_{n-2k,\alpha+1}(x;q), \tag{2.78}$$

where

$$h^{(r)}_k(q^2) = \sum_{m=1}^{\infty} \frac{-2J^{(r)}_{\alpha+1}(j^{(r)}_{m,\alpha};q^2)}{\frac{d}{dz} J^{(r)}_{\alpha}(z;q^2)\mid_{z=j^{(r)}_{m,\alpha}}} \left(\frac{1}{j^{(r)}_{m,\alpha}}\right)^{2k},$$

and $(j^{(r)}_{m,\alpha})^{\infty}_{m=1}$ $(r=2,3)$ *are the positive zero of* $J^{(r)}_{\alpha}(\cdot;q^2)$.

Proof. We start with the proof of (2.77). From [6,13], we have the identity

$$\frac{J^{(2)}_{\alpha+1}(t;q)}{J^{(2)}_{\alpha}(t;q)} = \sum_{n=1}^{\infty} h^{(2)}_n(q) t^{2n-1}, \tag{2.79}$$

where

$$h^{(2)}_n(q) = \sum_{m=1}^{\infty} \frac{-2J^{(2)}_{\alpha+1}(j^{(2)}_{m,\alpha};q^2)}{\frac{d}{dz} J^{(2)}_{\alpha}(z;q^2)\mid_{z=j^{(2)}_{m,\alpha}}} \left(\frac{1}{j^{(2)}_{m,\alpha}}\right)^{2n}.$$

Replacing t by $it(1-q)$ and q by q^2 in (2.79), we obtain

$$\frac{1}{J^{(2)}_{\alpha}(it(1-q);q^2)} = \frac{1}{J^{(2)}_{\alpha+1}(it(1-q);q^2)} \sum_{n=1}^{\infty} h^{(2)}_n (it(1-q))^{2n-1}. \tag{2.80}$$

Multiplying (2.80) by $E_q(xt)E_q(\frac{-t}{2})$ to obtain

$$\frac{E_q(xt)E_q(\frac{-t}{2})}{J^{(2)}_{\alpha}(it(1-q);q^2)} = \frac{E_q(xt)E_q(\frac{-t}{2})}{J^{(2)}_{\alpha+1}(it(1-q);q^2)} \sum_{n=1}^{\infty} h^{(2)}_n(q^2)(it(1-q))^{2n-1}. \tag{2.81}$$

Substituting from (2.8) into (2.81), we get

$$\frac{E_q(xt)E_q(\frac{-t}{2})}{g^{(2)}_{\alpha}(it;q)} = \frac{(1+q)[\alpha+1]_{q^2}}{(\frac{it}{2})} \frac{E_q(xt)E_q(\frac{-t}{2})}{g^{(2)}_{\alpha+1}(it;q)} \sum_{n=1}^{\infty} h^{(2)}_n(q^2)(it(1-q))^{2n-1}. \tag{2.82}$$

Consequently,

$$\begin{aligned}
&\sum_{n=0}^{\infty} B_{n,\alpha}^{(2)}(x;q)\frac{t^n}{[n]_q!}\\
&\quad= \frac{2(1+q)}{it}[\alpha+1]_{q^2}\left(\sum_{n=0}^{\infty} B_{n,\alpha+1}^{(2)}(x;q)\frac{t^n}{[n]_q!}\right)\left(\sum_{n=1}^{\infty} h_n^{(2)}(q^2)(it(1-q))^{2n-1}\right)\\
&\quad= 2(1-q^{2\alpha+2})\left(\sum_{n=0}^{\infty} B_{n,\alpha+1}^{(2)}(x;q)\frac{t^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty} h_{n+1}^{(2)}(q^2)t^{2n}(i(1-q))^{2n}\right)\\
&\quad= 2(1-q^{2\alpha+2})\sum_{n=0}^{\infty} t^n \sum_{k=0}^{[\frac{n}{2}]}(-1)^k \frac{(1-q)^{2k}\, h_{k+1}^{(2)}(q^2)}{[n-2k]_q!} B_{n-2k,\alpha+1}^{(2)}(x;q).
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{n=0}^{\infty} B_{n,\alpha}^{(2)}(x;q)\frac{t^n}{[n]_q!}\\
&\quad= 2(1-q^{2\alpha+2})\sum_{n=0}^{\infty} t^n \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k(1-q)^{2k}\, h_{k+1}^{(2)}(q^2)}{[n-2k]_q!} B_{n-2k,\alpha+1}^{(2)}(x;q).
\end{aligned} \tag{2.83}$$

Equating the coefficients of t^n in (2.83), we get (2.77). The proof of (2.78) follows from the identity (see [1, Eq. (4.3), p. 1201]),

$$\frac{J_{\alpha+1}^{(3)}(t;q)}{J_{\alpha}^{(3)}(t;q)} = \sum_{n=1}^{\infty} h_n^{(3)}(q)t^{2n-1},$$

where

$$h_n^{(3)}(q) = \sum_{m=1}^{\infty} \frac{-2J_{\alpha+1}^{(3)}(j_{m,\alpha}^{(3)};q^2)}{\frac{d}{dz}J_{\alpha}^{(3)}(z;q^2)\mid_{z=j_{m,\alpha}^{(3)}}}\left(\frac{1}{j_{m,\alpha}^{(3)}}\right)^{2n},$$

and by using the same technique. □

3. Asymptotic Relations for the Generalized q-Bernoulli Numbers

In this section, we derive asymptotic relations for the generalized q-Bernoulli numbers defined in (2.10).

