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Results in Mathematics

Weighted Approximation of Functions by the Szász–Mirakjan–Kantorovich Operator

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Abstract. We investigate the weighted approximation of functions in L_p-norm by Kantorovich modifications of the classical Szász–Mirakjan operator, with weights of type $(1+x)^\alpha$, $\alpha \in \mathbb{R}$. By defining an appropriate K-functional we prove direct inequality for them.

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1. Introduction

The classical Szász–Mirakjan operator (see [9, 10]) is defined for bounded functions $f(x)$ in $[0,\infty)$ by the formula

$$S_n f(x) = S_n(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) s_{n,k}(x), \quad x \geq 0, \tag{1.1}$$

where

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$$

and the Kantorovich modification of S_n is defined (see, for instance, [3, Chapter 9]) by

$$\tilde{S}_n f(x) = \tilde{S}_n(f;x) = \sum_{k=0}^{\infty} s_{n,k}(x)\, n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(u)du, \quad x \geq 0. \tag{1.2}$$

This operator is well-defined for every function $f(x)$, which is summable on any finite closed subinterval of $[0,\infty)$.

There are many papers about weighted approximation of functions by S_n in uniform norm—see, for instance, the bibliography of [6]. That is not the case about weighted approximation of functions by Kantorovich modifications of S_n. The best results, to our knowledge, are the next inequalities of weak type in terminology of [2], proved in [3, p.159, Theorem 10.1.3.].

Let $w^*(x) = x^{\gamma(0)}(1+x)^{\gamma(\infty)}$ where $\gamma(\infty)$ is arbitrary and $-1/p < \gamma(0) < 1 - 1/p$ for $1 \le p \le \infty$, for $p = 1$ and $p = \infty$ $\gamma(0)$ may also be equal to zero.

Theorem 1.1. *Suppose $w^* f \in L_p[0,\infty)$ and either $1 \le p \le \infty$ and $\alpha < 1$, or $1 < p < \infty$ and $\alpha \le 1$. Then for $\tilde{S}_n^*$ the next equivalency is true.*

$$\left\| w^* \left(\tilde{S}_n f - f\right) \right\|_{L_p[0,\infty)} = O\Big(n^{-\alpha}\Big) \Leftrightarrow \left\| w^* \Delta^2_{h\sqrt{\varphi}} f \right\|_{L_p[2h^2,\infty)} = O\Big(h^{2\alpha}\Big).$$

Here $\| \circ \|_{p(J)}$ stands for the usual L_p-norm on the interval J, $\varphi(x) = x$ and

$$\Delta^2_{h\sqrt{\varphi(x)}}(f,x) = f\left(x - h\sqrt{\varphi(x)}\right) - 2f(x) + f\left(x + h\sqrt{\varphi(x)}\right).$$

Our goal in this paper is to investigate the approximation of functions in the L_p-norm by the Szász–Mirakjan–Kantorovich operator. We prove Jackson-type inequality for the weighted error of approximation and by defining an appropriate K-functional we prove direct inequality for it.

Before stating our main result, let us introduce the needed notations. The weights under consideration in our survey are

$$w(x) = (1+x)^{\alpha}, \quad \alpha \in \mathbb{R}. \tag{1.3}$$

By $\varphi(x) = x$ we denote the weight which is naturally connected with the second moment of the Szász–Mirakjan operator. The first derivative operator is denoted by $D = \frac{d}{dx}$. Thus, $Dg(x) = g'(x)$ and $D^k g(x) = g^{(k)}(x)$ for every natural k. We define the second order differential operator $\tilde{D}$ by the formula

$$\tilde{D}g(x) = D(\varphi D g)(x) = x g''(x) + g'(x).$$

The space $AC_{loc}(0,\infty)$ consists of the functions which are absolutely continuous in $[a,b]$ for every $[a,b] \subset (0,\infty)$. We also set

$$L_p(w) = \{f : f, Df \in AC_{loc}(0,\infty),\ w(x)f(x) \in L_p[0,\infty)\},$$

$$W_p(w) = \begin{cases} \{f : f, Df \in AC_{loc}(0,\infty), w(x)\tilde{D}f \in L_p[0,\infty), \lim\limits_{x\to 0_+} \varphi(x)Df(x) = 0\}, \alpha \le 0 \\ \{f : f, Df \in AC_{loc}(0,\infty), w(x)\tilde{D}f \in L_p[0,\infty), \lim\limits_{x\to 0_+,\infty} \varphi(x)Df(x) = 0\}, \alpha > 0 \end{cases},$$

$$L_p(w) + W_p(w) = \{f : f = f_1 + f_2,\ f_1 \in L_p(w),\ f_2 \in W_p(w)\}.$$

Also, we define a K-functional $K_w(f,t)_p$ for $t > 0$, by

$$K_w(f,t)_p = \inf\left\{ \|w(f-g)\|_p + t\left\|w\tilde{D}g\right\|_p : f - g \in L_p(w),\ g \in W_p(w) \right\}. \tag{1.4}$$

The relation "$\theta_1(f,t)$ is equivalent to $\theta_2(f,t)$", in notation: $\theta_1(f,t) \sim \theta_2(f,t)$, means that there exists a positive constant C independent of f and t such that

$$C^{-1}\theta_1(f,t) \le \theta_2(f,t) \le C\theta_1(f,t).$$

Above and throughout C denotes a positive constant, not necessarily the same at each occurrence, which is independent of the function $f(x)$ (or $g(x)$), and the parameter n (or t) in the specified range.

Our main results are the following theorems. The first one is a Jackson-type inequality. It shows that the rate of convergence of $\tilde{S}_n$ is at least n^{-1} if the approximated function is smooth enough.

