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Results in Mathematics

Relationship Between the Analytic Generalized Fourier–Feynman Transform and the Function Space Integral

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Abstract. In this paper we investigated a relationship between the analytic generalized Fourier–Feynman transform associated with Gaussian process and the function space integral for exponential type functionals on the function space $C_{a,b}[0,T]$. The function space $C_{a,b}[0,T]$ can be induced by a generalized Brownian motion process. The Gaussian processes used in this paper are neither centered nor stationary.

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Keywords. generalized Brownian motion process, analytic generalized Fourier–Feynman transform, Gaussian process, exponential type functional.

1. Introduction

Let $C_0[0,T]$ denote the classical Wiener space. In [4], Cameron and Storvick defined the "*sequential*" Feynman integral by means of finite dimensional approximations for functionals on the Wiener space $C_0[0,T]$. The sequential definition for the Feynman path integral was intended to interpret the Feynman's uniform measure [12] on continuous paths space $C_0[0,T]$, because there is no countably additive measure as Lebesgue measure. It is well known that there is generally no quasi-invariant measure on infinite-dimensional linear spaces, see [13]. Thus, the Cameron and Storvick's sequential Feynman integral is a rigorous mathematical formulation for the Feynman's path integral. On the other hand, the concept of the "*analytic*" Feynman integral on the Wiener space $C_0[0,T]$ was introduced by Cameron [1]. We refer to the reference [5,

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Section 1] for a heuristic structure of the analytic Feynman integral of functionals on $C_0[0,T]$. The analytic Feynman integral is not defined in terms of a countably additive nonnegative measure. Rather, they are defined in terms of a process of analytic continuation and a limiting procedure. In this reason, Cameron and Storvick provided the Banach algebra $\mathcal{S}$ of analytic Feynman integrable functionals in [2]. Since then, in [3], Cameron and Storvick expressed the analytic Feynman integral of functionals in $\mathcal{S}$ as the limit of a sequence of Wiener integrals.

Let $D = [0,T]$ and let $(\Omega, \mathcal{B}, P)$ be a probability space. A generalized Brownian motion process (GBMP) on $\Omega \times D$ is a Gaussian process $Y \equiv \{Y_t\}_{t\in D}$ such that $Y_0 = c$ almost surely for some constant $c \in \mathbb{R}$ (in this paper we set $c = 0$), and for any $0 \le s < t \le T$,

$$Y_t - Y_s \sim N\big(a(t) - a(s),\, b(t) - b(s)\big),$$

where $N(m, \sigma^2)$ denotes the normal distribution with mean m and variance σ^2, $a(t)$ is a continuous real-valued function on $[0,T]$, and $b(t)$ is a monotonically increasing continuous real-valued function on $[0,T]$. Thus, the GBMP Y is determined by the functions $a(t)$ and $b(t)$. For more details, see [14,15]. Note that when $a(t) \equiv 0$ and $b(t) = t$, the GBMP is a standard Brownian motion (Wiener process). We are obliged to point out that a standard Brownian motion is stationary in time, whereas a GBMP is generally not stationary in time, and is subject to a drift $a(t)$.

In [8,10], the authors defined the analytic generalized Feynman integral and the analytic generalized Fourier–Feynman transform (GFFT) on the function space $C_{a,b}[0,T]$, and studied their properties and related topics. The function space $C_{a,b}[0,T]$, induced by a GBMP, was introduced by Yeh in [14], and was used extensively in [5–11].

In this paper we extend the ideas of [3] to the functionals on the very general function space $C_{a,b}[0,T]$. But our purpose of this paper is to obtain an expression of the analytic GFFT as a limit of a sequence of function space integrals on $C_{a,b}[0,T]$. The result in this paper enables us that the analytic GFFTs of functionals on the function space $C_{a,b}[0,T]$ can be interpreted as a limit of (non-analytic) function space transform.

The Wiener process used in [1–4] is centered and stationary in time and is free of drift. However, the Gaussian processes used in this paper, as well as in [6,7], are neither centered nor stationary.

2. Preliminaries

In this section we first provide a brief background and some well-known results about the function space $C_{a,b}[0,T]$ induced by the GBMP.

Let $a(t)$ be an absolutely continuous real-valued function on $[0,T]$ with $a(0) = 0$ and $a'(t) \in L^2[0,T]$, and let $b(t)$ be an increasing and continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each

$t \in [0,T]$. The GBMP Y determined by $a(t)$ and $b(t)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s), b(t)\}$. For more details, see [5,7,8,10,14,15]. By Theorem 14.2 in [15], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0) = 0$ under the sup norm). Hence, $(C_{a,b}[0,T], \mathcal{B}(C_{a,b}[0,T]), \mu)$ is the function space induced by Y where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel σ-field of $C_{a,b}[0,T]$. We then complete this function space to obtain the measure space $(C_{a,b}[0,T], \mathcal{W}(C_{a,b}[0,T]), \mu)$ where $\mathcal{W}(C_{a,b}[0,T])$ is the set of all μ-Carathéodory measurable subsets of $C_{a,b}[0,T]$.

A subset B of $C_{a,b}[0,T]$ is said to be scale-invariant measurable provided ρB is $\mathcal{W}(C_{a,b}[0,T])$-measurable for all $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $\mu(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). A functional F is said to be scale-invariant measurable provided F is defined on a scale-invariant measurable set and $F(\rho\,\cdot\,)$ is $\mathcal{W}(C_{a,b}[0,T])$-measurable for every $\rho > 0$. If two functionals F and G defined on $C_{a,b}[0,T]$ are equal s-a.e., we write $F \approx G$.

Remark 2.1. The function space $C_{a,b}[0,T]$ reduces to the Wiener space $C_0[0,T]$, considered in papers [1–4] if and only if $a(t) \equiv 0$ and $b(t) = t$ for all $t \in [0,T]$.

