Results Math (2021) 76:73

1422-6383/21/020001-14
published online March 22, 2021
https://doi.org/10.1007/s00025-021-01382-w

Results in Mathematics

Global Existence and Large Time Behavior of Solutions to 3D MHD System Near Equilibrium

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Abstract. In this paper, we consider the stability problem on perturbation near a physically steady state solution of the 3D generalized incompressible magnetohydrodynamic system in Lei-Lin space. The global stability and analytic estimates for small perturbation are established by the semigroup method in the critical space $\chi^{1-2\alpha}(\mathbb{R}^3)$ with $\frac{1}{2} \leq \alpha \leq 1$, where linear terms from perturbation incur much difficulty. By introducing a diagonalization process we successfully eliminate the linear terms. Then, by virtue of the analytic estimates for a solution, the temporal decay rate $(1+t)^{-(\frac{5}{4\alpha}-1)}$ of the global solution is obtained.

Mathematics Subject Classification. 35Q35, 35B40, 35B20.

Keywords. magnetohydrodynamic system, global stability, large-time behavior, Lei-Lin space.

1. Introduction

The 3D generalized incompressible magnetohydrodynamic (MHD) system can be written as:

$$\begin{cases} \partial_t U + \Lambda^{2\alpha} U + (U \cdot \nabla) U + \nabla P = (B \cdot \nabla) B, \\ \partial_t B + \Lambda^{2\alpha} B + (U \cdot \nabla) B = (B \cdot \nabla) U, \\ \nabla \cdot U = \nabla \cdot B = 0, \end{cases} \tag{1.1}$$

for $t \geq 0$, $x \in \mathbb{R}^3$. We denote $U = U(t,x)$, $B = B(t,x)$ and $P = P(t,x)$ the velocity field, magnetic field and scalar pressure of the fluid, respectively. The fractional Laplacian operator $\Lambda^{2\alpha} = (-\Delta)^\alpha$ is defined through a Fourier transform, namely $\widehat{\Lambda^{2\alpha} f}(\xi) = |\xi|^{2\alpha}\hat{f}(\xi)$. For simplicity, we set the kinematic viscosity, the magnetic Reynolds number and the corresponding coefficients to be equal to 1.

This paper considers the stability problem on perturbations near the special steady solution $(\tilde{u}, \tilde{b})$ given by the background magnetic field

$$\tilde{u}(t,x) \equiv (0,0,0), \quad \tilde{b}(t,x) \equiv (1,0,0). \tag{1.2}$$

It is clear that $(\tilde{u}, \tilde{b})$ is a solution of system (1.1). This special equilibrium has physical significance and the stability of (1.2) for the MHD system was initiated by Alfvén [1]. To understand the stability problem focused here, we consider the perturbation (u, b) around this equilibrium with $u = U - \tilde{u}$ and $b = B - \tilde{b}$, and (u, b) satisfies

$$\begin{cases} \partial_t u + \Lambda^{2\alpha} u + (u \cdot \nabla)u + \nabla P = (b \cdot \nabla)b + \partial_1 b, \\ \partial_t b + \Lambda^{2\alpha} b + (u \cdot \nabla)b = (b \cdot \nabla)u + \partial_1 u, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(0,x) = u_0(x),\ b(0,x) = b_0(x). \end{cases} \tag{1.3}$$

When $\alpha = 1$, the system (1.1) reduces to the classical MHD system, which describes the motion of electrically conducting fluids such as plasmas, liquid metals, and electrolytes. Due to its physical background, mathematical complexity and wide range of applications, MHD and related models have attracted the interest of many researchers and several important global regularity results have been established in different spaces (see, e.g., [6–8,13,14,19,23–25]). For 3D MHD system, Duvaut and Lions [8] established the local existence and uniqueness in the Sobolev space $H^s(\mathbb{R}^3)$ with $s \geq 3$. Miao et al. [14] established the global well-posedness in $BMO^{-1}(\mathbb{R}^3)$ and local well-posedness in $bmo^{-1}(\mathbb{R}^3)$ for small initial data. Wang and Wang [19] established the global existence of mild solutions in Lei-Lin space $\chi^{-1}(\mathbb{R}^3)$. When $b = 0$, it is reduced to the Navier-Stokes equation. The global well-posedness and long time behavior of solutions have attracted much attention (see, e.g., [3–5,11,12,16]).

Recently, the MHD system with initial data near equilibrium (1.3) has gained renewed interests. Ren et al. [15] proved that the global existence and temporal decay rate of smooth solutions for general perturbations to the system in $\mathbb{R}^2$. Ji et al. [10] established the asymptotic linear stability and the global stability on the 2D MHD system (1.3) with partial dissipation in Sobolev space. For 3D case, Abidi and Zhang [2], Deng and Zhang [9] obtained the global solution and large time behavior for (1.3) without magnetic dissipation via different approaches. Wu and Zhu [20] established the global regularity for (1.3) with mixed partial dissipation cases.

For the generalized MHD system (1.1), Wu [17,18] studied the global existence for (1.1) with $\alpha \geq \frac{5}{4}$ in Sobolev space. Ye [26,27] established a small global solution and large time behavior in the critical space $\chi^{1-2\alpha}(\mathbb{R}^3)$ with $\frac{1}{2} \leq \alpha \leq 1$. Later, Wang et al. [21] proved that the global well-posedness and analyticity results and we [22] obtained a explicit decay rate $(1+t)^{-(\frac{5}{4\alpha}-1)}$ to the system (1.1).

The main goal of this paper is to establish the global well-posedness, analytic and decay estimates for small initial data to the system (1.3) in the critical space $\chi^{1-2\alpha}(\mathbb{R}^3)$ with $\frac{1}{2} \le \alpha \le 1$. Our main results can be stated as follows.