Theorem 1.2. *Let $\tilde{S}_n$ be defined by (1.2), $w(x)$ by (1.3) and $1 \le p \le \infty$. Then there exist absolute constant $C > 0$ such that for all $f \in W_p(w)$ and all $n \in \mathbb{N}$ there hold*

$$\left\| w\left(\tilde{S}_n f - f\right)\right\|_p \le \frac{C(\alpha)}{n}\left\|w\tilde{D}f\right\|_p.$$

Theorem 1.3. *Let $\tilde{S}_n$ be defined by (1.2), the K-functional be given by (1.4), $w(x)$ by (1.3) and $1 \le p \le \infty$. Then there exist absolute constant $C > 0$ such that for all $f \in L_p(w) + W_p(w)$ and all $n \in \mathbb{N}$ there hold*

$$\|w(\tilde{S}_n f - f)\|_p \le CK_w\left(f, \frac{1}{n}\right)_p.$$

Remark 1.4. The inequalities in Theorems 1.2 and 1.3 are stronger than the results mentioned above. They are stronger even for $w(x) = 1$ - see [8, p. 4]

Remark 1.5. Very important question is how to characterize the K-functionals $K_w(f,t)_p$ by appropriate moduli of smoothness. To our knowledge it is completely open even for $w(x) = 1$. In series of papers [4,5,7] the authors introduced new moduli of smoothness and characterized the next weighted K-functionals

$$K^*_w(f,t)_p = \inf\left\{\|w(f-g)\|_p + t\left\|wD^2g\right\|_p : f-g, D^2g \in L_p(w)\right\},$$

which are different from $K_w(f,t)_p$. But under some additional restrictions on the functions f, for $p > 1$, they could be used in order to characterize the K-functionals $K_w(f,t)_p$. For $p = 1$ probably new moduli are needed, even in the unweighted case, i.e. $w(x) = 1$.

2. Auxiliary Results

In this section we collect some properties of S_n, $\tilde{S}_n$ and $s_{n,k}$, which can be found in [3,11,12], or verified by direct computation. Here we also prove all the lemmas we need to establish the main result.

We begin with the relations:

$$\sum_{k=0}^{\infty} s_{n,k}(x) = 1, \quad x \geq 0, \tag{2.1}$$

$$\sum_{k=0}^{\infty} k\, s_{n,k}(x) = nx, \quad x \geq 0, \tag{2.2}$$

$$\int_0^{\infty} s_{n,k}(x)\, dx = \frac{1}{n}. \tag{2.3}$$

The first several moments of the operators S_n and $\tilde{S}_n$ are

$$S_n(1,x) = 1, \quad S_n(\circ - x, x) = 0, \quad S_n\left((\circ - x)^2, x\right) = \frac{\varphi(x)}{n}; \tag{2.4}$$

$$\tilde{S}_n(1,x) = 1, \quad \tilde{S}_n(\circ - x, x) = \frac{1}{2n}, \quad \tilde{S}_n((\circ - x)^2, x) = \frac{\varphi(x)}{n} + \frac{1}{2n^2}. \tag{2.5}$$

Generally, it was shown in [3, (9.4.14)]

$$S_n\left((\circ - x)^{2m}, x\right) \leq C(m)\left(\frac{\varphi(x)}{n}\right)^m \quad \text{for} \quad x \geq \frac{1}{n}, \quad m \in \mathbb{N}. \tag{2.6}$$

Also, in [3] the next inequalities are proved. In [3, p.161, section 10.2] the next inequality about the boundedness of $\tilde{S}_n$ in weighted norm is proved, i.e. for every function $f \in L_p(w)$ the next inequality is true

$$\left\| w\tilde{S}_n f \right\|_p \leq C(\alpha) \left\| wf \right\|_p, \tag{2.7}$$

and in [3, p.163]

$$\sum_{k=0}^{\infty} s_{n,k}(x)\left(1 + \frac{k}{n}\right)^m \leq C(1+x)^m \quad \text{where} \quad m \in \mathbb{Z}. \tag{2.8}$$

We need to prove some additional lemmas. The first one is a simple generalization of inequality (2.8).

Lemma 2.1. *For $\alpha \in \mathbb{R}$ there exists a constant $C(\alpha)$ such that for every natural $n \geq |\alpha|$ and every $x \in [0, \infty)$*

$$\sum_{k=0}^{\infty} \left(1 + \frac{k}{n}\right)^{\alpha} s_{n,k}(x) \leq C(\alpha)(1+x)^{\alpha}. \tag{2.9}$$

Proof. Let $m \in \mathbb{N}$ be the smallest integer such that $m > |\alpha|$. By Holder's inequality we have

$$\sum_{k=0}^{\infty} s_{n,k}(x)\left(1 + \frac{k}{n}\right)^{\alpha} \leq \left\{\sum_{k=0}^{\infty} s_{n,k}(x)\left(1 + \frac{k}{n}\right)^{sign(\alpha)m}\right\}^{\frac{|\alpha|}{m}} \left\{\sum_{k=0}^{\infty} s_{n,k}(x)\right\}^{1-\frac{|\alpha|}{m}}.$$

and the lemma follows from (2.8) and (2.4). □

We need the next very important technical result.

Lemma 2.2. *For every integer m there exists a constant $C(m)$ such that for every naturals n and k, $n > |m|$ the next inequalities are true*

$$1 - \frac{C(m)}{n\left(1+\frac{k}{n}\right)^2} \le \frac{n^{m+1}(n+k)!}{(n+k+m)!}\int_0^\infty s_{n,k}(x)(1+x)^m dx \le 1. \tag{2.10}$$

Proof. We consider two cases.

1. $m \ge 0$.

Obviously the inequalities are true for $m = 0$. And for $m > 0$ we have

$$\int_0^\infty s_{n,k}(x)(1+x)^{m+1}dx = \int_0^\infty s_{n,k}(x)(1+x)^m dx + \frac{k+1}{n}\int_0^\infty s_{n,k+1}(x)(1+x)^m dx$$

and consequently

$$\begin{aligned}
&\frac{n^{m+2}(n+k)!}{(n+k+m+1)!}\int_0^\infty s_{n,k}(x)(1+x)^{m+1}dx \\
&\quad = \frac{n}{n+k+m+1}\frac{n^{m+1}(n+k)!}{(n+k+m)!}\int_0^\infty s_{n,k}(x)(1+x)^m dx \\
&\qquad + \frac{k+1}{n+k+1}\frac{n^{m+1}(n+k+1)!}{(n+k+m+1)!}\int_0^\infty s_{n,k+1}(x)(1+x)^m dx.
\end{aligned}$$

Now, inductively we have

$$\frac{n^{m+2}(n+k)!}{(n+k+m+1)!}\int_0^\infty s_{n,k}(x)(1+x)^{m+1}dx < \frac{n}{n+k+m+1} + \frac{k+1}{n+k+1} < 1$$

and

$$\begin{aligned}
&\frac{n^{m+2}(n+k)!}{(n+k+m+1)!}\int_0^\infty s_{n,k}(x)(1+x)^{m+1}dx \\
&\quad > \frac{n}{n+k+m+1}\left[1 - \frac{C_1(m)}{n\left(1+\frac{k}{n}\right)^2}\right] + \frac{k+1}{n+k+1}\left[1 - \frac{C_2(m)}{n\left(1+\frac{k+1}{n}\right)^2}\right] \\
&\quad = \frac{n}{n+k+m+1} + \frac{k+1}{n+k+1} - \frac{C_3(m)}{n\left(1+\frac{k}{n}\right)^2} \\
&\quad = 1 - \frac{m}{n\left(1+\frac{k+1}{n}\right)\left(1+\frac{k+m+1}{n}\right)} - \frac{C_3(m)}{n\left(1+\frac{k}{n}\right)^2} = 1 - \frac{C(m)}{n\left(1+\frac{k}{n}\right)^2}.
\end{aligned}$$

2. $m < 0$.

