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Results in Mathematics

Myers' Type Theorem for Integral Bakry–Émery Ricci Tensor Bounds

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Abstract. In this paper we first discuss weighted mean curvature and volume comparisons on smooth metric measure space $(M, g, e^{-f}dv)$ under the integral Bakry–Émery Ricci tensor bounds. In particular, we add an additional condition on the potential function f to ensure the validity of previous conclusions for some cases proved by the second author. Then, we apply the comparison results to get a new diameter estimate and a fundamental group finiteness under the integral Bakry–Émery Ricci tensor bounds, which sharpens Theorem 1.6 in Wu (J Geom Anal 29:828–867, 2019) and can be viewed as the extension of the works of Myers and Aubry.

Mathematics Subject Classification. Primary 53C20.

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1. Introduction

Myers' theorem [18] is one of classical theorems in Riemannian geometry. It states that if the Ricci curvature of a complete n-dimensional connected Riemannian manifold (M, g) satisfies $\mathrm{Ric} \geq (n-1)H$ for some constant $H > 0$, then manifold M is compact with finite fundamental group and its diameter is at most $\pi/\sqrt{H}$. Since then, many works generalize Myers' theorem, see for example [1,4,10,31].

In 1990s, Myers' theorem was generalized to a integral bound setting for the Ricci tensor. Petersen and Wei [22,23] extended many classical comparison theorems and geometrical results of pointwise Ricci tensor condition to the integral Ricci tensor bounds. Petersen and Sprouse [21] applied Petersen-Wei's comparison results to get a rough diameter bound. Aubry [2] improved this diameter bound by using Petersen–Wei's comparison results to star-shaped domains. For more related results, see [3,5–7,9,11,19,24,26,35] and references therein.

In 2000s, Myers' theorem was also generalized by Wei and Wylie [30] to the smooth metric measure space. Recall that an n-dimensional smooth metric measure space, denoted by $(M, g, e^{-f}dv)$, is a complete n-dimensional Riemannian manifold (M, g) coupled with a weighted volume $e^{-f}dv$ for some $f \in C^\infty(M)$, where dv is the usual Riemannian volume element on (M, g). The associated Bakry–Émery Ricci tensor, introduced by Bakry and Émery, is defined as

$$\mathrm{Ric}_f := \mathrm{Ric} + \mathrm{Hess}\, f,$$

where $\mathrm{Hess}\, f$ is the Hessian of f. If $\mathrm{Ric}_f = \rho g$ for some $\rho \in \mathbb{R}$, then $(M, g, e^{-f}dv)$ is a gradient Ricci soliton, which generalizes an Einstein manifold and plays a fundamental role in the Ricci flow [20]. Wei and Wylie [30] proved that if

$$\mathrm{Ric}_f \geq (n-1)H > 0 \quad \text{and} \quad |f| \leq k$$

for some constant $k \geq 0$, then M is compact and its diameter has an explicit upper bound depending only on n, H and k. This upper bound was improved by Limoncu [14] for large k and furthermore sharpened by Tadano [27] for any k. Fernández-López and García-Río [8] proved that if

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_V g \geq (n-1)H > 0 \quad \text{and} \quad |V| \leq a$$

for some constant $a \geq 0$, where $\mathcal{L}_V$ is the Lie derivative in the direction of smooth vector field V on M, then M is compact. Under the same conditions, Limoncu [13] gave an explicit upper bound to the diameter of M. Later, the upper bound was improved by Tadano [28] and then sharpened by the second author [32] to be

$$\mathrm{diam}(M) \leq \frac{2a}{(n-1)H} + \frac{\pi}{\sqrt{H}}. \tag{1}$$

Besides, there have been more Myers' type theorems involving the Bakry–Émery Ricci tensor; see [12,15–17,25,36] and references therein for details.

Recently, the second author [33] extended Wei–Wylie's comparison results of pointwise Bakry–Émery Ricci tensor [30] condition to the integral Bakry–Émery Ricci tensor bounds, and also extended the integral Ricci tensor case [2,22]. Wang and Wei [29] applied comparison results [33] to get local Sobolev constant estimates and gradient estimates under the integral Bakry–Émery Ricci tensor bounds.

From the above works, it is natural to ask.

Question. *For a complete smooth metric measure space* $(M, g, e^{-f}dv_g)$*, does there exist a Myers' type theorem under the integral Bakry–Émery Ricci tensor bounds?*

In order to study this question, inspired by Aubry's work [2], we see that the weighted mean curvature, and the weighted area (volume) comparison theorems will play an important role in the argument. However, the authors recently found that there exist some errors in comparison Theorems 1.1 and 3.1 of [33] for the case $\frac{\pi}{2\sqrt{H}} < r < \frac{\pi}{\sqrt{H}}$, $H > 0$. In this paper, we first correct these errors by adding an additional condition on f to ensure that comparison Theorems 1.1 and 3.1 of [33] remain true for the case $\frac{\pi}{2\sqrt{H}} < r < \frac{\pi}{\sqrt{H}}$, $H > 0$. Meanwhile we correct Theorem 1.6 of [33] correspondingly. These detailed description and further development will be given in Sect. 2. Then, we apply the corrected comparison results to obtain a new Myers' type theorem under the integral Bakry–Émery Ricci tensor bounds, which improves Theorem 1.6 in [33]. Our result, in some special cases, obviously includes Myers' and Aubry's diameter estimates.

To state the results, we fix some notations. Fix $H \in \mathbb{R}$, and consider at each point x of $(M, g, e^{-f}dv)$ with the smallest eigenvalue $\lambda(x)$ of $\mathrm{Ric}_f : T_xM \to T_xM$. We define the amount of Ric_f lying below $(n-1)H$

$$\mathrm{Ric}^H_{f\ -} := ((n-1)H - \lambda(x))_+ = \max\{0, (n-1)H - \lambda(x)\}.$$

For a constant $a \geq 0$ and a geodesic ball $B(x, r) \subset M$ with radius $r > 0$, center at $x \in M$, a weighted L^p norm of function ϕ on $(M, g, e^{-f}dv)$ is defined as

$$\|\phi\|_{p,f,a}(r) := \sup_{x\in M} \left(\int_{B(x,r)} |\phi|^p \mathcal{A}_f e^{-at}\, dt d\theta_{n-1} \right)^{\frac{1}{p}},$$

where $\mathcal{A}_f = \mathcal{A}_f(t, \theta) = e^{-f}\mathcal{A}(t, \theta)$ is the volume element of weighted form

$$e^{-f}dv = \mathcal{A}_f(t, \theta) dt \wedge d\theta_{n-1}$$

in polar coordinate, $\mathcal{A}(t, \theta)$ is the standard volume element of the metric g and $d\theta_{n-1}$ is the volume element on the unit sphere S^{n-1}. Clearly, $\|\mathrm{Ric}^H_{f\ -}\|_{p,f,a}(r) = 0$ if and only if $\mathrm{Ric}_f \geq (n-1)H$. The following normalized norm of ϕ is useful in this paper,

$$\overline{\|\phi\|}_{p,f,a}(r) := \sup_{x\in M} \left(\frac{1}{V_f(x, r)} \int_{B(x,r)} |\phi|^p \mathcal{A}_f e^{-at} dt d\theta_{n-1} \right)^{\frac{1}{p}},$$

where $V_f(x, r) := \int_{B(x,r)} e^{-f}dv$. If $r \to \infty$ above and the limit exists, then we have a global curvature quantity on M

$$\overline{\|\phi\|}_{p,f,a}(M) := \lim_{r\to\infty} \overline{\|\phi\|}_{p,f,a}(r).$$

Now we can apply weighted volume comparison estimates under the integral Bakry–Émery Ricci tensor bounds (see Sect. 2) to give a new Myers' type theorem under the integral Bakry–Émery Ricci tensor bounds.

Theorem 1.1. *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete smooth metric measure space. Given $p > n/2$ and $a \geq 0$, there exists an $0 < \epsilon(n,p,a) < 1$ such that if*

$$\overline{\|\mathrm{Ric}_{f\,-}^{1}\|}_{p,f,a}(M) \leq \epsilon(n,p,a)$$

and

$$\partial_r f = -a \quad \text{or} \quad \partial_r f \geq -a - 2(n-1)\cot\left(\pi - \overline{\|\mathrm{Ric}_{f\,-}^{1}\|}_{p,f,a}^{\frac{p}{2p-1}}(M)\right)$$

along all minimal geodesic segments from any $x \in M$, then M is compact with finite fundamental group $\pi_1(M)$ and

$$\mathrm{diam}(M) \leq \pi\left(1 + c(n,p,a)\overline{\|\mathrm{Ric}_{f\,-}^{1}\|}_{p,f,a}^{b}(M)\right).$$

for some positive constants $c(n,p,a)$ and $b = b(n,p,a)$.