Theorem 1.1. *Let $\frac{1}{2} \le \alpha \le 1$. Assume that $(u_0, b_0) \in \chi^{1-2\alpha}(\mathbb{R}^3)$ with the smallness condition $\|(u_0, b_0)\|_{\chi^{1-2\alpha}} < \epsilon_0$, then the system (1.3) admits a unique global solution $(u, b) \in \mathcal{C}(\mathbb{R}^+; \chi^{1-2\alpha}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+; \chi^1(\mathbb{R}^3))$, and for any $0 \le t < \infty$, it holds*

$$\|(u, b)\|_{\tilde{L}^\infty(\chi^{1-2\alpha})} + \|(u, b)\|_{L^1(\chi^1)} \lesssim \|(u_0, b_0)\|_{\chi^{1-2\alpha}}. \tag{1.4}$$

Furthermore the solution is analytic in the sense that

$$\|\mathrm{e}^{|D|^\alpha \sqrt{t}}(u, b)\|_{\tilde{L}^\infty(\chi^{1-2\alpha})} + \|\mathrm{e}^{|D|^\alpha \sqrt{t}}(u, b)\|_{L^1(\chi^1)} \lesssim \|(u_0, b_0)\|_{\chi^{1-2\alpha}}, \tag{1.5}$$

where $\mathrm{e}^{|D|^\alpha \sqrt{t}}$ is a Fourier multiplier whose symbol is given by $\mathrm{e}^{|\xi|^\alpha \sqrt{t}}$.

Compared with the Navier–Stokes equation [3,4] and MHD system [21], the difficulty to obtain a priori estimate (1.4) and the analyticity (1.5) for system (1.3) is the presence of linear terms $\partial_1 u$ and $\partial_1 b$. In fact, if we write the system (1.3) by Duhamel's formula in Fourier space, namely

$$\hat{u}(t, \xi) = \mathrm{e}^{-t|\xi|^{2\alpha}} \hat{u}_0(\xi) + \int_0^t \mathrm{e}^{-(t-\tau)|\xi|^{2\alpha}} \left[i\xi \cdot \left(-\widehat{u \otimes u} + \widehat{b \otimes b} \right) + i\xi \hat{P} + i\xi_1 \hat{b} \right] (\tau, \xi) \mathrm{d}\tau, \tag{1.6}$$

$$\hat{b}(t, \xi) = \mathrm{e}^{-t|\xi|^{2\alpha}} \hat{b}_0(\xi) + \int_0^t \mathrm{e}^{-(t-\tau)|\xi|^{2\alpha}} \left[i\xi \cdot \left(-\widehat{u \otimes b} + \widehat{b \otimes u} \right) + i\xi_1 \hat{u} \right] (\tau, \xi) \mathrm{d}\tau. \tag{1.7}$$

and estimate a small global solution directly by the semigroup method, we would have trouble in dealing with the linear term

$$\int_0^t \mathrm{e}^{-(t-\tau)|\xi|^{2\alpha}} |\xi| \left(|\hat{u}| + |\hat{b}| \right) (\tau, \xi) \mathrm{d}\tau. \tag{1.8}$$

Indeed, from (1.8), we are unable to simply estimate $\|(u, b)\|_{\tilde{L}^\infty(\chi^{1-2\alpha})}$ and $\|(u, b)\|_{L^1(\chi^1)}$. Therefore, we have to introduce a diagonalization process Proposition 3.1 to eliminate the linear terms. The process appears to be complex, but it offers a general framework for handling similar and more general situations. Once we get the global well-posedness and analytic estimates, combined with the property of a continuous function Lemma 2.2, we can immediately obtain the decay of $\|(u, b)(t)\|_{\chi^{1-2\alpha}}$.

Theorem 1.2. *Let $(u_0, b_0) \in \chi^{1-2\alpha}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, $(u, b) \in \mathcal{C}(\mathbb{R}^+; \chi^{1-2\alpha}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+; \chi^1(\mathbb{R}^3))$ with $\frac{1}{2} \le \alpha \le 1$ be a small global solution to (1.3) constructed* *in Theorem 1.1, then*

$$\|(u,b)(t)\|_{\chi^{1-2\alpha}} \lesssim (1+t)^{-(\frac{5}{4\alpha}-1)}.$$

The rest of this paper is organized in three sections. In Sect. 2, we introduce two definitions and two lemmas, which will play a key role in this paper. In Sect. 3, we prove Proposition 3.1 and Theorem 1.1 and in Sect. 4, we give the proof of Theorem 1.2. Throughout the paper, C stands for a generic positive constant, which may be different from line to line. Furthermore, we will use the notation $a \lesssim d$ to denote the relation $a \leq Cd$, $\|\cdot\|_X$ to denote $\|\cdot\|_{X(\mathbb{R}^3)}$, $\|(u,b)\|_X^p$ to denote $\|u\|_X^p + \|b\|_X^p$ and $\|\cdot\|_{\tilde{L}^q(\chi^s)}$ to denote $\|\cdot\|_{\tilde{L}^q(\mathbb{R}^+;\chi^s(\mathbb{R}^3))}$ for conciseness.

2. Preliminaries

In this section, we present two definitions and two preliminary lemmas that will be used in our proofs. The first one is the definition of Lei-Lin space $\chi^s(\mathbb{R}^3)$ (see [12]).

Definition 2.1. For $s \in \mathbb{R}$, The functional space $\chi^s(\mathbb{R}^3)$ is defined by

$$\chi^s := \left\{ f \in \mathcal{D}'(\mathbb{R}^3) \;\middle|\; \int_{\mathbb{R}^3} |\xi|^s |\hat{f}(\xi)| \mathrm{d}\xi < \infty \right\}$$

which is equipped with the norm

$$\|f\|_{\chi^s} \triangleq \int_{\mathbb{R}^3} |\xi|^s |\hat{f}(\xi)| \mathrm{d}\xi.$$

The next is the following time dependent spaces.