Theorem 3.1. *Let n be a non negative integer and α be a complex number such that $Re\,\alpha > -1$. Then for $n \in \mathbb{N}$,*

$$\begin{aligned}\beta_{2n,\alpha}(q) &= 2(-1)^{n+1}(q;q)_{2n}\sum_{k=1}^{\infty}\frac{Cos_q(\frac{j_{k,\alpha}^{(2)}}{2(1-q)})}{(j_{k,\alpha}^{(2)})^{2n+1}\,\frac{d}{dz}\mathcal{J}_{\alpha}^{(2)}(z;q^2)\mid_{z=j_{k,\alpha}^{(2)}}},\\ \beta_{2n+1,\alpha}(q) &= 2(-1)^{n}(q;q)_{2n+1}\sum_{k=1}^{\infty}\frac{Sin_q(\frac{j_{k,\alpha}^{(2)}}{2(1-q)})}{(j_{k,\alpha}^{(2)})^{2n+2}\,\frac{d}{dz}\mathcal{J}_{\alpha}^{(2)}(z;q^2)\mid_{z=j_{k,\alpha}^{(2)}}},\end{aligned} \tag{3.1}$$

where $\mathcal{J}_{\alpha}^{(2)}(z;q)$ is defined in (1.7).

Proof. Since

$$G(z) := \frac{E_q(\frac{-z}{2})}{g_{\alpha}^{(2)}(iz;q)} = \sum_{n=0}^{\infty}\beta_{n,\alpha}(q)\frac{z^n}{[n]_q!}, \quad |\, z \,| < \frac{j_{1,\alpha}^{(2)}}{1-q},$$

then

$$\frac{\beta_{n,\alpha}(q)}{[n]_q!} = \frac{G^{(n)}(0)}{n!}, \quad n \in \mathbb{N}_0.$$

Now, we integrate $f(z) := \frac{G(z)}{z^{n+1}}$, $G(z) = \frac{E_q(\frac{-z}{2})}{g_{\alpha}^{(2)}(iz;q)}$ on the contour Γ_m, where Γ_m is a circle of radius R_m, $|\, z_m \,| < R_m < |\, z_{m+1} \,|$. From the Cauchy Residue Theorem, see [2],

$$\int_{\Gamma_m} f(z)\,dz = 2\pi i\sum Res(f, z_k),$$

where $\{z_k\}$ are the poles of f that lie inside Γ_m. The function $f(z)$ has a pole at $z = 0$ of order $n+1$ and simple poles at $\pm z_k$ where $z_k = \frac{ij_{k,\alpha}^{(2)}}{1-q}$, $k \in \mathbb{N}$. Consequently,

$$I_m = \frac{1}{2\pi i}\int_{\Gamma_m} f(z)\,dz = Res(f(z),0) + \sum_{k=1}^{m} Res(f(z), \pm z_k). \tag{3.2}$$

Since

$$\begin{aligned} Res(f,0) &= \frac{f^n(0)}{n!} = \frac{\beta_{n,\alpha}(q)}{[n]_q!},\\ Res(f,z_k) &= \frac{E_q(\frac{-z_k}{2})}{\frac{d}{dz}g_{\alpha}^{(2)}(iz;q)\mid_{z=z_k}}\,\frac{1}{(z_k)^{n+1}}\\ &= \frac{E_q(\frac{-ij_{k,\alpha}^{(2)}}{2(1-q)})}{\frac{d}{dz}\mathcal{J}_{\alpha}^{(2)}(z;q^2)\mid_{z=j_{k,\alpha}^{(2)}}}\,\frac{(i)^{-n}(1-q)^n}{(j_{k,\alpha}^{(2)})^{n+1}},\end{aligned}$$

and

$$Res(f,-z_k)=\frac{E_q(\frac{z_k}{2})}{\frac{d}{dz}g_\alpha^{(2)}(iz;q)\mid_{z=-z_k}}\frac{1}{(-z_k)^{n+1}}$$
$$=\frac{E_q(\frac{ij_{k,\alpha}^{(2)}}{2(1-q)})}{\frac{d}{dz}\mathcal{J}_\alpha^{(2)}(z;q^2)\mid_{z=j_{k,\alpha}^{(2)}}}\frac{(-i)^{-n}(1-q)^n}{(j_{k,\alpha}^{(2)})^{n+1}}.$$

Then Eq. (3.2) can be written as

$$I_m=\frac{\beta_{n,\alpha}(q)}{[n]_q!}+\sum_{k=1}^{m}2Re\left((-i)^{-n}E_q\left(\frac{ij_{k,\alpha}^{(2)}}{2(1-q)}\right)\right)\frac{(1-q)^n}{(j_{k,\alpha}^{(2)})^{n+1}\frac{d}{dz}\mathcal{J}_\alpha^{(2)}(z;q^2)\mid_{z=j_{k,\alpha}^{(2)}}},\tag{3.3}$$

substituting into (3.3) with $-i=e^{-\frac{i\pi}{2}}$ gives

$$I_m=\frac{\beta_{n,\alpha}(q)}{[n]_q!}+2(1-q)^n\cos\frac{n\pi}{2}\sum_{k=1}^{m}\frac{Cos_q\left(\frac{j_{k,\alpha}^{(2)}}{2(1-q)}\right)}{\frac{d}{dz}\mathcal{J}_\alpha^{(2)}(z;q^2)\mid_{z=j_{k,\alpha}^{(2)}}}\frac{1}{(j_{k,\alpha}^{(2)})^{n+1}}$$
$$-2(1-q)^n\sin\frac{n\pi}{2}\sum_{k=1}^{m}\frac{Sin_q\left(\frac{j_{k,\alpha}^{(2)}}{2(1-q)}\right)}{\frac{d}{dz}\mathcal{J}_\alpha^{(2)}(z;q^2)\mid_{z=j_{k,\alpha}^{(2)}}}\frac{1}{(j_{k,\alpha}^{(2)})^{n+1}}.$$