Let us denote

$$I_{k,m} = \int_0^\infty s_{n,k}(x)(1+x)^m dx.$$

We have

$$I_{k,m} = \frac{n}{k} I_{k-1,m+1} - \frac{n}{k} I_{k-1,m} \tag{2.11}$$

and after integrating by parts

$$I_{k,m} = I_{k-1,m} + \frac{m}{n} I_{k,m-1}. \tag{2.12}$$

Multiplying (2.12) by $\frac{n}{k}$ and summing with (2.11) we obtain

$$I_{k,m} = \frac{n}{n+k} I_{k-1,m+1} + \frac{m}{n+k} I_{k,m-1} \tag{2.13}$$

and consequently

$$I_{k,m} \leq \frac{n}{n+k} I_{k-1,m+1}. \tag{2.14}$$

2a. $k \geq |m|$.

Applying (2.14) m times and using (2.3) we have

$$I_{k,m} \leq \frac{n}{n+k} \cdots \frac{n}{n+k+m+1} I_{k+m,0} = \frac{(n+k+m)!}{n^{m+1}(n+k)!} \tag{2.15}$$

i.e

$$\frac{n^{m+1}(n+k)!}{(n+k+m)!} \int_0^\infty s_{n,k}(x)(1+x)^m dx \leq 1.$$

From (2.15) we have

$$I_{k,m-1} \leq \frac{(n+k+m-1)!}{n^m(n+k)!}$$

and than from (2.13) it follows

$$I_{k,m} \geq \frac{n}{n+k} I_{k-1,m+1} + \frac{m}{n+k} \frac{(n+k+m-1)!}{n^m(n+k)!}$$

or

$$\begin{aligned} \frac{n^{m+1}(n+k)!}{(n+k+m)!} I_{k,m} \geq & \frac{n^{m+2}(n+k-1)!}{(n+k+m)!} I_{k-1,m+1} \\ & + \frac{mn}{(n+k)(n+k+m)} \frac{(n+k+m-1)!}{n^m(n+k)!} \end{aligned}$$

i.e.

$$\frac{n^{m+1}(n+k)!}{(n+k+m)!} I_{k,m} \geq \frac{n^{m+2}(n+k-1)!}{(n+k+m)!} I_{k-1,m+1} + \frac{C(m)}{n\left(1+\frac{k}{n}\right)^2}.$$

Applying this inequality m times we obtain (2.10).

2b. $k < |m|$.

Applying (2.14) k times

$$I_{k,m} \le \frac{n^k n!}{(n+k)!} I_{0,m+k}.$$

But

$$I_{0,m+k} = \int_0^\infty s_{n,0}(x)(1+x)^{m+k}dx < \int_0^\infty s_{n,0}(x)dx = \frac{1}{n}$$

and consequently

$$I_{k,m} \le \frac{n^{k-1} n!}{(n+k)!} \le \frac{(n+k+m)!}{n^{m+1}(n+k)!},$$

i.e.

$$\frac{n^{m+1}(n+k)!}{(n+k+m)!} \int_0^\infty s_{n,k}(x)(1+x)^m dx \le 1.$$

We have

$$\begin{aligned} n\int_0^\infty s_{n,k}(x)(1+x)^m dx > n\int_0^\infty \frac{e^{-(n-m)}(nx)^k}{k!}dx \\ = \left(\frac{n}{n-m}\right)^k n\int_0^\infty s_{n-m,k}(x)dx \\ = \left(\frac{n}{n-m}\right)^k > 1 + \frac{km}{n-m} > 1 - \frac{C(m)}{n}. \end{aligned}$$

Then

$$\begin{aligned} \frac{n^{m+1}(n+k)!}{(n+k+m)!} \int_0^\infty s_{n,k}(x)(1+x)^m dx > \prod_{k=0}^{|m|-1} \left(1 + \frac{k-i}{n}\right)\left(1 - \frac{C(m)}{n}\right) \\ > 1 - \frac{C(m)}{n}. \end{aligned}$$

The lemma is proved.

□

The next lemma is an elementary consequence of this lemma.

Lemma 2.3. *For $w(x)$ defined by (1.3) the next inequality is true*

$$\int_0^\infty w(x)s_{n,k}(x)dx \le \frac{C(\alpha)}{n} w\left(\frac{k}{n}\right). \tag{2.16}$$

Proof. By Holder's inequality applied for the smallest integer m such that $m > |\alpha|$ we have

$$\int_0^\infty w(x)s_{n,k}(x)dx \le \left\{\int_0^\infty s_{n,k}(x)(1+x)^{sign(\alpha)m}dx\right\}^{\frac{|\alpha|}{m}} \left\{\int_0^\infty s_{n,k}(x)dx\right\}^{1-\frac{|\alpha|}{m}}$$

and the lemma follows from (2.10) and (2.3). □

We need more technical results. Let us denote by ϕ the function $\phi(z) = \ln z$. In [8, p.7, 2.20 and p.11, 2.26] the next two estimations for $\tilde{S}$ are proved

$$\left\|\tilde{S}_n\phi - \phi\right\|_1 \le \frac{C}{n} \tag{2.17}$$

and

$$\left\|x\left(\tilde{S}_n\phi - \phi\right)\right\|_\infty \le \frac{C}{n}. \tag{2.18}$$

Lemma 2.4. *Let $1 \le p \le \infty$. Then for all $f \in W_p(w)$ and $n \in \mathbb{N}$ there holds*

$$\left\|\varphi w\left(\tilde{S}_n\phi - \phi\right)Df\right\|_p \le \frac{C(\alpha)}{n}\left\|w\tilde{D}f\right\|_p.$$

Proof. We consider the cases $\alpha \le 0$ and $\alpha > 0$ separately.

1. $\alpha \le 0$.