Definition 2.2. Let $s \in \mathbb{R}$ and $p \in [1,\infty]$. $f(t,x) \in L^p([0,T];\chi^s(\mathbb{R}^3))$ if and only if

$$\|f\|_{L^p(\chi^s)} \triangleq \Big(\int_0^T \|f(t,\cdot)\|_{\chi^s}^p \mathrm{d}t \Big)^{\frac{1}{p}} < \infty;$$

$f(t,x) \in \tilde{L}^p([0,T];\chi^s(\mathbb{R}^3))$ if and only if

$$\|f\|_{\tilde{L}^p(\chi^s)} \triangleq \Big\| \Big(\int_0^T \Big(|\xi|^s |\hat{f}(t,\xi)| \Big)^p \mathrm{d}t \Big)^{\frac{1}{p}} \Big\|_{L^1} < \infty.$$

By the Minkowski inequality it can be deduced that if $p > 1$ then $\tilde{L}^p([0,T];\chi^s(\mathbb{R}^3)) \hookrightarrow L^p([0,T];\chi^s(\mathbb{R}^3))$. Clearly if $p=1$ then $L^1([0,T];\chi^s(\mathbb{R}^3)) \equiv \tilde{L}^1([0,T];\chi^s(\mathbb{R}^3))$.

Finally, two lemmas will be applied in the proof of Theorems 1.1 and 1.2.

Lemma 2.1. ([26, Lemma 2.1]) *Assume that $\frac{1}{2} \leq \alpha \leq 1$, then the following inequality holds*

$$|\xi|^{2(1-\alpha)} \leq \frac{2^{2(1-\alpha)}}{2} \left(|\eta||\xi-\eta|^{1-2\alpha} + |\eta|^{1-2\alpha}|\xi-\eta| \right). \tag{2.1}$$

for any $\xi,\ \eta \in \mathbb{R}^3$.

Lemma 2.2. ([5]) *Let* $T > 0$ *and* $f : [0,T] \to \mathbb{R}^+$ *be a continuous function such that*

$$f(t) \leq M_0 + \theta_1 f(\theta_2 t); \quad \forall\ 0 \leq t \leq T \tag{2.2}$$

with $M_0 \geq 0$ *and* $\theta_1, \theta_2 \in (0,1)$. *Then*

$$f(t) \leq \frac{M_0}{1-\theta_1}; \quad \forall\ 0 \leq t \leq T.$$

Proof. Because f is a positive and continuous function, then there is a time $t_0 \in [0,T]$ such that

$$0 \leq f(t_0) = \max_{0\leq t\leq T} f(t),$$

applying (2.2) at $t = t_0$, we derive that

$$f(t_0) \leq M_0 + \theta_1 f(\theta_2 t_0) \leq M_0 + \theta_1 f(t_0),$$

which implies $f(t_0) \leq \frac{M_0}{1-\theta_1}$. We thus complete the proof of Lemma 2.2. □

3. Proof of Theorem 1.1

In this section, our aim is to establish the global well-posedness and analytic estimates with small initial data. First, we need to prove Proposition 3.1, which will play an important role in the proof of Theorem 1.1.

Proposition 3.1. *The system* (1.3) *can be represented in the following integral form*

$$\begin{aligned}\hat{u}(t,\xi) &= \frac{1}{2}\left(\mathrm{e}^{\lambda_1(\xi)t} + \mathrm{e}^{\lambda_2(\xi)t}\right)\hat{u}_0(\xi) + sgn(\xi_1)\frac{1}{2}\left(\mathrm{e}^{\lambda_2(\xi)t} - \mathrm{e}^{\lambda_1(\xi)t}\right)\hat{b}_0(\xi)\\ &\quad + \frac{1}{2}\int_0^t \Big[\left(\mathrm{e}^{\lambda_1(\xi)(t-\tau)} + \mathrm{e}^{\lambda_2(\xi)(t-\tau)}\right)\hat{F}\\ &\quad + sgn(\xi_1)\left(\mathrm{e}^{\lambda_2(\xi)(t-\tau)} - \mathrm{e}^{\lambda_1(\xi)(t-\tau)}\right)\hat{E}\Big](\tau,\xi)\,d\tau, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\hat{b}(t,\xi) &= sgn(\xi_1)\frac{1}{2}\left(\mathrm{e}^{\lambda_2(\xi)t} - \mathrm{e}^{\lambda_1(\xi)t}\right)\hat{u}_0(\xi) + \frac{1}{2}\left(\mathrm{e}^{\lambda_1(\xi)t} + \mathrm{e}^{\lambda_2(\xi)t}\right)\hat{b}_0(\xi)\\ &\quad + \frac{1}{2}\int_0^t \Big[sgn(\xi_1)\left(\mathrm{e}^{\lambda_2(\xi)(t-\tau)} - \mathrm{e}^{\lambda_1(\xi)(t-\tau)}\right)\hat{F}\\ &\quad + \left(\mathrm{e}^{\lambda_1(\xi)(t-\tau)} + \mathrm{e}^{\lambda_2(\xi)(t-\tau)}\right)\hat{E}\Big](\tau,\xi)\,d\tau, \end{aligned} \tag{3.2}$$

where $\lambda_1 = -|\xi|^{2\alpha} - i|\xi_1|$, $\lambda_2 = -|\xi|^{2\alpha} + i|\xi_1|$, $F = -u\cdot\nabla u + b\cdot\nabla b - \nabla P$, $E = -u\cdot\nabla b + b\cdot\nabla u$ *and*

$$sgn(\xi_1) = \begin{cases} 1, & \xi_1 > 0,\\ -1, & \xi_1 < 0.\end{cases}$$

Proof. We rewrite (1.3) in the Fourier space as

$$\begin{bmatrix} \partial_t \hat{u}_j(\xi) \\ \partial_t \hat{b}_j(\xi) \end{bmatrix} = \begin{bmatrix} -|\xi|^{2\alpha} & i\xi_1 \\ i\xi_1 & -|\xi|^{2\alpha} \end{bmatrix} \begin{bmatrix} \hat{u}_j(\xi) \\ \hat{b}_j(\xi) \end{bmatrix} + \begin{bmatrix} \hat{F}_j(\xi) \\ \hat{E}_j(\xi) \end{bmatrix}, \quad j = 1, 2, 3, \tag{3.3}$$

where we have suppressed the t-variable for notational brevity. To diagonalize the coefficient matrix, we seek the eigenvalues and eigenvectors. Clearly, the eigenvalues satisfy the corresponding characteristic equation

$$\lambda^2 + 2|\xi|^{2\alpha}\lambda + |\xi|^{4\alpha} + |\xi_1|^2 = 0, \tag{3.4}$$

and by calculation

$$\lambda_1 = -|\xi|^{2\alpha} - i|\xi_1|, \ \lambda_2 = -|\xi|^{2\alpha} + i|\xi_1|. \tag{3.5}$$

