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Results in Mathematics

Sums of Averages of GCD-Sum Functions II

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Abstract. Let $\gcd(k,j)$ denote the greatest common divisor of the integers k and j, and let r be any fixed positive integer. Define

$$M_r(x;f) := \sum_{k\le x} \frac{1}{k^{r+1}} \sum_{j=1}^{k} j^r f(\gcd(j,k))$$

for any large real number $x \ge 5$, where f is any arithmetical function. Let ϕ, and ψ denote the Euler totient and the Dedekind function, respectively. In this paper, we refine asymptotic expansions of $M_r(x;\mathrm{id})$, $M_r(x;\phi)$ and $M_r(x;\psi)$. Furthermore, under the Riemann Hypothesis and the simplicity of zeros of the Riemann zeta-function, we establish the asymptotic formula of $M_r(x;\mathrm{id})$ for any large positive number $x > 5$ satisfying $x = [x] + \frac{1}{2}$.

Mathematics Subject Classification. 11A25, 11N37.

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1. Introduction and Statement of Results

Let $\gcd(k,j)$ be the greatest common divisor of the integers k and j. The gcd-sum function, which is also known as Pillai's arithmetical function, is defined

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by

$$P(n)=\sum_{k=1}^{n}\gcd(k,n).$$

This function has been studied by many authors such as Broughan [4], Bordellés [3],Tanigawa and Zhai [18], Tóth [19], and others. Analytic properties for partial sums of the gcd-sum function $f(\gcd(j,k))$ were recently studied by Inoue and Kiuchi [8]. We recall that the symbol $*$ denotes the Dirichlet convolution of two arithmetical functions f and g defined by $f*g(n)=\sum_{d|n}f(d)g(n/d)$, for every positive integer n. For any arithmetical function f, the second author [11] showed, that for any fixed positive integer r and any large positive number $x\geq 2$ we have

$$\begin{aligned}M_r(x;f)&:=\sum_{k\leq x}\frac{1}{k^{r+1}}\sum_{j=1}^{k}j^r f(\gcd(k,j))\\&=\frac{1}{2}\sum_{n\leq x}\frac{f(n)}{n}+\frac{1}{r+1}\sum_{d\ell\leq x}\frac{\mu*f(d)}{d}\\&\quad+\frac{1}{r+1}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\sum_{d\ell\leq x}\frac{\mu*f(d)}{d}\frac{1}{\ell^{2m}}.\end{aligned}\tag{1}$$

Here, as usual, the function μ denotes the Möbius function and $B_m=B_m(0)$ are the Bernoulli numbers, with $B_m(x)$ being the Bernoulli polynomials defined by the generating function

$$\frac{ze^{xz}}{e^z-1}=\sum_{m=0}^{\infty}B_m(x)\frac{z^m}{m!}$$

with $|z|<2\pi$. Many applications of Eq. (1) have been given in [10], [12] and [13].

In [11], Eq. (1) was used to establish asymptotic formulas for $M_r(x;f)$ for specific choices of f such as the identity function id, the Euler totient function $\phi=\mathrm{id}*\mu$ or the Dedekind function $\psi=\mathrm{id}*|\mu|$. More precisely, let $\zeta(s)$ denote the Riemann zeta-function, then for $f=\mathrm{id}$, it was proved that

$$\begin{aligned}M_r(x;\mathrm{id})&=\frac{1}{(r+1)\zeta(2)}x\log x+\frac{x}{2}\\&\quad+\frac{1}{(r+1)\zeta(2)}\left(2\gamma-1-\frac{\zeta'(2)}{\zeta(2)}+\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\zeta(2m+1)\right)\\&\quad x+K_r(x),\end{aligned}$$

where

$$K_r(x)=\frac{1}{r+1}\sum_{n\leq x}\frac{\mu(n)}{n}\Delta\left(\frac{x}{n}\right)+O_r\left(\log x\right).\tag{2}$$

For $f=\phi$, it was shown that

$$M_r(x;\phi)=\frac{1}{(r+1)\zeta^2(2)}x\log x+\frac{x}{2\zeta(2)}$$
$$+\frac{1}{(r+1)\zeta^2(2)}\left(2\gamma-1-2\frac{\zeta'(2)}{\zeta(2)}+\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\zeta(2m+1)\right)$$
$$x+L_r(x),$$

where

$$L_r(x):=\frac{1}{r+1}\sum_{n\le x}\frac{\mu*\mu(n)}{n}\Delta\left(\frac{x}{n}\right)+O_r\left((\log x)^2\right). \tag{3}$$

Lastly, for $f=\psi$ it was proved that

$$M_r(x;\psi)=\frac{1}{(r+1)\zeta(4)}x\log x+\frac{\zeta(2)}{2\zeta(4)}x$$
$$+\frac{1}{(r+1)\zeta(4)}\left(2\gamma-1-2\frac{\zeta'(4)}{\zeta(4)}+\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\zeta(2m+1)\right)x+U_r(x),$$

where

$$U_r(x):=\frac{1}{r+1}\sum_{n\le x}\frac{\mu*|\mu|(n)}{n}\Delta\left(\frac{x}{n}\right)+O_r\left((\log x)^2\right). \tag{4}$$

The function $\Delta(x)$ denotes the error term of the Dirichlet divisor problem: Let $\tau=\mathbf{1}*\mathbf{1}$ be the divisor function, then for any large positive number $x\ge 2$,

$$\sum_{n\le x}\tau(n)=x\log x+(2\gamma-1)x+\Delta(x), \tag{5}$$

where γ is the Euler constant and $\Delta(x)$ can be estimated by $\Delta(x)=O\left(x^{\theta+\varepsilon}\right)$. It is known that one can take $1/4\le\theta\le 1/3$. More precisely, the Dirichlet divisor problem is to find the smallest value of θ for which the above estimate holds, for any $\epsilon>0$. This problem is still unsolved. The best estimate to date is

$$O\left(x^{131/416}(\log x)^{26947/8320}\right),$$

obtained by Huxley [7] in 2003.