We have

$$\begin{aligned}|\varphi(x)w(x)Df(x)| &= |xw(x)Df(x)| \\ &= \left|w(x)\int_0^x \tilde{D}f(t)dt\right| \le \int_0^x \left|w(t)\tilde{D}f(t)\right|dt,\end{aligned}$$

and consequently

$$|\varphi(x)w(x)Df(x)| \le \left|\int_0^\infty w(t)\tilde{D}f(t)dt\right| = \left\|w\tilde{D}f\right\|_1$$

and

$$|\varphi(x)w(x)Df(x)| \le x\left\|w\tilde{D}f\right\|_\infty.$$

Then for $p = 1$

$$\left\|w\varphi\left(\tilde{S}_n\phi - \phi\right)Df\right\|_1 \le \left\|w\tilde{D}f\right\|_1\left\|\tilde{S}_n\phi - \phi\right\|_1 \le \frac{C}{n}\left\|w\tilde{D}f\right\|_1$$

where for the last inequality we used the estimation (2.17).
For $p = \infty$

$$\left\|w\varphi\left(\tilde{S}_n\phi - \phi\right)Df\right\|_\infty \le \left\|w\tilde{D}f\right\|_1\left\|\varphi\left(\tilde{S}_n\phi - \phi\right)\right\|_\infty \le \frac{C}{n}\left\|w\tilde{D}f\right\|_\infty$$

where for the last inequality we used the estimation (2.18).

2. $\alpha > 0$.

For L_1 we use the representation

$$|\varphi(x)w(x)Df(x)| = \left| w(x)\int_x^\infty \tilde{D}f(t)dt \right| \le \int_v^\infty \left| w(t)\tilde{D}f(t)\right| dt.$$

Then as above

$$\left\| w\varphi \left(\tilde{S}_n\phi - \phi\right) Df \right\|_1 \le \left\| w\tilde{D}f \right\|_1 \left\| \tilde{S}_n\phi - \phi \right\|_1 \le \frac{C}{n} \left\| w\tilde{D}f \right\|_1 .$$

For L_∞ we consider two cases.

2a. $x \le 1$.

In this case $(1+x)^\alpha \le 2^\alpha(1+t)^\alpha$ for $0 \le t \le 1$. Consequently,

$$\begin{aligned}\left|\varphi(x)w(x)\left(\tilde{S}_n(\phi;x) - \phi(x)\right) Df(x)\right| &= \left| w(x)\int_0^x \tilde{D}f(t)dt \left[\tilde{S}_n(\phi;x) - \phi(x)\right]\right| \\ &\le 2^\alpha \int_0^x \left|w(t)\tilde{D}f(t)\right| dt \left|\tilde{S}_n(\phi;x) - \phi(x)\right| \\ &\le C(\alpha)\left\|w\tilde{D}f\right\|_\infty \left|x\left(\tilde{S}_n(\phi;x) - \phi(x)\right)\right| \\ &\le C(\alpha)\left\|w\tilde{D}f\right\|_\infty \left\|\varphi\left(\tilde{S}_n\phi - \phi\right)\right\|_\infty \\ &\le \frac{C(\alpha)}{n}\end{aligned}$$

where we used again the estimation (2.18).

2b. $x > 1$.

In this case $(1+x)^\alpha \sim x^\alpha$. And for $0 < \alpha < 1$ by using the Hardy's inequality and the estimation (2.18) we obtain

$$\begin{aligned}&\left|w(x)\varphi(x)\left(\tilde{S}_n(\phi;x) - \phi(x)\right) Df(x)\right| \\ &\quad\le \left|\frac{w(x)}{x}\int_0^x \tilde{D}f(t)dt\right| \left|x\left(\tilde{S}_n(\phi;x) - \phi(x)\right)\right| \\ &\quad\le C\left(\frac{1}{x^{1-\alpha}}\int_0^x \left|t\tilde{D}f(t)\right|\frac{dt}{t}\right)\left\|\varphi\left(\tilde{S}_n\phi - \phi\right)\right\|_\infty \\ &\quad\le \frac{C(\alpha)}{n}\left\|w\tilde{D}f\right\|_\infty .\end{aligned}$$

For $\alpha \geq 1$ again, using the Hardy's inequality and the estimation (2.18), we obtain

$$\begin{aligned}
&\left| w(x)\varphi(x)\left(\tilde{S}_n(\phi,x)-\phi(x)\right)Df(x)\right| \\
&\leq \left|\frac{w(x)}{x}\int_x^\infty \tilde{D}f(t)dt\right|\left|x\left(\tilde{S}_n(\phi;x)-\phi(x)\right)\right| \\
&\leq C\left(x^{\alpha-1}\int_x^\infty \left|t\tilde{D}f(t)\right|\frac{dt}{t}\right)\left\|\varphi\left(\tilde{S}_n\phi-\phi\right)\right\|_\infty \\
&\leq \frac{C(\alpha)}{n}\left\|w\tilde{D}f\right\|_\infty .
\end{aligned}$$

□

Lemma 2.5. *For every $x \in [0,\infty)$ and $n \in \mathbb{N}$ there holds*

$$\left|w(x)\tilde{S}_n\left(\frac{(\cdot)-x}{w(\cdot)};x\right)\right| \leq \frac{C}{n}.$$

Proof. We have by using the Lagrange's formula

$$\begin{aligned}
\left|w(x)\tilde{S}_n\left(\frac{(\cdot)-x}{w(\cdot)};x\right)\right| &= \left|w(x)\sum_{k=0}^\infty s_{n,k}(x)n\int_{\frac{k}{n}}^{\frac{k+1}{n}}\frac{t-x}{w(t)}dt\right| \\
&\leq \left|\sum_{k=0}^\infty s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)}n\int_{\frac{k}{n}}^{\frac{k+1}{n}}(t-x)\,dt\right| \\
&\quad + w(x)\sum_{k=0}^\infty s_{n,k}(x)n\int_{\frac{k}{n}}^{\frac{k+1}{n}}|t-x|\left|t-\frac{k}{n}\right|\frac{|w'(\xi)|}{w^2(\xi)}dt
\end{aligned}$$

where

$$\xi \in \left[\frac{k}{n},\frac{k+1}{n}\right].$$

Now

$$\begin{aligned}
&\left|\sum_{k=0}^\infty s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)}n\int_{\frac{k}{n}}^{\frac{k+1}{n}}(t-x)\,dt\right| \\
&\quad\leq \frac{1}{2n}\sum_{k=0}^\infty s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)} + \left|\sum_{k=0}^\infty s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)}\left(\frac{k}{n}-x\right)\right|.
\end{aligned}$$

