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Results in Mathematics

Penumbras and Separation of Convex Sets

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Abstract. The concept of penumbra of a convex set with respect to another convex set was introduced by Rockafellar (1970). We study various geometric and topological properties of penumbras, their role in proper and strong separation of convex sets, and their relation to polyhedra and M-decomposable sets.

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1. Introduction

Separation of convex sets by hyperplanes is among the core topics of convexity. Initiated by Minkowski [7] on the turn of 20th century, it became a useful tool in many mathematical disciplines, especially in convex analysis and optimization, convex geometry, and linear analysis.

In what follows, we consider separation of convex sets in the Euclidean space R^n. Existing results in this domain either deal with sufficient conditions for the existence of hyperplanes separating a given pair of convex sets, say K_1 and K_2 (see, for instance, [9,10]), or with analytical descriptions of all hyperplanes separating K_1 and K_2 (see, e.g., [1,3,4,8] for the case of polytopes or polyhedra, and [13] for the case of arbitrary convex sets).

In this paper, we describe in geometric terms the locus of all hyperplanes separating (properly, or strongly) a given pair of nonempty convex sets in R^n. This description generalizes the existing results, obtained for the case of polytopes and polyhedral sets, and gives a new geometric insight into the separation properties of convex sets.

Our approach is based on the properties of penumbras introduced by Rockafellar [9, p. 22] for the case of disjoint sets. We recall that the *penumbra*

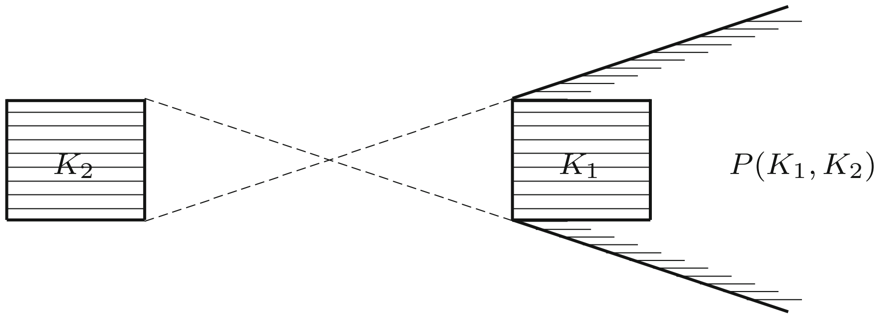

FIGURE 1. Penumbra of K_1 with respect to K_2

of a convex set K_1 with respect to another convex set K_2, denoted below $P(K_1, K_2)$, is defined by

$$P(K_1, K_2) = \cup\,(\mu K_1 + (1-\mu)K_2 : \mu \geq 1)$$
$$= \{\mu x_1 + (1-\mu)x_2 : \mu \geq 1, x_1 \in K_1, x_2 \in K_2\}.$$

Geometrically, $P(K_1, K_2)$ is the union of all closed halflines initiated at the points of K_1 in the directions of vectors from $K_1 - K_2$ (see Fig. 2).

We study various geometric and topological properties of penumbras (Sect. 2), their role in proper and strong separation of convex sets (Sects. 3–5), and their relation to polyhedra and M-decomposable sets (Sect. 6).

We conclude this section with necessary definitions, notation, and auxiliary results (see, e. g., [9,10] for details). The elements of R^n are called vectors (or points). In what follows, o stands for the zero vector of R^n. We denote by $[u, v]$ and (u, v) the closed and open line segments with endpoints $u, v \in \mathrm{R}^n$. Also, $u \cdot v$ will mean the dot product of u and v.

By an r-dimensional *plane* L in R^n, where $0 \leq r \leq n$, we mean a translate of a suitable r-dimensional subspace S of R^n: $L = c + S$, where $c \in \mathrm{R}^n$. It is known that a nonempty set $M \subset \mathrm{R}^n$ is a plane if and only if $(1-\lambda)x + \lambda y \in M$ whenever $x, y \in M$ and $\lambda \in \mathrm{R}$.

