



Partition Bounded Sets Into Sets Having Smaller Diameters

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Abstract. For each positive integer m and each real finite dimensional Banach space X , we set $\beta(X, m)$ to be the infimum of $\delta \in (0, 1]$ such that each set $A \subset X$ having diameter 1 can be represented as the union of m subsets of A whose diameters are at most δ . Elementary properties of $\beta(X, m)$, including its stability with respect to X in the sense of Banach-Mazur metric, are presented. Two methods for estimating $\beta(X, m)$ are introduced. The first one estimates $\beta(X, m)$ using the knowledge of $\beta(Y, m)$, where Y is a Banach space sufficiently close to X . The second estimation uses the information about $\beta_X(K, m)$, the infimum of $\delta \in (0, 1]$ such that $K \subset X$ is the union of m subsets having diameters not greater than δ times the diameter of K , for certain classes of convex bodies K in X . In particular, we show that $\beta(l_p^3, 8) \leq 0.925$ holds for each $p \in [1, +\infty]$ by applying the first method, and we proved that $\beta(X, 8) < 1$ whenever X is a three-dimensional Banach space satisfying $\beta_X(B_X, 8) < \frac{221}{328}$, where B_X is the unit ball of X , by applying the second method. These results and methods are closely related to the extension of Borsuk's problem in finite dimensional Banach spaces and to C. Zong's computer program for Borsuk's conjecture.

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1. Introduction

Let $X = (\mathbb{R}^n, \|\cdot\|)$ be an n -dimensional Banach space with unit ball B_X . For each $A \subset X$, we denote by $\text{int } A$ and $\text{bd } A$ the *interior* and the *boundary* of A , respectively. When A is nonempty and bounded, we denote by $\delta(A)$ the *diameter* of A . I.e., $\delta(A) = \sup \{\|x - y\| \mid x, y \in A\}$. A compact convex subset of X having interior points is called a *convex body*. Since any two norms on \mathbb{R}^n induce the same topology, K is a convex body (is bounded, resp.) in X if and only if it is a convex body (is bounded, resp.) in $\mathbb{E}^n = (\mathbb{R}^n, \|\cdot\|_E)$, where $\|\cdot\|_E$ is the Euclidean norm. Let \mathcal{K}^n be the set of all convex bodies in \mathbb{E}^n . A bounded subset A of X is said to be *complete* if $x \notin A \Rightarrow \delta((A \cup \{x\})) > \delta(A)$. It is clear that each complete set is closed and convex. For each bounded subset A of X , there always exists a complete set A^c (cf. the beginning of Section 2 in [24] for the existence in the finite-dimensional situation), called a *completion* of A , with diameter $\delta(A)$ containing A . Note that A may have different completions. Put

$$\mathcal{B}^n = \{A \subset \mathbb{R}^n \mid A \text{ is bounded and } \delta(A) > 0\}, \quad \mathcal{C}_X = \{A \in \mathcal{B}^n \mid A \text{ is complete}\}.$$

In 1933, Borsuk [5] posed the following:

Problem 1 (Borsuk’s Problem). *Is it possible to partition every bounded subset of \mathbb{E}^n into $n + 1$ sets of smaller diameter?*

The answer is affirmative when $n \leq 3$ (cf. [10, 14, 17], or [16]), is negative for all $n \geq 64$ (cf. [4] and Theorem 1 in [19]).

Progress has been made by providing upper bounds for $b(n) = \sup \{b(A) \mid A \in \mathcal{B}^n\}$, where $b(A)$, called the *Borsuk number* of A , is the minimal positive integer m such that A is the union of m subsets having smaller diameter. For example, L. Danzer (cf. [9]), M. Lassak (cf. [21]), and O. Schramm (cf. [26] and [6]) showed that

$$b(n) \leq \sqrt{\frac{(n+2)^3(2+\sqrt{2})^{n-1}}{3}}, \quad b(n) \leq 2^{n-1} + 1, \quad \text{and} \quad b(n) \leq 5n^{\frac{3}{2}}(4 + \log n) \left(\frac{3}{2}\right)^{\frac{n}{2}},$$

respectively.

One can also study Problem 1 via estimating $\beta(n, m) = \sup \{\beta(A, m) \mid A \in \mathcal{B}^n\}$, where m is a positive integer and, for each $A \in \mathcal{B}^n$,

$$\beta(A, m) = \inf \left\{ \frac{1}{\delta(A)} \max_{k \in [m]} \delta(A_k) \mid A = \bigcup_{k=1}^m A_k \right\}.$$

Here we used the shorthand notation $[m] := \{i \in \mathbb{Z}^+ \mid 1 \leq i \leq m\}$.