The first purpose of this paper is to refine the error terms $K_r(x), L_r(x)$ and $U_r(x)$ from the above formulas. Therefore, let $\sigma_u=\mathrm{id}_u*\mathbf{1}$ be the generalized divisor function for any real number u and let $m\ge 1$ be an integer. Then for any large positive number $x\ge 2$, the function $\Delta_{-2m}(x)$ denotes the error term of the generalized divisor problem given by

$$\sum_{n\le x}\sigma_{-2m}(n)=\zeta(1+2m)x-\frac{1}{2}\zeta(2m)+\Delta_{-2m}(x). \tag{6}$$

For more details about the functions $\Delta(x), \Delta_{-2m}(x)$, see [1]. We have the following results:

Theorem 1. *Let $\Delta(x)$ and $\Delta_{-2m}(x)$ be the error terms given by Eqs. (5) and (6), respectively. For any large positive number $x > 5$ and fixed positive integer r, we have*

$$K_r(x) = \frac{1}{r+1}\sum_{d\le x}\frac{\mu(d)}{d}\Delta\left(\frac{x}{d}\right) + \frac{1}{r+1}\sum_{d\le x}\frac{\mu(d)}{d}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\Delta_{-2m}\left(\frac{x}{d}\right) + O_r\left(\delta(x)\right),$$

where the function $\delta(x)$ is defined by

$$\delta(x) := \exp\left(-C\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right) \tag{7}$$

with C being a positive constant. Moreover, we have

$$L_r(x) = \frac{1}{r+1}\sum_{n\le x}\frac{\mu*\mu(n)}{n}\Delta\left(\frac{x}{n}\right) + \frac{1}{r+1}\sum_{n\le x}\frac{\mu*\mu(n)}{n}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\Delta_{-2m}\left(\frac{x}{n}\right) + O_r\left((\log x)^{2/3}(\log\log x)^{1/3}\right).$$

and

$$U_r(x) = \frac{1}{r+1}\sum_{n\le x}\frac{\mu*|\mu|(n)}{n}\Delta\left(\frac{x}{n}\right) + \frac{1}{r+1}\sum_{n\le x}\frac{\mu*|\mu|(n)}{n}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\Delta_{-2m}\left(\frac{x}{n}\right) - \frac{1}{4\zeta(2)}\log x + O_r\left((\log x)^{2/3}\right).$$

Remark 1. It is easily checked that using the weakest estimate $\Delta_{-2m}(x) = O_m(1)$ in the results Theorem 1 yields much better results than the previously known formulas for $K_r(x)$, $L_r(x)$ and $M_r(x)$ from Eqs. (2), (3), and (4).

Furthermore, even better estimates of $K_r(x)$ can be achieved by additional assumptions on the Riemann zeta-function. Under the Riemann Hypothesis, Maier and Montgomery [15] gave a sharper estimate of the partial sum of the Möbius function, which was later improved by Soundararajan [17].

The author proved that

$$M(x) := \sum_{n\le x} \mu(n) = O\left(x^{1/2}\eta(x)\right)$$

where

$$\eta(x) := \exp\left((\log x)^{1/2}(\log\log x)^{14}\right), \tag{8}$$

for any large positive number $x > 5$ satisfying $x = [x] + \frac{1}{2}$. This latter has been improved slightly by Balazard and de Roton [2]. By using the above result on $M(x)$, we obtain the next statement.

Theorem 2. *Assume the Riemann Hypothesis and let $\Delta(x)$ and $\Delta_{-2m}(x)$ be the error terms given by Eqs. (5) and (6), respectively. Then for any large positive number $x > 5$ such that $x = [x] + \frac{1}{2}$ and fixed positive integer r, we have*

$$\begin{aligned} K_r(x) &= \frac{1}{r+1}\sum_{d\le x}\frac{\mu(d)}{d}\Delta\left(\frac{x}{d}\right) \\ &\quad + \frac{1}{r+1}\sum_{d\le x}\frac{\mu(d)}{d}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\Delta_{-2m}\left(\frac{x}{d}\right) + O_r\left(\frac{\eta(x)\log x}{x^{1/2}}\right). \end{aligned}$$

For our further considerations, let $\rho = \alpha + i\beta$ denote the generic non-trivial zeros of the Riemann zeta-function. Under the assumption that all zeros ρ in the critical strip of $\zeta(s)$ are simple, we are able to prove an additional refinement for the error term $K_r(x)$.