The open ρ-neighborhood of a nonempty $X \subset \mathrm{R}^n$, denoted $U_\rho(X)$, is the union of all open balls $U_\rho(x)$ of radius $\rho > 0$ centered at $x \in X$. Nonempty sets X_1 and X_2 in R^n are called *strongly disjoint* provided $U_\rho(X_1) \cap U_\rho(X_2) = \emptyset$ for a suitable $\rho > 0$; the latter occurs if and only if the *inf*-distance $\delta(X_1, X_2)$, defined by

$$\delta(X_1, X_2) = \inf\{\|x_1 - x_2\| : x_1 \in X_1, x_2 \in X_2\},$$

is positive. For a nonempty set $X \subset \mathrm{R}^n$, the notations $\mathrm{cl}\, X$ and $\mathrm{int}\, X$ stand, respectively, for the closure and interior of X. In a standard way, $\mathrm{conv}\, X$ and $\mathrm{span}\, X$ mean, respectively, the convex hull and the span of X, and the affine span of X, denoted $\mathrm{aff}\, X$, is defined as the intersection of all planes containing X. Clearly, $\operatorname{aff} X = \operatorname{span} X$ if $o \in X$. The dimension of a nonempty set $X \subset \mathrm{R}^n$, denoted $\dim X$, is defined by the equality $\dim X = \dim(\operatorname{aff} X)$.

A *hyperplane* in R^n is a plane which can be described as

$$H = \{x \in \mathrm{R}^n : x{\cdot}e = \gamma\}, \quad e \neq o, \quad \gamma \in \mathrm{R}. \tag{1}$$

Every hyperplane of the form (1) determines the opposite *closed halfspaces*

$$V_1 = \{x \in \mathrm{R}^n : x{\cdot}e \leq \gamma\} \ \text{ and } \ V_2 = \{x \in \mathrm{R}^n : x{\cdot}e \geq \gamma\} \tag{2}$$

and the opposite *open halfspaces*

$$W_1 = \{x \in \mathrm{R}^n : x{\cdot}e < \gamma\} \ \text{ and } \ W_2 = \{x \in \mathrm{R}^n : x{\cdot}e > \gamma\}. \tag{3}$$

We recall that a nonempty set C in R^n is a *cone* with apex $z \in \mathrm{R}^n$ if $z + \lambda(x - z) \in C$ whenever $\lambda \geq 0$ and $x \in C$. (Obviously, this definition implies that $z \in C$, although a stronger condition $\lambda > 0$ can be beneficial; see, e. g., [6].) The cone C is called convex if it is a convex set. For a convex set $K \subset \mathrm{R}^n$ and a point $z \in \mathrm{R}^n$, the *generated cone* $C_z(K)$ with apex z is defined by

$$C_z(K) = \{z + \lambda(x - z) : x \in K, \ \lambda \geq 0\}.$$

It is known that both sets $C_z(K)$ and $\operatorname{cl} C_z(K)$ are convex cones with apex z.

2. Properties of Penumbras

Theorem 1. *For convex sets K_1 and K_2 in R^n, the assertions below hold.*

1) *The set $P(K_1, K_2)$ is convex and $K_1 \subset P(K_1, K_2)$.*
2) $\operatorname{aff} P(K_1, K_2) = \operatorname{aff}(K_1 \cup K_2)$.
3) *$P(K_1, K_2) \cap P(K_2, K_1) = \emptyset$ if and only if $K_1 \cap K_2 = \emptyset$.*
4) *$\delta(P(K_1, K_2), P(K_2, K_1)) = \delta(K_1, K_2)$.*