In 2020, C. Zong (cf. [32]) proved that $\beta(A, m)$ is uniformly continuous on the space \mathcal{K}^n endowed with the Hausdorff metric, and reformulated Problem 1 as the following:

TABLE 1. Known estimations on $\beta(n, m)$

m										
$\beta(n, m)$										
n	3	4			5			6	7	8
2	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$			$\geq \sin \frac{\pi}{5}$			$\leq \frac{\sqrt{3}}{3}$	$\frac{1}{2}$?
3	–	$\in [0.8880, 0.9887]$			$\in \left[\frac{\sqrt{3}}{2}, 0.9425 \right]$			$\geq \frac{\sqrt{6}}{3}$	$\geq \frac{\sqrt{2}}{2}$	$\geq \frac{\sqrt{2}}{2}$

Problem 2. Does there exists a positive number $\alpha_n < 1$ such that

$$\beta(K, n + 1) \leq \alpha_n, \quad \forall K \in \mathcal{D}^n,$$

where

$$\mathcal{D}^n = \left\{ K \in \mathcal{K}^n \mid B_2^n \subseteq K \subseteq \frac{\sqrt{n/(2n+2)}}{1 - \sqrt{n/(2n+2)}} B_2^n \right\}?$$

Some known estimations of $\beta(n, m)$ are listed in Table 1 below (cf., [12, 14, 20], and [13]).

Grünbaum extended Borsuk’s problem to the case of Banach spaces and asked the following (cf. [15]):

Problem 3. Let $A \subset X = (\mathbb{R}^n, \|\cdot\|)$ be bounded. What is the smallest positive integer m , denoted by $b_X(A)$, such that A can be represented as the union of m sets having smaller diameter.

Put

$$b(X) = \max \{b_X(A) \mid A \in \mathcal{B}^n\} = \max \{b_X(A) \mid A \in \mathcal{K}^n\},$$

$$B(n) = \max \{b((\mathbb{R}^n, \|\cdot\|)) \mid \|\cdot\| \text{ is a norm on } \mathbb{R}^n\}.$$

It is clear that

$$b_X(K) \leq c(K), \quad \forall K \in \mathcal{K}^n, \tag{1}$$

where $c(K)$ is the least number of smaller homothetic copies of K needed to cover K . Indeed, if $\{c_i + \gamma_i K \mid i \in [m]\}$ is a collection of smaller homothetic copies of K that can cover K , then we have

$$K = \bigcup_{i \in [m]} (K \cap (c_i + \lambda_i K)) \quad \text{and} \quad \delta(K \cap (c_i + \lambda_i K)) \leq \lambda_i \delta(K), \quad \forall i \in [m].$$

Since Hadwiger’s covering conjecture (see, e.g., [1, 3, 8, 23, 31]) asserts that $c(K) \leq 2^n, \forall K \in \mathcal{K}^n$, it is reasonable to make the following conjecture (cf. [2, p. 75]):

Conjecture 1. For each integer $n \geq 3, B(n) = 2^n$.

When X is two-dimensional and $A \in \mathcal{B}^2$, $b_X(A) \in \{2, 3, 4\}$ (cf. [3, §33]). Therefore, $B(2) = 4$. Let $l_p^n = (\mathbb{R}^n, \|\cdot\|_p)$, where

$$\|(\alpha_1, \dots, \alpha_n)\|_p = \left(\sum_{i \in [n]} |\alpha_i|^p \right)^{\frac{1}{p}}.$$

L. Yu and C. Zong [30] proved that

$$b(l_p^3) \leq 8, \quad \forall p \in [1, +\infty]. \tag{2}$$

By the main result of [18] and (1), there exist universal constants c_1 and $c_2 > 0$ such that $B(n) \leq c_1 4^n e^{-c_2 \sqrt{n}}$, $\forall n \geq 2$. Despite this progress, Conjecture 1 is still open when $n \geq 3$.

In this paper, we study Conjecture 1 by estimating

$$\beta_X(A, m) = \inf \left\{ \frac{1}{\delta(A)} \max \{ \delta(A_k) \mid k \in [m] \} \mid A = \bigcup_{k=1}^m A_k \right\}$$

for $A \in \mathcal{B}^n$, and

$$\beta(X, m) = \sup \{ \beta_X(A, m) \mid A \in \mathcal{B}^n \}.$$

We focus mainly, but not only, on the case $n = 3$.