Let $L^2_{a,b}[0,T]$ (see [8] and [10]) be the space of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue–Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$; i.e.,

$$L^2_{a,b}[0,T] = \left\{ v : \int_0^T v^2(s)db(s) < +\infty \text{ and } \int_0^T v^2(s)d|a|(s) < +\infty \right\}$$

where $|a|(\cdot)$ denotes the total variation function of $a(\cdot)$. Then $L^2_{a,b}[0,T]$ is a separable Hilbert space with inner product defined by

$$(u,v)_{a,b} = \int_0^T u(t)v(t)dm_{|a|,b}(t) \equiv \int_0^T u(t)v(t)d[b(t) + |a|(t)],$$

where $m_{|a|,b}$ denotes the Lebesgue–Stieltjes measure induced by $|a|(\cdot)$ and $b(\cdot)$. In particular, note that $\|u\|_{a,b} \equiv \sqrt{(u,u)_{a,b}} = 0$ if and only if $u(t) = 0$ a.e. on $[0,T]$. For more details, see [8,10].

Let

$$C'_{a,b}[0,T] = \left\{ w \in C_{a,b}[0,T] : w(t) = \int_0^t z(s)db(s) \text{ for some } z \in L^2_{a,b}[0,T] \right\}.$$

For $w \in C'_{a,b}[0,T]$, with $w(t) = \int_0^t z(s)db(s)$ for $t \in [0,T]$, let $D : C'_{a,b}[0,T] \to L^2_{a,b}[0,T]$ be defined by the formula

$$Dw(t) = z(t) = \frac{w'(t)}{b'(t)}. \tag{2.1}$$

Then $C'_{a,b} \equiv C'_{a,b}[0,T]$ with inner product

$$(w_1, w_2)_{C'_{a,b}} = \int_0^T Dw_1(t)Dw_2(t)db(t)$$

is a separable Hilbert space. For more details, see [6,7].

Note that the two separable Hilbert spaces $L^2_{a,b}[0,T]$ and $C'_{a,b}[0,T]$ are (topologically) homeomorphic under the linear operator given by Eq. (2.1). The inverse operator of D is given by

$$(D^{-1}z)(t) = \int_0^t z(s)db(s), \quad t \in [0,T].$$

In the case that $a(t) \equiv 0$, then the operator $D : C'_{0,b}[0,T] \to L^2_{0,b}[0,T]$ is an isometry.

In this paper, in addition to the conditions put on $a(t)$ above, we now add the condition

$$\int_0^T |a'(t)|^2 d|a|(t) < +\infty \tag{2.2}$$

from which it follows that

$$\begin{aligned}\int_0^T |Da(t)|^2 d[b(t) + |a|(t)] &= \int_0^T \left|\frac{a'(t)}{b'(t)}\right|^2 d[b(t) + |a|(t)] \\ &< M\|a'\|_{L^2[0,T]} + M^2 \int_0^T |a'(t)|^2 d|a|(t) < +\infty,\end{aligned}$$

where $M = \sup_{t\in[0,T]}(1/b'(t))$. Thus, the function $a : [0,T] \to \mathbb{R}$ satisfies the condition (2.2) if and only if $a(\cdot)$ is an element of $C'_{a,b}[0,T]$. Under the condition (2.2), we observe that for each $w \in C'_{a,b}[0,T]$ with $Dw = z$,

$$(w, a)_{C'_{a,b}} = \int_0^T Dw(t)Da(t)db(t) = \int_0^T z(t)da(t).$$

Let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal set in $(C'_{a,b}[0,T], \|\cdot\|_{C'_{a,b}})$ such that the De_n's are of bounded variation on $[0,T]$. For $w \in C'_{a,b}[0,T]$ and $x \in C_{a,b}[0,T]$, we define the Paley–Wiener–Zygmund (PWZ) stochastic integral $(w,x)^{\sim}$ as follows:

$$(w,x)^{\sim} = \lim_{n\to\infty} \int_0^T \sum_{j=1}^{n} (w, e_j)_{C'_{a,b}} De_j(t)dx(t)$$

if the limit exists.

We will emphasize the following fundamental facts. For each $w \in C'_{a,b}[0,T]$, the PWZ stochastic integral $(w,x)^{\sim}$ exists for a.e. $x \in C_{a,b}[0,T]$. If $Dw = z \in L^2_{a,b}[0,T]$ is of bounded variation on $[0,T]$, then the PWZ stochastic integral $(w,x)^{\sim}$ equals the Riemann–Stieltjes integral $\int_0^T z(t)dx(t)$. Furthermore, for each $w \in C'_{a,b}[0,T]$, $(w,x)^{\sim}$ is a Gaussian random variable with mean

$(w,a)_{C'_{a,b}}$ and variance $\|w\|^2_{C'_{a,b}}$. Thus, for an orthogonal set $\{g_1,\ldots,g_n\}$ of nonzero functions in $(C'_{a,b}[0,T],\|\cdot\|_{C'_{a,b}})$ and a Lebesgue measurable function $f:\mathbb{R}^n\to\mathbb{C}$, it follows that

$$\begin{aligned}&\int_{C_{a,b}[0,T]} f\big((g_1,x)^\sim,\ldots,(g_n,x)^\sim\big)d\mu(x)\\&=\bigg(\prod_{j=1}^n 2\pi\|g_j\|^2_{C'_{a,b}}\bigg)^{-n/2}\int_{\mathbb{R}^n} f(u_1,\ldots,u_n)\\&\qquad\times\exp\bigg\{-\sum_{j=1}^n\frac{[u_j-(g_j,a)_{C'_{a,b}}]^2}{2\|g_j\|^2_{C'_{a,b}}}\bigg\}du_1\cdots du_n\end{aligned}\tag{2.3}$$

in the sense that if either side of Eq. (2.3) exists, both sides exist and equality holds. Also we note that for $w,x\in C'_{a,b}[0,T]$, $(w,x)^\sim=(w,x)_{C'_{a,b}}$.

The following integration formula on the function space $C_{a,b}[0,T]$ is also used in this paper:

$$\int_{\mathbb{R}}\exp\big\{-\alpha u^2+\beta u\big\}du=\sqrt{\frac{\pi}{\alpha}}\exp\Big\{\frac{\beta^2}{4\alpha}\Big\}\tag{2.4}$$

for complex numbers α and β with $\mathrm{Re}(\alpha)>0$.