Now, we show that the integral $I_m\to 0$ as $m\to\infty$. Bergweiller and Hayman [7] introduced the asymptotic relation for $E_q(z)$,

$$\mid M(r;E_q)\mid:=\sup\{\mid E_q(z)\mid:\mid z\mid=r\}\sim e^{\frac{-(\log r)^2}{2\log q}},\qquad\text{when}\quad r=\mid z\mid\to\infty.$$

In [4], Annaby and Mansour proved that for $r=\mid z\mid\to\infty$

$$z^{-\nu}J_\nu^{(2)}(z;q)\sim\exp\left(-\frac{(\log r)^2}{2\log q}-\frac{\log 2}{\log q}\log r\right).$$

Hayman in [15] introduced the higher order asymptotics of $J_\nu^{(2)}(z;q)$. Then, Annaby and Mansour, see [4], pointed out that the first order asymptotics of the zeros of $J_\nu^{(2)}(z;q^2)$ is given by

$$j_{m,\nu}^{(2)}=2q^{-2m}q^{-\nu+1}(1+O(q^{2m})),\quad(m\to\infty).$$

Hence if $(z_m)_m$ are the positive zeros of $g_\alpha^{(2)}(iz;q)$, then

$$\lim_{m\to\infty}\frac{z_m}{z_{m+1}}=\lim_{m\to\infty}\frac{j_{m,\nu}^{(2)}}{j_{m+1,\nu}^{(2)}}=q^2,\quad\lim_{m\to\infty}z_m=\infty.\tag{3.4}$$

Let $0 < \epsilon < (q^{-1} - 1)$. There exists $M_0 \in \mathbb{N}$ such that if $m \in \mathbb{N}$, $m \geq M_0$, then

$$q^2(1-\epsilon) < \frac{z_m}{z_{m+1}} < q^2(1+\epsilon).$$

Hence $z_m < qz_{m+1}$ for all $m \geq M_0$. We can choose R_m, $\delta := q^{-1} \sup_{m \geq M_0} \frac{z_m}{z_{m+1}}$ such that $(z_m < \delta R_m < qz_{m+1} < R_m)$. Indeed,

$$\delta = q^{-1} \sup_{m \geq M_0} \frac{z_m}{z_{m+1}} \geq q^{-1} \frac{z_m}{z_{m+1}}, \; m \geq M_0.$$

But $qz_{m+1} < R_m$ leads to $\delta > \frac{z_m}{R_m}$ and so $z_m < \delta R_m$. Now,

$$\delta = q^{-1} \sup_{m \geq M_0} \frac{z_m}{z_{m+1}} \geq q^{-1} \lim_{m \to \infty} \frac{z_m}{z_{m+1}} = q^{-1} q^2 = q.$$

Also $\delta = q^{-1} \sup_{m \geq M_0} \frac{z_m}{z_{m+1}} \leq q(1+\epsilon) < 1$. Hence $1 > \delta \geq q$ and so by

$$z_m \leq R_m \leq \frac{q}{\delta} z_{m+1} \leq z_{m+1}, \tag{3.5}$$

the annulus $\delta R_m < |z| < R_m$ has no zeros of the function $g_\alpha^{(2)}(iz;q)$. Hence, from the minimum modulus principle we have

$$\begin{aligned} \mid g_\alpha^{(2)}(iz;q) \mid &\geq c_1 \, e^{-\frac{(\log \delta R_m)^2}{2 \log q} - \frac{\log 2}{\log q} \log \delta R_m}, \quad c_1 > 0. \\ \mid E_q(\frac{-z}{2}) \mid &\leq c_2 \, e^{\frac{-(\log \frac{R_m}{2})^2}{2 \log q}}, \quad c_2 > 0. \end{aligned} \tag{3.6}$$

Therefore, from (3.6), we conclude that

$$\begin{aligned} \mid \frac{E_q(\frac{-z}{2})}{g_\alpha^{(2)}(iz;q)} \mid &\leq \frac{c_2}{c_1} \frac{e^{\frac{-(\log \frac{R_m}{2})^2}{2 \log q}}}{e^{-\frac{(\log \delta R_m)^2}{2 \log q} - \frac{\log 2}{\log q} \log \delta R_m}} \\ &\leq \frac{c_2}{c_1} \, e^{\frac{1}{2 \log q}\left(((\log \delta R_m)^2 - (\log \frac{R_m}{2})^2)\right) + \frac{\log 2}{\log q} \log \delta R_m} \\ &\leq \frac{c_2}{c_1} \, e^K \, e^{\frac{2 \log 2 \log R_m}{\log q} + \frac{\log \delta \log R_m}{\log q}}, \end{aligned}$$

where

$$K = \frac{1}{2 \log q} \left((\log \delta)^2 - (\log 2)^2 + 2 \log 2 \log \delta \right).$$