Theorem 3. *Assume that the zeros of $\zeta(s)$ are simple. Let $T_* \ge x^6$ be some positive number satisfying the inequality*

$$\frac{1}{\zeta(\sigma + iT_*)} \ll T_*^{\varepsilon}$$

for $\frac{1}{2} \le \sigma \le 2$. For any large positive number $x > 5$ with $x = [x] + \frac{1}{2}$ we then have

$$\begin{aligned} K_r(x) &= \frac{1}{r+1}\sum_{n\le x}\frac{\mu(n)}{n}\Delta\left(\frac{x}{n}\right) + \frac{1}{r+1}\sum_{n\le x}\frac{\mu(n)}{n}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\Delta_{-2m}\left(\frac{x}{n}\right) \\ &\quad + \frac{2\gamma + C_{\text{odd}}(r) - 1}{r+1}\sum_{|\beta|\le T_*}\frac{x^{\rho-1}}{(\rho-2)\zeta'(\rho)} - \frac{C_{\text{even}}(r)}{2(r+1)}\sum_{|\beta|\le T_*}\frac{x^{\rho-1}}{(\rho-1)\zeta'(\rho)} \\ &\quad + \frac{1}{r+1}\sum_{|\beta|\le T_*}\frac{x^{\rho-1}}{(\rho-2)^2\zeta'(\rho)} + O_r\left(x^{-3}\right), \end{aligned}$$

where the functions $C_{\mathrm{odd}}(r)$ *and* $C_{\mathrm{even}}(r)$ *are given by*

$$C_{\mathrm{odd}}(r) := \sum_{m=1}^{[r/2]} \binom{r+1}{2m} B_{2m}\zeta(2m+1),$$

and

$$C_{\mathrm{even}}(r) := \sum_{m=1}^{[r/2]} \binom{r+1}{2m} B_{2m}\zeta(2m)$$

for any fixed positive integer r*.*

Finally, define the sum

$$J_{-\lambda}(T) := \sum_{0<\beta\leq T} \frac{1}{|\zeta'(\rho)|^{2\lambda}}$$

which is intimately connected to Mertens function. Assuming the simplicity of the zeros of $\zeta(s)$, Gonek [5] and Hejhal [6] independently conjectured that for any real number $\lambda < 3/2$, we have

$$J_{-\lambda}(T) \asymp T(\log T)^{(\lambda-1)^2} \tag{9}$$

We use this conjecture to prove the following:

Theorem 4. *Assume that the Riemann Hypothesis and Gonek-Hejhal conjecture. Then*

$$K_r(x) = \frac{1}{r+1}\sum_{n\leq x}\frac{\mu(n)}{n}\Delta\left(\frac{x}{n}\right) + \frac{1}{r+1}\sum_{d\leq x}\frac{\mu(d)}{d}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\Delta_{-2m}\left(\frac{x}{d}\right) + O_r\left(\frac{(\log x)^{5/4}}{x^{1/2}}\right),$$

for any large positive number $x > 5$ *satisfying* $x = [x] + \frac{1}{2}$*.*

2. Proofs of Theorems 1 and 2

In order to prove our main results, we first show some necessary lemmas.

2.1. Auxiliary Lemmas

Lemma 1. *For any large positive number* $x > 5$*, we have*

$$\sum_{n\leq x}\frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)} + O\left(\frac{\delta(x)}{x}\right), \tag{10}$$

$$\sum_{n\leq x}\frac{\mu(n)}{n^2}\log n = \frac{\zeta'(2)}{\zeta^2(2)} + O\left(\frac{\delta(x)}{x}\right), \tag{11}$$

and

$$\sum_{n\leq x}\frac{\mu(n)}{n}=O\left(\delta(x)\right), \tag{12}$$

where $\delta(x)$ is given by Eq. (7). Assume that $x=[x]+\frac{1}{2}$. Under the Riemann Hypothesis we have

$$\sum_{n\leq x}\frac{\mu(n)}{n^2}=\frac{1}{\zeta(2)}+O\left(\frac{\eta(x)}{x^{3/2}}\right), \tag{13}$$

$$\sum_{n\leq x}\frac{\mu(n)}{n^2}\log n=\frac{\zeta'(2)}{\zeta^2(2)}+O\left(\frac{\eta(x)\log x}{x^{3/2}}\right), \tag{14}$$

and

$$\sum_{n\leq x}\frac{\mu(n)}{n}=O\left(\frac{\eta(x)}{x^{1/2}}\right). \tag{15}$$

for any large positive number $x>5$. Here $\eta(x)$ is given by Eq. (8).

Proof. Eqs. (10) and (11) follow from Lemmas 2.2 and 2.3 in [16]. The proof of Eq. (12) can be found in [9]. The formulas (13)–(15) follow from Lemma 2.1 in [8]. □

Lemma 2. *For any large positive number $x>5$, we have*

$$\sum_{n\leq x}\frac{\phi(n)}{n}=\frac{x}{\zeta(2)}+O\left((\log x)^{2/3}(\log\log x)^{1/3}\right).$$

Proof. For any large positive number $x\geq 5$, we use the result of Liu in [14]

$$\sum_{\ell\leq x}\frac{\mu(\ell)}{\ell}\vartheta\left(\frac{x}{\ell}\right)=O\left((\log x)^{2/3}(\log\log x)^{1/3}\right),$$

the fact that $\phi=\mathrm{id}*\mu$, and Eqs. (10), (12) to obtain the formula

$$\begin{aligned}\sum_{n\leq x}\frac{\phi(n)}{n}&=\sum_{\ell\leq x}\frac{\mu(\ell)}{\ell}\left(\frac{x}{\ell}-\vartheta\left(\frac{x}{\ell}\right)-\frac{1}{2}\right)\\&=\frac{x}{\zeta(2)}+O\left((\log x)^{2/3}(\log\log x)^{1/3}\right).\end{aligned}$$