We give some remarks about Theorem 1.1.

Remark 1.2. For any constant $H > 0$, a renormalization argument readily shows that we can replace $\|\mathrm{Ric}_{f\,-}^{1}\|_{p,f,a}$ and $\cot(\pi - \overline{\|\mathrm{Ric}_{f\,-}^{1}\|}_{p,f,a}^{\frac{p}{2p-1}}(M))$ by $\|\mathrm{Ric}_{f\,-}^{H}\|_{p,f,a}$ and $\sqrt{H}\cot(\pi - \overline{\|\mathrm{Ric}_{f\,-}^{H}\|}_{p,f,a}^{\frac{p}{2p-1}}(M))$, respectively in Theorem 1.1 provided $c(n,p,a)$ and π are replaced by $c(n,p,a,H)$ and $\pi/\sqrt{H}$, respectively.

Remark 1.3. When f is constant and $a = 0$, Theorem 1.1 returns to the Aubry's result [2]. If we further assume $\overline{\|\mathrm{Ric}_{f\,-}^{1}\|}_{p,f,a}(M) \equiv 0$, i.e., $\mathrm{Ric} \geq n-1$, then Theorem 1.1 recovers the classical Myers' theorem [18].

Remark 1.4. In view of the diameter estimate (1), it remains an interesting question if there exists a Myers's type theorem for a bound of integral Bakry–Émery Ricci tensor under only $\partial_r f \geq -a$ for some constant $a \geq 0$, along all minimal geodesic segments r from $x \in M$.

Remark 1.5. We also improve the result of Theorem 1 in [34]. In [34], the number ϵ depends on R and $\epsilon \to 0$ as $R \to \infty$, and hence it essentially applies to compact smooth metric measure spaces; while the small number ϵ in Theorem 1.1 is independent of R.

The rest of paper is organized as follows. In Sect. 2, we give some notations and recall some integral comparison theorems [33] on smooth metric measure spaces. In particular, we give some minor corrections of Theorems 1.1 and 3.1 of [33] for $\frac{\pi}{2\sqrt{H}} < r \leq R < \frac{\pi}{\sqrt{H}}$ when $H > 0$; see Theorems 2.1 and

2.2 below. In Sect. 3, we apply corrected comparison theorems of Sect. 2 to give some weighted measures for star-shaped domains and geodesic balls under integral Bakry–Émery Ricci tensor bounds. In Sect. 4, following the Aubry's argument in [2], we first give a key local diameter estimate. Then we apply the local diameter estimate to prove Theorem 1.1. In Appendix, we give another Myers' theorem under only the integral of m-Bakry–Émery Ricci tensor bounds.

2. Preliminary

In this section, we recall weighted area and volume comparisons [33] under the integral Bakry–Émery Ricci tensor bounds. These results can be regarded as the generalizations of the integral Ricci tensor [2,22]. In particular, we give some minor corrections of Theorems 1.1 and 3.1 of [33] for $\frac{\pi}{2\sqrt{H}} < r \leq R < \frac{\pi}{\sqrt{H}}$ when $H > 0$. These results will be used in the proof of our main result.

The weighted volume of the geodesic sphere $S(x,r) = \{y \in M \,|\, d(x,y) = r\}$ is defined as

$$A_f(x,r) = \int_{S^{n-1}} \mathcal{A}_f(r,\theta)d\theta_{n-1}.$$

For a constant $a \geq 0$, the model space (M_H, g_H), the n-dimensional simply connected space with constant sectional curvature H, can be modified to the pointed weighted model space $M_{H,a} := (M_H, g_H, e^{-h}dv_{g_H}, O)$, where $O \in M_H$ is a base point and $h(x) = -a \cdot d(x,O)$. Let $\mathcal{A}_H^a(r) := e^{ar}\mathcal{A}_H(r)$ be the h-volume element in (M_H, g_H), where $\mathcal{A}_H$ is the volume element in (M_H, g_H). We let

$$A_H^a(r) = \int_{S^{n-1}} \mathcal{A}_H^a(r,\theta)d\theta_{n-1}$$

be the weighted volume of the geodesic sphere in the weighted model space $M_{H,a}$. It is easy to see that $A_H^a(r) = e^{ar}A_H(r)$, where $A_H(r)$ is the usual volume of the geodesic sphere in (M_H, g_H). Moreover, the weighted volume of ball $B(x,r) \subset M$ and the h-volume of ball $B(O,r) \subset M_H$ are defined respectively by

$$V_f(x,r) = \int_0^r A_f(x,t)dt \quad \text{and} \quad V_H^a(r) = \int_0^r A_H^a(t)dt.$$

Obviously, $V_H(r) \leq V_H^a(r) \leq e^{ar}V_H(r)$. When f is constant (and $a = 0$), all above notations recover the usual integral quantities on manifolds [22,23].

In $(M, g, e^{-f}dv)$, we assume that $\partial_r f \geq -a$ for some constant $a \geq 0$, along a minimal geodesic segment from $x \in M$. As in [33], we consider the error function

$$\varphi := (m_f - m_H - a)_+,$$

where $m_f = m - \partial_r f$, m is the mean curvature of the geodesic sphere in the outer normal direction and m_H is the mean curvature of the geodesic sphere in (M_H, g_H).

The correct statement of comparison Theorem 1.1 in [33] is the following, also see [34].

Theorem 2.1. *Let $(M, g, e^{-f}dv)$ be an n-dimensional smooth metric measure space. Assume that*

$$\partial_r f \geq -a$$

for some constant $a \geq 0$, along a minimal geodesic segment from $x \in M$. For any $p > n/2$, $H \in \mathbb{R}$ (assume $r \leq \frac{\pi}{2\sqrt{H}}$ when $H > 0$),

$$\|\varphi\|_{2p,f,a}(r) \leq \left[\frac{(n-1)(2p-1)}{(2p-n)} \left\|\mathrm{Ric}_{f\ -}^{H}\right\|_{p,f,a}(r)\right]^{\frac{1}{2}}$$

and

$$\varphi^{2p-1}\mathcal{A}_f\, e^{-ar} \leq (2p-1)^p \left(\frac{n-1}{2p-n}\right)^{p-1} \int_0^r (\mathrm{Ric}_{f\ -}^{H})^p \mathcal{A}_f e^{-at} dt$$

along that minimal geodesic segment from x.

Moreover, if $\frac{\pi}{2\sqrt{H}} < r < \frac{\pi}{\sqrt{H}}$ when $H > 0$ and f further satisfies

$$\partial_r f = -a \quad \text{or} \quad \partial_r f \geq -a - 2(n-1)\sqrt{H}\cot(\sqrt{H}r), \tag{2}$$

then we have

$$\left\|\sin^{\frac{4p-n-1}{2p}}(\sqrt{H}t)\cdot\varphi\right\|_{2p,f,a}(r) \leq \left[\frac{(n-1)(2p-1)}{(2p-n)} \left\|\mathrm{Ric}_{f\ -}^{H}\right\|_{p,f,a}(r)\right]^{\frac{1}{2}}$$

and

$$\sin^{4p-n-1}(\sqrt{H}r)\varphi^{2p-1}\mathcal{A}_f e^{-ar} \leq (2p-1)^p \left(\frac{n-1}{2p-n}\right)^{p-1} \int_0^r (\mathrm{Ric}_{f\ -}^{H})^p \mathcal{A}_f e^{-at} dt$$

along that minimal geodesic segment from x.

Proof of Theorem 2.1. Compared with Theorem 1.1 of [33], we add an additional condition (2) for the case $\frac{\pi}{2\sqrt{H}} < r < \frac{\pi}{\sqrt{H}}$, $H > 0$, to make corrections. Indeed, in the proof of this case (see p. 837 in [33]), the second author made a mistake from

$$\varphi' + \frac{1}{n-1}\Big[(\varphi + a + \partial_r f)(\varphi + 2m_H + a + \partial_r f)\Big] \leq \mathrm{Ric}_{f\ -}^{H}$$

to

$$\varphi' + \frac{\varphi^2}{n-1} + \frac{2m_H\varphi}{n-1} \leq \mathrm{Ric}_{f\ -}^{H}, \tag{3}$$

because the second author overlooked a fact that m_H is negative at this case. To correct this error, we add an additional condition (2) to ensure (3) still holds. Note that in the course of discussion, we have used $m_H =$

$(n-1)\sqrt{H}\cot(\sqrt{H}r)$. Then the rest of proof remains correct without any modification. □

The correction impacts comparison Theorem 3.1 of [33] for the case $\frac{\pi}{2\sqrt{H}} < r \leq R < \frac{\pi}{\sqrt{H}}$, $H > 0$. The corresponding modification in accordance with the correction is that.