Now, using the ML-inequality (see [2]) to obtain

$$\begin{aligned} \mid I_m \mid = \mid \int_{\Gamma_m} f(z) dz \mid &\leq (2\pi R_m) \mid M(r; f(z)) \mid \\ &\leq \frac{2\pi R_m c_2}{c_1} \, e^K \, e^{\frac{2 \log 2 \log R_m}{\log q} + \frac{\log \delta \log R_m}{\log q}} \frac{1}{R_m^{n+1}} \\ &\leq \frac{2\pi c_2}{c_1} \, e^K \, R_m^{\frac{2 \log 2}{\log q}} \, R_m^{\frac{\log \delta}{\log q} - n}. \end{aligned} \tag{3.7}$$

From (3.4) and (3.5), we have $\lim_{m\to\infty} R_m = \infty$. Also, since $0 < q < 1$ and $1 > \delta \geq q$ then

$$R_m^{\frac{2\log 2}{\log q}} \to 0 \quad \textit{and} \quad R_m^{\frac{\log\delta}{\log q}-n} \to 0 \quad \textit{as } m \to \infty.$$

Hence $\lim_{m\to\infty} I_m = 0$. Consequently,

$$\frac{\beta_{n,\alpha}(q)}{[n]_q!} = -2(1-q)^n \cos\frac{n\pi}{2} \sum_{k=1}^{\infty} \frac{Cos_q\left(\frac{j^{(2)}_{k,\alpha}}{2(1-q)}\right)}{\frac{d}{dz}\mathcal{J}^{(2)}_\alpha(z;q^2)\mid_{z=j^{(2)}_{k,\alpha}}} \frac{1}{(j^{(2)}_{k,\alpha})^{n+1}}$$
$$+ 2(1-q)^n \sin\frac{n\pi}{2} \sum_{k=1}^{\infty} \frac{Sin_q\left(\frac{j^{(2)}_{k,\alpha}}{2(1-q)}\right)}{\frac{d}{dz}\mathcal{J}^{(2)}_\alpha(z;q)\mid_{z=j^{(2)}_{k,\alpha}}} \frac{1}{(j^{(2)}_{k,\alpha})^{n+1}}.$$

Therefore,

$$\beta_{2n,\alpha}(q) = 2(-1)^{n+1}(q;q)_{2n} \sum_{k=1}^{\infty} \frac{Cos_q\left(\frac{j^{(2)}_{k,\alpha}}{2(1-q)}\right)}{(j^{(2)}_{k,\alpha})^{2n+1}\,\frac{d}{dz}\mathcal{J}^{(2)}_\alpha(z;q^2)\mid_{z=j^{(2)}_{k,\alpha}}},$$

$$\beta_{2n+1,\alpha}(q) = 2(-1)^{n}(q;q)_{2n+1} \sum_{k=1}^{\infty} \frac{Sin_q\left(\frac{j^{(2)}_{k,\alpha}}{2(1-q)}\right)}{(j^{(2)}_{k,\alpha})^{2n+2}\,\frac{d}{dz}\mathcal{J}^{(2)}_\alpha(z;q^2)\mid_{z=j^{(2)}_{k,\alpha}}},$$

which completes the proof of the theorem. □

Remark 3.2. If we substitute with $\alpha = \frac{1}{2}$ in the second equation in (3.1), then $(z_k)_k$ will be the positive zeros of $Sin_q(z)$ and consequently, the series in the left hand side vanishes which coincide with the known result that the odd Bernoulli numbers vanish ($\beta_{2n+1}(q) = 0$, $n \geq 1$) (see [19]). Similarly, if we set $\alpha = -\frac{1}{2}$ in the first equation in (3.1), the series in the left hand side vanishes and this coincide with the fact that the even Euler's numbers are zero ($E_{2n}(q) = 0$, $n \geq 1$) (see [19]).

Corollary 3.3. *The asymptotic relations of the generalized q-Bernoulli numbers* $(\beta_{n,\alpha}(q))_n$,

$$\beta_{2n,\alpha}(q) = 2(-1)^{n+1}(q;q)_{2n} \frac{Cos_q\left(\frac{j^{(2)}_{1,\alpha}}{2(1-q)}\right)}{(j^{(2)}_{1,\alpha})^{2n+1}\,\frac{d}{dz}\mathcal{J}^{(2)}_\alpha(z;q^2)\mid_{z=j^{(2)}_{1,\alpha}}}\left(1+o(1)\right),$$

$$\beta_{2n+1,\alpha}(q) = 2(-1)^{n}(q;q)_{2n+1} \frac{Sin_q\left(\frac{j^{(2)}_{1,\alpha}}{2(1-q)}\right)}{(j^{(2)}_{1,\alpha})^{2n+2}\,\frac{d}{dz}\mathcal{J}^{(2)}_\alpha(z;q^2)\mid_{z=j^{(2)}_{1,\alpha}}}\left(1+o(1)\right),$$

where $\mathcal{J}^{(2)}_\alpha(z;q)$ *is defined in* (1.7).

Proof. The proof follows directly from Theorem 3.1. □

4. Applications of the Generalized q-Bernoulli Polynomials

In this section, we introduce connection relations between the generalized q-Bernoulli polynomials $B_{n,\alpha}^{(k)}(x;q)$ $(k=1,2,3)$ and the q-Laguerre and the little q-Legendre polynomials.