3. Gaussian Processes

Let $C^*_{a,b}[0,T]$ be the set of functions k in $C'_{a,b}[0,T]$ such that Dk is continuous except for a finite number of finite jump discontinuities and is of bounded variation on $[0,T]$. For any $w\in C'_{a,b}[0,T]$ and $k\in C^*_{a,b}[0,T]$, let the operation $\odot$ between $C'_{a,b}[0,T]$ and $C^*_{a,b}[0,T]$ be defined by

$$w\odot k=D^{-1}(DwDk),$$

where $DwDk$ denotes the pointwise multiplication of the functions Dw and Dk. Then $(C^*_{a,b}[0,T],\odot)$ is a commutative algebra with the identity b. For a more detailed study of the operation $\odot$, see [7].

For each $t\in[0,T]$, let $\varPhi_t(\tau)=D^{-1}\chi_{[0,t]}(\tau)=\int_0^\tau\chi_{[0,t]}(u)db(u)$, $\tau\in[0,T]$, and for $k\in C'_{a,b}[0,T]$ with $Dk\neq 0$ m_L-a.e. on $[0,T]$ (m_L denotes the Lebesgue measure on $[0,T]$), let $\mathcal{Z}_k(x,t)$ be the PWZ stochastic integral

$$\mathcal{Z}_k(x,t)=(k\odot\varPhi_t,x)^\sim,\tag{3.1}$$

let $\beta_k(t)=\int_0^t\{Dk(u)\}^2db(u)$, and let $\alpha_k(t)=\int_0^t Dk(u)da(u)$. Then $\mathcal{Z}_k:C_{a,b}[0,T]\times[0,T]\to\mathbb{R}$ is a Gaussian process with mean function

$$\int_{C_{a,b}[0,T]}\mathcal{Z}_k(x,t)d\mu(x)=\int_0^t Dk(u)da(u)=\alpha_k(t)$$

and covariance function

$$\int_{C_{a,b}[0,T]} \big(\mathcal{Z}_k(x,s)-\alpha_k(s)\big)\big(\mathcal{Z}_k(x,t)-\alpha_k(t)\big)d\mu(x)$$
$$=\int_0^{\min\{s,t\}}\{Dk(u)\}^2 db(u)=\beta_k(\min\{s,t\}).$$

In addition, by [15, Theorem 21.1], $\mathcal{Z}_k(\cdot,t)$ is stochastically continuous in t on $[0,T]$. If Dk is of bounded variation on $[0,T]$, then, for all $x\in C_{a,b}[0,T]$, $\mathcal{Z}_k(x,t)$ is continuous in t. Also, for any functions k_1 and k_2 in $C'_{a,b}[0,T]$,

$$\int_{C_{a,b}[0,T]} \mathcal{Z}_{k_1}(x,s)\mathcal{Z}_{k_2}(x,t)d\mu(x)$$
$$=\int_0^{\min\{s,t\}} Dk_1(u)Dk_2(u)db(u)+\int_0^s Dk_1(u)da(u)\int_0^t Dk_2(u)da(u).$$

Of course if $k(t)\equiv b(t)$, then $\mathcal{Z}_b(x,t)=x(t)$, the continuous sample paths of the GBMP Y, of which the function space $C_{a,b}[0,T]$ consists. Choosing $a(t)\equiv 0$ and $b(t)=t$ on $[0,T]$, as commented in Remark 2.1 above, the function space $C_{a,b}[0,T]$ reduces to the classical Wiener space $C_0[0,T]$, and thus the Gaussian process (3.1) with $k(t)\equiv t$ is an ordinary Wiener process.

From the properties of the PWZ stochastic integral and the operation $\odot$ between $C'_{a,b}[0,T]$ and $C^*_{a,b}[0,T]$, it follows that for all $\rho\in\mathbb{R}$,

$$\rho\mathcal{Z}_k(x,t)=\mathcal{Z}_{\rho k}(x,t)=\mathcal{Z}_k(\rho x,t),$$

and for any $w\in C'_{a,b}[0,T]$ and each $k\in C^*_{a,b}[0,T]$,

$$(w,\mathcal{Z}_k(x,\cdot))^\sim=(w\odot k,x)^\sim \tag{3.2}$$

for μ-a.e. $x\in C_{a,b}[0,T]$. Thus, throughout the remainder of this paper, we require k to be in $C^*_{a,b}[0,T]$ for each process $\mathcal{Z}_k$.

We define a class of those functions as follows: let

$$\mathrm{Supp}_{C^*_{a,b}}[0,T]=\{k\in C^*_{a,b}[0,T]: Dk\neq 0\ m_L\text{-a.e on }[0,T]\}.$$

Then for any $k\in \mathrm{Supp}_{C^*_{a,b}}[0,T]$, the Lebesgue–Stieltjes integrals

$$\|w\odot k\|^2_{C'_{a,b}}=\int_0^T (Dw(t))^2(Dk(t))^2db(t)$$

and

$$(w\odot k,a)_{C'_{a,b}}=\int_0^T Dw(t)Dk(t)Da(t)db(t)=\int_0^T Dw(t)Dk(t)da(t)$$

exist for all $w\in C'_{a,b}[0,T]$.

4. Transforms on the Class of Exponential-Type Functionals

Given a function $k \in \text{Supp}_{C_{a,b}^*}[0,T]$, we define the generalized $\mathcal{Z}_k$-function space integral (namely, the function space integral associated with the Gaussian paths $\mathcal{Z}_k(x,\cdot)$) for functionals F on $C_{a,b}[0,T]$ by the formula

$$I_k[F] \equiv I_{k,x}[F(\mathcal{Z}_k(x,\cdot))] = \int_{C_{a,b}[0,T]} F\big(\mathcal{Z}_k(x,\cdot)\big)d\mu(x).$$

Throughout this paper, let $\mathbb{C}$, $\mathbb{C}_+$ and $\widetilde{\mathbb{C}}_+$ denote the set of complex numbers, complex numbers with positive real part, and non-zero complex numbers with nonnegative real part, respectively. Furthermore, for each $\lambda \in \mathbb{C}$, $\lambda^{1/2}$ denotes the principal square root of λ, i.e., $\lambda^{1/2}$ is always chosen to have non-negative real part.