In Sect. 2, we present elementary properties of $\beta(X, m)$, including its stability with respect to X in the sense of Banach-Mazur metric. In Sect. 3, we provide an estimation of $\beta(X, 8)$ for three-dimensional Banach spaces X such that $\beta_X(B_X, 8)$ is sufficiently small. In Sect. 4, we show that

$$\beta(l_p^3, 8) \leq 0.925, \quad \forall p \in [1, +\infty],$$

which can be viewed as a quantitative version of Yu and Zong’s result (2).

2. Elementary properties of $\beta(X, m)$

Proposition 1. *For each finite dimensional Banach space $X = (\mathbb{R}^n, \|\cdot\|)$ and each positive integer m , we have*

$$\beta(X, m) = \sup \{ \beta_X(A, m) \mid A \in \mathcal{C}_X \}.$$

Proof. Put $\beta = \sup \{ \beta_X(A, m) \mid A \in \mathcal{C}_X \}$. We only need to show that $\beta(X, m) \leq \beta$. Let A be an arbitrary set in \mathcal{B}^n , and A^c be a completion of A . For each $\varepsilon > 0$, there exists a collection $\{B_i \mid i \in [m]\}$ of subsets of A^c such that

$$A^c = \bigcup_{i \in [m]} B_i, \quad \text{and} \quad \frac{1}{\delta(A)} \max \{ \delta_X(B_k) \mid k \in [m] \} \leq \beta_X(A^c, m) + \varepsilon.$$

It follows that

$$A = A \cap A^c = \bigcup_{i \in [m]} (A \cap B_i).$$

Thus

$$\beta_X(A, m) \leq \frac{1}{\delta(A)} \max \{ \delta(A \cap B_k) \mid k \in [m] \} \leq \beta_X(A^c, m) + \varepsilon,$$

which implies that $\beta_X(A, m) \leq \beta_X(A^c, m)$. Thus $\beta(X, m) \leq \beta$ as claimed. \square

Let \mathcal{T}^n be the set of all non-singular linear transformations on \mathbb{R}^n . The (multiplicative) Banach-Mazur metric $d_{BM}^M : \mathcal{K}^n \times \mathcal{K}^n \mapsto \mathbb{R}$ is defined by

$$d_{BM}^M(K_1, K_2) = \inf \{ \gamma \geq 1 \mid \exists T \in \mathcal{T}^n, v \in \mathbb{R}^n \text{ s.t. } T(K_2) \subset K_1 \subset \gamma T(K_2) + v, \forall K_1, K_2 \in \mathcal{K}^n \}.$$

The infimum is clearly attained. When both K_1 and K_2 are symmetric with respect to o , we have

$$d_{BM}^M(K_1, K_2) = \inf \{ \gamma \geq 1 \mid \exists T \in \mathcal{T}^n \text{ s.t. } T(K_2) \subset K_1 \subset \gamma T(K_2) \}.$$

In this situation, $d_{BM}^M(K_1, K_2)$ equals to the Banach-Mazur distance between the Banach spaces X and Y having K_1 and K_2 as unit balls, respectively. I.e.,

$$d_{BM}^M(K_1, K_2) = d_{BM}^M(X, Y) := \inf \{ \|T\| \cdot \|T^{-1}\| \mid T \text{ is an isomorphism from } X \text{ onto } Y \}.$$

We have the following result showing the stability of $\beta(X, m)$ with respect to X in the sense of Banach-Mazur metric.

Theorem 2. *If $X = (\mathbb{R}^n, \|\cdot\|_X)$ and $Y = (\mathbb{R}^n, \|\cdot\|_Y)$ are two Banach spaces satisfying $d_{BM}^M(X, Y) \leq \gamma$ for some $\gamma \geq 1$, then*

$$\beta(X, m) \leq \gamma \beta(Y, m), \forall m \in \mathbb{Z}^+.$$

Proof. By applying a suitable linear transformation if necessary, we may assume that

$$B_Y \subseteq B_X \subseteq \gamma B_Y.$$

In this situation we have, for each $x \in \mathbb{R}^n$,

$$\begin{aligned} \|x\|_X &= \inf \{ \lambda > 0 \mid x \in \lambda B_X \} \leq \inf \{ \lambda > 0 \mid x \in \lambda B_Y \} \\ &= \|x\|_Y \\ &= \inf \{ \lambda \gamma > 0 \mid x \in \lambda \gamma B_Y \} \\ &= \gamma \inf \{ \lambda > 0 \mid x \in \lambda \gamma B_Y \} \\ &\leq \gamma \inf \{ \lambda > 0 \mid x \in \lambda B_X \} = \gamma \|x\|_X. \end{aligned}$$