The q-Laguerre polynomials $L_n^{\alpha}(x;q)$ of degree n are defined by

$$\begin{aligned} L_n^{\alpha}(x;q) :&= \frac{1}{(q;q)_n}\,{}_2\phi_1\left(\begin{matrix} q^{-n}, -x \\ 0 \end{matrix} q; q^{n+\alpha+1}\right) \\ &= \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q^{\alpha+1};q)_k}(-1)^k(q^{n+\alpha+1})^k x^k. \end{aligned} \tag{4.1}$$

The Rodrigues formula is given by

$$L_n^{\alpha}(x;q) = \frac{(1-q)^n}{(q;q)_n}(-x;q)_\infty\, x^{-\alpha}\, D_q^n\left(\frac{x^{\alpha+n}}{(-x;q)_\infty}\right), \tag{4.2}$$

and the orthogonality relation is

$$\begin{aligned} &\int_0^\infty \frac{x^\alpha}{(-x;q)_\infty}L_m^{\alpha}(x;q)L_n^{\alpha}(x;q)dx \\ &\quad = \frac{(q^{-\alpha};q)_\infty}{(q;q)_\infty}\frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n}\Gamma_q(-\alpha)\Gamma_q(\alpha+1)\delta_{mn}, \end{aligned} \tag{4.3}$$

$\alpha > -1$, where δ_{mn} is the Kronecker delta function, see [17,21]. The q-Laguerre polynomials $L_n^{\alpha}(x;q)$ satisfy three term recurrence relation

$$-xa_nL_n^{\alpha}(x;q) = L_{n+1}^{\alpha}(x;q) - b_nL_n^{\alpha}(x;q) + d_nL_{n-1}^{\alpha}(x;q),$$

where

$$a_n = \frac{q^{2n+\alpha+1}}{1-q^{n+1}}, \quad b_n = 1 + \frac{q(1-q^{n+\alpha}}{1-q^{n+1}}, \quad d_n = \frac{q(1-q^{n+\alpha})}{1-q^{n+1}}.$$

In the following, let $\alpha > -1$ and $\mathbb{P}_n = \{p(x) : \deg p(x) \le n\}$ with the inner product

$$\langle p(x), g(x)\rangle = \int_0^\infty \frac{x^\alpha}{(-x;q)_\infty}p(x)g(x)dx,$$

where $p(x),\ g(x) \in \mathbb{P}_n$.

Theorem 4.1. *Let $p(x) \in \mathbb{P}_n$. Then $p(x)$ can be expanded as*

$$p(x) = \sum_{m=0}^{n} C_m L_m^{\alpha}(x;q),$$

where

$$C_m = \frac{q^m(1-q)^{m-1}(q^{\alpha+m+1};q)_\infty}{(q;q)_\infty} \int_0^\infty D_q^m\left(\frac{x^{\alpha+m}}{(-x;q)_\infty}\right) p(x)dx.$$

Proof. Since

$$p(x) = \sum_{m=0}^{n} C_m L_m^\alpha(x;q),$$

in order to calculate the constant C_m, we use (4.3) to obtain

$$\begin{aligned}\langle p(x), L_k^\alpha(x;q)\rangle &= \left\langle \sum_{m=0}^{n} C_m L_m^\alpha(x;q), L_k^\alpha(x;q)\right\rangle \\ &= \sum_{m=0}^{n} C_m \langle L_m^\alpha(x;q), L_k^\alpha(x;q)\rangle.\end{aligned}$$

Then

$$\begin{aligned}\langle p(x), L_m^\alpha(x;q)\rangle &= C_m \langle L_m^\alpha(x;q), L_m^\alpha(x;q)\rangle \\ &= C_m \frac{(q^{\alpha+1};q)_m}{q^m(q;q)_m}(1-q)^{1+\alpha}\Gamma_q(\alpha+1).\end{aligned}$$

Therefore,

$$C_m = \frac{q^m(q;q)_m}{(q^{\alpha+1};q)_m(1-q)^{1+\alpha}\Gamma_q(\alpha+1)} \int_0^\infty \frac{x^\alpha}{(-x;q)_\infty} L_m^\alpha(x;q)p(x)dx. \quad (4.4)$$

Using (4.2) with n replaced by m, we obtain

$$C_m = \frac{q^m(1-q)^{m-1}(q^{\alpha+m+1};q)_\infty}{(q;q)_\infty} \int_0^\infty D_q^m\left(\frac{x^{\alpha+m}}{(-x;q)_\infty}\right) p(x)dx,$$

and the theorem follows. □

The following Lemma, see [18], is essential in the proof of Theorem 4.3.

Lemma 4.2. *Let the functions f and g be defined and continuous on $[0,\infty]$. Assume that the improper Riemann integrals of the functions $f(x)g(x)$ and $f(x/q)g(x)$ exist on $[0,\infty]$. Then*

$$\begin{aligned}\int_0^\infty f(x)D_q g(x)dx &= \frac{f(0)g(0)}{1-q}\ln q - \frac{1}{q}\int_0^\infty g(x)D_{q^{-1}}f(x)dx \\ &= \frac{f(0)g(0)}{1-q}\ln q - \int_0^\infty g(qx)D_q f(x)dx.\end{aligned}$$