The associated eigenvectors are given by

$$m = \begin{bmatrix} i\xi_1 \\ \lambda_1 + |\xi|^{2\alpha} \end{bmatrix}, \quad n = \begin{bmatrix} i\xi_1 \\ \lambda_2 + |\xi|^{2\alpha} \end{bmatrix}. \tag{3.6}$$

m and n are independent and

$$\begin{bmatrix} -|\xi|^{2\alpha} & i\xi_1 \\ i\xi_1 & -|\xi|^{2\alpha} \end{bmatrix} g = g \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \tag{3.7}$$

where $g = \begin{bmatrix} m \ n \end{bmatrix}$, and the inverse of the matrix g follows that

$$g^{-1} = \begin{bmatrix} \frac{1}{2i\xi_1} & -\frac{1}{2i|\xi_1|} \\ \frac{1}{2i\xi_1} & \frac{1}{2i|\xi_1|} \end{bmatrix}. \tag{3.8}$$

If we define

$$\begin{bmatrix} \hat{A}_j(\xi) \\ \hat{D}_j(\xi) \end{bmatrix} = g^{-1} \begin{bmatrix} \hat{u}_j(\xi) \\ \hat{b}_j(\xi) \end{bmatrix}, \tag{3.9}$$

then $\hat{A}_j(\xi)$ and $\hat{D}_j(\xi)$ satisfy

$$\begin{bmatrix} \partial_t \hat{A}_j(\xi) \\ \partial_t \hat{D}_j(\xi) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \hat{A}_j(\xi) \\ \hat{D}_j(\xi) \end{bmatrix} + g^{-1} \begin{bmatrix} \hat{F}_j(\xi) \\ \hat{E}_j(\xi) \end{bmatrix}. \tag{3.10}$$

We further write (3.10) in the integral form

$$\begin{bmatrix} \hat{A}_j(t,\xi) \\ \hat{D}_j(t,\xi) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{\lambda_1(\xi)t} & 0 \\ 0 & \mathrm{e}^{\lambda_2(\xi)t} \end{bmatrix} \begin{bmatrix} \hat{A}_{0j}(\xi) \\ \hat{D}_{0j}(\xi) \end{bmatrix} + \int_0^t \begin{bmatrix} \mathrm{e}^{\lambda_1(\xi)(t-\tau)} & 0 \\ 0 & \mathrm{e}^{\lambda_2(\xi)(t-\tau)} \end{bmatrix} g^{-1} \begin{bmatrix} \hat{F}_j(\tau,\xi) \\ \hat{E}_j(\tau,\xi) \end{bmatrix} \mathrm{d}\tau, \tag{3.11}$$

or

$$\begin{aligned} \begin{bmatrix} \hat{u}_j(t,\xi) \\ \hat{b}_j(t,\xi) \end{bmatrix} &= g \begin{bmatrix} \mathrm{e}^{\lambda_1(\xi)t} & 0 \\ 0 & \mathrm{e}^{\lambda_2(\xi)t} \end{bmatrix} g^{-1} \begin{bmatrix} \hat{u_0}_j(\xi) \\ \hat{b_0}_j(\xi) \end{bmatrix} \\ &\quad + \int_0^t g \begin{bmatrix} \mathrm{e}^{\lambda_1(\xi)(t-\tau)} & 0 \\ 0 & \mathrm{e}^{\lambda_2(\xi)(t-\tau)} \end{bmatrix} g^{-1} \begin{bmatrix} \hat{F}_j(\tau,\xi) \\ \hat{E}_j(\tau,\xi) \end{bmatrix} \mathrm{d}\tau, \end{aligned} \tag{3.12}$$

where

$$g\begin{bmatrix} e^{\lambda_1(\xi)t} & 0 \\ 0 & e^{\lambda_2(\xi)t} \end{bmatrix} g^{-1} = \begin{bmatrix} \frac{e^{\lambda_1(\xi)t}+e^{\lambda_2(\xi)t}}{2} & \mathrm{sgn}(\xi_1)\frac{e^{\lambda_2(\xi)t}-e^{\lambda_1(\xi)t}}{2} \\ \mathrm{sgn}(\xi_1)\frac{e^{\lambda_2(\xi)t}-e^{\lambda_1(\xi)t}}{2} & \frac{e^{\lambda_1(\xi)t}+e^{\lambda_2(\xi)t}}{2} \end{bmatrix}, \quad (3.13)$$

and

$$g\begin{bmatrix} e^{\lambda_1(\xi)(t-\tau)} & 0 \\ 0 & e^{\lambda_2(\xi)(t-\tau)} \end{bmatrix} g^{-1} = \begin{bmatrix} \frac{e^{\lambda_1(\xi)(t-\tau)}+e^{\lambda_2(\xi)(t-\tau)}}{2} & \mathrm{sgn}(\xi_1)\frac{e^{\lambda_2(\xi)(t-\tau)}-e^{\lambda_1(\xi)(t-\tau)}}{2} \\ \mathrm{sgn}(\xi_1)\frac{e^{\lambda_2(\xi)(t-\tau)}-e^{\lambda_1(\xi)(t-\tau)}}{2} & \frac{e^{\lambda_1(\xi)(t-\tau)}+e^{\lambda_2(\xi)(t-\tau)}}{2} \end{bmatrix}. \quad (3.14)$$

We are then led to the desired representation in Proposition 3.1. □

With Lemma 2.1 and Proposition 3.1 at our disposal, we start to prove Theorem 1.1.