From (2.9)

$$\frac{1}{2n}\sum_{k=0}^\infty s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)} \leq \frac{C}{n}. \tag{2.19}$$

Since

$$s_{n,k}(x)\frac{k}{n} = xs_{n,k-1}(x)$$

it follows again from (2.9) that

$$\left|\sum_{k=0}^{\infty} s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)}\left(\frac{k}{n}-x\right)\right| \le xw(x)\sum_{k=0}^{\infty} s_{n,k}(x)\left|w^{-1}\left(\frac{k+1}{n}\right)-w^{-1}\left(\frac{k}{n}\right)\right|$$
$$\le \frac{Cxw(x)}{n}\sum_{k=0}^{\infty}\frac{s_{n,k}(x)}{\left(1+\frac{k}{n}\right)^{\alpha+1}} \le \frac{Cx}{n(1+x)} \le \frac{C}{n}. \tag{2.20}$$

The second inequality above follows from the easily proved inequality

$$\left|\left(1+\frac{k+1}{n}\right)^{\beta}-\left(1+\frac{k}{n}\right)^{\beta}\right| \le \frac{C}{n}\left(1+\frac{k}{n}\right)^{\beta-1} \quad \text{where} \quad \beta\in\mathbb{R}.$$

From (2.19) and (2.20) we obtain

$$\left|\sum_{k=0}^{\infty} s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)} n\int_{\frac{k}{n}}^{\frac{k+1}{n}}(t-x)\,dt\right| \le \frac{C}{2n}. \tag{2.21}$$

Now, for $t\in\left[\frac{k}{n},\frac{k+1}{n}\right]$

$$\left|t-\frac{k}{n}\right|\frac{|w'(\xi)|}{w^2(\xi)} \le \frac{C}{n\left(1+\frac{k}{n}\right)w\left(\frac{k}{n}\right)}$$

and

$$|t-x| \le \left|\frac{k}{n}-x\right| + \left|\frac{k+1}{n}-x\right|.$$

So

$$\left|w(x)\sum_{k=0}^{\infty} s_{n,k}(x) n\int_{\frac{k}{n}}^{\frac{k+1}{n}}|t-x|\left|t-\frac{k}{n}\right|\frac{|w'(\xi)|}{w^2(\xi)}dt\right|dt$$
$$\le \frac{Cw(x)}{n}\sum_{k=0}^{\infty}\frac{s_{n,k}(x)}{\left(1+\frac{k}{n}\right)^{\alpha+1}}\left|\frac{k}{n}-x\right| + \frac{Cw(x)}{n}\sum_{k=0}^{\infty}\frac{s_{n,k}(x)}{\left(1+\frac{k}{n}\right)^{\alpha+1}}\left|\frac{k+1}{n}-x\right|.$$

We will estimate the first term. The estimation of the second is similar.

By applying Cauchy's inequality we get

$$\sum_{k=0}^{\infty}\frac{s_{n,k}(x)}{\left(1+\frac{k}{n}\right)^{\alpha+1}}\left|\frac{k}{n}-x\right| \le \left[\sum_{k=0}^{\infty}\frac{s_{n,k}(x)}{\left(1+\frac{k}{n}\right)^{2(\alpha+1)}}\right]^{1/2}\left[\sum_{k=0}^{\infty}s_{n,k}(x)\left(\frac{k}{n}-x\right)^2\right]^{1/2}.$$

From (2.9)

$$\sum_{k=0}^{\infty}\frac{s_{n,k}(x)}{\left(1+\frac{k}{n}\right)^{2(\alpha+1)}} \le \frac{C}{(1+x)^{2(\alpha+1)}},$$

from (2.4)

$$\sum_{k=0}^{\infty} s_{n,k}(x)\left(\frac{k}{n}-x\right)^2=\frac{x}{n},$$

and consequently

$$\frac{Cw(x)}{n}\sum_{k=0}^{\infty}\frac{s_{n,k}(x)}{\left(1+\frac{k}{n}\right)^{\alpha+1}}\left|\frac{k}{n}-x\right|\le\frac{C\sqrt{x}}{(1+x)n^{3/2}}\le\frac{C}{n}.$$

□

Lemma 2.6. *For every $x\in[0,\infty)$ and $n\in\mathbb{N}$ there holds*

$$\left|xw(x)\tilde{S}_n\left(\frac{\ln x-\ln(\cdot)}{w(\cdot)};x\right)\right|\le\frac{C}{n}.$$

Proof. Again, by using the Lagrange's formula we have

$$\begin{aligned}&\left|xw(x)\tilde{S}_n\left(\frac{(\cdot)-x}{w(\cdot)};x\right)\right|\\&\quad\le x\left|\sum_{k=0}^{\infty}s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)}n\int_{\frac{k}{n}}^{\frac{k+1}{n}}(\ln x-\ln t)\,dt\right|\\&\qquad+xw(x)\sum_{k=0}^{\infty}s_{n,k}(x)n\int_{\frac{k}{n}}^{\frac{k+1}{n}}|\ln x-\ln t|\left|t-\frac{k}{n}\right|\frac{|w'(\xi)|}{w^2(\xi)}dt.\end{aligned}$$

For the last term in the RHS using

$$x|\ln x-\ln t|\le|t-x|$$

we have

$$\begin{aligned}&xw(x)\sum_{k=0}^{\infty}s_{n,k}(x)n\int_{\frac{k}{n}}^{\frac{k+1}{n}}|\ln x-\ln t|\left|t-\frac{k}{n}\right|\frac{|w'(\xi)|}{w^2(\xi)}dt\\&\quad\le w(x)\sum_{k=0}^{\infty}s_{n,k}(x)n\int_{\frac{k}{n}}^{\frac{k+1}{n}}|t-x|\left|t-\frac{k}{n}\right|\frac{|w'(\xi)|}{w^2(\xi)}dt\end{aligned}$$

and we already estimated it in the previous lemma.