Theorem 4.3. *If $n \in \mathbb{N}$ and $x \in \mathbb{C}$, then*

$$B^{(1)}_{n,\alpha}(x;q) = \sum_{m=0}^{n} A_m \left(\sum_{k=m}^{n} q^{\frac{k(2n-k+1)}{2}} \frac{(q^{-n}, q^{-\alpha-k};q)_k (q^{-k};q)_m}{(q;q)_k} \beta_{n-k,\alpha}(q) \right) L^{\alpha}_m(x;q),$$

$$B^{(2)}_{n,\alpha}(x;q) = \sum_{m=0}^{n} A_m \left(\sum_{k=m}^{n} \frac{q^{nk}(q^{-n}, q^{-\alpha-k};q)_k (q^{-k};q)_m}{(q;q)_k} \beta_{n-k,\alpha}(q) \right) L^{\alpha}_m(x;q),$$

$$B^{(3)}_{n,\alpha}(x;q) = \sum_{m=0}^{n} A_m \left(\sum_{k=m}^{n} q^{\frac{k(4n-k+1)}{4}} \frac{(q^{-n}, q^{-\alpha-k};q)_k (q^{-k};q)_m}{(q;q)_k} \beta^{(3)}_{n-k,\alpha}(q) \right) L^{\alpha}_m(x;q),$$

where

$$A_m = \frac{-q^m (q^{\alpha+m+1}, q^{-\alpha};q)_\infty}{(1-q)(q,q;q)_\infty} \frac{\pi}{sin(\alpha\pi)}.$$

Proof. We prove the identity for $B^{(1)}_{n,\alpha}(x;q)$ and the proofs for $B^{(k)}_{n,\alpha}(x;q)$ $(k = 2, 3)$ are similar. Substitute with $p(x) = B^{(1)}_{n,\alpha}(x;q)$ in (4.4). This gives

$$C_m = \frac{q^m (q;q)_m}{(q^{\alpha+1};q)_m (1-q)^{1+\alpha} \Gamma_q(\alpha+1)} \int_0^\infty \frac{x^\alpha}{(-x;q)_\infty} L^{\alpha}_m(x;q) B^{(1)}_{n,\alpha}(x;q) dx. \tag{4.5}$$

Since $\{L^{\alpha}_m(x;q)\}_{n\in\mathbb{N}}$ is an orthogonal polynomials sequence then $C_m = 0$ for $m > n$, and

$$B^{(1)}_{n,\alpha}(x;q) = \sum_{m=0}^{n} C_m L^{\alpha}_m(x;q).$$

Now, we calculate C_m. Using (2.12) in (4.5) gives

$$C_m = \frac{q^m (q;q)_m}{(q^{\alpha+1};q)_m (1-q)^{1+\alpha} \Gamma_q(\alpha+1)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \beta_{n-k,\alpha}(q) \int_0^\infty \frac{x^{\alpha+k}}{(-x;q)_\infty} L^{\alpha}_m(x;q) dx.$$

Since

$$\int_0^\infty \frac{x^\alpha}{(-x;q)_\infty} L^{\alpha}_m(x;q) x^k dx = 0, \quad \text{for } k < m,$$

then

$$C_m = \frac{q^m (q;q)_m}{(q^{\alpha+1};q)_m (1-q)^{1+\alpha} \Gamma_q(\alpha+1)} \sum_{k=m}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \beta_{n-k,\alpha}(q) \int_0^\infty \frac{x^{\alpha+k}}{(-x;q)_\infty} L^{\alpha}_m(x;q) dx.$$

From (4.2), we get

$$C_m = \frac{q^m (1-q)^{m-1} (q^{\alpha+m+1};q)_\infty}{(q;q)_\infty} \sum_{k=m}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \beta_{n-k,\alpha}(q) \int_0^\infty D^m_q \left(\frac{x^{\alpha+m}}{(-x;q)_\infty} \right) x^k dx,$$

then applying the q-integration by part introduced in Lemma 4.2 m times, we obtain

$$C_m = \frac{(-1)^m q^m (1-q)^{m-1} (q^{\alpha+m+1}; q)_\infty}{(q; q)_\infty} \times \sum_{k=m}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\prod_{i=0}^{m-1} q^{i-k} \right) \frac{[k]_q!}{[k-m]_q!} \beta_{n-k,\alpha}(q) \int_0^\infty \frac{x^{\alpha+k}}{(-x; q)_\infty} dx.$$

From [16, Eq. (5.4), p. 465],

$$\frac{1}{\Gamma_q(z)} = \frac{sin\pi z}{\pi} \int_0^\infty \frac{t^{-z}}{(-t(1-q); q)_\infty} dt, \quad Re\, z > 0.$$

Then

$$\int_0^\infty \frac{x^{\alpha+k}}{(-x; q)_\infty} dx = \frac{\pi}{sin(-\alpha-k)\pi} \frac{1}{\Gamma_q(-\alpha-k)} (1-q)^{\alpha+k+1}.$$

Therefore,

$$C_m = \frac{(-1)^m q^m (q^{\alpha+m+1}; q)_\infty}{(1-q)(q, q; q)_\infty} \times \sum_{k=m}^{n} \left(\prod_{i=0}^{m-1} q^{i-k} \right) \frac{(q; q)_n (q^{-\alpha-k}; q)_\infty}{(q; q)_{n-k} (q; q)_{k-m}} \frac{\pi}{sin(-\alpha-k)\pi} \beta_{n-k,\alpha}(q). \tag{4.6}$$

Since

$$\frac{\pi}{sin(-\alpha-k)\pi} = (-1)^{k-1} \frac{\pi}{sin(\alpha\pi)}, \quad \prod_{i=0}^{m-1} q^{i-k} = q^{\frac{m(m-1)}{2}} q^{-km}, \tag{4.7}$$

then substituting from (4.7) into (4.6), we get

$$C_m = \frac{(-1)^m q^m q^{m(m-1)/2} (q^{\alpha+m+1}; q)_\infty (q^{-\alpha}; q)_\infty}{(1-q)(q, q; q)_\infty} \frac{\pi}{sin(\alpha\pi)} \times \sum_{k=m}^{n} (-1)^{k-1} q^{-km} \frac{(q; q)_n (q^{-\alpha-k}; q)_k}{(q; q)_{n-k} (q; q)_{k-m}} \beta_{n-k,\alpha}(q).$$