Definition 4.1. Given a function $k \in \text{Supp}_{C_{a,b}^*}[0,T]$, let $\mathcal{Z}_k$ be the Gaussian process given by (3.1) and let F be a $\mathbb{C}$-valued scale-invariant measurable functional on $C_{a,b}[0,T]$ such that

$$J_F(\mathcal{Z}_k;\lambda) = I_{k,x}[F(\lambda^{-1/2}\mathcal{Z}_k(x,\cdot))]$$

exists and is finite for all $\lambda > 0$. If there exists a function $J_F^*(\mathcal{Z}_k;\lambda)$ analytic on $\mathbb{C}_+$ such that $J_F^*(\mathcal{Z}_k;\lambda) = J_F(\mathcal{Z}_k;\lambda)$ for all $\lambda \in (0,+\infty)$, then $J_F^*(\mathcal{Z}_k;\lambda)$ is defined to be the analytic $\mathcal{Z}_k$-function space integral (namely, the analytic function space integral associated with the Gaussian paths $\mathcal{Z}_k(x,\cdot)$) of F over $C_{a,b}[0,T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$I_k^{\text{an}_\lambda}[F] \equiv I_{k,x}^{\text{an}_\lambda}[F(\mathcal{Z}_k(x,\cdot))] \equiv \int_{C_{a,b}[0,T]}^{\text{an}_\lambda} F\big(\mathcal{Z}_k(x,\cdot)\big)d\mu(x) := J_F^*(\mathcal{Z}_k;\lambda). \tag{4.1}$$

Let q be a non-zero real number and let F be a measurable functional whose analytic $\mathcal{Z}_k$-function space integral $J_F^*(\mathcal{Z}_k;\lambda)$ exists for all λ in $\mathbb{C}_+$. If the following limit exists, we call it the analytic generalized $\mathcal{Z}_k$-Feynman integral (namely, the analytic generalized Feynman integral associated with the paths $\mathcal{Z}_k(x,\cdot)$) of F with parameter q and we write

$$I_k^{\text{anf}_q}[F] \equiv I_{k,x}^{\text{anf}_q}[F(\mathcal{Z}_k(x,\cdot))] = \lim_{\lambda\to -iq} I_{k,x}^{\text{an}_\lambda}[F(\mathcal{Z}_k(x,\cdot))], \tag{4.2}$$

where λ approaches $-iq$ through values in $\mathbb{C}_+$.

Next we state the definition of the analytic GFFT associated with Gaussian process on function space.

Definition 4.2. Given a function $k \in \text{Supp}_{C_{a,b}^*}[0,T]$, let $\mathcal{Z}_k$ be the Gaussian process given by (3.1) and let F be a scale-invariant measurable functional on $C_{a,b}[0,T]$ such that for all $\lambda \in \mathbb{C}_+$ and $y \in C_{a,b}[0,T]$, the following analytic $\mathcal{Z}_k$-function space integral

$$T_{\lambda,k}(F)(y) = I_{k,x}^{\text{an}_\lambda}[F(y + \mathcal{Z}_k(x,\cdot))]$$

exists. Let q be a non-zero real number. For $p \in (1,2]$, we define the L_p analytic $\mathcal{Z}_k$-GFFT (namely, the GFFT associated with the paths $\mathcal{Z}_k(x,\cdot)$), $T_{q,k}^{(p)}(F)$ of F, by the formula,

$$T_{q,k}^{(p)}(F)(y) = \underset{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}}{\text{l.i.m.}}\, T_{\lambda,k}(F)(y)$$

if it exists; i.e., for each $\rho > 0$,

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} \int_{C_{a,b}[0,T]} \left|T_{\lambda,k}(F)(\rho y) - T_{q,k}^{(p)}(F)(\rho y)\right|^{p'} d\mu(y) = 0$$

where $1/p + 1/p' = 1$. We define the L_1 analytic $\mathcal{Z}_k$-GFFT, $T_{q,k}^{(1)}(F)$ of F, by the formula

$$T_{q,k}^{(1)}(F)(y) = \lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} T_{\lambda,k}(F)(y) = I_{k,x}^{\text{anf}_q}[F(y+\mathcal{Z}_k(x,\cdot))] \tag{4.3}$$

for s-a.e. $y \in C_{a,b}[0,T]$, if it exists.

We note that for $1 \le p \le 2$, $T_{q,k}^{(p)}(F)$ is defined only s-a.e.. We also note that if $T_{q,k}^{(p)}(F)$ exists and if $F \approx G$, then $T_{q,k}^{(p)}(G)$ exists and $T_{q,k}^{(p)}(G) \approx T_{q,k}^{(p)}(F)$. Moreover, from Eqs. (4.2), (4.1), and (4.3), it follows that

$$I_k^{\text{anf}_q}[F] \equiv I_{k,x}^{\text{anf}_q}[F(\mathcal{Z}_k(x,\cdot))] = T_{q,k}^{(1)}(F)(0) \tag{4.4}$$

in the sense that if either side exists, then both sides exist and equality holds.

Remark 4.3. Note that if $k \equiv b$ on $[0,T]$, then the analytic generalized $\mathcal{Z}_b$-Feynman integral, $I_b^{\text{anf}_q}[F]$, and the L_p analytic $\mathcal{Z}_b$-GFFT, $T_{q,b}^{(p)}(F)$ agree with the previous definitions of the analytic generalized Feynman integral and the analytic GFFT respectively [5,8,10].

Let $\mathcal{E}$ be the class of all functionals which have the form

$$\Psi_w(x) = \exp\{(w,x)^{\sim}\} \tag{4.5}$$

for some $w \in C'_{a,b}[0,T]$ and for s-a.e. $x \in C_{a,b}[0,T]$. More precisely, since we shall identify functionals which coincide s-a.e. on $C_{a,b}[0,T]$, the class $\mathcal{E}$ can be regarded as the space of all s-equivalence classes of functionals of the form (4.5).