For the first term we have

$$\begin{aligned}&x\left|\sum_{k=0}^{\infty}s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)}n\int_{\frac{k}{n}}^{\frac{k+1}{n}}(\ln x-\ln t)\,dt\right|\\&\quad\le x\left|\sum_{k=0}^{\infty}s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)}\left(\ln\frac{k+1}{n}-\ln x\right)\right|+Cx\sum_{k=0}^{\infty}s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)}\frac{1}{k}.\end{aligned}$$

The last inequality follows from

$$n\int_{\frac{k}{n}}^{\frac{k+1}{n}}(\ln x-\ln t)\,dt=\ln x-\ln\frac{k+1}{n}+O\left(\frac{1}{k}\right).$$

Since

$$x\sum_{k=0}^{\infty} s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)}\frac{1}{k} \le \frac{C}{n}$$

it follows that

$$xw(x)\left|\tilde{S}_n\left(\frac{\ln x - \ln(\cdot)}{w(\cdot)} : x\right)\right| \le x\left|\sum_{k=0}^{\infty} s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)}\left(\ln\frac{k+1}{n} - \ln x\right)\right| + \frac{C}{n}.$$

Now we consider two cases.

1. $x \le \frac{1}{n}$.

   In this case:

$$\begin{aligned} x\left|\sum_{k=0}^{\infty} s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)}\left(\ln\frac{k+1}{n} - \ln x\right)\right| &\le x\sum_{k=0}^{\infty} s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)}\left(\ln(k+1) - \ln(nx)\right) \\ &\le x\sum_{k=0}^{\infty} s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)}\ln(k+1) + x|\ln(nx)|\sum_{k=0}^{\infty} s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)} \\ &\le x\sum_{k=0}^{\infty} s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)}(k+1) + Cx|\ln(nx)| \le C\left(nx^2 + x + \frac{1}{n}\right) \le \frac{C}{n} \end{aligned}$$

   because of (2.9), (2.2) and the simple inequalities

$$nx^2 \le \frac{1}{n} \quad \text{and} \quad x|\ln(nx)| \le \frac{1}{n}.$$

2. $x > \frac{1}{n}$.

   By Taylor's formula

$$\ln\frac{k+1}{n} = \ln x + \frac{1}{x}\left(\frac{k+1}{n} - x\right) - \int_x^{\frac{k+1}{n}}\left(\frac{k+1}{n} - t\right)\frac{dt}{t^2}$$

   and consequently

$$\begin{aligned} &x\left|\sum_{k=0}^{\infty} s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)}\left(\ln\frac{k+1}{n} - \ln x\right)\right| \\ &\quad\le \left|\sum_{k=0}^{\infty} s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)}\left(\frac{k+1}{n} - x\right)\right| \\ &\qquad + x\left|\sum_{k=0}^{\infty} s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)}\int_x^{\frac{k+1}{n}}\left(\frac{k+1}{n} - t\right)\frac{dt}{t^2}\right|. \end{aligned} \tag{2.22}$$

The estimation of the first term in RHS of (2.22) is analogous to the estimation of

$$\left|\sum_{k=0}^{\infty} s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)}\left(\frac{k}{n}-x\right)\right|$$

in the previous lemma.

For the second term in RHS of (2.22) since

$$\int_x^{\frac{k+1}{n}}\left(\frac{k+1}{n}-t\right)\frac{dt}{t^2}\le\left|\frac{k+1}{n}-x\right|\int_x^{\frac{k+1}{n}}\frac{dt}{t^2}=\frac{n}{(k+1)x}\left(\frac{k+1}{n}-x\right)^2$$

we have by applying the Cauchy's inequality, (2.9) and (2.6)

$$\begin{aligned}
&x\left|\sum_{k=0}^{\infty} s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)}\int_x^{\frac{k+1}{n}}\left(\frac{k+1}{n}-t\right)\frac{dt}{t^2}\right|\\
&\quad\le\sum_{k=0}^{\infty} s_{n,k}(x)\frac{w(x)}{w\left(\frac{k}{n}\right)}\frac{n}{k+1}\left(\frac{k+1}{n}-x\right)^2\\
&\quad=\frac{w(x)}{x}\sum_{k=0}^{\infty}\frac{s_{n,k+1}(x)}{w\left(\frac{k}{n}\right)}\left(\frac{k+1}{n}-x\right)^2\\
&\quad\le\frac{w(x)}{x}\left[\sum_{k=0}^{\infty}\frac{s_{n,k+1}(x)}{w\left(\frac{k}{n}\right)}\right]^{1/2}\left[\sum_{k=0}^{\infty}s_{n,k+1}(x)\left(\frac{k+1}{n}-x\right)^4\right]^{1/2}\\
&\quad\le\frac{w(x)}{x}\left[\sum_{k=0}^{\infty}\frac{s_{n,k}(x)}{w\left(\frac{k}{n}\right)}\right]^{1/2}\left[\sum_{k=0}^{\infty}s_{n,k+1}(x)\left(\frac{k+1}{n}-x\right)^4\right]^{1/2}\\
&\quad\le\frac{w(x)}{x}\left[\frac{C}{w(x)}\right]^{1/2}\left[C\left(\frac{x}{n}\right)^2\right]^{1/2}=\frac{C}{n}.
\end{aligned}$$

□

Lemma 2.7. *Let $1\le p\le\infty$. Then for all $f\in W_p(w)$ and $n\in\mathbb{N}$ there holds*

$$\left\|w(x)\tilde{S}_n\left(\int_x^{(\cdot)}[\phi(\cdot)-\phi(u)]\,\tilde{D}f(u)du\right)\right\|_p\le\frac{C(\alpha)}{n}\left\|w\tilde{D}f\right\|_p.$$

Proof. We have

$$\begin{aligned}
&\left|w(x)\tilde{S}_n\left(\int_x^{(\cdot)}\left[\phi(\cdot)-\phi(u)\right]\tilde{D}f(u)du\right)\right|\\
&\quad = w(x)\sum_{k=0}^{\infty}s_{n,k}(x)n\int_{\frac{k}{n}}^{\frac{k+1}{n}}\left(\int_x^t\left[\ln t-\ln u\right]\frac{w(u)\tilde{D}f(u)}{w(u)}du\right)dt\\
&\quad \leq C\sum_{k=0}^{\infty}s_{n,k}(x)\frac{w(x)}{w(k/n)}n\int_{\frac{k}{n}}^{\frac{k+1}{n}}\left(\int_x^t\left[\ln t-\ln u\right]\left|w(u)\tilde{D}f(u)\right|du\right)dt\\
&\quad = C\sum_{k=0}^{\infty}s_{n,k}(x)\left[\frac{n(1+x)}{n+k}\right]^{\alpha}b_{k,1}
\end{aligned}$$

where

$$b_{k,1}=n\int_{\frac{k}{n}}^{\frac{k+1}{n}}\left(\int_x^t\left[\ln t-\ln u\right]\left|w(u)\tilde{D}f(u)\right|du\right)dt.$$