Theorem 2.2. *Let $(M, g, e^{-f}dv)$ be an n-dimensional smooth metric measure space. Assume that*

$$\partial_r f \geq -a$$

for some constant $a \geq 0$, along all minimal geodesic segments from $x \in M$. Let $H \in \mathbb{R}$ and $p > n/2$ be given, and when $H > 0$ assume that $R \leq \frac{\pi}{2\sqrt{H}}$. For $0 < r \leq R$, we have

$$\left(\frac{A_f(x,R)}{A_H^a(R)}\right)^{\frac{1}{2p-1}} - \left(\frac{A_f(x,r)}{A_H^a(r)}\right)^{\frac{1}{2p-1}} \leq C(n,p,H,R)\left(\|\mathrm{Ric}_{f\,-}^H\|_{p,f,a}(R)\right)^{\frac{p}{2p-1}},$$

where

$$C(n,p,H,R) := \left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}} \int_0^R A_H(t)^{-\frac{1}{2p-1}}\,dt.$$

Moreover, if $\frac{\pi}{2\sqrt{H}} < r \leq R < \frac{\pi}{\sqrt{H}}$ when $H > 0$ and f further satisfies (2), then we have

$$\left(\frac{A_f(x,R)}{A_H^a(R)}\right)^{\frac{1}{2p-1}} - \left(\frac{A_f(x,r)}{A_H^a(r)}\right)^{\frac{1}{2p-1}} \leq \left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}} \left(\|\mathrm{Ric}_{f\,-}^H\|_{p,f,a}(R)\right)^{\frac{p}{2p-1}} \int_r^R \frac{(\sqrt{H})^{\frac{n-1}{2p-1}}}{\sin^2(\sqrt{H}t)}\,dt. \tag{4}$$

Moreover, from Theorem 1.3 of [33], we have a weighted volume comparison estimate under the integral Bakry–Émery Ricci tensor bounds.

Theorem 2.3. *Let $(M, g, e^{-f}dv)$ be an n-dimensional smooth metric measure space. Assume that*

$$\partial_r f \geq -a$$

for some constant $a \geq 0$, along all minimal geodesic segments from $x \in M$. Let $H \in \mathbb{R}$ and $p > n/2$. For $0 < r \leq R$ (assume $R \leq \frac{\pi}{2\sqrt{H}}$ when $H > 0$),

$$\left(\frac{V_f(x,R)}{V_H^a(R)}\right)^{\frac{1}{2p-1}} - \left(\frac{V_f(x,r)}{V_H^a(r)}\right)^{\frac{1}{2p-1}} \leq C(n,p,H,a,R)\left(\|\mathrm{Ric}_{f\,-}^H\|_{p,f,a}^p(R)\right)^{\frac{1}{2p-1}}, \tag{5}$$

where

$$C(n,p,H,a,R) := \left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}} \int_0^R A_H(t)\left(\frac{t\,e^{at}}{V_H^a(t)}\right)^{\frac{2p}{2p-1}}\,dt.$$

Proof of Theorem 2.3. Compared with Theorem 3.1 of [33], the only difference is that we add an additional condition (2) for the case $\frac{\pi}{2\sqrt{H}} < r \leq R < \frac{\pi}{\sqrt{H}}$, $H > 0$, to make corrections. The proof is the same as in [33] except we use the present Theorem 2.1 instead of the previous Theorem 1.1 of [33]. □

3. Bounds of Weighted Volume of Geodesic Balls

In (M, g), a subset $T \subset M$ is called to be a star-shaped set at x, if for any $y \in T$, there exists a minimal geodesic connecting x to y contained in T. Clearly the geodesic ball $B(x, r)$ is a star-shaped set at x. By integrating only along the direction lies in the start-shaped set at x, we can prove the same comparison estimates as comparison Theorems 2.1 and 2.2 in Sect. 2 for any star-shaped set at x, where $\mathrm{Ric}_{f\,-}^{H}$ only needs to integrate on the same star-shaped set. This will be useful in the following discussion.

In this section we will apply Theorem 2.2 to prove several weighted measures for star-shaped domains and geodesic balls. When f is constant and $a = 0$, these results recover the Aubry's case [2].

First, we will give a weighted area estimate of a star-shaped domain T when the Bakry–Émery Ricci tensor concentrates sufficiently above $n-1$ on T, which is the first step of the proof of main theorem.

Lemma 3.1. *Let $p > n/2$. Assume that $(M, g, e^{-f}dv)$ contains a subset T, star-shaped at a point x, which satisfies*

$$\epsilon = R_T^2 \overline{\|\mathrm{Ric}_{f\,-}^{1}\|}_{p,f,a}(T) \leq \left(\frac{\pi}{6}\right)^{2-\frac{1}{p}}$$

and

$$\partial_r f = -a \quad \text{or} \quad \partial_r f \geq -a - 2(n-1)\cot\left(\pi - \epsilon^{\frac{p}{2p-1}}\right)$$

along all minimal geodesic segments from $x \in M$ for some constant $a \geq 0$, where R_T is constant such that $T \subset B(x, R_T)$. There exists an explicit constant $C(n, p)$ such that for all radius $R_T \geq r \geq \pi$, we have

$$A_{f,T}(x, r) \leq C(n, p)\frac{e^{ar}}{r}\epsilon^{\frac{p(n-1)}{2p-1}} V_f(T), \tag{6}$$

where $A_{f,T}(x, r)$ is the weighted volume of $S(x, r) \bigcap T$.

Proof of Lemma 3.1. For $R_T \geq r \geq \pi$, we choose the model space with constant sectional curvature $H_r = (\frac{\pi-\epsilon'}{r})^2 < 1$, where $\epsilon' = \epsilon^{\frac{p}{2p-1}}$. For $t \in [\frac{\pi}{2(\pi-\epsilon')}r, r]$, since cotangent function is monotonic decreasing, then

$$\begin{aligned} \partial_t f + a = 0 \quad \text{or} \quad \partial_t f + a &\geq -2(n-1)\cot\left(\pi - \epsilon'\right) \\ &\geq -2(n-1)\cot\left(\frac{\pi-\epsilon'}{r}t\right). \end{aligned}$$

In each case of the above, it implies

$$(\partial_t f + a)^2 \geq -2(n-1)(\partial_t f + a)\cot(\sqrt{H_r}t)$$

for $t \in [\frac{\pi}{2(\pi-\epsilon')}r, r]$. Then by the weighted area comparison estimate (4) of Theorem 2.2 for the star-shaped set, we have

$$\begin{aligned}&\left(\frac{A_{f,T}(x,r)}{e^{ar}\sin^{n-1}(\sqrt{H_r}r)}\right)^{\frac{1}{2p-1}} - \left(\frac{A_{f,T}(x,t)}{e^{at}\sin^{n-1}(\sqrt{H_r}t)}\right)^{\frac{1}{2p-1}}\\ &\quad\leq \left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}} \left(\|\mathrm{Ric}^1_{f\,-}\|_{p,f,a}(T)\right)^{\frac{p}{2p-1}} \int_t^r \frac{ds}{\sin^2(\sqrt{H_r}s)}.\end{aligned}$$

By concavity of the sine function on $\sqrt{H_r}s \in [\frac{\pi}{2}, \pi]$, we have

$$\int_t^r \frac{ds}{\sin^2(\sqrt{H_r}s)} \leq \frac{\pi^2(r-t)}{4(\pi-\sqrt{H_r}t)(\pi-\sqrt{H_r}r)} \leq \frac{\pi^2(r-t)}{4(\pi-\sqrt{H_r}t)\epsilon'} \leq \frac{\pi r}{4\epsilon'},$$

where we used a fact that $\frac{r-t}{\pi-\sqrt{H_r}t}$ is decreasing in t. Thus, by the theorem's assumption, we have

$$\begin{aligned}&\left(\frac{A_{f,T}(x,r)}{e^{ar}\sin^{n-1}(\sqrt{H_r}r)}\right)^{\frac{1}{2p-1}} - \left(\frac{A_{f,T}(x,t)}{e^{at}\sin^{n-1}(\sqrt{H_r}t)}\right)^{\frac{1}{2p-1}}\\ &\quad\leq \left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}} \frac{\pi}{4}\left(\frac{V_f(T)}{r}\right)^{\frac{1}{2p-1}}.\end{aligned}$$

Since

$$\sin\left(\sqrt{H_r}r\right) = \sin(\pi-\epsilon') = \sin(\epsilon') \leq \epsilon'$$

and

$$\sin\left(\sqrt{H_r}t\right) = \sin\left(\frac{t}{r}(\pi-\epsilon')\right),$$

then for all $t \in [\frac{\pi r}{2(\pi-\epsilon')}, r]$, we further obtain

$$\begin{aligned}A_{f,T}(x,r)^{\frac{1}{2p-1}} &\leq A_{f,T}(x,t)^{\frac{1}{2p-1}}\left(\frac{e^{ar}}{e^{at}}\right)^{\frac{1}{2p-1}}\left(\frac{\epsilon'}{\sin[(\pi-\epsilon')\frac{t}{r}]}\right)^{\frac{n-1}{2p-1}}\\ &\quad+\frac{\pi}{4}\left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}}\left(\frac{V_f(T)e^{ar}}{r}\right)^{\frac{1}{2p-1}}\epsilon'^{\frac{n-1}{2p-1}}.\end{aligned} \tag{7}$$