Given $q \in \mathbb{R}\setminus\{0\}$, $\tau \in C'_{a,b}[0,T]$, and $k \in C^*_{a,b}[0,T]$, let $\mathcal{E}_{q,\tau,k}$ be the class of all functionals having the form

$$\Psi_w^{q,\tau,k}(x) = K_{q,\tau,k}^a \Psi_w(x) \tag{4.6}$$

for s-a.e. $x \in C_{a,b}[0,T]$, where Ψ_w is given by Eq. (4.5) and $K_{q,\tau,k}^a$ is a complex number given by

$$K_{q,\tau,k}^a = \exp\left\{\frac{i}{2q}\|\tau\odot k\|^2_{C'_{a,b}} + (-iq)^{-1/2}(\tau\odot k, a)_{C'_{a,b}}\right\}. \tag{4.7}$$

The functionals given by Eq. (4.6) and linear combinations (with complex coefficients) of the $\Psi_w^{q,\tau,k}$'s are called the (partially) exponential-type functionals on $C_{a,b}[0,T]$.

For notational convenience, let $\Psi_w^{0,\tau,k}(x) = \Psi_w(x)$ and let $\mathcal{E}_{0,\tau,k} = \mathcal{E}$. Then for any $(q,\tau,k) \in \mathbb{R} \times C'_{a,b}[0,T] \times C^*_{a,b}[0,T]$, the class $\mathcal{E}_{q,\tau,k}$ is dense in $L_2(C_{a,b}[0,T])$, see [9,11]. We then define the class $\mathcal{E}(C_{a,b}[0,T])$ to be the linear span of $\mathcal{E}$, i.e., $\mathcal{E}(C_{a,b}[0,T]) = \mathrm{Span}\mathcal{E}$.

Remark 4.4. (i) One can see that $\mathcal{E}(C_{a,b}[0,T]) = \mathrm{Span}\mathcal{E}_{q,\tau,k}$ for every $(q,\tau,k) \in \mathbb{R} \times C'_{a,b}[0,T] \times C^*_{a,b}[0,T]$.

(ii) The linear space $\mathcal{E}(C_{a,b}[0,T])$ is a commutative (complex) algebra under the pointwise multiplication and with identity $\Psi_0 \equiv 1$ because

$$\Psi_{w_1}^{q_1,\tau_1,k_1}(x)\Psi_{w_2}^{q_2,\tau_2,k_2}(x) = K^a_{q_1,\tau_1,k_1}K^a_{q_2,\tau_2,k_2}\Psi_{w_1+w_2}(x)$$

for μ-a.e. $x \in C_{a,b}[0,T]$.

(iii) Note that every exponential-type functional is scale-invariant measurable. Since we shall identify functionals which coincide s-a.e. on $C_{a,b}[0,T]$, $\mathcal{E}(C_{a,b}[0,T])$ can be regarded as the space of all s-equivalence classes of exponential-type functionals.

The following two theorems are due to by Chang and Choi [6].

Theorem 4.5. *Let $\Psi_w \in \mathcal{E}$ be given by Eq. (4.5). Then for all $p \in [1,2]$, any non-zero real number q, and each function k in $\mathrm{Supp}_{C^*_{a,b}}[0,T]$, the L_p analytic $\mathcal{Z}_k$-GFFT of Ψ_w, $T^{(p)}_{q,k}(\Psi_w)$ exists and is given by the formula*

$$T^{(p)}_{q,k}(\Psi_w) \approx \Psi_w^{q,w,k}, \tag{4.8}$$

where $\Psi_w^{q,w,k}$ is given by Eq. (4.6) with τ replaced with w. Thus, $T^{(p)}_{q,k}(\Psi_w)$ is an element of $\mathcal{E}(C_{a,b}[0,T])$.

Let F be a functional in $\mathcal{E}(C_{a,b}[0,T])$. Since $\mathcal{E}(C_{a,b}[0,T]) = \mathrm{Span}\mathcal{E}$, there exist a finite sequence $\{w_1,\dots,w_n\}$ of functions in $C'_{a,b}[0,T]$, and a sequence $\{c_1,\dots,c_n\}$ in $\mathbb{C}\setminus\{0\}$ such that

$$F \approx \sum_{j=1}^n c_j\Psi_{w_j}. \tag{4.9}$$

Then for all $p \in [1,2]$, any non-zero real number q, and each function k in $\mathrm{Supp}_{C^*_{a,b}}[0,T]$, the L_p analytic $\mathcal{Z}_k$-GFFT of F, $T^{(p)}_{q,k}(F)$ exists and is given by the formula

$$T^{(p)}_{q,k}(F) \approx \sum_{j=1}^n c_jT^{(p)}_{q,k}(\Psi_{w_j}) \approx \sum_{j=1}^n c_j\Psi_{w_j}^{q,w_j,k},$$

where $\Psi_{w_j}^{q,w_j,k}$ is given by Eq. (4.6) with τ and w replaced with w_j and w_j, for each $j \in \{1,\dots,n\}$, respectively.

Theorem 4.6. *For all* $p \in [1,2]$, *any* $q \in \mathbb{R} \setminus \{0\}$, *and each* $k \in \mathrm{Supp}_{C^*_{a,b}}[0,T]$, *the* L_p *analytic* $\mathcal{Z}_k$*-GFFT*, $T^{(p)}_{q,k} : \mathcal{E}(C_{a,b}[0,T]) \to \mathcal{E}(C_{a,b}[0,T])$ *is an onto transform.*

5. Relationship Between the $\mathcal{Z}_k$-Fourier–Feynman Transform and the Function Space Integral

In this section, we establish a relationship between the analytic $\mathcal{Z}_k$-GFFT and the $\mathcal{Z}_k$-function space integral of functionals in the class $\mathcal{E}(C_{a,b}[0,T])$.