Proof. 1) For the convexity of $P(K_1, K_2)$, choose any points $u, v \in P(K_1, K_2)$ and a scalar $\lambda \in [0, 1]$. Then

$$u = \eta x_1 + (1 - \eta)x_2 \quad \text{and} \quad v = \theta y_1 + (1 - \theta)y_2$$

for suitable scalars $\eta, \theta \geq 1$ and points $x_i, y_i \in K_i$, $i = 1, 2$. One has

$$\begin{aligned}(1 - \lambda)u + \lambda v &= (1 - \lambda)(\eta x_1 + (1 - \eta)x_2) + \lambda(\theta y_1 + (1 - \theta)y_2) \\ &\in (1 - \lambda)(\eta K_1 + (1 - \eta)K_2) + \lambda(\theta K_1 + (1 - \theta)K_2) \\ &= (1 - \lambda)\eta K_1 + \lambda\theta K_1 + (1 - \lambda)(1 - \eta)K_2 + \lambda(1 - \theta)K_2.\end{aligned}$$

Put $\gamma = (1 - \lambda)\eta + \lambda\theta$. Clearly, $\gamma \geq 1$. Since $\alpha K + \beta K = (\alpha + \beta)K$ whenever K is convex and $\alpha\beta \geq 0$ (see, e. g., [9, Theorem 3.2]), we have

$$\begin{aligned}&(1 - \lambda)\eta K_1 + \lambda\theta K_1 = \gamma K_1, \\ &(1 - \lambda)(1 - \eta)K_2 + \lambda(1 - \theta)K_2 = (1 - \gamma)K_2.\end{aligned}$$

Therefore,

$$(1 - \lambda)u + \lambda v \in \gamma K_1 + (1 - \gamma)K_2 \subset P(K_1, K_2),$$

which proves the convexity of $P(K_1, K_2)$.

Expressing any point $x_1 \in K_1$ as $x_1 = 1x_1 + (1-1)x_2$, where $x_2 \in K_2$, we obtain the inclusion $x_1 \in P(K_1, K_2)$. Hence $K_1 \subset P(K_1, K_2)$.

2) The inclusion $P(K_1, K_2) \subset \operatorname{aff}(K_1 \cup K_2)$ follows from the fact that any point $x \in P(K_1, K_2)$ is an affine combination of points from $K_1 \cup K_2$. Consequently,

$$\operatorname{aff} P(K_1, K_2) \subset \operatorname{aff}(\operatorname{aff}(K_1 \cup K_2)) = \operatorname{aff}(K_1 \cup K_2).$$

For the opposite inclusion, choose points $x_1 \in K_1$ and $x_2 \in K_2$. Put $z = 2x_1 - x_2$. Since $z \in P(K_1, K_2)$ and $K_1 \subset P(K_1, K_2)$, one has

$$x_2 = 2x_1 + (1-2)z \in \operatorname{aff}(K_1 \cup P(K_1, K_2)) = \operatorname{aff} P(K_1, K_2).$$

So, $K_2 \subset \operatorname{aff} P(K_1, K_2)$. Thus $K_1 \cup K_2 \subset \operatorname{aff} P(K_1, K_2)$, which gives the desired inclusion

$$\operatorname{aff}(K_1 \cup K_2) \subset \operatorname{aff}(\operatorname{aff} P(K_1, K_2)) = \operatorname{aff} P(K_1, K_2).$$

3) If $P(K_1, K_2) \cap P(K_2, K_1) = \emptyset$, then $K_1 \cap K_2 = \emptyset$ due to part 1). Conversely, let $K_1 \cap K_2 = \emptyset$. Assume for a moment the existence of a point

$$u \in P(K_1, K_2) \cap P(K_2, K_1).$$

Then

$$u = \gamma x_1 + (1-\gamma)x_2 = \mu y_2 + (1-\mu)y_1$$

for suitable scalars $\gamma, \mu \geq 1$ and points $x_1, y_1 \in K_1$ and $x_2, y_2 \in K_2$. Clearly,

$$x_1 = (1-\gamma^{-1})x_2 + \gamma^{-1}u, \quad y_2 = (1-\mu^{-1})y_1 + \mu^{-1}u. \tag{4}$$