Hence,

$$\|x\|_X \leq \|x\|_Y \leq \gamma \|x\|_X. \tag{3}$$

In the rest of this proof, we denote by $\delta_X(A)$ and $\delta_Y(A)$ the diameter of a bounded subset A of \mathbb{R}^n with respect to $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. By (3), we have

$$\delta_X(A) \leq \delta_Y(A) \leq \gamma\delta_X(A), \quad \forall A \in \mathcal{B}^n. \tag{4}$$

Let A be a bounded subset of X . Then A is also bounded in Y . Let A^c be a completion of A in Y . For any $\varepsilon > 0$, there exists a collection $\{B_i \mid i \in [m]\}$ of subsets of A^c such that A^c is the union of this collection and that

$$\frac{1}{\delta_Y(A^c)} \max \{\delta_Y(B_i) \mid i \in [m]\} \leq \beta_Y(A^c, m) + \varepsilon.$$

Then

$$A = A \cap A^c = \bigcup_{i \in [m]} (B_i \cap A),$$

and, by (4),

$$\begin{aligned} \frac{1}{\delta_X(A)} \max \{\delta_X(B_i \cap A) \mid i \in [m]\} &\leq \frac{\gamma}{\delta_Y(A)} \max \{\delta_Y(B_i) \mid i \in [m]\} \\ &= \frac{\gamma}{\delta_Y(A^c)} \max \{\delta_Y(B_i) \mid i \in [m]\} \\ &\leq \gamma(\beta_Y(A^c, m) + \varepsilon). \end{aligned}$$

Therefore, $\beta_X(A, m) \leq \gamma\beta_Y(A^c, m)$. It follows that $\beta(X, m) \leq \gamma\beta(Y, m)$. \square

Corollary 3. *If $X = (\mathbb{R}^n, \|\cdot\|_X)$ and $Y = (\mathbb{R}^n, \|\cdot\|_Y)$ are isometric, then $\beta(X, m) = \beta(Y, m)$.*

Proposition 4. *Let $l_\infty^n = (\mathbb{R}^n, \|\cdot\|_\infty)$. Then $\beta(l_\infty^n, 2^n) = \frac{1}{2}$.*

Proof. Put $X = l_\infty^n$. Then every complete set in X is a homothetic copy of B_X , see [11] and [27]. Therefore,

$$\beta_X(A, 2^n) = \beta_X(B_X, 2^n), \quad \forall A \in \mathcal{C}_X.$$

Thus it suffices to show $\beta_X(B_X, 2^n) = \frac{1}{2}$.

On the one hand, $B_X = \frac{1}{2}B_X + \frac{1}{2}V$, where V is the set of vertices of B_X . Since the cardinality of V is 2^n , we have $\beta_X(B_X, 2^n) \leq \frac{1}{2}$.

On the other hand, suppose that B_X is the union of 2^n of its subsets B_1, \dots, B_{2^n} . For each $i \in [2^n]$, let B_i^c be a completion of B_i . Then

$$B_X \subseteq \bigcup_{i \in [2^n]} B_i^c.$$

It follows that

$$\text{vol } B_X \leq \sum_{i \in [2^n]} \text{vol } B_i^c \leq \sum_{i \in [2^n]} \left(\frac{1}{2^n}\right) \max \{\delta_X(B_i^c) \mid i \in [2^n]\} \text{vol } B_X,$$

which implies that $\max \{\delta_X(B_i^c) \mid i \in [2^n]\} \geq 1$. Thus $\beta_X(B_X, 2^n) \geq \frac{1}{2}$, which completes the proof. \square

Corollary 5. *Let $X = (\mathbb{R}^n, \|\cdot\|)$. If $d_{BM}^M(X, l_\infty^n) < 2$, then $\beta(X, 2^n) < 1$.*

We end this section with the following result.

Proposition 6. $\sup \{\beta(X, 4) \mid X \text{ is a two-dimensional Banach space}\} = \frac{\sqrt{2}}{2}$.

Proof. Put $\eta = \sup \{\beta(X, 4) \mid X \text{ is a two-dimensional Banach space}\}$. Let $K \subset \mathbb{R}^2$ be a planar convex body. By the main result in [22], K can be covered by four translates of $\frac{\sqrt{2}}{2}K$. It follows that $\beta_X(K, 4) \leq \frac{\sqrt{2}}{2}$ holds for each two-dimensional Banach space X . Thus, $\eta \leq \frac{\sqrt{2}}{2}$.