For $t \in [\frac{\pi r}{2(\pi-\epsilon')}, \frac{5\pi r}{6(\pi-\epsilon')}]$, we have

$$\sin\left((\pi-\epsilon')\frac{t}{r}\right) \geq \sin\frac{\pi}{6} = \frac{1}{2}.$$

When $\epsilon' \leq \frac{\pi}{6}$, we have

$$\frac{5\pi}{6(\pi-\epsilon')}r \leq r.$$

Combining these estimates and applying the inequality

$$(a+b)^{2p-1} \leq 2^{2p-2}(a^{2p-1}+b^{2p-1})$$

for any $a, b \geq 0$ to the above inequality (7), we get

$$\begin{aligned} A_{f,T}(x,r) &\leq 2^{2p+n-3}\epsilon'^{(n-1)}\frac{e^{ar}}{e^{at}}A_{f,T}(x,t) \\ &\quad + \left(\frac{n-1}{(2p-1)(2p-n)}\right)^{p-1}\frac{\pi^{2p-1}e^{ar}}{4^p\, r}\epsilon'^{(n-1)}V_f(T) \end{aligned}$$

for all $t \in [\frac{\pi\, r}{2(\pi-\epsilon')}, \frac{5\pi\, r}{6(\pi-\epsilon')}]$.

By the mean value theorem, there exists $t_0 \in [\frac{\pi\, r}{2(\pi-\epsilon')}, \frac{5\pi\, r}{6(\pi-\epsilon')}]$ such that

$$\begin{aligned} A_{f,T}(x,t_0) &= \frac{3(\pi-\epsilon')}{\pi r}\int_{\frac{\pi r}{2(\pi-\epsilon')}}^{\frac{5\pi r}{6(\pi-\epsilon')}} A_{f,T}(x,t)dt \\ &\leq \frac{3}{r}\int_0^{R_T} A_{f,T}(x,t)dt \\ &= \frac{3}{r}V_f(T). \end{aligned}$$

Hence,

$$A_{f,T}(x,r) \leq \left[3\cdot 2^{2p+n-3} + \frac{\pi^{2p-1}}{4^p}\left(\frac{n-1}{(2p-1)(2p-n)}\right)^{p-1}\right]\frac{e^{ar}}{r}\epsilon^{\frac{p(n-1)}{2p-1}}V_f(T),$$

which completes the estimate. □

Then we will apply Theorem 2.3 to give some estimates for the weighted relative volume of geodesic balls, which is the second step of the proof of main theorem.

Lemma 3.2. *Let $(M, g, e^{-f}dv)$ be an n-dimensional smooth metric measure space. Assume that*

$$\partial_r f \geq -a$$

for some constant $a \geq 0$ along all minimal geodesic segments from $x \in M$. For any $R_T > 0$, there exist (computable) constants $C(n,p,aR_T) > 0$ and $B(n,p,aR_T)$ such that when M contains a star-shaped subset $T \subset B(x, R_T)$ which satisfies

$$\epsilon_0 = R_T^2\overline{\|\mathrm{Ric}^0_{f\_}\|}_{p,f,a}(T) \leq B(n,p,aR_T),$$

then we have

(i) for $0 < r \leq R \leq R_T$,

$$\frac{V_{f,T}(x,r)}{V_{f,T}(x,R)} \geq \left(1 - C(n,p,aR_T)\epsilon_0^{\frac{p}{2p-1}}\right)^{2p-1}\frac{r^n}{e^{aR}R^n},$$

where $V_{f,T}(x,r)$ *is the weighted volume of* $B_T(x,r)\bigcap T$*, and*

$$C(n,p,aR_T) := \frac{2n(2p-1)}{2p-n}\left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}}(e^{aR_T})^{\frac{2p+2}{2p-1}}.$$

(ii) if $T=B(x,R_0), y\in T$ *and* $r\geq 0$ *satisfy* $d(x,y)+r\leq R_0$*, then*

$$\left(\frac{V_{f,T}(y,r)}{V_{f,T}(x,R_0)}\right)^{\frac{1}{2p-1}} \geq \left(\frac{r^n}{e^{aR_0}R_0^n}\right)^{\frac{1}{2p'-1}} \times\left[\left(\frac{2}{3}-C(n,p,aR_0)\epsilon_0^{\frac{p'}{2p'-1}}\right)\left(\frac{r}{R_0}\right)^{\frac{2n+b(n,p,aR_0)}{2p'-1}} - D(n,p,aR_0)\epsilon_0^{\frac{p'}{2p'-1}}\right],$$

where $p':=\max\{n,p\}$*, constant* $b(n,p,aR_0)>\frac{(2p-n)aR_0}{(2p-1)^2\log 3/2}$ *and*

$$D(n,p,aR_0) := \frac{\frac{(2p-1)n}{2p-n}\left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}}(e^{aR_0})^{\frac{2p+2}{2p-1}}}{1-(3/2)^{\frac{2p}{2p-n}}(e^{aR_0})^{-\frac{1}{2p-1}}}.$$

Proof of Lemma 3.2. For any $t\leq r\leq R\leq R_T$, by the weighted volume comparison (5) for a star-sharped set, we have

$$\left(\frac{V_{f,T}(x,R)}{V_0^a(R)}\right)^{\frac{1}{2p-1}} - \left(\frac{V_{f,T}(x,r)}{V_0^a(r)}\right)^{\frac{1}{2p-1}} \leq \left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}}\left(\|\mathrm{Ric}_{f\,-}^0\|_{p,f,a}^p(R)\right)^{\frac{1}{2p-1}}\int_0^R A_0(t)\left(\frac{t\,e^{at}}{V_0^a(t)}\right)^{\frac{2p}{2p-1}}dt$$

Using the facts $V_0(r)\leq V_0^a(r)\leq e^{ar}V_0(r)$, $A_0(t)=nt^{n-1}\omega_n$ and $V_0(t)=t^n\omega_n$, we have

$$\begin{aligned}\int_0^R A_0(t)\left(\frac{t\,e^{at}}{V_0^a(t)}\right)^{\frac{2p}{2p-1}}dt &\leq \int_0^R A_0(t)\left(\frac{t\,e^{at}}{V_0(t)}\right)^{\frac{2p}{2p-1}}dt\\ &= \int_0^R (nt^{n-1}\omega_n)\left(\frac{t\,e^{at}}{t^n\omega_n}\right)^{\frac{2p}{2p-1}}dt\\ &\leq \frac{(2p-1)n}{2p-n}(e^{aR})^{\frac{2p}{2p-1}}(\omega_n)^{-\frac{1}{2p-1}}R^{\frac{2p-n}{2p-1}}\end{aligned}$$

and

$$\begin{aligned}\left(\frac{V_{f,T}(x,R)}{V_0^a(R)}\right)^{\frac{1}{2p-1}} - \left(\frac{V_{f,T}(x,r)}{V_0^a(r)}\right)^{\frac{1}{2p-1}} &\geq \left(\frac{V_{f,T}(x,R)}{e^{aR}V_0(R)}\right)^{\frac{1}{2p-1}} - \left(\frac{V_{f,T}(x,r)}{V_0(r)}\right)^{\frac{1}{2p-1}}\\ &= \left(\frac{V_{f,T}(x,R)}{e^{aR}\omega_n R^n}\right)^{\frac{1}{2p-1}} - \left(\frac{V_{f,T}(x,r)}{\omega_n r^n}\right)^{\frac{1}{2p-1}},\end{aligned}$$

where ω_n is the volume of the Euclidean unit n-ball. Hence, we have

$$\begin{aligned}&\left(\frac{V_{f,T}(x,R)}{e^{aR}R^n}\right)^{\frac{1}{2p-1}}-\left(\frac{V_{f,T}(x,r)}{r^n}\right)^{\frac{1}{2p-1}}\\&\quad\leq D(n,p)(e^{aR})^{\frac{2p}{2p-1}}R^{\frac{2p-n}{2p-1}}\left(\|\mathrm{Ric}^0_{f\_}\|^p_{p,f,a}(R)\right)^{\frac{1}{2p-1}},\end{aligned}\tag{8}$$

where

$$D(n,p):=\frac{(2p-1)n}{2p-n}\left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}}.$$

By letting $R=R_T$ and $r=R$ in the above inequality and using the definition of ϵ_0, we deduce that

$$\left(\frac{V_{f,T}(x,R)}{V_{f,T}(T)}\right)^{\frac{1}{2p-1}}\geq\left[1-D(n,p)(e^{aR_T})^{\frac{2p+1}{2p-1}}\epsilon_0^{\frac{p}{2p-1}}\right]\left(\frac{R^n}{e^{aR_T}R_T^n}\right)^{\frac{1}{2p-1}}$$

for any $R\leq R_T$. If we further let

$$\epsilon_0^{\frac{p}{2p-1}}\leq\frac{1}{2D(n,p)}(e^{aR_T})^{-\frac{2p+1}{2p-1}},$$

then

$$\left(\frac{V_{f,T}(x,R)}{V_{f,T}(T)}\right)^{\frac{1}{2p-1}}\geq\frac{1}{2}\left(\frac{R^n}{e^{aR_T}R_T^n}\right)^{\frac{1}{2p-1}}\tag{9}$$

for any $R\leq R_T$.