Throughout this section, for convenience, we use the following notation: for $\zeta \in \widetilde{\mathbb{C}}_+$ and $n = 1, 2, \ldots$, let

$$G_n(\zeta, x) = \exp\left\{\left[\frac{1-\zeta}{2}\right]\sum_{j=1}^{n}[(e_j, x)^\sim]^2 + (\zeta^{1/2} - 1)\sum_{j=1}^{n}(e_j, a)_{C'_{a,b}}(e_j, x)^\sim\right\}, \tag{5.1}$$

where $\{e_n\}_{n=1}^{\infty}$ is a complete orthonormal set of functions in $C'_{a,b}[0,T]$.

Lemma 5.1. *Let* k *be a function in* $\mathrm{Supp}_{C^*_{a,b}}[0,T]$, *let* $\{e_1, \ldots, e_n\}$ *be an orthonormal set of functions in* $C'_{a,b}[0,T]$, *and let* w *be a function in* $C'_{a,b}[0,T]$. *Then for each* $\zeta \in \mathbb{C}_+$, *and* $n \in \mathbb{N}$, *the functional* $\exp\{(w \odot k, x)^\sim\}G_n(\zeta, x)$ *is* μ*-integrable, where* G_n *is given by (5.1). Also, it follows that*

$$\begin{aligned}\mathcal{W}_n(w;k;\zeta) &\equiv \int_{C_{a,b}[0,T]} \exp\{(w \odot k, x)^\sim\}G_n(\zeta, x)\,d\mu(x)\\ &= \zeta^{-n/2}\exp\left\{\frac{1}{2\zeta}\sum_{j=1}^{n}(e_j, w\odot k)^2_{C'_{a,b}} + \frac{1}{2}\left[\|w\odot k\|^2_{C'_{a,b}} - \sum_{j=1}^{n}(e_j, w\odot k)^2_{C'_{a,b}}\right]\right.\\ &\quad + \zeta^{-1/2}\sum_{j=1}^{n}(e_j, a)_{C'_{a,b}}(e_j, w\odot k)_{C'_{a,b}}\\ &\quad \left. + (e^{w\odot k}_{n+1}, a)_{C'_{a,b}}\left[\|w\odot k\|^2_{C'_{a,b}} - \sum_{j=1}^{n}(e_j, w\odot k)^2_{C'_{a,b}}\right]^{1/2}\right\},\end{aligned} \tag{5.2}$$

where

$$e^{w\odot k}_{n+1} = \left[\|w\odot k\|^2_{C'_{a,b}} - \sum_{j=1}^{n}(e_j, w\odot k)^2_{C'_{a,b}}\right]^{-1/2} \times \left\{w\odot k - \sum_{j=1}^{n}(e_j, w\odot k)_{C'_{a,b}}e_j\right\}.$$

Proof. We note that given two functions $k \in \mathrm{Supp}_{C^*_{a,b}}[0,T]$ and $w \in C'_{a,b}[0,T]$, $w \odot k$ is an element of $C'_{a,b}[0,T]$. Using the Gram–Schmidt process, we obtain

$e_{n+1}^{w\odot k} \in C'_{a,b}[0,T]$ such that $\{e_1,\dots,e_n,e_{n+1}^{w\odot k}\}$ forms an orthonormal set in $C'_{a,b}[0,T]$ and

$$w\odot k = \sum_{j=1}^{n} c_j e_j + c_{n+1} e_{n+1}^{w\odot k}$$

where

$$c_j = \begin{cases} (e_j, w\odot k)_{C'_{a,b}} & , \quad j=1,\dots,n \\ \big[\|w\odot k\|^2_{C'_{a,b}} - \sum_{j=1}^{n}(e_j, w\odot k)^2_{C'_{a,b}}\big]^{1/2} & , \quad j=n+1 \end{cases}.$$

Next, for $\zeta \in \mathbb{C}_+$, using (5.1), (2.3), the Fubini theorem, and (2.4), it follows that

$$\begin{aligned}
&\int_{C_{a,b}[0,T]} \exp\{(w\odot k, x)^{\sim}\} G_n(\zeta,x)\,d\mu(x) \\
&= (2\pi)^{-(n+1)/2} \int_{\mathbb{R}^{n+1}} \exp\bigg\{\bigg[\frac{1-\zeta}{2}\bigg]\sum_{j=1}^{n} u_j^2 + (\zeta^{1/2}-1)\sum_{j=1}^{n}(e_j,a)_{C'_{a,b}} u_j \\
&\qquad + \sum_{j=1}^{n+1} c_j u_j - \frac{1}{2}\sum_{j=1}^{n}[u_j - (e_j,a)_{C'_{a,b}}]^2 - \frac{1}{2}[u_{n+1} - (e_{n+1}^{w\odot k},a)_{C'_{a,b}}]^2\bigg\} \\
&\qquad \times du_1\cdots du_n du_{n+1} \\
&= \bigg(\prod_{j=1}^{n}(2\pi)^{-1/2}\int_{\mathbb{R}} \exp\bigg\{-\frac{\zeta}{2}u_j^2 + [\zeta^{1/2}(e_j,a)_{C'_{a,b}} + c_j]u_j\bigg\}du_j\bigg) \\
&\quad \times \bigg((2\pi)^{-1/2}\int_{\mathbb{R}} \exp\bigg\{-\frac{1}{2}u_{n+1}^2 + [(e_{n+1}^{w\odot k},a)_{C'_{a,b}} + c_{n+1}]u_{n+1}\bigg\}du_{n+1}\bigg) \\
&\quad \times \exp\bigg\{-\frac{1}{2}\sum_{j=1}^{n}(e_j,a)^2_{C'_{a,b}} - \frac{1}{2}(e_{n+1}^{w\odot k},a)^2_{C'_{a,b}}\bigg\} \\
&= \zeta^{-n/2}\exp\bigg\{\zeta^{-1/2}\sum_{j=1}^{n}(e_j,a)_{C'_{a,b}} c_j + \frac{1}{2\zeta}\sum_{j=1}^{n} c_j^2 \\
&\quad + (e_{n+1}^{w\odot k},a)_{C'_{a,b}} c_{n+1} + \frac{1}{2}c_{n+1}^2\bigg\} \\
&= \zeta^{-n/2}\exp\bigg\{\zeta^{-1/2}\sum_{j=1}^{n}(e_j,a)_{C'_{a,b}}(e_j,w\odot k)_{C'_{a,b}} + \frac{1}{2\zeta}\sum_{j=1}^{n}(e_j,w\odot k)^2_{C'_{a,b}} \\
&\quad + (e_{n+1}^{w\odot k},a)_{C'_{a,b}}\bigg[\|w\odot k\|^2_{C'_{a,b}} - \sum_{j=1}^{n}(e_j,w\odot k)^2_{C'_{a,b}}\bigg]^{1/2} \\
&\quad + \frac{1}{2}\bigg[\|w\odot k\|^2_{C'_{a,b}} - \sum_{j=1}^{n}(e_j,w\odot k)^2_{C'_{a,b}}\bigg]\bigg\}
\end{aligned}$$