Let $X = l_2^2$ and B_X be the unit disk of l_2^2 . To show that $\eta \geq \frac{\sqrt{2}}{2}$, we only need to prove $\beta_X(B_X, 4) \geq \frac{\sqrt{2}}{2}$. Suppose the contrary that B_X is the union of A_1, A_2, A_3, A_4 , where

$$\max \{\delta(A_i) \mid i \in [4]\} < \frac{\sqrt{2}}{2}.$$

Let v_1, v_2, v_3 , and v_4 be the vertices of a square inscribed in the unit circle S_X of X . Then for any $\{i, j\} \subset [4]$

$$\|v_i - v_j\| \geq \sqrt{2}.$$

Assume without loss of generality that $v_1 \in A_1$ and $v_2 \in A_2$, then

$$\frac{v_1 + v_2}{\|v_1 + v_2\|} \notin A_1 \cup A_2.$$

Assume that $\frac{v_1 + v_2}{\|v_1 + v_2\|} \in A_3$. Then $v_3, v_4 \notin A_1 \cup A_2 \cup A_3$, and A_4 cannot contain both v_3 and v_4 , a contradiction. Thus, $\beta_X(B_X, 4) \geq \frac{\sqrt{2}}{2}$ as claimed. \square

3. Estimating $\beta(X, m)$ via $\beta_X(B_X, m)$

Let S be an n -dimensional simplex in $X = (\mathbb{R}^n, \|\cdot\|)$. If the distance between each pair of vertices of S all equals to $\delta(S)$, then we say that S is *equilateral*.

Proposition 7. *Let T be a triangle in \mathbb{R}^2 and $X = (\mathbb{R}^2, \|\cdot\|)$. Then $\beta_X(T, 4) \leq \frac{1}{2}$. If T is equilateral in X , then $\beta_X(T, 4) = \frac{1}{2}$.*

Proof. We only need to consider the case when T is equilateral. Assume without loss of generality that $\delta(T) = 1$. It is clear that $\beta_X(T, 4) \leq \frac{1}{2}$. Denote by $\{a, b, c\}$ the set of vertices of T , and by $\{p, q, r\}$ the set of midpoints of three sides of T . Then $\|p - q\| = \|p - r\| = \|q - r\| = \frac{1}{2}$.

Suppose the contrary that $\beta_X(T, 4) < \frac{1}{2}$. Then there exist four subsets T_1, T_2, T_3, T_4 of T such that $T = \bigcup_{i \in [4]} T_i$ and that $\max \{\delta(T_i) \mid i \in [4]\} < \frac{1}{2}$. We may assume that $a \in T_1, b \in T_2$, and $c \in T_3$. Since $\{p, q, r\} \cap \bigcup_{i \in [3]} T_i = \emptyset$, we have $\{p, q, r\} \subseteq T_4$, which is impossible. Thus $\beta_X(T, 4) = \frac{1}{2}$ as claimed. \square

Proposition 8. *Let T be a 3-dimensional simplex in $X = (\mathbb{R}^3, \|\cdot\|)$. Then*

$$\beta_X(T, 8) \leq \frac{9}{16}.$$

Proof. Denote by $\{v_i \mid i \in [4]\}$ the set of vertices of T . Without loss of generality we may assume that $o = \frac{1}{4} \sum_{i \in [4]} v_i$. For each $i \in [4]$, put $T_i = \frac{7}{16}v_i + \frac{9}{16}T$. Then the portion of T not covered by $\bigcup_{i \in [4]} T_i$ is

$$T_5 = \left\{ \sum_{i \in [4]} \lambda_i v_i \mid \lambda_i \in \left[0, \frac{7}{16}\right], \forall i \in [4], \sum_{i \in [4]} \lambda_i = 1 \right\}.$$

Suppose that $\sum_{i \in [4]} \lambda_i v_i \in T_5$. Then

$$\begin{aligned} \frac{4}{7} \sum_{i \in [4]} \lambda_i v_i - o &= \frac{4}{7} \sum_{i \in [4]} \lambda_i v_i - \frac{1}{4} \sum_{i \in [4]} v_i \\ &= - \sum_{i \in [4]} \left(\frac{1}{4} - \frac{4}{7} \lambda_i \right) v_i \\ &= - \left(\sum_{j \in [4]} \left(\frac{1}{4} - \frac{4}{7} \lambda_j \right) \right) \sum_{i \in [4]} \frac{\frac{1}{4} - \frac{4}{7} \lambda_i}{\sum_{j \in [4]} \left(\frac{1}{4} - \frac{4}{7} \lambda_j \right)} v_i \\ &\in -\frac{3}{7}T. \end{aligned}$$