Here $\vartheta(x)$ is the oscillatory function defined by $x-[x]-\frac{1}{2}$. This completes the proof. □

Lemma 3. *For any large positive number $x>5$, we have*

$$\sum_{n\leq x}\frac{\psi(n)}{n}=\frac{\zeta(2)}{\zeta(4)}x-\frac{1}{2\zeta(2)}\log x+O\left((\log x)^{2/3}\right).$$

Proof. The proof can be found in [20, Satz 3]. □

Lemma 4. *For any large positive number $x > 5$, we have*

$$\sum_{n\le x} \frac{\mu * \mu(n)}{n^2} = \frac{1}{\zeta^2(2)} + O\left(\frac{\kappa(x)}{x}\right), \tag{16}$$

$$\sum_{n\le x} \frac{\mu * \mu(n)}{n^2} \log n = 2\frac{\zeta'(2)}{\zeta^3(2)} + O\left(\frac{\kappa(x)}{x}\right), \tag{17}$$

and

$$\sum_{n\le x} \frac{\mu * \mu(n)}{n} = O\left(\kappa(x)\right), \tag{18}$$

where $\kappa(x)$ is given by

$$\kappa(x) = \exp\left(-D(\log x \log\log x)^{1/3}\right)$$

with D being a positive constant.

Proof. Eqs. (16), (17) and (18) follow from Eqs. (3.5), (3.6) and (3.3) in [8], respectively. □

Lemma 5. *For any large positive number $x > 5$, we have*

$$\sum_{n\le x} \frac{|\mu| * \mu(n)}{n^2} = \frac{1}{\zeta(4)} + O\left(\frac{\delta(x)}{x^{3/2}}\right), \tag{19}$$

$$\sum_{n\le x} \frac{|\mu| * \mu(n)}{n^2} \log n = 2\frac{\zeta'(4)}{\zeta^2(4)} + O\left(\frac{\delta(x)}{x^{3/2}}\right), \tag{20}$$

and

$$\sum_{n\le x} \frac{|\mu| * \mu(n)}{n} = \frac{1}{\zeta(2)} + O\left(\frac{\delta(x)}{x^{1/2}}\right). \tag{21}$$

Proof. Eqs. (19) and (20) follow from Eqs. (3.7) and (3.8) in [8], respectively. It is known that

$$\sum_{n=1}^{\infty} \frac{|\mu| * \mu(n)}{n} = \frac{1}{\zeta(2)},$$

Now, we write our sums as follows

$$\sum_{n\le x} \frac{|\mu| * \mu(n)}{n} = \sum_{n=1}^{\infty} \frac{|\mu| * \mu(n)}{n} - \sum_{n>x} \frac{|\mu| * \mu(n)}{n}$$
$$= \frac{1}{\zeta(2)} - \sum_{n>x} \frac{|\mu| * \mu(n)}{n}.$$

To complete the proof, it remains to estimate the last sum above. Notice that

$$\sum_{n>x} \frac{|\mu| * \mu(n)}{n} = \int_x^{\infty} \frac{\sum_{x<n\le t} |\mu| * \mu(n)}{t^2}\, dt$$

and that

$$\sum_{n\le x}|\mu|*\mu(n)=\sum_{n\le\sqrt{x}}\mu(n)+O\left(x^{\epsilon}\right)=O\left(x^{1/2}\delta(x)\right),$$

where we used Eq. (3.4) from [8]. Therefore, we have

$$\sum_{n>x}\frac{|\mu|*\mu(n)}{n}=O\left(\int_x^{\infty}\frac{t^{1/2}\delta(t)}{t^2}\,dt\right)=O\left(\frac{\delta(x)}{x^{1/2}}\right),$$

and Eq. (21) is proved. □

Lemma 6. *For any large positive number $x>5$, we have*

$$\sum_{n\le x}\sum_{d|n}\frac{\phi(d)}{d}=\frac{1}{\zeta(2)}x\log x+\frac{1}{\zeta(2)}\left(2\gamma-1-\frac{\zeta'(2)}{\zeta(2)}\right)x$$
$$+\sum_{d\le x}\frac{\mu(d)}{d}\Delta\left(\frac{x}{d}\right)+O\left(\delta(x)\right), \tag{22}$$

and

$$\sum_{d\ell\le x}\frac{\phi(d)}{d}\frac{1}{\ell^{2m}}=\frac{\zeta(1+2m)}{\zeta(2)}x+\sum_{d\le x}\frac{\mu(d)}{d}\Delta_{-2m}\left(\frac{x}{d}\right)+O_m\left(\delta(x)\right) \tag{23}$$

for any positive integer m. Suppose that $x=[x]+\frac{1}{2}$. Under the Riemann Hypothesis, we have

$$\sum_{n\le x}\sum_{d|n}\frac{\phi(d)}{d}=\frac{1}{\zeta(2)}x\log x+\frac{1}{\zeta(2)}\left(2\gamma-1-\frac{\zeta'(2)}{\zeta(2)}\right)x$$
$$+\sum_{d\le x}\frac{\mu(d)}{d}\Delta\left(\frac{x}{d}\right)+O\left(\frac{\eta(x)\log x}{x^{1/2}}\right), \tag{24}$$

and

$$\sum_{d\ell\le x}\frac{\phi(d)}{d}\frac{1}{\ell^{2m}}=\frac{\zeta(1+2m)}{\zeta(2)}x+\sum_{d\le x}\frac{\mu(d)}{d}\Delta_{-2m}\left(\frac{x}{d}\right)+O_m\left(\frac{\eta(x)}{x^{1/2}}\right) \tag{25}$$