Proof of Theorem 1.1. Thanks to (3.1) and (3.2) in Proposition 3.1, we have

$$|\hat{u}(t,\xi)| + |\hat{b}(t,\xi)| \lesssim e^{-|\xi|^{2\alpha}t}\left(|\hat{u}_0(\xi)| + |\hat{b}_0(\xi)|\right) + \int_0^t e^{-|\xi|^{2\alpha}(t-\tau)}\left(|\hat{F}(\tau,\xi)| + |\hat{E}(\tau,\xi)|\right)d\tau, \quad (3.15)$$

where we have used the fact $\lambda_{1,2} = -|\xi|^{2\alpha} \pm i|\xi_1|$ and $|e^{\pm i|\xi_1|t}| = 1$.

Now we divide the proof of Theorem 1.1 into three steps. The first step estimates $\|(u,b)\|_{\tilde{L}^\infty(\chi^{1-2\alpha})}$, the second step estimates $\|(u,b)\|_{L^1(\chi^1)}$, and the third step establishes the analytic estimate.

Step 1. $\|(u,b)\|_{\tilde{L}^\infty(\chi^{1-2\alpha})}$ estimate. Multiplying (3.15) by $|\xi|^{1-2\alpha}$, we get

$$\begin{aligned} &|\xi|^{1-2\alpha}|\hat{u}(t,\xi)| + |\xi|^{1-2\alpha}|\hat{b}(t,\xi)| \\ &\quad \le e^{-|\xi|^{2\alpha}t}|\xi|^{1-2\alpha}\left(|\hat{u}_0(\xi)| + |\hat{b}_0(\xi)|\right) \\ &\qquad + \int_0^t e^{-|\xi|^{2\alpha}(t-\tau)}|\xi|^{2-2\alpha}\left(|\widehat{u\otimes u}| + |\widehat{b\otimes b}| + |\widehat{u\otimes b}| + |\widehat{b\otimes u}|\right)(\tau,\xi)d\tau. \end{aligned} \quad (3.16)$$

Under the definition of $\tilde{L}^\infty(\chi^{1-2\alpha})$, we take the $L^\infty-$norm in time to (3.16). By Lemma 2.1 and Young inequality it yields

$$\begin{aligned} &\int_0^\infty e^{-|\xi|^{2\alpha}(t-\tau)}|\xi|^{2-2\alpha}|\widehat{u\otimes u}|(\tau,\xi)d\tau \\ &\quad \lesssim \int_0^\infty \left(\int_{\mathbb{R}^3}\left(|\eta||\xi-\eta|^{1-2\alpha} + |\eta|^{1-2\alpha}|\xi-\eta|\right)|\hat{u}(t,\xi-\eta)||\hat{u}(t,\eta)|d\eta\right)dt \\ &\quad \lesssim \int_0^\infty \left(|\xi|^{1-2\alpha}|\hat{u}(t,\xi)| *_\xi |\xi||\hat{u}(t,\xi)|\right)dt \\ &\quad \lesssim \left(\sup_{0\le t<\infty}\left(|\xi|^{1-2\alpha}|\hat{u}(t,\xi)|\right)\right) *_\xi \left(\int_0^\infty |\xi||\hat{u}(t,\xi)|dt\right), \end{aligned} \quad (3.17)$$

similarly

$$\begin{aligned}
&\int_0^\infty \mathrm{e}^{-|\xi|^{2\alpha}(t-\tau)}|\xi|^{2-2\alpha}\left(|\widehat{b\otimes b}|+|\widehat{u\otimes b}|+|\widehat{b\otimes u}|\right)(\tau,\xi)\mathrm{d}\tau\\
&\lesssim \left(\sup_{0\le t<\infty}(|\xi|^{1-2\alpha}|\hat{b}(t,\xi)|)\right) *_\xi \left(\int_0^\infty |\xi||\hat{b}(t,\xi)|\mathrm{d}t\right)\\
&\quad+\left(\sup_{0\le t<\infty}(|\xi|^{1-2\alpha}|\hat{u}(t,\xi)|)\right) *_\xi \left(\int_0^\infty |\xi||\hat{b}(t,\xi)|\mathrm{d}t\right)\\
&\quad+\left(\sup_{0\le t<\infty}(|\xi|^{1-2\alpha}|\hat{b}(t,\xi)|)\right) *_\xi \left(\int_0^\infty |\xi||\hat{u}(t,\xi)|\mathrm{d}t\right). \qquad (3.18)
\end{aligned}$$

We conclude from the estimates (3.17) and (3.18) that

$$\begin{aligned}
&\sup_{0\le t<\infty}|\xi|^{1-2\alpha}|\hat{u}(t,\xi)|+\sup_{0\le t<\infty}|\xi|^{1-2\alpha}|\hat{b}(t,\xi)|\\
&\lesssim |\xi|^{1-2\alpha}\left(|\hat{u}_0(\xi)|+|\hat{b}_0(\xi)|\right)+\left(\sup_{0\le t<\infty}(|\xi|^{1-2\alpha}|\hat{u}(t,\xi)|)\right) *_\xi \left(\int_0^\infty |\xi||\hat{u}(t,\xi)|\mathrm{d}t\right)\\
&\quad+\left(\sup_{0\le t<\infty}(|\xi|^{1-2\alpha}|\hat{b}(t,\xi)|)\right) *_\xi \left(\int_0^\infty |\xi||\hat{b}(t,\xi)|\mathrm{d}t\right)\\
&\quad+\left(\sup_{0\le t<\infty}(|\xi|^{1-2\alpha}|\hat{u}(t,\xi)|)\right) *_\xi \left(\int_0^\infty |\xi||\hat{b}(t,\xi)|\mathrm{d}t\right)\\
&\quad+\left(\sup_{0\le t<\infty}(|\xi|^{1-2\alpha}|\hat{b}(t,\xi)|)\right) *_\xi \left(\int_0^\infty |\xi||\hat{u}(t,\xi)|\mathrm{d}t\right). \qquad (3.19)
\end{aligned}$$