It follows that $\sum_{i \in [4]} \lambda_i v_i \in -\frac{3}{4}T$. Thus $T_5 \subset -\frac{3}{4}T$. It is not difficult to verify that T can be covered by 4 translates of $\frac{3}{4}T$, which implies that T_5 can be covered by 4 translates of $-\frac{9}{16}T$. Therefore, $\beta_X(T, 8) \leq \frac{9}{16}$. \square

Proposition 9. *Let T be a 3-dimensional simplex in $X = (\mathbb{R}^3, \|\cdot\|)$. Then*

$$\beta_X(T, 9) \leq \frac{9}{17}.$$

Proof. We use the idea in the proof of Proposition 8. For each $i \in [4]$, put $T_i = \frac{8}{17}v_i + \frac{9}{17}T$. Then the portion of T not covered by $\bigcup_{i \in [4]} T_i$ is

$$T_5 = \left\{ \sum_{i \in [4]} \lambda_i v_i \mid \lambda_i \in \left[0, \frac{8}{17}\right], \forall i \in [4], \sum_{i \in [4]} \lambda_i = 1 \right\}.$$

As in the proof of Proposition 8, $T_5 \subset -\frac{15}{17}T$. By using the idea in the proof of Proposition 8 again, one can show that $\beta_X(T, 5) \leq \frac{3}{5}$. Therefore, T_5 is the union of 5 subsets of T_5 whose diameters are not larger than $\frac{9}{17}\delta(T)$. It follows that $\beta_X(T, 9) \leq \frac{9}{17}$. \square

Remark 10. The estimations in Proposition 8 and Proposition 9 are independent of the choice of norm on \mathbb{R}^3 .

For a convex body K , the Minkowski measure of symmetry, denoted by $s(K)$, is defined as

$$s(K) = \min \{ \lambda > 0 \mid \exists x \in X \text{ s.t. } K + x \subset -\lambda K \}.$$

It is known that

$$1 \leq s(K) \leq n, \forall K \in \mathcal{K}^n;$$

the equality on the left holds if and only if K is centrally symmetric, and the equality on the right holds if and only if K is an n -dimensional simplex (cf. [29]).

The following lemma shows the stability of $\beta_X(K, m)$ with respect to K .

Lemma 11. *Let $X = (\mathbb{R}^n, \|\cdot\|)$, and K and L be two convex bodies in X . If there exist a number $\gamma \geq 1$ and a point $c \in \mathbb{R}^n$ such that*

$$K \subseteq L \subseteq \gamma K + c,$$

then, for each $m \in \mathbb{Z}^+$, we have

$$\beta_X(L, m) \leq \gamma \beta_X(K, m).$$

Proof. For each $\varepsilon > 0$, there exists a collection $\{K_i \mid i \in [m]\}$ of subsets of $\gamma K + c$ such that

$$\gamma K + c = \bigcup_{i \in [m]} K_i$$

and

$$\delta(K_i) \leq \gamma \delta(K) \beta_X(K, m) + \varepsilon \leq \gamma \delta(L) \beta_X(K, m) + \varepsilon, \forall i \in [m].$$

Since

$$L = L \cap (\gamma K + c) = \bigcup_{i \in [m]} (L \cap K_i),$$

we have

$$\beta_X(L, m) \leq \gamma \beta_X(K, m) + \frac{\varepsilon}{\delta(L)}.$$

Since ε is arbitrary, $\beta_X(L, m) \leq \gamma \beta_X(K, m)$ as claimed. □

Theorem 12. *Let $X = (\mathbb{R}^3, \|\cdot\|)$, $m \in \mathbb{Z}^+$, and*

$$\eta = \sup \{ \beta_X(T, m) \mid T \text{ is a 3-dimensional simplex in } \mathbb{R}^3 \}.$$

We have

$$\beta(X, m) \leq \min_{\varepsilon \in (0, 1/3)} \max \left\{ \left(1 + \frac{4\varepsilon}{1 - 3\varepsilon} \right) \eta, \frac{2(3 - \varepsilon)}{4 - \varepsilon} \beta_X(B_X, m) \right\}.$$

Proof. Let K be a complete set in X , ε be a number in $(0, \frac{1}{3})$. We distinguish two cases.