Proof. We recall the identity $\frac{\phi}{\mathrm{id}}*\mathbf{1}=\frac{\mu}{\mathrm{id}}*\tau$. Using Eqs. (5), (10) and (11), we obtain

$$\begin{aligned}\sum_{n\le x}\sum_{d|n}\frac{\phi(d)}{d}&=\sum_{d\le x}\frac{\mu(d)}{d}\sum_{\ell\le x/d}\tau(\ell)\\&=x\left(\log x+2\gamma-1\right)\sum_{d\le x}\frac{\mu(d)}{d^2}-x\sum_{d\le x}\frac{\mu(d)}{d^2}\log d+\sum_{d\le x}\frac{\mu(d)}{d}\Delta\left(\frac{x}{d}\right)\\&=\frac{1}{\zeta(2)}x\log x+\frac{1}{\zeta(2)}\left(2\gamma-1-\frac{\zeta'(2)}{\zeta(2)}\right)x+\sum_{d\le x}\frac{\mu(d)}{d}\Delta\left(\frac{x}{d}\right)+O\left(\delta(x)\right),\end{aligned}$$

which completes the proof of Eq. (22). Further, we recall the identity $\frac{\phi}{\mathrm{id}} * \mathrm{id}_{-2m} = \frac{\mu}{\mathrm{id}} * \sigma_{-2m}$, and use Eqs. (6), (10) and (12) to get

$$\begin{aligned}\sum_{d\ell \le x} \frac{\phi(d)}{d} \frac{1}{\ell^{2m}} &= \sum_{d\le x} \frac{\mu(d)}{d} \sum_{\ell \le x/d} \sigma_{-2m}(\ell) \\ &= \sum_{d\le x} \frac{\mu(d)}{d} \left(\zeta(1+2m)\frac{x}{d} - \frac{1}{2}\zeta(2m) + \Delta_{-2m}\left(\frac{x}{d}\right)\right) \\ &= \frac{\zeta(1+2m)}{\zeta(2)} x + \sum_{d\le x} \frac{\mu(d)}{d} \Delta_{-2m}\left(\frac{x}{d}\right) + O_m\left(\delta(x)\right).\end{aligned}$$

This completes the proof of Eq. (23). Similarly, we use Eqs. (13), (14) and (15) to deduce Eqs. (24) and (25). □

Lemma 7. *For any large positive number $x > 5$, we have*

$$\begin{aligned}\sum_{n\le x} \sum_{d|n} \frac{\mu * \phi(d)}{d} = \frac{1}{\zeta^2(2)} x\log x + \frac{1}{\zeta^2(2)} \left(2\gamma - 1 - 2\frac{\zeta'(2)}{\zeta(2)}\right) x \\ + \sum_{d\le x} \frac{\mu*\mu(d)}{d} \Delta\left(\frac{x}{d}\right) + O\left(\kappa(x)\right),\end{aligned} \tag{26}$$

and

$$\sum_{d\ell\le x} \frac{\mu*\phi(d)}{d} \frac{1}{\ell^{2m}} = \frac{\zeta(1+2m)}{\zeta^2(2)} x + \sum_{d\le x} \frac{\mu*\mu(d)}{d} \Delta_{-2m}\left(\frac{x}{d}\right) + O_m\left(\kappa(x)\right) \tag{27}$$

for any positive integer m. Here $\kappa(x)$ is defined above in Lemma 4.

Proof. We use the identity $\frac{\mu*\phi}{\mathrm{id}} * \mathbf{1} = \frac{\mu*\mu}{\mathrm{id}} * \tau$, Eqs. (5), (16), and (17) to obtain

$$\begin{aligned}\sum_{k\le x} \sum_{d|k} \frac{\mu*\phi(d)}{d} &= \sum_{d\le x} \frac{\mu*\mu(d)}{d} \sum_{\ell\le x/d} \tau(\ell) \\ &= x\left(\log x + 2\gamma - 1\right) \sum_{d\le x} \frac{\mu*\mu(d)}{d^2} - x \sum_{d\le x} \frac{\mu*\mu(d)}{d^2} \log d + \sum_{d\le x} \frac{\mu*\mu(d)}{d} \Delta\left(\frac{x}{d}\right) \\ &= \frac{x\log x}{\zeta^2(2)} + \frac{1}{\zeta^2(2)} \left(2\gamma - 1 - 2\frac{\zeta'(2)}{\zeta(2)}\right) x + \sum_{d\le x} \frac{\mu*\mu(d)}{d} \Delta\left(\frac{x}{d}\right) + O\left(\kappa(x)\right),\end{aligned}$$

which completes the proof of Eq. (26). By using the fact that $\frac{\mu * \phi}{\mathrm{id}} * \mathrm{id}_{-2m} = \frac{\mu * \mu}{\mathrm{id}} * \sigma_{-2m}$, together with Eqs. (6), (16), and (18) we get

$$\begin{aligned}\sum_{d\ell \le x} \frac{\mu * \phi(d)}{d} \frac{1}{\ell^{2m}} &= \sum_{d \le x} \frac{\mu * \mu(d)}{d} \sum_{\ell \le x/d} \sigma_{-2m}(\ell)\\ &= \frac{\zeta(1+2m)}{\zeta^2(2)} x + \sum_{d \le x} \frac{\mu * \mu(d)}{d} \Delta_{-2m}\left(\frac{x}{d}\right) + O_m\left(\kappa(x)\right).\end{aligned}$$