On the other hand, (8) also implies

$$\begin{aligned}\left(\frac{V_{f,T}(x,r)}{V_{f,T}(x,R)}\right)^{\frac{1}{2p-1}}&\geq\left(\frac{r^n}{e^{aR}R^n}\right)^{\frac{1}{2p-1}}\\&\quad\times\left[1-D(n,p)R^{\frac{2p}{2p-1}}(e^{aR})^{\frac{2p+1}{2p-1}}\left(\frac{\|\mathrm{Ric}^0_{f\_}\|^p_{p,f,a}(R)}{V_{f,T}(x,R)}\right)^{\frac{1}{2p-1}}\right].\end{aligned}$$

Using a easy fact $\|\mathrm{Ric}^0_{f\_}\|_{p,f,a}(R)\leq\|\mathrm{Ric}^0_{f\_}\|_{p,f,a}(R_T)$ for any $R\leq R_T$, we have

$$\begin{aligned}&R^{\frac{2p}{2p-1}}(e^{aR})^{\frac{2p+1}{2p-1}}\left(\frac{\|\mathrm{Ric}^0_{f\_}\|^p_{p,f,a}(R)}{V_{f,T}(x,R)}\right)^{\frac{1}{2p-1}}\\&\quad\leq\left(\frac{R}{R_T}\right)^{\frac{2p}{2p-1}}(e^{aR})^{\frac{2p+1}{2p-1}}R_T^{\frac{2p}{2p-1}}\left(\frac{\|\mathrm{Ric}^0_{f\_}\|^p_{p,f,a}(R_T)}{V_{f,T}(x,R_T)}\right)^{\frac{1}{2p-1}}\left(\frac{V_{f,T}(x,R_T)}{V_{f,T}(x,R)}\right)^{\frac{1}{2p-1}}.\end{aligned}$$

According to the definition of ϵ_0 and (9) and using $R \leq R_T$, the above inequality can be further simplified as

$$\left(\frac{V_{f,T}(x,r)}{V_{f,T}(x,R)}\right)^{\frac{1}{2p-1}} \geq \left(\frac{r^n}{e^{aR}R^n}\right)^{\frac{1}{2p-1}} \left[1 - 2D(n,p)(e^{aR})^{\frac{2p+1}{2p-1}}\epsilon_0^{\frac{p}{2p-1}}(e^{aR_T})^{\frac{1}{2p-1}}\right]$$
$$\geq \left(\frac{r^n}{e^{aR}R^n}\right)^{\frac{1}{2p-1}} \left[1 - 2D(n,p)\epsilon_0^{\frac{p}{2p-1}}(e^{aR_T})^{\frac{2p+2}{2p-1}}\right],$$

which implies (i) of the theorem.

Next, adapting the argument of Aubry [2], we will apply the iteration trick to get the comparison for non-concentric balls. Let $z \in B(x,R_0)$, r and R such that $0 < r \leq R \leq R_0 - d(x,z)$. Since $B(z,R) \subset B(x,R_0)$, dividing by $V_{f,T}(x,R_0)^{1/(2p-1)}$ in (8), we get

$$\left(\frac{V_{f,T}(z,R)}{V_{f,T}(x,R_0)}\right)^{\frac{1}{2p-1}} \leq D(n,p)(e^{aR})^{\frac{2p}{2p-1}}R^{\frac{2p-n}{2p-1}}\left(\|\mathrm{Ric}^0_{f\,\_}\|^p_{p,f,a}(R)\right)^{\frac{1}{2p-1}}$$
$$\times \left(\frac{e^{aR}R^n}{V_{f,T}(x,R_0)}\right)^{\frac{1}{2p-1}} + \left(\frac{V_{f,T}(z,r)}{r^n}\right)^{\frac{1}{2p-1}}$$
$$\times \left(\frac{e^{aR}R^n}{V_{f,T}(x,R_0)}\right)^{\frac{1}{2p-1}}.$$

Again using $\|\mathrm{Ric}^0_{f\,\_}\|_{p,f,a}(R) \leq \|\mathrm{Ric}^0_{f\,\_}\|_{p,f,a}(R_T)$ and the definition of ϵ_0, we have

$$\left(\frac{V_{f,T}(z,R)}{V_{f,T}(x,R_0)}\right)^{\frac{1}{2p-1}} \leq D(n,p)(e^{aR})^{\frac{2p+1}{2p-1}}\left(\frac{R^2\epsilon_0}{R_0^2}\right)^{\frac{p}{2p-1}}$$
$$+ \left(\frac{e^{aR}R^n}{r^n}\right)^{\frac{1}{2p-1}}\left(\frac{V_{f,T}(z,r)}{V_{f,T}(x,R_0)}\right)^{\frac{1}{2p-1}}. \tag{10}$$

To iterate this estimate with a sequence of balls of increasing size, we will construct a sequence of increasing balls centered on a minimizing geodesic $B_i = B(y_i,R_i)$ such that $B_1 = B(y,r)$, B_k is concentric to $B(x,R_0)$, and B_i contains a ball centered at y_{i+1} and of radius r_{i+1} close to R_i. Let $\gamma : [0,d(x,y)] \to M$ be a minimizing geodesic from x to y and for some $\alpha = \alpha(n,p) \in [1/2,1)$ such that

$$-\log\alpha \leq 2\log(2-\alpha),\ \ (2-\alpha)^{2p-n}\alpha^n < 1,\ \ \alpha^{\frac{n}{2p-1}} \geq \frac{2}{3},\ \ (2-\alpha)^{\frac{2p-n}{2p-1}} < \frac{3}{2}. \tag{11}$$

Denoted $\lfloor x \rfloor$ by the floor function of a real number x which gives the greatest integer less than or equal to x. We let

$$k = \left\lfloor \frac{\log(1+\frac{d(x,y)}{r})}{\log(2-\alpha)} \right\rfloor + 2, \quad \text{and} \quad y_i = \gamma(d(x,y) + r - (2-\alpha)^{i-1}r)$$

for $i \leq k-1$, $y_k = x$, $r_i = \alpha(2-\alpha)^{i-2}r$ and $R_i = (2-\alpha)^{i-1}r$. Then, we know

$$B(y_{i+1}, r_{i+1}) \subset B(y_i, R_i) \subset B(x, R_0)$$

for any $i \leq k-1$. Setting $z = y_{i+1}, R = R_{i+1}$ and $r = r_{i+1}$ in (10), by iteration we have

$$\begin{aligned}\left(\frac{V_{f,T}(y_{i+1}, R_{i+1})}{V_{f,T}(x, R_0)}\right)^{\frac{1}{2p-1}} &\leq D(n,p)(e^{aR_{i+1}})^{\frac{2p+1}{2p-1}}\left(\frac{R_{i+1}^2\epsilon_0}{R_0^2}\right)^{\frac{p}{2p-1}} \\ &\quad + (e^{aR_{i+1}})^{\frac{1}{2p-1}}\left(\frac{R_{i+1}}{r_{i+1}}\right)^{\frac{n}{2p-1}}\left(\frac{V_{f,T}(y_{i+1}, r_{i+1})}{V_{f,T}(x, R_0)}\right)^{\frac{1}{2p-1}} \\ &\leq D(n,p)(e^{aR_0})^{\frac{2p+1}{2p-1}}\left(\frac{r^2\epsilon_0}{R_0^2}\right)^{\frac{p}{2p-1}}(2-\alpha)^{\frac{2pi}{2p-1}} \\ &\quad + (e^{aR_0})^{\frac{1}{2p-1}}\left[\frac{(2-\alpha)^n V_{f,T}(y_i, R_i)}{\alpha^n V_{f,T}(x, R_0)}\right]^{\frac{1}{2p-1}}.\end{aligned}$$