Case 1. The Banach-Mazur distance from K to three-dimensional simplices is bounded from the above by

$$1 + \frac{4\varepsilon}{1 - 3\varepsilon}.$$

Then there exist a tetrahedron T and a point $c \in \mathbb{R}^3$ such that

$$T \subset K \subset \left(1 + \frac{4\varepsilon}{1 - 3\varepsilon}\right)T + c.$$

By Lemma 11, we have

$$\beta_X(K, m) \leq \left(1 + \frac{4\varepsilon}{1 - 3\varepsilon}\right)\eta.$$

Case 2. The Banach-Mazur distance from K to three-dimensional simplex is at least

$$1 + \frac{4\varepsilon}{1 - 3\varepsilon}.$$

From Theorem 2.1 in [25], it follows that

$$s(K) \leq 3 - \varepsilon.$$

Denote by $R(K)$ the circumradius of K . Theorem 1.1 in [7] shows that

$$s(K) = \frac{R(K)/\delta(K)}{1 - R(K)/\delta(K)}.$$

It follows that

$$\frac{R(K)}{\delta(K)} \leq \frac{3 - \varepsilon}{4 - \varepsilon}.$$

By a suitable translation if necessary, we may assume that

$$K \subseteq \frac{3 - \varepsilon}{4 - \varepsilon} \delta(K) B_X.$$

For each $\gamma > 0$, there exists a collection $\{B_i \mid i \in [8]\}$ such that

$$B_X = \bigcup_{i \in [8]} B_i \quad \text{and} \quad \delta(B_i) \leq 2\beta_X(B_X, m) + \gamma, \quad \forall i \in [m].$$

It follows that

$$\beta_X(K, m) \leq \frac{2(3 - \varepsilon)}{4 - \varepsilon} \beta_X(B_X, m) + \frac{3 - \varepsilon}{4 - \varepsilon} \gamma.$$

Hence

$$\beta_X(K, m) \leq \frac{2(3 - \varepsilon)}{4 - \varepsilon} \beta_X(B_X, m).$$

This completes the proof. □

Corollary 13. *Let $X = (\mathbb{R}^3, \|\cdot\|)$. If $\beta_X(B_X, 8) < \frac{221}{328}$, then $\beta(X, 8) < 1$.*

Proof. Since $\frac{2(3-\frac{7}{57})}{4-\frac{7}{57}} \frac{221}{328} = 1$ and

$$\frac{2(3-\varepsilon)}{4-\varepsilon} = 2 - \frac{2}{4-\varepsilon}$$

is continuous with respect to ε on $(0, \frac{1}{3})$, there exists a number $\varepsilon_0 < \frac{7}{57}$ such that

$$\frac{2(3-\varepsilon_0)}{4-\varepsilon_0} \beta_X(B_X, 8) < 1$$

It follows that

$$\beta(X, 8) \leq \max \left\{ \left(1 + \frac{4\varepsilon_0}{1-3\varepsilon_0} \right) \frac{9}{16}, \frac{2(3-\varepsilon_0)}{4-\varepsilon_0} \beta_X(B_X, 8) \right\} < 1.$$

□

In particular, Corollary 13 shows that $\beta(l_1^3, 8) < 1$ since the unit ball of l_1^3 can be covered by 8 balls having radius $\frac{2}{3} < \frac{221}{328}$. By solving the optimization problem

$$\min_{\varepsilon \in (0, 1/3)} \max \left\{ \left(1 + \frac{4\varepsilon}{1-3\varepsilon} \right) \frac{9}{16}, \frac{2(3-\varepsilon)}{4-\varepsilon} \frac{2}{3} \right\},$$

one can show that $\beta(l_1^3, 8) \leq 0.989\dots$. This estimation can be improved, see the next section.

4. An estimation of $\beta(l_p^3, 8)$

Lemma 14. For each $p \in [1, 2]$, $d_{BM}^M(l_p^3, l_\infty^3) \leq \frac{\sqrt{18 \cdot 19}}{10} \approx 1.85$.