Therefore, Eq. (27) is proved. □

Lemma 8. *For any large positive number $x > 5$, we have*

$$\begin{aligned}\sum_{n \le x} \sum_{d \mid n} \frac{\mu * \psi(d)}{d} &= \frac{1}{\zeta(4)} x \log x + \frac{1}{\zeta(4)} \left(2\gamma - 1 - 2\frac{\zeta'(4)}{\zeta(4)}\right) x\\ &\quad + \sum_{d \le x} \frac{|\mu| * \mu(d)}{d} \Delta\left(\frac{x}{d}\right) + O\left(\frac{\delta(x)}{x^{1/2}}\right),\end{aligned} \tag{28}$$

and

$$\begin{aligned}\sum_{d\ell \le x} \frac{\mu * \psi(d)}{d} \frac{1}{\ell^{2m}} &= \frac{\zeta(1+2m)}{\zeta(4)} x\\ &\quad + \sum_{d \le x} \frac{|\mu| * \mu(d)}{d} \Delta_{-2m}\left(\frac{x}{d}\right) - \frac{\zeta(2m)}{2\zeta(2)} + O_m\left(\frac{\delta(x)}{x^{1/2}}\right)\end{aligned} \tag{29}$$

for any positive integer m.

Proof. From the identity $\frac{\mu * \psi}{\mathrm{id}} * \mathbf{1} = \frac{\mu * |\mu|}{\mathrm{id}} * \tau$, we have

$$\sum_{k \le x} \sum_{d \mid k} \frac{\mu * \psi(d)}{d} = \sum_{d \le x} \frac{\mu * |\mu|(d)}{d} \sum_{\ell \le x/d} \tau(\ell).$$

Using Eqs. (5), (19) and (20), we obtain the formula Eq. (28). Now, we use the identity $\frac{\mu * \psi}{\mathrm{id}} * \mathrm{id}_{-2m} = \frac{\mu * |\mu|}{\mathrm{id}} * \sigma_{-2m}$ to write our second sums as follows

$$\sum_{d\ell \le x} \frac{\mu * \psi(d)}{d} \frac{1}{\ell^{2m}} = \sum_{d \le x} \frac{\mu * |\mu|(d)}{d} \sum_{\ell \le x/d} \sigma_{-2m}(\ell).$$

Again, we use Eq. (6) to get

$$\sum_{d\ell \le x} \frac{\mu * \psi(d)}{d} \frac{1}{\ell^{2m}} = \sum_{d \le x} \frac{\mu * |\mu|(d)}{d} \left(\zeta(1+2m)\frac{x}{d} - \frac{1}{2}\zeta(2m) + \Delta_{-2m}\left(\frac{x}{d}\right)\right).$$

Applying Eqs. (19) and (21) to the above, we deduce the desired result. □

Now we are ready to prove our main theorems.

2.2. Proofs of the Theorems

Proof of Theorem 1. First, we take $f = \mathrm{id}$ into Eq. (1) to get

$$\begin{aligned} M_r(x;\mathrm{id}) &= \frac{1}{2}\sum_{n\le x} 1 + \frac{1}{r+1}\sum_{d\ell\le x}\frac{\mu*\mathrm{id}(d)}{d} \\ &\quad + \frac{1}{r+1}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\sum_{d\ell\le x}\frac{\mu*\mathrm{id}(d)}{d}\frac{1}{\ell^{2m}} \\ &= \frac{1}{2}\sum_{n\le x} 1 + \frac{1}{r+1}\sum_{n\le x}\sum_{d\mid n}\frac{\phi(d)}{d} \\ &\quad + \frac{1}{r+1}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\sum_{d\ell\le x}\frac{\phi(d)}{d}\frac{1}{\ell^{2m}}. \end{aligned} \tag{30}$$

Applying Eqs. (22) and (23) above yields

$$\begin{aligned} K_r(x) &= \frac{1}{r+1}\sum_{d\le x}\frac{\mu(d)}{d}\Delta\left(\frac{x}{d}\right) \\ &\quad + \frac{1}{r+1}\sum_{d\le x}\frac{\mu(d)}{d}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\Delta_{-2m}\left(\frac{x}{d}\right) + O_r\left(\delta(x)\right), \end{aligned}$$

which gives the desired result. We take $f = \phi$ into Eq. (1) to get

$$\begin{aligned} M_r(x;\phi) &= \frac{1}{2}\sum_{n\le x}\frac{\phi(n)}{n} + \frac{1}{r+1}\sum_{n\le x}\sum_{d\mid n}\frac{\mu*\phi(d)}{d} \\ &\quad + \frac{1}{r+1}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\sum_{d\ell\le x}\frac{\mu*\phi(d)}{d}\frac{1}{\ell^{2m}}. \end{aligned}$$