Then taking L^1-norm in ξ to (3.19), it follows that

$$\|(u,b)\|_{\tilde{L}^\infty(\chi^{1-2\alpha})}\lesssim \|(u_0,b_0)\|_{\chi^{1-2\alpha}}+\|(u,b)\|_{\tilde{L}^\infty(\chi^{1-2\alpha})}\|(u,b)\|_{L^1(\chi^1)}. \qquad (3.20)$$

Step 2. $\|(u,b)\|_{L^1(\chi^1)}$ **estimate.** Multiplying (3.15) by $|\xi|$, we have

$$\begin{aligned}
&|\xi||\hat{u}(t,\xi)|+|\xi||\hat{b}(t,\xi)|\\
&\le |\xi|^{2\alpha}\mathrm{e}^{-|\xi|^{2\alpha}t}|\xi|^{1-2\alpha}\left(|\hat{u}_0(\xi)|+|\hat{b}_0(\xi)|\right)\\
&\quad+\int_0^t |\xi|^{2\alpha}\mathrm{e}^{-|\xi|^{2\alpha}(t-\tau)}|\xi|^{2-2\alpha}\left(|\widehat{u\otimes u}|+|\widehat{b\otimes b}|+|\widehat{u\otimes u}|+|\widehat{b\otimes b}|\right)(\tau,\xi)\mathrm{d}\tau. \qquad (3.21)
\end{aligned}$$

Taking the L^1-norm in time to (3.21), applying $\int_0^\infty |\xi|^{2\alpha}\mathrm{e}^{-|\xi|^{2\alpha}t}\mathrm{d}t \leq 1$ and arguing similarly to the above, we deduce

$$\begin{aligned}
&\int_0^\infty |\xi||\hat{u}(t,\xi)|\mathrm{d}t + \int_0^\infty |\xi||\hat{b}(t,\xi)|\mathrm{d}t \\
&\lesssim |\xi|^{1-2\alpha}\left(|\hat{u}_0(\xi)| + |\hat{b}_0(\xi)|\right) + \left(\sup_{0\leq t<\infty}(|\xi|^{1-2\alpha}|\hat{u}(t,\xi)|)\right) *_\xi \left(\int_0^\infty |\xi||\hat{u}(t,\xi)|\mathrm{d}t\right) \\
&\quad + \left(\sup_{0\leq t<\infty}(|\xi|^{1-2\alpha}|\hat{b}(t,\xi)|)\right) *_\xi \left(\int_0^\infty |\xi||\hat{b}(t,\xi)|\mathrm{d}t\right) \\
&\quad + \left(\sup_{0\leq t<\infty}(|\xi|^{1-2\alpha}|\hat{u}(t,\xi)|)\right) *_\xi \left(\int_0^\infty |\xi||\hat{b}(t,\xi)|\mathrm{d}t\right) \\
&\quad + \left(\sup_{0\leq t<\infty}(|\xi|^{1-2\alpha}|\hat{b}(t,\xi)|)\right) *_\xi \left(\int_0^\infty |\xi||\hat{u}(t,\xi)|\mathrm{d}t\right). \qquad (3.22)
\end{aligned}$$

Taking the L^1-norm in ξ to (3.22), one obtains

$$\|(u,b)\|_{L^1(\chi^1)} \lesssim \|(u_0,b_0)\|_{\chi^{1-2\alpha}} + \|(u,b)\|_{\tilde{L}^\infty(\chi^{1-2\alpha})}\|(u,b)\|_{L^1(\chi^1)}. \qquad (3.23)$$

Combining (3.20) and (3.23) it gives

$$\begin{aligned}
&\|(u,b)\|_{\tilde{L}^\infty(\chi^{1-2\alpha})} + \|(u,b)\|_{L^1(\chi^1)} \\
&\quad \lesssim \|(u_0,b_0)\|_{\chi^{1-2\alpha}} + \left(\|(u,b)\|_{\tilde{L}^\infty(\chi^{1-2\alpha})} + \|(u,b)\|_{L^1(\chi^1)}\right)^2. \qquad (3.24)
\end{aligned}$$

By bootstrap argument, if taking $\|(u_0,b_0)\|_{\chi^{1-2\alpha}} < \epsilon_0$, we deduce that $\|(u,b)\|_{\tilde{L}^\infty(\chi^{1-2\alpha})} + \|(u,b)\|_{L^1(\chi^1)} < 2\epsilon_0$, which implies the existence of a global solution in $\mathcal{C}(\mathbb{R}^+;\chi^{1-2\alpha}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+;\chi^1(\mathbb{R}^3))$ for small initial data in $\chi^{1-2\alpha}(\mathbb{R}^3)$. And we complete the proof of (1.4).