For the above inequality, letting $\alpha_i = \left(\frac{V_{f,T}(y_i,R_i)}{V_{f,T}(x,R_0)}\right)^{\frac{1}{2p-1}}$, $\beta = (2-\alpha)^{\frac{2p}{2p-1}}$, $C = D(n,p)(e^{aR_0})^{\frac{2p+1}{2p-1}}\left(\frac{r^2\epsilon_0}{R_0^2}\right)^{\frac{p}{2p-1}}$ and $d = (e^{aR_0})^{\frac{1}{2p-1}}(\frac{2-\alpha}{\alpha})^{\frac{n}{2p-1}}$, then the above inequality becomes a simple form

$$a_{i+1} \leq C\beta^i + d\,a_i$$

for any $0 \leq i \leq k-1$. Therefore,

$$a_i \leq d^{i-1}\left(a_1 + \frac{C}{1-\beta/d}\right).$$

This implies that

$$\begin{aligned}\left(\frac{V_{f,T}(y_{k-1}, R_{k-1})}{V_{f,T}(x, R_0)}\right)^{\frac{1}{2p-1}} &\leq \left(\frac{2-\alpha}{\alpha}\right)^{\frac{n(k-2)}{2p-1}}(e^{aR_0})^{\frac{k-2}{2p-1}} \\ &\times \left[\left(\frac{V_{f,T}(y, r)}{V_{f,T}(x, R_0)}\right)^{\frac{1}{2p-1}} + \frac{D(n,p)(e^{aR_0})^{\frac{2p+1}{2p-1}}\left(\frac{r^2\epsilon_0}{R_0^2}\right)^{\frac{p}{2p-1}}}{1-(2-\alpha)^{\frac{2p-n}{2p-1}}\alpha^{\frac{n}{2p-1}}(e^{aR_0})^{-\frac{1}{2p-1}}}\right].\end{aligned}$$

On the other hand, by (i), we have

$$
\begin{aligned}
&\left(\frac{V_{f,T}(y_{k-1},R_{k-1})}{V_{f,T}(x,R_0)}\right)^{\frac{1}{2p-1}}\\
&\quad\geq\left(\frac{V_{f,T}(y_k,r_k)}{V_{f,T}(x,R_0)}\right)^{\frac{1}{2p-1}}\\
&\quad\geq\left(\frac{r_k^n}{e^{aR_0}R_0^n}\right)^{\frac{1}{2p-1}}\left[1-C(n,p,aR_T)\epsilon_0^{\frac{p}{2p-1}}\right]^{2p-1}\\
&\quad\geq\alpha^{\frac{n}{2p-1}}(2-\alpha)^{\frac{n(k-2)}{2p-1}}\left(\frac{r^n}{e^{aR_0}R_0^n}\right)^{\frac{1}{2p-1}}\left[1-C(n,p,aR_T)\epsilon_0^{\frac{p}{2p-1}}\right]^{2p-1}.
\end{aligned}
$$

Combining the above two estimates on $\frac{V_{f,T}(y_{k-1},R_{k-1})}{V_{f,T}(x,R_0)}$, we conclude that there exist two constants $C(n,p,aR_T)>0$ and $B(n,p,aR_T)>0$ such that when $\epsilon_0\leq B(n,p,aR_T)$,

$$
\begin{aligned}
\left(\frac{V_{f,T}(y,r)}{V_{f,T}(x,R_0)}\right)^{\frac{1}{2p-1}}\geq&\left(\frac{\alpha^n}{e^{aR_0}}\right)^{\frac{k-1}{2p-1}}\left(\frac{r}{R_0}\right)^{\frac{n}{2p-1}}\left[1-C(n,p,aR_T)\epsilon_0^{\frac{p}{2p-1}}\right]^{2p-1}\\
&-\frac{D(n,p)(e^{aR_0})^{\frac{2p+1}{2p-1}}\left(\frac{r^2\epsilon_0}{R_0^2}\right)^{\frac{p}{2p-1}}}{1-(2-\alpha)^{\frac{2p}{2p-1}}(e^{aR_0})^{-\frac{1}{2p-1}}}.
\end{aligned}
$$

By our assumption, we observe that

$$
\begin{aligned}
\left(\frac{\alpha^n}{e^{aR_0}}\right)^{\frac{k-2}{2p-1}}&\geq\left(\frac{\alpha^n}{e^{aR_0}}\right)^{\frac{\log(1+\frac{d(x,y)}{r})}{(2p-1)\log(2-\alpha)}}\\
&\geq\left(\frac{r}{r+d(x,y)}\right)^{\frac{-n\log\alpha+aR_0}{(2p-1)\log(2-\alpha)}}\\
&\geq\left(\frac{r}{R_0}\right)^{\frac{2n+b(n,p,aR_0)}{2p-1}},
\end{aligned}
$$

where $b(n,p,aR_0)$ is constant satisfying

$$
b(n,p,aR_0)=\frac{aR_0}{\log(2-\alpha)}>\frac{aR_0(2p-n)}{(2p-1)^2\log 3/2}
$$

according to (11). Using this estimate and the last inequality of (11), we finally get

$$\left(\frac{V_{f,T}(y,r)}{V_{f,T}(x,R_0)}\right)^{\frac{1}{2p-1}} \geq \left(\frac{r^n}{e^{aR_0}R_0^n}\right)^{\frac{1}{2p-1}} \times \left[\left(\frac{2}{3} - C(n,p,aR_T)\epsilon_0^{\frac{p}{2p-1}}\right)\left(\frac{r}{R_0}\right)^{\frac{2n+b(n,p,aR_0)}{2p-1}} - D(n,p,aR_0)\epsilon_0^{\frac{p}{2p-1}}\right],$$

which finishes the proof of (ii). □

4. Diameter Estimate

In this section, we start to apply Lemmas 3.1 and 3.2 to prove a local diameter estimate, which is a critical step to prove the main theorem.

Theorem 4.1. *Assume that $(M, g, e^{-f}dv)$ contains a subset T satisfying the following conditions:*

(1) *T is star-shaped at a point of x;*
(2) *$B(x, R_0) \subset T \subset B(x, R_T)$ for some $\pi < R_0 \leq R_T$;*
(3) *$\epsilon = R_T^2 \overline{\|\mathrm{Ric}_{f\,-}^1\|}_{p,f,a}(T) \leq B(n, p, aR_T)$ for some constant $B(p, n, aR_T)$;*
(4) *$\partial_r f = -a$ or $\partial_r f \geq -a - 2(n-1)\cot\left(\pi - \epsilon^{\frac{p}{2p-1}}\right)$ along all minimal geodesic segments from $x \in M$, for some constant $a \geq 0$.*

Then $M \subset T$ and

$$\mathrm{diam}(M) \leq \pi\left[1 + C(n,p,aR_T)\epsilon^{\frac{p(n-1)}{(2p-1)(3n-1+b(n,p,a))}}\right]$$

for some constant $C(n, p, aR_T)$, where the constant $b(n, p, a) > \frac{2\pi a(2p-n)}{(2p-1)^2\log(3/2)}$.

Proof of Theorem 4.1. If constant $B(p, n, aR_T)$ is sufficiently small, by Lemma 3.2, we have

$$\frac{V_{f,T}(x,R)}{V_f(T)} \geq \frac{R^n}{2(e^{aR_T})R_T^n}.$$

Hence we may assume $T = B(x, R_0)$ and $\pi < R_0 \leq 2\pi$. Fix $\delta \in (0, \frac{R_0-\pi}{2})$. If $y \in M$ satisfies $d(x, y) \geq \pi + \delta$, then

$$B(y,\delta) \subset B(x,\pi+2\delta)\backslash B(x,\pi).$$

By Lemma 3.1,

$$V_{f,T}(y,\delta) \leq \int_{\pi}^{\pi+2\delta} A_{f,T}(x,r)dr \leq 2\delta\overline{C}(n,p,aR_T)\epsilon^{\frac{p(n-1)}{2p-1}}V_f(x,R_0),$$

where

$$\overline{C}(n,p,aR_T) := \left[3\cdot 2^{2p+n-3} + \frac{\pi^{2p-1}}{4^p}\left(\frac{n-1}{(2p-1)(2p-n)}\right)^{p-1}\right]e^{aR_T}.$$

On the other hand, since

$$\epsilon_0 = R_T^2\overline{\|\mathrm{Ric}_{f\,-}^0\|}_{p,f,a}(T) \leq R_T^2\overline{\|\mathrm{Ric}_{f\,-}^1\|}_{p,f,a}(T) = \epsilon,$$

by Lemma 3.2(ii), we get

$$\begin{aligned} &V_{f,T}(y,\delta)\\ &\geq \left(\frac{\delta^n}{e^{aR_0}R_0^n}\right)^{\frac{n}{2p'-1}}\left[\frac{1}{2}\left(\frac{\delta}{R_0}\right)^{\frac{2n+b(n,p,aR_0)}{2p'-1}} - D(n,p,aR_T)\epsilon^{\frac{p'}{2p'-1}}\right]^{2p'-1}V_f(x,R_0)\\ &\geq \left(\frac{\delta^n}{e^{2\pi a}(2\pi)^n}\right)^{\frac{n}{2p'-1}}\left[\frac{1}{2}\left(\frac{\delta}{2\pi a}\right)^{\frac{2n+b(n,p,aR_0)}{2p'-1}} - D(n,p,aR_T)\epsilon^{\frac{p'}{2p'-1}}\right]^{2p'-1}V_f(x,R_0), \end{aligned}$$

by taking $\frac{2}{3} - C(n,p,aR_T)\epsilon_0^{\frac{p}{2p-1}} \geq \frac{1}{2}$ and noting that $\cot\vartheta \leq 0$ for $\frac{\pi}{2} \leq \vartheta < \pi$.