Now, put

$$\xi = \frac{\gamma}{\gamma+\mu-1}, \quad \eta = \frac{\mu}{\gamma+\mu-1}, \tag{5}$$

$$a = \frac{\gamma-1}{\gamma+\mu-1}\,x_2 + \frac{\mu-1}{\gamma+\mu-1}\,y_1 + \frac{1}{\gamma+\mu-1}\,u. \tag{6}$$

Obviously, $\xi, \eta \in [0,1]$. Based on (4) and the convexity of K_1, one has

$$a = \xi\big((1-\gamma^{-1})x_2 + \gamma^{-1}u\big) + (1-\xi)y_1 = \xi x_1 + (1-\xi)y_1 \in K_1.$$

Similarly,

$$a = (1-\eta)x_2 + \eta\big((1-\mu^{-1})y_1 + \mu^{-1}u\big) = (1-\eta)x_2 + \eta y_2 \in K_2.$$

Thus $a \in K_1 \cap K_2$, in contradiction with the hypothesis $K_1 \cap K_2 = \emptyset$.

4) The inequality

$$\delta(P(K_1, K_2), P(K_2, K_1)) \leq \delta(K_1, K_2)$$

is trivial. For the opposite one, choose points $u \in P(K_1, K_2)$ and $v \in P(K_2, K_1)$. Then

$$u = \gamma x_1 + (1-\gamma)x_2 \quad \text{and} \quad v = \mu y_2 + (1-\mu)y_1$$

for suitable scalars $\gamma, \mu \geq 1$ and points $x_1, y_1 \in K_1$ and $x_2, y_2 \in K_2$. Now, put

$$a_1 = \frac{\gamma - 1}{\gamma + \mu - 1}\, x_2 + \frac{\mu - 1}{\gamma + \mu - 1}\, y_1 + \frac{1}{\gamma + \mu - 1}\, u,$$
$$a_2 = \frac{\gamma - 1}{\gamma + \mu - 1}\, x_2 + \frac{\mu - 1}{\gamma + \mu - 1}\, y_1 + \frac{1}{\gamma + \mu - 1}\, v.$$

With ξ and η defined by (5), we obtain, based on (4), that

$$a_1 = \xi x_1 + (1 - \xi) y_1 \in K_1, \quad a_2 = (1 - \eta) x_2 + \eta y_2 \in K_2.$$

Hence

$$\delta(K_1, K_2) \leq \|a_1 - a_2\| = \frac{1}{\gamma + \mu - 1} \|u - v\| \leq \|u - v\|,$$

which implies the desired inequality

$$\delta(K_1, K_2) \leq \delta(P(K_1, K_2), P(K_2, K_1)).$$

□

Definition 1. Given convex sets K_1 and K_2 in R^n, let

$$\begin{aligned} P_0(K_1, K_2) &= \cup\, (\mu K_1 + (1 - \mu) K_2 : \mu > 1) \\ &= \{\mu x_1 + (1 - \mu) x_2 : \mu > 1,\, x_1 \in K_1,\, x_2 \in K_2\}. \end{aligned}$$

We will need the following auxiliary lemma.

Lemma 1. ([9], §6) *For a convex set $K \subset \mathrm{R}^n$, the assertions below hold.*

1) *A point $x \in K$ belongs to $\operatorname{rint} K$ if and only if for any point $y \in K$ there is a scalar $\eta > 1$ such that $\eta x + (1 - \eta) y \in K$.*
2) *If $x \in \operatorname{rint} K$ and $y \in \operatorname{cl} K$, then $[x, y) \subset \operatorname{rint} K$. Consequently, $\operatorname{cl} K = \operatorname{cl}(\operatorname{rint} K)$.*

□

Topological properties of penumbra are described in the following theorem.