Let μ be the smallest integer such that $\mu>|\alpha|$. By Holder's inequality we have

$$\begin{aligned}
&\sum_{k=0}^{\infty}s_{n,k}(x)\left[\frac{n(1+x)}{n+k}\right]^{\alpha}b_{k,1}\\
&\quad \leq\left\{\sum_{k=0}^{\infty}s_{n,k}(x)\left[\frac{n(1+x)}{n+k}\right]^{sign(\alpha)\mu}b_{k,1}\right\}^{\frac{|\alpha|}{\mu}}\left\{\sum_{k=0}^{\infty}s_{n,k}(x)b_{k,1}\right\}^{1-\frac{|\alpha|}{\mu}}.
\end{aligned}$$

We define a new operator $\tilde{S}_{n,\alpha}$ by the formula

$$\tilde{S}_{n,\alpha}f(x)=\int_0^{\infty}K_n(x,t)f(t)dt=\sum_{k=0}^{\infty}s_{n,k}(x)(1+x)^m\frac{n^{m+1}(n+k)!}{(n+k+m)!}\int_{\frac{k}{n}}^{\frac{k+1}{n}}f(t)dt$$

where $m=sign(\alpha)\mu$.

For the estimation of the L_1-norm by applying the Holder's inequality again we have

$$\begin{aligned}
&\left\|\sum_{k=0}^{\infty}s_{n,k}(x)\left[\frac{n(1+x)}{n+k}\right]^{\alpha}b_{k,1}\right\|_1\\
&\quad \leq\left\|\sum_{k=0}^{\infty}s_{n,k}(x)\left[\frac{n(1+x)}{n+k}\right]^{m}b_{k,1}\right\|_1^{\frac{|\alpha|}{\mu}}\left\|\sum_{k=0}^{\infty}s_{n,k}(x)b_{k,1}\right\|_1^{1-\frac{|\alpha|}{\mu}}.
\end{aligned}$$

For the second term in RHS we can use the estimation (2.17) - simply replacing $\tilde{D}f$ by $w\tilde{D}f$. For the first term since

$$\left(\frac{n}{n+k}\right)^m\leq C\frac{n^m(n+k)!}{(n+k+m)!}$$

we have

$$\begin{aligned}&\left\|\sum_{k=0}^{\infty} s_{n,k}(x)\left[\frac{n(1+x)}{n+k}\right]^m b_{k,1}\right\|_1\\&\quad\le C\left\|\tilde{S}_{n,\alpha}\left(\int_x^{(\cdot)}\left[\ln(\cdot)-\ln u\right]\left|w(u)\tilde{D}f(u)\right|du\right)\right\|_1\\&\quad= C\left(\int_0^x+\int_x^\infty\right)=C\left(I_1(x)+I_2(x)\right).\end{aligned}$$

Changing the order of integration twice in both of the integrals we obtain

$$\int_0^\infty I_1(x)dx=\int_0^\infty\left|w(u)\tilde{D}f(u)\right|\left(\int_u^\infty \tilde{S}_{n,\alpha}\left([\ln u-\ln(\cdot)]_+;x\right)dx\right)du$$

and

$$\int_0^\infty I_2(x)dx=\int_0^\infty\left|w(u)\tilde{D}f(u)\right|\left(\int_0^u \tilde{S}_{n,\alpha}\left([\ln(\cdot)-\ln u]_+;x\right)dx\right)du$$

where

$$[f(z)]_+=\frac{1}{2}\left(|f(z)|+f(z)\right).$$

Consequently,

$$\int_0^\infty I_1(x)dx+\int_0^\infty I_2(x)dx=\int_0^\infty\left|w(u)\tilde{D}f(u)\right|I(u)du$$

where

$$I(u)=\int_u^\infty \tilde{S}_{n,\alpha}\left([\ln u-\ln(\cdot)]_+;x\right)dx+\int_0^u \tilde{S}_{n,\alpha}\left([\ln(\cdot)-\ln u]_+;x\right)dx.$$

We have by equation (2.10) of Lemma 2.2 for every function $h(x)\in\mathbb{L}_1[0,\infty)$

$$\left\|\tilde{S}_{n,\alpha}h\right\|_1=\sum_{k=0}^\infty\frac{n^m(n+k)!}{(n+k+m)!}\int_0^\infty s_{n,k}(x)(1+x)^m dx\int_{\frac{k}{n}}^{\frac{k+1}{n}}h(t)dt\le\|h\|_1$$

and

$$\begin{aligned}\left\|\tilde{S}_{n,\alpha}h\right\|_1&\ge\sum_{k=0}^\infty\left(1-\frac{C}{n\left(1+\frac{k}{n}\right)^2}\right)\int_{\frac{k}{n}}^{\frac{k+1}{n}}h(t)dt\\&=\|h\|_1-\frac{C}{n}\sum_{k=0}^\infty\frac{1}{\left(1+\frac{k}{n}\right)^2}\int_{\frac{k}{n}}^{\frac{k+1}{n}}|h(t)|dt.\end{aligned}$$

Also,

$$\sum_{k=0}^{\infty} \frac{1}{\left(1+\frac{k}{n}\right)^2} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |h(t)|dt \leq \sum_{k=0}^{\infty} \frac{1}{\left(1+\frac{k}{n}\right)^2} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |\ln t|dt$$
$$\leq \frac{1}{n} \sum_{k=0}^{\infty} \frac{\left|\ln \frac{k+1}{n}\right|}{\left(1+\frac{k}{n}\right)^2} \leq \frac{C}{n}$$

i.e.

$$\int_0^{\infty} \tilde{S}_{n,\alpha}\left([\ln u - \ln(\cdot)]_+; x\right) dx = \int_0^u (\ln u - \ln x)\, dx + O\left(\frac{1}{n}\right).$$

Since

$$\int_0^{\infty} \tilde{S}_{n,\alpha} dx = \int_0^u \tilde{S}_{n,\alpha} dx + \int_u^{\infty} \tilde{S}_{n,\alpha} dx$$

it follows that

$$\int_u^{\infty} \tilde{S}_{n,\alpha}\left([\ln u - \ln(\cdot)]_+; x\right) dx$$
$$= \int_0^u (\ln u - \ln x)\, dx - \int_0^u \tilde{S}_{n,\alpha}\left([\ln u - \ln(\cdot)]_+; x\right) dx + O\left(\frac{1}{n}\right).$$