From the above lower estimates on $V_{f,T}(y,\delta)$, we can distinguish two cases:

(i) either

$$\left(\frac{\delta}{2\pi}\right)^{\frac{2n+b(n,p,aR_0)}{2p'-1}} \leq 4D(n,p,aR_T)\epsilon^{\beta},$$

where

$$\beta := \frac{p(n-1)(2n+b(n,p,aR_0))}{(2p-1)(2p'-1)(3n-1+b(n,p,aR_0))} \leq \frac{p'}{2p'-1},$$

(ii) or the above inequality becomes

$$V_{f,T}(y,\delta) \geq D(n,p,aR_T)\left(\frac{\delta}{2\pi}\right)^n\epsilon^{(2p'-1)\beta}V_f(x,R_0).$$

Combining the above two estimates about $V_{f,T}(y,\delta)$ gives a bound on δ:

$$\delta \leq \widetilde{C}(n,p,aR_T)\epsilon^{\frac{p(n-1)}{(2p-1)(3n-1+b(n,p,aR_0))}} \tag{12}$$

for some constant $\widetilde{C}(n,p,aR_T)$ which only depends on n, p and aR_T. Therefore we can infer that $M \subset B(x,R_0)$. Indeed, if there exists a point $y \in M$ such that $d(x,y) > \pi + \delta'$, where

$$\widetilde{C}(n,p,aR_T)\epsilon^{\frac{p(n-1)}{(2p-1)(3n-1+b(n,p,aR_0))}} < \delta' < \frac{R_0-\pi}{2},$$

then by the connected property of M, along a minnizing geodesic from x to y, there exists a point $y' \in M$ which exactly equals to $(\pi + \delta')$ from x, i.e., $d(x, y') = \pi + \delta'$. By the estimate (12), we have

$$\delta' \leq \widetilde{C}(n, p, aR_T)\epsilon^{\frac{p(n-1)}{(2p-1)(3n-1+b(n,p,aR_0))}},$$

which contradicts our choice of δ'.

Now let z be any point of $(M, g, e^{-f}dv)$. Since $M \subset B(x, R_0)$, then

$$\begin{aligned} R_0^2\overline{\|\mathrm{Ric}_{f\,-}^1\|}_{p,f,a}(B(z,R_0)) &\leq \left(\frac{V_{f,T}(x,R_0)}{V_{f,T}(z,R_0)}\right)^{\frac{1}{p}} R_0^2\overline{\|\mathrm{Ric}_{f\,-}^1\|}_{p,f,a}(T) \\ &= \left(\frac{V_{f,T}(x,R_0)}{V_{f,T}(z,R_0)}\right)^{\frac{1}{p}} \epsilon. \end{aligned}$$

We also observe that

$$B\left(x, R_0 - \pi - \widetilde{C}(n,p,aR_T)\epsilon^{\frac{p(n-1)}{(2p-1)(3n-1+b(n,p,aR_0))}}\right) \subset B(z, R_0).$$

Therefore, by Lemma 3.2(i), since $\pi < R_0 \leq 2\pi$, we have

$$\begin{aligned} \frac{V_{f,T}(z,R_0)}{V_{f,T}(x,R_0)} &\geq \frac{\left[R_0 - \pi - \widetilde{C}(n,p,aR_T)\epsilon^{\frac{p(n-1)}{(2p-1)(3n-1+b(n,p,aR_0))}}\right]^n}{2e^{aR_0}(2\pi)^n} \\ &\geq \frac{(R_0-\pi)^n}{4e^{aR_0}(2\pi)^n} \end{aligned}$$

as long as $B(p, n, aR_T)$ in Theorem 4.1 is sufficiently small. Substituting this into the above inequality yields

$$R_0^2\overline{\|\mathrm{Ric}_{f\,-}^1\|}_{p,f,a}(B(z,R_0)) \leq \left(\frac{4e^{aR_0}(2\pi)^n}{(R_0-\pi)^n}\right)^{\frac{1}{p}} \epsilon.$$

This shows that the above argument for the point x can also be suitable for any point $z \in M$ by replacing ϵ to $\frac{(4e^{aR_0})^{1/p}(2\pi)^{n/p}}{(R_0-\pi)^{n/p}}\epsilon$. So we indeed prove that

$$d(y, z) < \pi + \delta < R_0$$

for any $y, z \in M$. This completes the proof. □

Finally we will apply Theorem 4.1 and the universal cover argument to prove Theorem 1.1.

Proof of Theorem 1.1. For complete smooth metric measure space $(M, g, e^{-f}dv)$, we assume that $\overline{\|\mathrm{Ric}_{f\,-}^1\|}_{p,f,a}(M)$ is finite for some constant $a \geq 0$. Let $\{(B(x_i, 2\pi))\}_{i\in I}$ denote be a maximal family of disjoint balls in $(M, g, e^{-f}dv)$. Consider the Dirichlet domains

$$T_i := \{y \in M | d(x_i, y) < d(x_j, y), \forall j \neq i\}.$$

If $y \in T_i \backslash B(x_i, 4\pi)$, then there exists x_j such that $B(x_j, 2\pi) \bigcap B(y, 2\pi) \neq \emptyset$ by the maximum of $\{(B(x_i, 2\pi))\}$. If $z \in B(x_j, 2\pi) \bigcap B(y, 2\pi)$, then

$$d(x_j, y) \leq d(x_j, z) + d(x_j, z) < 4\pi.$$

Hence T_i satisfies the following three facts:

(1) $B(x_i, 2\pi) \subset T_i \subset B(x_i, 4\pi)$;
(2) T_i is star-shaped at the x_i;
(3) $M = \bigcup_i T_i$ up to a set of zero measure.

Therefore, we have

$$\begin{aligned} \int_M \left(\mathrm{Ric}^1_{f\,-}\right)^p \mathcal{A}_f e^{-at} &= \sum_{i \in I} \int_{T_i} \left(\mathrm{Ric}^1_{f\,-}\right)^p \mathcal{A}_f e^{-at} \\ &\geq \alpha^p \sum_{i \in I} V_f(T_i) \\ &= \alpha^p V_f(M), \end{aligned}$$

where $\alpha := \inf_{i \in I} \overline{\|\mathrm{Ric}^1_{f\,-}\|}_{p,f,a}(T_i)$. If $\alpha > \frac{B(n,p,a)}{32\pi^2}$, where $B(n, p, a) = B(n, p, 4\pi a)$ is constant defined as in Theorem 4.1, then

$$V_f(M) \leq \left(\frac{B(n, p, a)}{32\pi^2}\right)^{-p} \int_M \left(\mathrm{Ric}^1_{f\,-}\right)^p \mathcal{A}_f e^{-at}.$$

Elsewhere, there exists a star-shaped set T_i satisfying the assumptions of Theorem 4.1. In particular,

$$R^2_{T_i} \overline{\|\mathrm{Ric}^1_{f\,-}\|}_{p,f,a}(T_i) \leq 16\pi^2 \overline{\|\mathrm{Ric}^1_{f\,-}\|}_{p,f,a}(T_i) \leq \frac{B(n, p, a)}{2}.$$

So we bound the diameter of M by Theorem 4.1.