Using Lemma 2, as well as Eqs. (26) and (27), we get

$$\begin{aligned} L_r(x) &= \frac{1}{r+1}\sum_{n\le x}\frac{\mu*\mu(n)}{n}\Delta\left(\frac{x}{n}\right) \\ &\quad + \frac{1}{r+1}\sum_{n\le x}\frac{\mu*\mu(n)}{n}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\Delta_{-2m}\left(\frac{x}{n}\right) \\ &\quad + O_r\left((\log x)^{2/3}(\log\log x)^{1/3}\right), \end{aligned}$$

as desired. Taking $f=\psi$ into Eq. (1) we get

$$M_r(x;\psi)=\frac{1}{2}\sum_{n\le x}\frac{\psi(n)}{n}+\frac{1}{r+1}\sum_{d\ell\le x}\frac{\mu*\psi(d)}{d}$$
$$+\frac{1}{r+1}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\sum_{d\ell\le x}\frac{\mu*\psi(d)}{d}\frac{1}{\ell^{2m}}. \tag{31}$$

Applying Lemma 3, as well as Eqs. (28) and (29) in the above formula yields

$$U_r(x)=\frac{1}{r+1}\sum_{n\le x}\frac{\mu*|\mu|(n)}{n}\Delta\left(\frac{x}{n}\right)$$
$$+\frac{1}{r+1}\sum_{n\le x}\frac{\mu*|\mu|(n)}{n}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\Delta_{-2m}\left(\frac{x}{n}\right)$$
$$-\frac{1}{4\zeta(2)}\log x+O_r\left((\log x)^{2/3}\right).$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. By assuming the Riemann Hypothesis, and applying Eqs. (24) and (25) in Eq. (30), we immediately deduce that

$$K_r(x)=\frac{1}{r+1}\sum_{d\le x}\frac{\mu(d)}{d}\Delta\left(\frac{x}{d}\right)$$
$$+\frac{1}{r+1}\sum_{d\le x}\frac{\mu(d)}{d}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\Delta_{-2m}\left(\frac{x}{d}\right)+O_r\left(\frac{\eta(x)\log x}{x^{1/2}}\right),$$

which completes the proof of Theorem 2. □

3. Proofs of Theorems 3 and 4

To prove Theorems 3 we just need the following lemma.

Lemma 9. *Under the hypotheses of Theorem 3, we have*

$$\sum_{n\le x}\frac{\mu(n)}{n^2}=\frac{1}{\zeta(2)}+\sum_{|\beta|\le T_*}\frac{x^{\rho-2}}{(\rho-2)\zeta'(\rho)}+\frac{\pi^2}{\zeta(3)}x^{-4}+O\left(x^{-5}\right),$$
$$\sum_{n\le x}\frac{\mu(n)}{n^2}\log\frac{x}{n}=\frac{1}{\zeta(2)}\left(\log x-\frac{\zeta'(2)}{\zeta(2)}\right)$$
$$+\sum_{|\beta|\le T_*}\frac{x^{\rho-2}}{(\rho-2)^2\zeta'(\rho)}-\frac{\pi^2}{4\zeta(3)}x^{-4}+O\left(x^{-5}\right),$$

and

$$\sum_{n\le x}\frac{\mu(n)}{n}=\sum_{|\beta|\le T_*}\frac{x^{\rho-1}}{(\rho-1)\zeta'(\rho)}+\frac{4\pi^2}{3\zeta(3)}x^{-3}+O\left(x^{-5}\right).$$

Proof. The proof of the lemma can be found in [8, Lemma 3.5]. $\square$

Proof of Theorem 3. We recall that

$$\begin{aligned}M_r(x;\mathrm{id})&=\frac{1}{2}\sum_{n\le x}1+\frac{1}{r+1}\sum_{d\ell\le x}\frac{\mu*\mathrm{id(d)}}{d}\\&\quad+\frac{1}{r+1}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\sum_{d\ell\le x}\frac{\mu*\mathrm{id(d)}}{d}\frac{1}{\ell^{2m}}.\end{aligned}$$

Using the fact that

$$\frac{\mu*\mathrm{id}}{\mathrm{id}}*\mathbf{1}=\frac{\mu}{\mathrm{id}}*\tau,\qquad\qquad\frac{\mu*\mathrm{id}}{\mathrm{id}}*\mathrm{id}_{-2m}=\frac{\mu}{\mathrm{id}}*\sigma_{-2m},$$

and Eqs. (5) and (6), we get

$$\begin{aligned}M_r(x;\mathrm{id})&=\frac{[x]}{2}+\frac{x}{r+1}\sum_{n\le x}\frac{\mu(n)}{n^2}\log\frac{x}{n}+\frac{x}{r+1}\left(2\gamma-1+C_{\mathrm{odd}}(r)\right)\sum_{n\le x}\frac{\mu(n)}{n^2}\\&\quad-\frac{C_{\mathrm{even}}(r)}{2(r+1)}\sum_{n\le x}\frac{\mu(n)}{n}+\frac{1}{r+1}\sum_{n\le x}\frac{\mu(n)}{n}\Delta\left(\frac{x}{n}\right)\\&\quad+\frac{1}{r+1}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\sum_{n\le x}\frac{\mu(n)}{n}\Delta_{-2m}\left(\frac{x}{n}\right).\end{aligned}$$