Step 3. Analytic estimate. Let $\hat{v}(t,\xi) = \mathrm{e}^{|\xi|^\alpha\sqrt{t}}\hat{u}(t,\xi)$ and $\hat{w}(t,\xi) = \mathrm{e}^{|\xi|^\alpha\sqrt{t}}\hat{b}(t,\xi)$, then $\hat{v}(t,\xi)$ and $\hat{w}(t,\xi)$ satisfy that

$$\begin{aligned}
&|\hat{v}(t,\xi)| + |\hat{w}(t,\xi)| \\
&\leq \mathrm{e}^{|\xi|^\alpha\sqrt{t}-|\xi|^{2\alpha}t}\left(|\hat{u}_0(\xi)| + |\hat{b}_0(\xi)|\right) \\
&\quad + \int_0^t \mathrm{e}^{|\xi|^\alpha\sqrt{t}-|\xi|^{2\alpha}(t-\tau)}|\xi|\left(|\widehat{u\otimes u}| + |\widehat{b\otimes b}| + |\widehat{u\otimes b}| + |\widehat{b\otimes u}|\right)(\tau,\xi)\mathrm{d}\tau \\
&\leq \mathrm{e}^{|\xi|^\alpha\sqrt{t}-\frac{1}{2}|\xi|^{2\alpha}t}\mathrm{e}^{-\frac{1}{2}|\xi|^{2\alpha}t}\left(|\hat{u}_0(\xi)| + |\hat{b}_0(\xi)|\right) \\
&\quad + \int_0^t \mathrm{e}^{|\xi|^\alpha(\sqrt{t}-\sqrt{\tau})-\frac{1}{2}|\xi|^{2\alpha}(t-\tau)}\mathrm{e}^{-\frac{1}{2}|\xi|^{2\alpha}(t-\tau)} \\
&\mathrm{e}^{|\xi|^\alpha\sqrt{\tau}}|\xi|\left(|\widehat{u\otimes u}| + |\widehat{b\otimes b}| + |\widehat{u\otimes b}| + |\widehat{b\otimes u}|\right)(\tau,\xi)\mathrm{d}\tau. \qquad (3.25)
\end{aligned}$$

Since $e^{|\xi|^{\alpha}\sqrt{t}-\frac{1}{2}|\xi|^{2\alpha}t}$ and $e^{|\xi|^{\alpha}(\sqrt{t}-\sqrt{\tau})-\frac{1}{2}|\xi|^{2\alpha}(t-\tau)}$ are uniformly bounded in time, then we have

$$
\begin{aligned}
&|\hat{v}(t,\xi)|+|\hat{w}(t,\xi)| \\
&\lesssim e^{-\frac{1}{2}|\xi|^{2\alpha}t}\left(|\hat{u}_0(\xi)|+|\hat{b}_0(\xi)|\right)+\int_0^t e^{-\frac{1}{2}|\xi|^{2\alpha}(t-\tau)}|\xi|\Big[\int_{\mathbb{R}^3} e^{|\xi-\eta|^{\alpha}\sqrt{\tau}}|\hat{u}(\xi-\eta)|e^{|\eta|^{\alpha}\sqrt{\tau}}|\hat{u}(\eta)|d\eta \\
&\quad+\int_{\mathbb{R}^3} e^{|\xi-\eta|^{\alpha}\sqrt{\tau}}|\hat{b}(\xi-\eta)|e^{|\eta|^{\alpha}\sqrt{\tau}}|\hat{b}(\eta)|d\eta+\int_{\mathbb{R}^3} e^{|\xi-\eta|^{\alpha}\sqrt{\tau}}|\hat{u}(\xi-\eta)|e^{|\eta|^{\alpha}\sqrt{\tau}}|\hat{b}(\eta)|d\eta \\
&\quad+\int_{\mathbb{R}^3} e^{|\xi-\eta|^{\alpha}\sqrt{\tau}}|\hat{b}(\xi-\eta)|e^{|\eta|^{\alpha}\sqrt{\tau}}|\hat{u}(\eta)|d\eta\Big]d\tau \\
&\lesssim e^{-\frac{1}{2}|\xi|^{2\alpha}t}\left(|\hat{u}_0(\xi)|+|\hat{b}_0(\xi)|\right) \\
&\quad+\int_0^t e^{-\frac{1}{2}|\xi|^{2\alpha}(t-\tau)}|\xi|\left(|\widehat{v\otimes v}|+|\widehat{w\otimes w}|+|\widehat{v\otimes w}|+|\widehat{w\otimes v}|\right)(\tau,\xi)d\tau. \qquad (3.26)
\end{aligned}
$$

Therefore, with (3.26) at our disposal, the proof of analyticity (1.5) can be obtained in the same manner as that of (1.4). Thus, we complete the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

The proof of Theorem 1.2 follows from that of Theorem 1.4 in our other literature [22] with some suitable modifications. In this section, for completeness and reader's convenience, we prove it concisely. The method lies in decomposing the spectrum of the solution into low and high frequency parts. The details are as follows.

Proof. We write $\|(u,b)(t)\|_{\chi^{1-2\alpha}} = I_1 + I_2$, for any $\lambda > 0$, $t > 0$, where

$$
\begin{aligned}
I_1 &= \int_{|\xi|\le\lambda} |\xi|^{1-2\alpha}\left(|\hat{u}|+|\hat{b}|\right)(t,\xi)d\xi \\
&\le \left(\int_{|\xi|\le\lambda} |\xi|^{2(1-2\alpha)}d\xi\right)^{\frac{1}{2}} \|(u_0,b_0)\|_{L^2} \\
&\lesssim \lambda^{\frac{5-4\alpha}{2}}\|(u_0,b_0)\|_{L^2} \qquad (4.1)
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \int_{|\xi|>\lambda} e^{-\sqrt{t/2}|\xi|^{\alpha}}e^{\sqrt{t/2}|\xi|^{\alpha}}|\xi|^{1-2\alpha}\left(|\hat{u}|+|\hat{b}|\right)(t,\xi)d\xi \\
&\lesssim e^{-\sqrt{t/2}\lambda^{\alpha}}\int_{\mathbb{R}^3} e^{\sqrt{t/2}|\xi|^{\alpha}}|\xi|^{1-2\alpha}\left(|\hat{u}|+|\hat{b}|\right)(t,\xi)d\xi. \qquad (4.2)
\end{aligned}
$$