Using the relation (2.47), we obtain

$$C_m = \frac{-q^m (q^{\alpha+m+1}; q^{-\alpha}; q)_\infty}{(1-q)(q, q; q)_\infty} \frac{\pi}{sin(\alpha\pi)} \times \sum_{k=m}^{n} q^{\frac{k(2n-k+1)}{2}} \frac{(q^{-n}; q)_k (q^{-k}; q)_m (q^{-\alpha-k}; q)_k}{(q; q)_k} \beta_{n-k,\alpha}(q),$$

and this completes the proof of the theorem. □

The little q-Legendre polynomials $(P_n(x \mid q))_n$ are defined by

$$P_n(x \mid q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{n+1} \\ q \end{matrix}\; q; qx\right) = \sum_{k=0}^{n} \frac{(q^{-n}; q)_k (q^{n+1}; q)_k}{(q; q)_k} \frac{q^k x^k}{(q; q)_k}.$$

They satisfy the Rodrigues formula

$$P_n(x \mid q) = \frac{q^{n(n-1)/2}(1-q)^n}{(q; q)_n} D_{q^{-1}}^n (x^n (qx; q)_n), \quad \text{for } n \geq 0, \tag{4.8}$$

and the orthogonality relation

$$\int_0^1 P_m(x \mid q) P_n(x \mid q) d_q x = \frac{(1-q)}{(1-q^{2n+1})} \delta_{mn}, \quad \text{for } m, n \geq 0, \tag{4.9}$$

see [21]. Let $\mathbb{P}_n = \{g(x) : \deg g(x) \leq n\}$ with the inner product

$$\langle g(x), p(x) \rangle = \int_0^1 g(x) p(x) d_q x,$$

where $p(x)$, $g(x) \in \mathbb{P}_n$.

Theorem 4.4. *Let $g(x) \in \mathbb{P}_n$. Then $g(x)$ can be represented by*

$$g(x) = \sum_{k=0}^{n} C_k P_k(x \mid q),$$

where

$$C_k = \frac{q^{k(k-1)/2}(1-q)^{k-1}(1-q^{2k+1})}{(q; q)_k} \int_0^1 D_{q^{-1}}^k (x^k (qx; q)_k) g(x) d_q x.$$

Proof. Since

$$g(x) = \sum_{k=0}^{n} C_k P_k(x \mid q),$$

then by the orthogonality relation (4.9), we obtain

$$C_k = \frac{(1-q^{2k+1})}{(1-q)} \langle g(x), P_k(x \mid q) \rangle = \frac{(1-q^{2k+1})}{(1-q)} \int_0^1 P_k(x \mid q) g(x) d_q x. \tag{4.10}$$

By using (4.8), we get

$$C_k = \frac{q^{k(k-1)/2}(1-q)^{k-1}(1-q^{2k+1})}{(q; q)_k} \int_0^1 D_{q^{-1}}^k (x^k (qx; q)_k) g(x) d_q x,$$

which readily gives the result. □

Theorem 4.5. *For $n \in \mathbb{N}$ and $x \in \mathbb{C}$,*

$$B^{(1)}_{n,\alpha}(x;q) = \sum_{k=0}^{n} \lambda_k \left(\sum_{m=k}^{n} (-1)^m q^{\frac{m(2n-m+1)}{2}+1} \frac{(q^{-n};q)_m (q^{-m};q)_k}{(q;q)_{m+k+1}} \beta_{n-m,\alpha}(q) \right) P_k(x \mid q),$$

$$B^{(2)}_{n,\alpha}(x;q) = \sum_{k=0}^{n} \lambda_k \left(\sum_{m=k}^{n} (-1)^m q^{nm+1} \frac{(q^{-n};q)_m (q^{-m};q)_k}{(q;q)_{m+k+1}} \beta_{n-m,\alpha}(q) \right) P_k(x \mid q),$$

$$B^{(3)}_{n,\alpha}(x;q) = \sum_{k=0}^{n} \lambda_k \left(\sum_{m=k}^{n} (-1)^m q^{\frac{m(4n-m+1)}{4}+1} \frac{(q^{-n};q)_m (q^{-m};q)_k}{(q;q)_{m+k+1}} \beta^{(3)}_{n-m,\alpha}(q) \right) P_k(x \mid q),$$

where

$$\lambda_k = q^{\frac{-k(k-3)}{2}} (1 - q^{2k+1}).$$

Proof. Substitute with $g(x) = B^{(1)}_{n,\alpha}(x;q)$ in (4.10), we obtain

$$C_k = \frac{(1-q^{2k+1})}{(1-q)} \int_0^1 P_k(x \mid q) B^{(1)}_{n,\alpha}(x;q) d_q x. \tag{4.11}$$

Since the polynomials $\{P_k(x \mid q)\}$ are orthogonal, then $C_k = 0$ for $k > n$, and

$$B^{(1)}_{n,\alpha}(x;q) = \sum_{k=0}^{n} C_k P_k(x \mid q). \tag{4.12}$$

Set

$$B^{(1)}_{n,\alpha}(x;q) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \beta_{n-m,\alpha}(q) x^m.$$