Consequently,

$$\begin{aligned} I(u) &= \int_0^u (\ln u - \ln x)\, dx + \int_0^u \tilde{S}_{n,\alpha}\left([\ln(\cdot) - \ln u]_+ - [\ln u - \ln(\cdot)]_+; x\right) dx \\ &\quad + O\left(\frac{1}{n}\right) \\ &= \int_0^u (\ln u - \ln x)\, dx + \int_0^u \tilde{S}_{n,\alpha}\left(\ln(\cdot) - \ln u; x\right) dx + O\left(\frac{1}{n}\right) \\ &= \int_0^u \left(\tilde{S}_{n,\alpha}\left(\ln(\cdot); x\right) - \ln x\right) dx + +O\left(\frac{1}{n}\right) \\ &\leq \int_0^{\infty} \left(\tilde{S}_{n,\alpha}\left(\ln(\cdot); x\right) - \ln x\right) dx + +O\left(\frac{1}{n}\right) \\ &= \left\|\tilde{S}_{n,\alpha} \ln(\cdot) - \ln(\cdot)\right\|_1 + O\left(\frac{1}{n}\right) \\ &\leq \left\|\tilde{S}_{n,\alpha} \ln(\cdot) - \tilde{S}_n \ln(\cdot)\right\|_1 + \left\|\tilde{S}_n \ln(\cdot) - \ln(\cdot)\right\|_1 + O\left(\frac{1}{n}\right) \end{aligned}$$

and by using the estimate (2.17) and Lemma 2.2 we complete the proof of lemma for $p = 1$.

Now we estimate the L_∞-norm. We have

$$\begin{aligned}&\left|w(x)\tilde{S}_n\left(\int_x^{(\cdot)}(\ln(\cdot)-\ln x)\tilde{D}f(u)du;x\right)\right|\\&\qquad\le\left\|w\tilde{D}f\right\|_\infty\left\|w(x)\tilde{S}_n\left(\int_x^{(\cdot)}\frac{\ln(\cdot)-\ln u}{w(u)}du;x\right)\right\|_\infty.\end{aligned}$$

From the obvious $w^{-1}(u)\le w^{-1}(\cdot)+w^{-1}(x)$ for u between $(\cdot)$ and x it follows that

$$\begin{aligned}&\left\|w(x)\tilde{S}_n\left(\int_x^{(\cdot)}\frac{\ln(\cdot)-\ln u}{w(u)}du;x\right)\right\|_\infty\\&\quad\le\left|\tilde{S}_n\left(\int_x^{(\cdot)}(\ln(\cdot)-\ln u)du;x\right)\right|+\left\|w(x)\tilde{S}_n\left(\int_x^{(\cdot)}\frac{\ln(\cdot)-\ln u}{w(\cdot)}du;x\right)\right\|_\infty.\end{aligned}\tag{2.23}$$

For the first term in the RHS of (2.23) we have

$$\tilde{S}_n\left(\int_x^{(\cdot)}(\ln(\cdot)-\ln u)du;x\right)=\frac{1}{2n}-x\left(\tilde{S}_n(\ln(\cdot);x)-\ln x\right)$$

and by using the estimation (2.18) we obtain the needed estimate.

For the second term in the RHS of (2.23) we have

$$\begin{aligned}&w(x)\tilde{S}_n\left(\int_x^{(\cdot)}\frac{\ln(\cdot)-\ln u}{w(\cdot)}du;x\right)\\&\quad=w(x)\left[x\tilde{S}_n\left(\frac{\ln x-\ln(\cdot)}{w(\cdot)};x\right)+\tilde{S}_n\left(\frac{(\cdot)-x}{w(\cdot)};x\right)\right]\\&\quad=xw(x)\tilde{S}_n\left(\frac{\ln x-\ln(\cdot)}{w(\cdot)};x\right)+w(x)\tilde{S}_n\left(\frac{(\cdot)-x}{w(\cdot)};x\right).\end{aligned}$$

By using Lemma 2.5 and Lemma 2.6 we complete the proof of lemma for $p=\infty$. □

3. Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2. We follow the argument in [1, pp. 41–42]. We have for $x,t>0$

$$f(t)=f(x)+\varphi(x)\left(\phi(t)-\phi(x)\right)Df(x)+\int_x^t\left(\phi(t)-\phi(u)\right)\tilde{D}f(u)du.$$

Applying $\tilde{S}_n$ to both sides with regard to t and using (2.5), we obtain

$$\tilde{S}_n(f;x) - f(x)$$
$$= \varphi(x)\left[\tilde{S}_n(\phi(\cdot);x) - \phi(x)\right] Df(x) + \tilde{S}_n\left(\int_x^{(\cdot)}\left[\phi(\cdot) - \phi(u)\right]\tilde{D}f(u)du\right)$$

and

$$w(x)\left[\tilde{S}_n(f,x) - f(x)\right] = w(x)\varphi(x)\left[\tilde{S}_n(\phi(\cdot);x) - \phi(x)\right] Df(x)$$
$$+w(x)\tilde{S}_n\left(\int_x^{(\cdot)}\left[\phi(\cdot) - \phi(u)\right]\tilde{D}f(u)du\right).$$

Consequently

$$\left\|w\left(\tilde{S}_n f - f\right)\right\|_p$$
$$\leq \left\|w\varphi\left(\tilde{S}_n\phi - \phi\right)Df\right\|_p + \left\|w(x)\tilde{S}_n\left(\int_x^{(\cdot)}\left[\phi(\cdot) - \phi(u)\right]\tilde{D}f(u)du\right)\right\|_p.$$

By using Lemmas 2.4 and 2.7 we complete the proof. □

Proof of Theorem 1.3. To recall, we denote by C positive constants, not necessarily the same at each occurrence, which are independent of f, g, n, and l. We prove the theorem by means of a standard argument.

Let $1 \leq p \leq \infty$. For any $g \in \tilde{W}_p(w)$ such that $f - g \in L_p(w)$ we have in virtue of (2.7) and Theorem 1.2

$$\left\|w(f - \tilde{S}_n f)\right\|_p \leq \|w(f-g)\|_p + \left\|w\tilde{S}_n(f-g)\right\|_p + \frac{C}{n}\left\|w\tilde{D}g\right\|_p$$
$$\leq C\left(\|w(f-g)\|_p + \frac{1}{n}\left\|w\tilde{D}g\right\|_p\right).$$

Taking the infimum on g we arrive at the left-hand side inequality in the theorem. □

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