as desired. □

In our next theorem we express the analytic $\mathcal{Z}_k$-GFFT of functionals in $\mathcal{E}(C_{a,b}[0,T])$ as the limit of a sequence of $\mathcal{Z}_k$-function space integrals.

Theorem 5.2. *Let $F \in \mathcal{E}(C_{a,b}[0,T])$ be given by Eq. (4.9). Given a non-zero real number q, let $\{\zeta_n\}$ be a sequence in $\mathbb{C}_+$ such that $\zeta_n \to -iq$. Then, for all $p \in [1,2]$, and each function k in $\mathrm{Supp}_{C^*_{a,b}}[0,T]$, it follows that*

$$T^{(p)}_{q,k}(F)(y) = \lim_{n\to\infty} \zeta_n^{n/2} \int_{C_{a,b}[0,T]} F\big(y + \mathcal{Z}_k(x,\cdot)\big) G_n(\zeta_n, x) d\mu(x), \quad (5.3)$$

for s-a.e. $y \in C_{a,b}[0,T]$, where G_n is given by Eq. (5.1).

Proof. In view of Theorems 4.5 and 4.6 , it will suffice to show that Eq. (5.3) with p and F replaced with 1 and Ψ_w holds true.

From Theorem 4.5, we know that the L_1 analytic $\mathcal{Z}_k$-GFFT of Ψ_w given by (4.5), $T^{(1)}_{q,k}(\Psi_w)$, exists. Using (4.5), (3.2), the Fubini theorem, and the first expression of (5.2) with ζ replaced with ζ_n, it follows that for all $n \in \mathbb{N}$,

$$\begin{aligned} &\zeta_n^{n/2} \int_{C_{a,b}[0,T]} \Psi_w\big(y + \mathcal{Z}_k(x,\cdot)\big) G_n(\zeta_n, x) d\mu(x) \\ &= \zeta_n^{n/2} \exp\{(w,y)^\sim\} \bigg[\int_{C_{a,b}[0,T]} \exp\{(w \odot k, x)^\sim\} G_n(\zeta_n, x) d\mu(x) \bigg] \\ &= \exp\{(w,y)^\sim\} \zeta_n^{n/2} \mathcal{W}_n(w; k; \zeta_n). \end{aligned} \quad (5.4)$$

Next, using (5.4), (5.2), Parseval's relation, (4.7), (4.5), (4.6) with τ replaced with w, and (4.8) with $p = 1$, it follows that

$$\begin{aligned} &\lim_{n\to\infty} \zeta_n^{n/2} \int_{C_{a,b}[0,T]} \Psi_w\big(y + \mathcal{Z}_k(x,\cdot)\big) G_n(\zeta_n, x) d\mu(x) \\ &= \exp\{(w,y)^\sim\} \lim_{n\to\infty} \zeta_n^{n/2} \mathcal{W}_n(w; k; \zeta_n) \\ &= \exp\bigg\{ (w,y)^\sim + \lim_{n\to\infty} \frac{1}{2\zeta_n} \sum_{j=1}^n (e_j, w \odot k)^2_{C'_{a,b}} \\ &\quad + \frac{1}{2} \lim_{n\to\infty} \bigg[\|w \odot k\|^2_{C'_{a,b}} - \sum_{j=1}^n (e_j, w \odot k)^2_{C'_{a,b}} \bigg] \\ &\quad + \lim_{n\to\infty} \zeta_n^{-1/2} \sum_{j=1}^n (e_j, a)_{C'_{a,b}} (e_j, w \odot k)_{C'_{a,b}} \\ &\quad + \lim_{n\to\infty} (e_{n+1}, a)_{C'_{a,b}} \bigg[\|w \odot k\|^2_{C'_{a,b}} - \sum_{j=1}^n (e_j, w \odot k)^2_{C'_{a,b}} \bigg]^{1/2} \bigg\} \\ &= \exp\{(w,y)^\sim\} K^a_{q,w,k} \\ &= \Psi_w(y) K^a_{q,w,k} \end{aligned}$$

$$= \Psi_w^{q,\tau,k}(y)$$
$$= T_{q,k}^{(1)}(\Psi_w)(y)$$

for s-a.e. $y \in C_{a,b}[0,T]$, as desired. □

The following corollary follows immediately from (4.4) and (5.3).

Corollary 5.3. *Let F, q and $\{\zeta_n\}$ be as in Theorem 5.2. Then, for each function k in $\mathrm{Supp}_{C^*_{a,b}}[0,T]$, it follows that*

$$I_k^{\mathrm{anf}_q}[F] = \lim_{n\to\infty} \zeta_n^{n/2} \int_{C_{a,b}[0,T]} F\big(\mathcal{Z}_k(x,\cdot)\big) G_n(\zeta_n, x) d\mu(x),$$

where G_n is given by Eq. (5.1).

We establish our next corollary after a careful examination of the proof of Theorem 5.2, and by using Eq. (4.1) instead of (4.4).