From (4.11),

$$\begin{aligned} C_k &= \frac{(1-q^{2k+1})}{(1-q)} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \beta_{n-m,\alpha}(q) \int_0^1 P_k(x \mid q) x^m d_q x \\ &= \frac{(1-q^{2k+1})}{(1-q)} \sum_{m=k}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \beta_{n-m,\alpha}(q) \int_0^1 P_k(x \mid q) x^m d_q x, \end{aligned}$$

since

$$\int_0^1 P_k(x \mid q) x^m d_q x = 0 \quad \mathit{for}\ m < k.$$

Hence, by the Rodrigues formula in (4.8), we obtain

$$\begin{aligned} C_k = \frac{(1-q^{2k+1}) q^{k(k-1)/2} (1-q)^{k-1}}{(q;q)_k} \sum_{m=k}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \beta_{n-m,\alpha}(q) \\ \times \int_0^1 D^k_{q^{-1}} (x^k (qx;q)_k) x^m d_q x. \end{aligned} \tag{4.13}$$

Using the q^{-1}-integration by parts

$$\int_0^a f\left(\frac{t}{q}\right) D_{q^{-1}}g(t)d_qt = q\left((fg)\left(\frac{a}{q}\right) - (fg)(0)\right) - \int_0^a g(t)D_{q^{-1}}f(t)d_qt, \tag{4.14}$$

where f and g are continuous functions at zero, see [5]. This gives

$$\begin{aligned}\int_0^1 D^k_{q^{-1}}(x^k(qx;q)_k)x^m d_qx =& q\left[x^m D^{k-1}_{q^{-1}}(x^k(qx;q)_k)\right]_0^{\frac{1}{q}} \\ &- [m]_q q^{1-m}\int_0^1 x^{m-1}D^{k-1}_{q^{-1}}(x^k(qx;q)_k)d_qx.\end{aligned} \tag{4.15}$$

The first term on the right hand side of (4.15) vanishes because

$$D_{q^{-1}}(x^k(qx;q)_k) = [k]_{q^{-1}}x^{k-1}(x;q)_k + x^k D_{q^{-1}}(qx;q)_k,$$

and

$$D^j_{q^{-1}}(qx;q)_k \mid_{x=\frac{1}{q}} = a^k \frac{[k]_q!}{[k-j]_q!}(1;q)_{k-j} = 0, \quad \text{for } j = 0,1,\ldots,k-1.$$

Therefore,

$$\int_0^1 D^k_{q^{-1}}(x^k(qx;q)_k)x^m d_qx = -[m]_q q^{1-m}\int_0^1 x^{m-1}D^{k-1}_{q^{-1}}(x^k(qx;q)_k)d_qx. \tag{4.16}$$

Now, applying (4.14) $k-1$ times on the right hand side of (4.16), and using that $D^m_{q^{-1}}(x^k(qx;q)_k = 0$ at $x=0$, $x=\frac{1}{q}$ $(m=0,1,\ldots,k-1)$ yields

$$\int_0^1 D^k_{q^{-1}}(x^k(qx;q)_k)x^m d_qx = (-1)^k\left(\prod_{j=0}^{k-1} q^{1-m-j}\right)\frac{[m]_q!}{[m-k]_q!}\int_0^1 x^m(qx;q)_k\, d_qx.$$

Since

$$\begin{aligned}B_q(x,y) &= \int_0^1 t^{x-1}(qt;q)_{y-1}d_qt \\ &= \int_0^1 t^{x-1}\frac{(tq;q)_\infty}{(tq^y;q)_\infty}d_qt, \quad Re\,(x)>0,\ Re\,(y)>0,\end{aligned}$$

see [5, Eq. (1.58), p. 22], then

$$\begin{aligned}\int_0^1 D^k_{q^{-1}}(x^k(qx;q)_k)x^m d_qx &= (-1)^k q^{\frac{-k(k-1)}{2}+k}q^{-mk}\frac{[m]_q!}{[m-k]_q!}B_q(m+1,k+1) \\ &= (-1)^k q^{\frac{-k(k-3)}{2}}q^{-mk}\frac{[m]_q!\Gamma_q(m+1)\Gamma_q(k+1)}{[m-k]_q!\Gamma_q(m+k+2)} \\ &= (-1)^k q^{\frac{-k(k-3)}{2}}q^{-mk}\frac{([m]_q!)^2[k]_q!}{[m-k]_q![m+k+1]_q!}.\end{aligned} \tag{4.17}$$

Substituting from (4.17) into (4.13) yields

$$\begin{aligned} C_k &= (-1)^k q^k (1-q^{2k+1}) \sum_{m=k}^{n} q^{-mk} \frac{(q;q)_n (q;q)_m}{(q;q)_{n-m}(q;q)_{m-k}(q;q)_{m+k+1}} \beta_{n-m,\alpha}(q) \\ &= q^{\frac{-k(k-3)}{2}} (1-q^{2k+1}) \sum_{m=k}^{n} (-1)^m \frac{q^{\frac{m(2n-m+1)}{2}+1}(q^{-n};q)_m (q^{-m};q)_k}{(q;q)_{m+k+1}} \beta_{n-m,\alpha}(q), \end{aligned} \tag{4.18}$$

where we used the identity in (2.47). Therefore, from (4.18) and (4.12), we get the required result for $B^{(1)}_{n,\alpha}(x;q)$. Similarly, we can prove the result for $B^{(k)}_{n,\alpha}(x;q)\,(k=2,3)$. □

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Declarations

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