Theorem 2. *For convex sets K_1 and K_2 in R^n, the assertions below hold.*

1) $\operatorname{rint} P(K_1, K_2) = P_0(\operatorname{rint} K_1, \operatorname{rint} K_2)$.
2) $P(\operatorname{cl} K_1, \operatorname{cl} K_2) \subset \operatorname{cl} P(K_1, K_2) = \operatorname{cl} P_0(\operatorname{rint} K_1, \operatorname{rint} K_2)$.
3) *If K_2 is bounded and $\operatorname{cl} K_1 \cap \operatorname{cl} K_2 = \emptyset$, then $P(\operatorname{cl} K_1, \operatorname{cl} K_2) = \operatorname{cl} P(K_1, K_2)$.*

Proof. 1) First, we will prove the inclusion

$$P_0(\operatorname{rint} K_1, \operatorname{rint} K_2) \subset \operatorname{rint} P(K_1, K_2).$$

For this, choose any point $x \in P_0(\operatorname{rint} K_1, \operatorname{rint} K_2)$. Then $x = \mu x_1 + (1 - \mu) x_2$ for a suitable scalar $\mu > 1$ and points $x_1 \in \operatorname{rint} K_1$ and $x_2 \in \operatorname{rint} K_2$. Let y be any point in $P(K_1, K_2)$. Then $y = \gamma y_1 + (1 - \gamma) y_2$, where $\gamma \geq 1$, $y_1 \in K_1$, and

$y_2 \in K_2$. The obvious inequality $\gamma - 1 > \gamma - \mu$ implies the existence of a scalar $\varphi_0 > 1$ such that

$$\gamma - 1 > \varphi(\gamma - \mu) \quad \forall\, \varphi \in (1, \varphi_0).$$

Consequently, the scalar $\eta = \varphi\mu + (1 - \varphi)\gamma$ satisfies the inequality $\eta > 1$ for all $1 < \varphi < \varphi_0$. We are going to show the existence of a scalar $\varphi^* \in (1, \varphi_0)$ such that

$$z^* := \varphi^* x + (1 - \varphi^*)y \in P(K_1, K_2).$$

Indeed, for any $1 < \varphi < \varphi_0$, let

$$\lambda_1(\varphi) = \frac{\varphi\mu}{\eta}, \quad \lambda_2(\varphi) = \frac{\varphi(1-\mu)}{1-\eta}.$$

It is easy to see that $\lambda_1(\varphi) > 1$ and $\lambda_2(\varphi) \geq 1$ for all $1 < \varphi < \varphi_0$, and that $\lim_{\varphi \to 1} \lambda_i(\varphi) = 1$, $i = 1, 2$. We observe that the inequality $\lambda_2(\varphi) \geq 1$ is equivalent to $(1 - \gamma)(1 - \varphi) \geq 0$; whence $\lambda_2(\varphi) = 1$ if and only if $\gamma = 1$.

Lemma 1 shows the existence of a sufficiently small $\varphi^* \in (1, \varphi_0)$ such that the scalars $\lambda_i^* = \lambda_i(\varphi^*)$ satisfy the conditions: $\lambda_1^* > 1$, $\lambda_2^* \geq 1$, and

$$z_i^* = \lambda_i^* x_i + (1 - \lambda_i^*)y_i \in K_i, \quad i = 1, 2.$$

Let

$$\eta^* = \varphi^*\mu + (1 - \varphi^*)\gamma \quad \text{and} \quad z^* = \eta^* z_1^* + (1 - \eta^*)z_2^*.$$