Proof. Put $c_1 = (3, 3, -2)$, $c_2 = (-2, 3, 3)$, $c_3 = (3, -2, 3)$. Denote by Q the parallelepiped having

$$\left\{ \sum_{i \in [3]} \sigma_i c_i \mid \sigma_i \in \{-1, 1\}, \forall i \in [3] \right\}$$

as the set of vertices. We have

$$\begin{aligned} & \max \left\{ \left\| \sum_{i \in [3]} \sigma_i c_i \right\|_p \mid i \in [3] \right\} \\ &= \max \left\{ \|(4, 4, 4)\|_p, \|(-2, 8, -2)\|_p, \|(8, -2, -2)\|_p, \|(2, 2, -8)\|_p \right\} \\ &= \max \left\{ \|(4, 4, 4)\|_p, \|(-2, 8, -2)\|_p \right\} \\ &= \|(-2, 8, -2)\|_p = 2 \|(-1, 4, -1)\|_p = 2 \|(1, 1, 4)\|_p. \end{aligned}$$

It follows that

$$\frac{1}{2\|(1, 1, 4)\|_p}Q \subset B_p^3,$$

where B_p^3 is the unit ball of l_p^3 . Let q be the number satisfying

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and let f_1, f_2, f_3 be linear functionals defined on l_p^3 such that, for any $(\alpha, \beta, \gamma) \in \mathbb{R}^3$,

$$\begin{aligned} f_1((\alpha, \beta, \gamma)) &= \frac{1}{100}(15\alpha - 5\beta + 15\gamma), \\ f_2((\alpha, \beta, \gamma)) &= \frac{1}{100}(-5\alpha + 15\beta + 15\gamma), \\ f_3((\alpha, \beta, \gamma)) &= \frac{1}{100}(15\alpha + 15\beta - 5\gamma). \end{aligned}$$

Then

$$\begin{aligned} c_3 + \text{span}\{c_1, c_2\} &= \{x \in \mathbb{R}^3 \mid f_1(x) = 1\}, \\ c_2 + \text{span}\{c_1, c_3\} &= \{x \in \mathbb{R}^3 \mid f_2(x) = 1\}, \\ c_1 + \text{span}\{c_2, c_3\} &= \{x \in \mathbb{R}^3 \mid f_3(x) = 1\}. \end{aligned}$$

Thus the distances from the origin o to the facets of Q all equals to

$$\frac{100}{\|(15, -5, 15)\|_q}.$$

It follows that

$$\frac{1}{2\|(1, 1, 4)\|_p}Q \subset B_p^3 \subset \frac{\|(15, -5, 15)\|_q}{100}Q = \frac{\|(1, 1, 4)\|_p \|(3, 1, 3)\|_q}{10} \frac{1}{2\|(1, 1, 4)\|_p}Q,$$

which implies that

$$d_{BM}^M(l_p^3, l_\infty^3) \leq \frac{\|(1, 1, 4)\|_p \|(3, 1, 3)\|_q}{10} \leq \frac{\|(1, 1, 4)\|_2 \|(3, 1, 3)\|_2}{10} = \frac{\sqrt{18 \cdot 19}}{10}.$$

□

Remark 15. The last inequality in Lemma 14 can be verified in the following way. Put

$$f(p) = (4^p + 2)^{\frac{1}{p}} \cdot \left(2 \cdot 3^{\frac{p}{p-1}} + 1\right)^{\frac{p-1}{p}}.$$

Numerical results show that $f'(p) = 0$ has a unique solution $p_0 \approx 1.320$ in $[1, 2]$, and

$$f(p_0) \approx 17.550 < f(2).$$

Moreover, $f(1) < f(2)$. Thus $f(p)$ is maximized at $p = 2$.

Numerical results show that when $p \in [1, 1.736)$,

$$\frac{\|(1, 1, 4)\|_p \|(3, 1, 3)\|_q}{10} \leq \frac{9}{5}.$$

The estimation in Lemma 14 could be improved by choosing points c_1, c_2, c_3 more carefully for different $p \in [1, 2]$.

Theorem 16. *We have the following estimation:*

$$\beta(l_p^3, 8) \leq \begin{cases} \frac{\sqrt{18-19}}{20}, & p \in [1, 2), \\ \frac{3^{1/p}}{2}, & p \in [2, +\infty]. \end{cases}$$

Proof. First we consider the case when $p \in [2, +\infty]$. By Proposition 37.6 in [28], $d_{BM}^M(l_p^3, l_\infty^3) = 3^{1/p}$, this together with Theorem 2, implies that $\beta(l_p^3, 8) \leq \frac{3^{1/p}}{2} \leq \frac{\sqrt{3}}{2}$.

The case when $p \in [1, 2]$ follows directly from Lemma 14 and Theorem 2. \square

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Declarations

Conflicts of interest The author declares no conflict of interest.

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