Next, we will prove the π_1-finiteness when $\overline{\|\mathrm{Ric}^1_{f\,-}\|}_{p,f,a}(M)$ is bounded. The proof of this result is essentially known in [2]. We give a proof for completeness. In fact we only need to justify that their universal covers are compact. Applying Theorem 4.1 to the universal Riemannian cover $(\widetilde{M}, \widetilde{g})$, we have to construct a good star-shaped subset of $\widetilde{M}$ on which the Ricci curvature is controlled by $\overline{\|\mathrm{Ric}^1_{f\,-}\|}_{p,f,a}(M)$. The fundamental group acts freely and isometrically on $\widetilde{M}$. For all $\tilde{x} \in \widetilde{M}$ and any subset $\widetilde{T} \subset \widetilde{M}$, which is union of fundamental domains, we let $\theta_{\widetilde{T}}(\tilde{x})$ denote the cardinality of $\widetilde{T} \bigcap \pi_1(\tilde{x})$. Set $\tilde{x}_0 \in \widetilde{M}$ and $\tilde{x} \in B(\tilde{x}_0, 2\pi)$ that maximizes $\theta_{B(\tilde{x}_0, 2\pi)}$. By the preceding discussion, we may assume $\mathrm{diam}(M) \leq 2\pi$ and then

$$1 \leq \theta_{B(\tilde{x}_0, 2\pi)}(\tilde{y}) \leq N \quad \text{and} \quad \theta_{B(\tilde{x}_0, 6\pi)}(\tilde{y}) \geq N$$

for all $\tilde{y} \in B(\tilde{x}_0, 2\pi)$, where $N := \theta_{B(\tilde{x}_0, 2\pi)}(\tilde{x})$. For all $\tilde{y} \in B(\tilde{x_0}, 2\pi)$, we choose N distinct points $\tilde{y}_1, \ldots, \tilde{y}_N$ in $\pi_1(\tilde{y})$ such that

$$d(\tilde{y}_i, \tilde{x}_0) \leq \inf_{\tilde{z} \in \pi_1(\tilde{y}) \backslash \{\tilde{y}_1, \ldots, \tilde{y}_N\}} d(\tilde{z}, \tilde{x}_0)$$

for any $1 \le i \le N$, and let $\widetilde{T}$ be the union of $\{\tilde{y}_1, \ldots, \tilde{y}_N\}$ for all $\tilde{y} \in B(\tilde{x}_0, 2\pi)$. So, on $\widetilde{M}$, we have

$$B(\tilde{x}_0, 2\pi) \subset \widetilde{T} \subset B(\tilde{x}_0, 6\pi) \quad \text{and} \quad \theta_T = N.$$

Hence,

$$\frac{1}{V_f(\widetilde{T})} \int_{\widetilde{T}} \left(\widetilde{\mathrm{Ric}^1_{f}}_-\right)^p \widetilde{\mathcal{A}}_f e^{-at} = \frac{1}{V_f(M)} \int_M \left(\mathrm{Ric}^1_{f\,-}\right)^p \mathcal{A}_f e^{-at}.$$

Now we show $\widetilde{T}$ is a star-shaped subset at $\tilde{x}$ of $(\widetilde{M}, \widetilde{g})$. Set $\tilde{y} \in \widetilde{T}$ and let γ be a minimizing geodesic from $\tilde{y}$ to $\tilde{x}_0$. Assume there exists $\tilde{z} \in \gamma \setminus \widetilde{T}$. Since $\theta_{\widetilde{T}}(\tilde{z}) = N$, there exist distinct nontrivial decktranformations $\sigma_1, \ldots, \sigma_N$ such that $\sigma_i(\tilde{z}) \in \widetilde{T}$ for all $1 \le i \le N$. But every element of $\pi_1(M)\backslash\{id\}$ acts without fixed point on $\widetilde{M}$, thus there exists $1 \le i_0 \le N$ such that $\sigma_{i_0}(\tilde{y}) \notin \widetilde{T}$. Since σ_{i_0} acts isometrically, then we have

$$d(\tilde{x}_0, \tilde{y}) \le d(\tilde{x}_0, \sigma_{i_0}(\tilde{y})), \quad d(\tilde{x}_0, \tilde{z}) \ge d(\tilde{x}_0, \sigma_{i_0}(\tilde{z})), \quad d(\tilde{y}, \tilde{z}) = d(\sigma_{i_0}(\tilde{y}), \sigma_{i_0}(\tilde{z})).$$

Now, we have

$$\begin{aligned} d(\tilde{x}_0, \tilde{y}) &= d(\tilde{x}_0, \tilde{z}) + d(\tilde{z}, \tilde{y}) \\ &\ge d(\tilde{x}_0, \sigma_{i_0}(\tilde{z})) + d(\sigma_{i_0}(\tilde{z}), \sigma_{i_0}(\tilde{y})) \\ &\ge d(\tilde{x}_0, \sigma_{i_0}(\tilde{y})). \end{aligned}$$

Combining above we have equalities everywhere. We have a minimal geodesics connecting $\tilde{x}$, $\sigma_{i_0}(\tilde{y})$ which contains $\sigma_{i_0}(\tilde{z})$. Hence the geodesic $\sigma_{i_0}(\gamma)$ contain $\tilde{x}$ and $\sigma_{i_0}(\tilde{x}) = \tilde{x}$, which contradicts the fact that σ_{i_0} has no fixed point. □

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5. Appendix: Myers' Type Theorem for Integral Bounds of m-Bakry–Émery Ricci Tensor

In this section, we will state Myers' type theorem under the only integral m-Bakry–Émery Ricci tensor bounds. Since the argument is almost the same as the Aubry's manifold case, we omit the proof here.

On a smooth metric measure space $(M, g, e^{-f} dv_g)$, we can define m-Bakry–Émery Ricci tensor

$$\mathrm{Ric}^m_f := \mathrm{Ric}_f - \frac{1}{m} df \otimes df$$

for some number $m > 0$, which is another natural generalization of the Ricci tensor. This curvature tensor is also introduced by Bakry and Émery. Here m is finite, and we have the Bochner formula for the m-Bakry–Émery Ricci tensor

$$\begin{aligned}\frac{1}{2}\Delta_f|\nabla u|^2 &= |\mathrm{Hess}u|^2 + \langle \nabla\Delta_f u, \nabla u\rangle + \mathrm{Ric}_f(\nabla u, \nabla u)\\ &\geq \frac{(\Delta_f u)^2}{m+n} + \langle \nabla\Delta_f u, \nabla u\rangle + \mathrm{Ric}_f^m(\nabla u, \nabla u)\end{aligned}$$

for some $u \in C^\infty(M)$, which is regarded as the Bochner formula of the Ricci curvature of an $(n+m)$-dimensional manifold. This property makes sure that many geometrical results for manifolds with Ricci tensor can be easily leads extended to smooth metric measure spaces with m-Bakry–Émery Ricci tensor (without any assumption on f), such as Wei and Wylie [30] and Wu [33].

Following Wu [33], on an n-dimensional smooth metric measure space $(M, g, e^{-f}dv_g)$, for each $x \in M$, $m > 0$, $H \in \mathbb{R}$ and let $\lambda(x)$ be the smallest eigenvalue of $\mathrm{Ric}_f^m : T_xM \to T_xM$. We define

$$\mathrm{Ric}_f^{m\,H}{}_- := \big((n+m-1)H - \lambda(x)\big)_+$$

and introduce a L_f^p-norm of function ϕ on the geodesic ball $B_x(r)$

$$\|\phi\|_{p,f}(r) := \sup_{x\in M}\Big(\int_{B_x(r)} |\phi|^p \cdot e^{-f}dv_g\Big)^{\frac{1}{p}}.$$

Clearly, $\|\mathrm{Ric}_f^{m\,H}{}_-\|_{p,f}(r) = 0$ iff $\mathrm{Ric}_f^m \geq (n+m-1)H$. The following normalized norm of ϕ is also useful,

$$\overline{\|\phi\|}_{p,f}(r) := \sup_{x\in M}\left(\frac{1}{V_f(x,r)}\int_{B(x,r)} |\phi|^p \cdot e^{-f}dv\right)^{\frac{1}{p}},$$

where $V_f(x,r) := \int_{B(x,r)} e^{-f}dv$. If $r \to \infty$ above and the limit exists, then we have another global curvature quantity on M

$$\overline{\|\phi\|}_{p,f}(M) := \lim_{r\to\infty}\overline{\|\phi\|}_{p,f}(r).$$

Applying the comparison theorems in [33], following the above discussion, we can similarly generalize Aubry's Myers' theorem to the case of smooth metric measure spaces with only the m-Bakry–Émery Ricci tensor integral bounds.

Theorem 5.1. *Let $(M, g, e^{-f}dv)$ be an n-dimensional complete smooth metric measure space. Given $p > n/2$ and $m > 0$, there exists a number $0 < \epsilon(n+m, p) < 1$ such that if*

$$\overline{\|\mathrm{Ric}_f^{m\,1}{}_-\|}_{p,f}(M) \leq \epsilon(n+m,p),$$

then M is compact with finite fundamental group $\pi_1(M)$ and

$$\mathrm{diam}(M) \leq \pi \left(1 + c(n+m,p)\overline{\|\mathrm{Ric}_f^{m\,1}{}_{-}\|}^{\frac{1}{10}}_{p,f}(M)\right).$$

for some constant $c(n+m,p)$.

We would like to point out that the above result may be regarded as the Aubry's result for $(n+m)$-dimensional manifolds. The main reason is that the Bochner formula for the m-Bakry–Émery Ricci tensor can be regarded as the Bochner formula of the Ricci curvature of an $(n+m)$-dimensional manifold.

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