Under the hypotheses of the theorem, we use Lemma 9 to obtain

$$\begin{aligned}M_r(x;\mathrm{id})&=\frac{[x]}{2}+\frac{x\log x}{(r+1)\zeta(2)}+\frac{x}{(r+1)\zeta(2)}\left(2\gamma-1-\frac{\zeta'(2)}{\zeta(2)}+C_{\mathrm{odd}}(r)\right)\\&\quad+\frac{1}{r+1}\sum_{n\le x}\frac{\mu(n)}{n}\Delta\left(\frac{x}{n}\right)+\frac{1}{r+1}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}\\&\quad B_{2m}\zeta(2m)\sum_{n\le x}\frac{\mu(n)}{n}\Delta_{-2m}\left(\frac{x}{n}\right)\\&\quad+\frac{1}{r+1}\left(2\gamma-1+C_{\mathrm{odd}}(r)\right)\sum_{|\beta|\le T_*}\frac{x^{\rho-1}}{(\rho-2)\zeta'(\rho)}\\&\quad+\frac{1}{r+1}\sum_{|\beta|\le T_*}\frac{x^{\rho-1}}{(\rho-2)^2\zeta'(\rho)}-\frac{C_{\mathrm{even}}(r)}{2(r+1)}\\&\quad\sum_{|\beta|\le T_*}\frac{x^{\rho-1}}{(\rho-1)\zeta'(\rho)}+O_r\left(x^{-3}\right),\end{aligned}$$

which completes the proof. □

Proof of Theorem 4. To prove our theorem it suffices to show that

$$\sum_{|\beta|\le T_*}\frac{x^{-\frac{1}{2}+i\beta}}{(-j+i\beta)\zeta'(\frac{1}{2}+i\beta)}=O\left(x^{-1/2}(\log x)^{5/4}\right)$$

with $j=1/2$ and $3/2$. We take $\lambda=-1/2$ into Eq. (9), then $J_{-1/2}(T_*)\ll T_*(\log T_*)^{1/4}$. Using the above and partial summation we have

$$\sum_{|\beta|\le T_*}\frac{1}{\beta|\zeta'(\frac{1}{2}+i\beta)|}\ll\left[\frac{J_{-1/2}(t)}{t}\right]_{14}^{T_*}+\int_{14}^{T_*}\frac{J_{-1/2}(t)}{t^2}dt\ll(\log T_*)^{5/4},$$

and the proof is complete. □

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References

[1] Apostol, T.M.: Introduction to Analytic Number Theory. Springer, Berlin (1976)

[2] Balazard, M., de Roton, A.: Notes de lecture de l'article " Partial sums of the Möbius function" de Kannan Soundararajan. arXiv:0810.3587v1 [math.NT] (2008)

[3] Bordellés, O.: The composition of the GCD and certain arithmetic functions. J. Integer Sequences **13**, Article 10.7.1 (2010)

[4] Broughan, K.A.: The average order of the Dirichlet series of the GCD-sum function. J. Integer Sequences **10**, Article 07.4.2 (2007)

[5] Gonek, S.M.: On negative moments of the Riemann zeta-function. Mathematika **36**, 71–88 (1989)

[6] Hejhal, D.: On the distribution of log $\left|\zeta'(\frac{1}{2}+it)\right|$. In: Aubert, K.E., Bombieri, E., Goldfeld, D. (eds.) Number theory, trace formula and discrete groups, Symposium in Honor of Atle Selberg, Oslo, Norway, vol. 1989, pp. 343–370 . Academic Press, San Diego (1987)

[7] Huxley, M.N.: Exponential sums and lattice points III. Proc. Lond. Math. Soc. **87**, 591–609 (2003)

[8] Inoue, S., Kiuchi, I.: On sums of GCD-sum functions. Preprint

[9] Jia, R.Q.: Estimation of partial sums of series $\sum \mu(n)/n$. Kexue Tongbao. **30**, 575–578 (1985)

[10] Kiuchi, I.: On sums of averages of generalized Ramanujan sums. Tokyo J. Math. **40**, 255–275 (2017)

[11] Kiuchi, I.: Sums of averages of GCD-sum functions. J. Number Theory **176**, 449–472 (2017)

[12] Kiuchi, I.: Sums of averages of generalized Ramanujan sums. J. Number Theory **180**, 310–348 (2017)

[13] Kiuchi, I., Saad Eddin, S.: Sums of weighted averages of GCD-sum functions. Int. J. Number Theory **14**, 2699–2728 (2018)

[14] Liu, H.-Q.: On Euler's function. Proc. R. Soc. Edinb. **146**, 769–775 (2016)

[15] Maier, H., Montgomery, H.L.: On the sum of the Möbius function. Bull. Lond. Math. Soc. **41**, 213–226 (2009)

[16] Sitaramachandra Rao, R., Suryanarayana, D.: The number of pairs of integers with L. C. M. $\leq x$. Arch. Math. (Basel) **21**, 490–497 (1970)

[17] Soundararajan, K.: Partial sums of the Möbius function. J. Reine Angew. Math. **631**, 141–152 (2009)

[18] Tanigawa, Y., Zhai, W.: On the GCD-sum functions. J. Integer Sequences **11**, Article 08.2.3 (2008)

[19] Tóth, L.: A survey of GCD-sum functions. J. Integer Sequences. **13**, Article 10.8.1 (2010)

[20] Walfisz, A.: Weylsche Exponentialsummen in der Neueren Zahlentheorie. Veb Deutscher Verlag Der Wissenschaften, Berlin (1963)

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