Corollary 5.4. *Let $F \in \mathcal{E}(C_{a,b}[0,T])$ be given by Eq. (4.9). Let $\lambda \in \mathbb{C}_+$, and let $\{\zeta_n\}$ be a sequence in $\mathbb{C}_+$ such that $\zeta_n \to \lambda$. Then, for each function k in $\mathrm{Supp}_{C^*_{a,b}}[0,T]$, it follows that*

$$I_k^{\mathrm{an}_\lambda}[F] = \lim_{n\to\infty} \zeta_n^{n/2} \int_{C_{a,b}[0,T]} F\big(\mathcal{Z}_k(x,\cdot)\big) G_n(\zeta_n, x) d\mu(x), \tag{5.5}$$

where G_n is given by Eq. (5.1).

Our next result, namely a change of scale formula for function space integrals, now follows easily by Corollary 5.4 above.

Corollary 5.5. *Let $F \in \mathcal{E}(C_{a,b}[0,T])$ be given by Eq. (4.9). Then for any $\rho > 0$, and each function k in $\mathrm{Supp}_{C^*_{a,b}}[0,T]$, it follows that*

$$\int_{C_{a,b}[0,T]} F(\rho\mathcal{Z}_k(x,\cdot)) d\mu(x) = \lim_{n\to\infty} \rho^{-n} \int_{C_{a,b}[0,T]} F(\mathcal{Z}_k(x,\cdot)) G_n(\rho^{-2}, x) d\mu(x), \tag{5.6}$$

where G_n is given by Eq. (5.1).

Proof. Note that for every $F \in \mathcal{E}(C_{a,b}[0,T])$ and all $\rho > 0$, the function space integral in the left-hand side of (5.6) exists. To ensure the equality in (5.6), simply choose $\lambda = \rho^{-2}$ and $\zeta_n = \rho^{-2}$ for every $n \in \mathbb{N}$ in (5.5). □

6. Concluding Remark

It is known that the class $\mathcal{E}(C_{a,b}[0,T])$ is a dense subspace of the space $L_2(C_{a,b}[0,T])$. For a related work, see [9,11]. Thus, using the L_2-approximation [11, Remark 4], one can develop the sequential approximation such as Eq. (5.3) for functionals F in $L_2(C_{a,b}[0,T])$ whose L_p analytic $\mathcal{Z}_k$-GFFT $T_{q,k}^{(p)}(F)$ exists. But, there exists a (bounded) functional F in $L_2(C_{a,b}[0,T])$ whose L_p $\mathcal{Z}_k$-GFFT $T_{q,k}^{(p)}(F)$ does not exist, see [5].

Indeed, the class $\mathcal{E}(C_{a,b}[0,T])$ is a very rich class of functionals on $C_{a,b}[0,T]$. It contains many meaningful functionals which discussed in quantum mechanics. We finish this paper with a very simple example for such functionals, which arises in quantum mechanics.

Example 6.1. Consider the functional F_S given by

$$F_S(x) = \exp\left\{ \int_0^T x(t)db(t) \right\}$$

for s-a.e. $x \in C_{a,b}[0,T]$. Then F_S is an element of $\mathcal{E}(C_{a,b}[0,T])$ because

$$F_S(x) = \exp\left\{(Sb, x)^{\sim})\right\}$$

where $S : C'_{a,b}[0,T] \to C'_{a,b}[0,T]$ is an operator defined by

$$Sw(t) = \int_0^t [w(T) - w(s)]db(s).$$

One can see that the adjoint operator S^* of S is given by

$$S^*w(t) = \int_0^t w(s)db(s).$$

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References

[1] Cameron, R.H.: The ILSTOW and Feynman integrals. J. D'Anal. Math. **10**, 287–361 (1962–63)

[2] Cameron, R.H., Storvick, D.A.: Some Banach algebras of analytic Feynman integrable functionals. In: Analytic Functions (Kozubnik, 1979), Lecture Notes in Mathematics 798, pp. 18–67. Springer, Berlin (1980)

[3] Cameron, R.H., Storvick, D.A.: Relationships between the Wiener integral and the analytic Feynman integral. Rend. Circ. Mat. Palermo **2**(Suppl. 17), 117–133 (1987)

[4] Cameron, R.H., Storvick, D.A.: A Simple Definition of the Feynman Integral With Applications. Mem. Amer. Math. Soc. **288**, Amer. Math. Soc. (1983)

[5] Chang, S.J., Choi, J.G.: Effect of drift of the generalized Brownian motion process: an example for the analytic Feynman integral. Arch. Math. **106**, 591–600 (2016)

[6] Chang, S.J., Choi, J.G.: Generalized transforms and generalized convolution products associated with Gaussian paths on function space. Commun. Pure Appl. Anal. **19**, 371–389 (2020)

[7] Chang, S.J., Choi, J.G., Ko, A.Y.: Multiple generalized analytic Fourier–Feynman transform via rotation of Gaussian paths on function space. Banach J. Math. Anal. **9**, 58–80 (2015)

[8] Chang, S.J., Choi, J.G., Skoug, D.: Integration by parts formulas involving generalized Fourier–Feynman transforms on function space. Trans. Am. Math. Soc. **355**, 2925–2948 (2003)

[9] Chang, S.J., Chung, H.S., Skoug, D.: Integral transforms of functionals in $L^2(C_{a,b}[0,T])$. J. Fourier Anal. Appl. **15**, 441–462 (2009)

[10] Chang, S.J., Skoug, D.: Generalized Fourier-Feynman transforms and a first variation on function space. Integral Transforms Spec. Funct. **14**, 375–393 (2003)

[11] Chang, S.J., Skoug, D., Chung, H.S.: Relationships for modified generalized integral transforms, modified convolution products and first variations on function space. Integral Transforms Spec. Funct. **25**, 790–804 (2014)

[12] Feynman, R.P.: Space-time approach to non-relativistic quantum mechanics. Rev. Mod. Phys. **20**, 367–387 (1948)

[13] Yamasaki, Y.: Measures on infinite dimensional spaces. World Sci. Ser. Pure Math. **5**, World Scientific Publishing, Singapore (1985)

[14] Yeh, J.: Singularity of Gaussian measures on function spaces induced by Brownian motion processes with non-stationary increments. Ill. J. Math. **15**, 37–46 (1971)

[15] Yeh, J.: Stochastic Processes and the Wiener Integral. Marcel Dekker Inc., New York (1973)

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