Because $\eta^* > 1$, we obtain $z^* \in P(K_1, K_2)$. Furthermore,

$$\begin{aligned} z^* &= \eta^* z_1^* + (1 - \eta^*)z_2^* \\ &= \eta^*(\lambda_1^* x_1 + (1 - \lambda_1^*)y_1) + (1 - \eta^*)(\lambda_2^* x_2 + (1 - \lambda_2^*)y_2) \\ &= \varphi^*\mu x_1 + (\eta^* - \varphi^*\mu)y_1 + \varphi^*(1 - \mu)x_2 + (1 - \eta^* - \varphi^* + \varphi^*\mu)y_2 \\ &= \varphi^*(\mu x_1 + (1 - \mu)x_2) + (1 - \varphi^*)(\gamma y_1 + (1 - \gamma)y_2) \\ &= \varphi^* x + (1 - \varphi^*)y. \end{aligned}$$

Since the point y was arbitrarily chosen in $P(K_1, K_2)$, Lemma 1 implies the inclusion $x \in \operatorname{rint} P(K_1, K_2)$.

For the opposite inclusion,

$$\operatorname{rint} P(K_1, K_2) \subset P_0(\operatorname{rint} K_1, \operatorname{rint} K_2),$$

choose any point $x \in \operatorname{rint} P(K_1, K_2)$. Let y be a point in $P_0(\operatorname{rint} K_1, \operatorname{rint} K_2)$. Then $y = \gamma y_1 + (1 - \gamma)y_2$, where $\gamma > 1$, $y_1 \in \operatorname{rint} K_1$ and $y_2 \in \operatorname{rint} K_2$. By the above argument, $y \in \operatorname{rint} P(K_1, K_2)$.

Since the case $x = y$ is trivial, we may assume that $x \neq y$. By Lemma 1, there is a scalar $\nu > 1$ such that the point $z = \nu x + (1 - \nu)y$ belongs to $P(K_1, K_2)$. With $\alpha = \nu^{-1}$, we can write $x = (1 - \alpha)y + \alpha z$, where $0 < \alpha < 1$. Because $z \in P(K_1, K_2)$, one has $z = \beta z_1 + (1 - \beta)z_2$, where $\beta \geq 1$, $z_1 \in K_1$, and $z_2 \in K_2$. Now, let $\mu = \alpha\beta + (1 - \alpha)\gamma$. Clearly, $\mu > 1$. Next, put

$$\alpha_1 = \frac{\alpha\beta}{\mu}, \quad \alpha_2 = \frac{\alpha(1-\beta)}{1-\mu}.$$

It is easy to see that $\alpha_1, \alpha_2 \in [0,1)$. Therefore, Lemma 1 implies the inclusions

$$x_i := (1-\alpha_i)y_i + \alpha_i z_i + \in \operatorname{rint} K_i, \quad i = 1, 2.$$

Furthermore, the equalities

$$\begin{aligned} x &= (1-\alpha)y + \alpha z \\ &= (1-\alpha)(\gamma y_1 + (1-\gamma)y_2) + \alpha(\beta z_1 + (1-\beta)z_2) \\ &= \mu((1-\alpha_1)y_1 + \alpha_1 z_1) + (1-\mu)((1-\alpha_2)y_2 + \alpha_2 z_2) \\ &= \mu x_1 + (1-\mu)x_2 \end{aligned}$$

give the desired inclusion $x \in P_0(\operatorname{rint} K_1, \operatorname{rint} K_2)$.

2) Let u be any point in $P(\operatorname{cl} K_1, \operatorname{cl} K_2)$. Then $u = \mu x_1 + (1-\mu)x_2$ for a suitable scalar $\mu \geq 1$ and points $x_1 \in \operatorname{cl} K_1$ and $x_2 \in \operatorname{cl} K_2$. Choose any scalar $\varepsilon > 0$ and let $\rho = \varepsilon/(2\mu - 1)$. There are points $x_1' \in K_1$ and $x_2' \in K_2$ such that $\|x_1 - x_1'\| < \rho$ and $\|x_2 - x_2'\| < \rho$. Put $u' = \mu x_1' + (1-\mu)x_2'$. Then $u' \in P(K_1, K_2)$ and

$$\|u - u'\| \leq \mu\|x_1 - x_1'\| + (\mu - 1)\|x_2 - x_2'\| < (2\mu - 1)\rho = \varepsilon.$$

So, $u \in \operatorname{cl} P(K_1, K_2)$, as desired.