Here we estimate $\int_{\mathbb{R}^3} e^{\sqrt{t/2}|\xi|^{\alpha}}|\xi|^{1-2\alpha}\left(|\hat{u}|+|\hat{b}|\right)(t,\xi)d\xi$. For a fixed time $t>0$, setting $V(z,x)=u(z+\frac{t}{2},x)$, $W(z,x)=b(z+\frac{t}{2},x)$ and $H(z,x)=P(z+\frac{t}{2},x)$, Let $\epsilon>0$ such that $\epsilon<\epsilon_0$, according to the assumption of Theorem 1.1, if

$\|(V,W)(0,x)\|_{\chi^{1-2\alpha}} < \epsilon$, without loss of generality, (V,W) is the unique global solution of the following system

$$\begin{cases} \partial_z V + \mu\Lambda^{2\alpha}V + (V\cdot\nabla)V + \nabla H = (W\cdot\nabla)W + \partial_1 W, \\ \partial_z W + \nu\Lambda^{2\alpha}W + (V\cdot\nabla)W = (W\cdot\nabla)V + \partial_1 V, \\ \nabla\cdot V = \nabla\cdot W = 0, \\ V(0,x) = u(\frac{t}{2},x),\ W(0,x) = b(\frac{t}{2},x). \end{cases}$$

By the analytic estimate (1.5), we deduce that

$$\int_{\mathbb{R}^3} \mathrm{e}^{\sqrt{z}|\xi|^\alpha}|\xi|^{1-2\alpha}\left(|\hat{V}|+|\hat{W}|\right)(z,\xi)\mathrm{d}\xi \lesssim \int_{\mathbb{R}^3}|\xi|^{1-2\alpha}\left(|\hat{u}|+|\hat{b}|\right)\left(\frac{t}{2},\xi\right)\mathrm{d}\xi, \tag{4.3}$$

or

$$\int_{\mathbb{R}^3} \mathrm{e}^{\sqrt{z}|\xi|^\alpha}|\xi|^{1-2\alpha}\left(|\hat{u}|+|\hat{b}|\right)\left(z+\frac{t}{2},\xi\right)\mathrm{d}\xi \lesssim \int_{\mathbb{R}^3}|\xi|^{1-2\alpha}\left(|\hat{u}|+|\hat{b}|\right)\left(\frac{t}{2},\xi\right)\mathrm{d}\xi. \tag{4.4}$$

For $z=\frac{t}{2}$, we have

$$\int_{\mathbb{R}^3} \mathrm{e}^{\sqrt{t/2}|\xi|^\alpha}|\xi|^{1-2\alpha}\left(|\hat{u}|+|\hat{b}|\right)(t,\xi)\mathrm{d}\xi \lesssim \|(u,b)\left(\frac{t}{2}\right)\|_{\chi^{1-2\alpha}}. \tag{4.5}$$

Inserting the estimate (4.5) into (4.2) leads to the result

$$I_2 \lesssim \mathrm{e}^{-\sqrt{t/2}\lambda^\alpha}\|(u,b)\left(\frac{t}{2}\right)\|_{\chi^{1-2\alpha}}. \tag{4.6}$$

Then

$$\|(u,b)(t)\|_{\chi^{1-2\alpha}} \lesssim \lambda^{\frac{5-4\alpha}{2}}\|(u_0,b_0)\|_{L^2} + \mathrm{e}^{-\sqrt{t/2}\lambda^\alpha}\|(u,b)\left(\frac{t}{2}\right)\|_{\chi^{1-2\alpha}}. \tag{4.7}$$

Multiplying (4.7) by $t^{\frac{5}{4\alpha}-1}$, we obtain

$$\begin{aligned} &t^{\frac{5}{4\alpha}-1}\|(u,b)(t)\|_{\chi^{1-2\alpha}} \\ &\quad\lesssim \lambda^{\frac{5-4\alpha}{2}}t^{\frac{5}{4\alpha}-1}\|(u_0,b_0)\|_{L^2} + 2^{\frac{5}{4\alpha}-1}\left(\frac{t}{2}\right)^{\frac{5}{4\alpha}-1}\mathrm{e}^{-\sqrt{t/2}\lambda^\alpha}\|(u,b)\left(\frac{t}{2}\right)\|_{\chi^{1-2\alpha}}. \end{aligned} \tag{4.8}$$

One chooses $\lambda>0$ such that $2^{\frac{5}{4\alpha}-1}\mathrm{e}^{-\sqrt{t/2}\lambda^\alpha}=\frac{1}{2}$, then $\lambda=\left(\frac{5\sqrt{2}\log 2}{4\alpha\sqrt{t}}\right)^{\frac{1}{\alpha}}$,

$$t^{\frac{5}{4\alpha}-1}\|(u,b)(t)\|_{\chi^{1-2\alpha}} \lesssim \left(\frac{5\sqrt{2}\log 2}{4\alpha}\right)^{\frac{5-4\alpha}{2\alpha}}\|(u_0,b_0)\|_{L^2} + \frac{1}{2}\left(\frac{t}{2}\right)^{\frac{5}{4\alpha}-1}\|(u,b)\left(\frac{t}{2}\right)\|_{\chi^{1-2\alpha}}. \tag{4.9}$$

Applying Lemma 2.2 with

$$M_0 = C\left(\frac{5\sqrt{2}\log 2}{4\alpha}\right)^{\frac{5-4\alpha}{2\alpha}} \|(u_0, b_0)\|_{L^2}, \quad \theta_1 = \theta_2 = \frac{1}{2},$$

and

$$f(t) = t^{\frac{5}{4\alpha}-1}\|(u,b)(t)\|_{\chi^{1-2\alpha}},$$

it yields that

$$\limsup_{t\to+\infty} t^{\frac{5}{4\alpha}-1}\|(u,b)(t)\|_{\chi^{1-2\alpha}} \leq \frac{M_0}{1-\theta_1} = 2M_0. \tag{4.10}$$

The proof of Theorem 1.2 is thus finished. □

Acknowledgements

The research of B Yuan was partially supported by the National Natural Science Foundation of China (No. 11471103).

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Received: June 29, 2020.
Accepted: March 9, 2021.