3) Suppose that K_2 is bounded and $\operatorname{cl} K_1 \cap \operatorname{cl} K_2 = \emptyset$. By the above argument, it suffices to prove the inclusion

$$\operatorname{cl} P(K_1, K_2) \subset P(\operatorname{cl} K_1, \operatorname{cl} K_2).$$

For this, choose any point $u \in \operatorname{cl} P(K_1, K_2)$ and a sequence of points $u_1, u_2, \ldots$ in $P(K_1, K_2)$ converging to u. We can write

$$u_i = \mu_i x_i + (1-\mu_i)y_i, \quad \text{where} \quad \mu_i \geq 1,\ x_i \in K_1,\ y_i \in K_2,\ i \geq 1.$$

Therefore,

$$x_i = \mu_i^{-1} u_i + (1-\mu_i^{-1})y_i, \quad \text{where} \quad \mu_i^{-1} \in (0,1],\ i \geq 1.$$

By a compactness argument, there is an increasing sequence of integers $i_1, i_2, \ldots$ such that $\mu_{i_r}^{-1} \to \alpha \in [0,1]$ and $y_{i_r} \to y \in \operatorname{cl} K_2$ as $r \to \infty$. Consequently, there exists the limit

$$x := \lim_{i_r \to \infty} x_{i_r} = \lim_{i_r \to \infty} (\mu_{i_r}^{-1} u_{i_r} + (1-\mu_{i_r}^{-1})y_{i_r}) = \alpha u + (1-\alpha)y. \qquad (7)$$

Clearly, $x \in \operatorname{cl} K_1$ due to the inclusions $x_{i_r} \in K_1$, $r \geq 1$.

We observe that $\alpha \neq 0$. Indeed, if $\alpha = 0$, then (7) would give $x = y$, contrary to the assumption $\operatorname{cl} K_1 \cap \operatorname{cl} K_2 = \emptyset$. So, $\alpha \neq 0$. Then $\alpha^{-1} > 1$, and (7) gives

$$u = \alpha^{-1}x + (1-\alpha^{-1})y \in P(\operatorname{cl} K_1, \operatorname{cl} K_2),$$

as desired. Finally, by Lemma 1,

$$\operatorname{cl} P(K_1, K_2) = \operatorname{cl}(\operatorname{rint} P(K_1, K_2)) = \operatorname{cl} P_0(\operatorname{rint} K_1, \operatorname{rint} K_2).$$

□

Remark 1. Generally, both inclusions

$$\operatorname{rint} P(K_1, K_2) \subset P(\operatorname{rint} K_1, \operatorname{rint} K_2),$$
$$P(\operatorname{cl} K_1, \operatorname{cl} K_2) \subset \operatorname{cl} P(K_1, K_2)$$

may be proper. Indeed, in the plane R^2, consider the closed convex sets

$$K_1 = \{(x, 1) : 0 \le x \le 1\} \quad \text{and} \quad K_2 = \{(x, 0) : x \in \mathrm{R}\}.$$

With $M = \{(x, y) : y > 1\}$, one has $P(K_1, K_2) = K_1 \cup M$ and

$$\operatorname{rint} P(K_1, K_2) = M \ne \operatorname{rint} K_1 \cup M = P(\operatorname{rint} K_1, \operatorname{rint} K_2),$$
$$P(\operatorname{cl} K_1, \operatorname{cl} K_2) = P(K_1, K_2) \ne \operatorname{cl} M = \operatorname{cl} P(K_1, K_2).$$

3. Penumbras and Separation

We recall that convex sets K_1 and K_2 in R^n are *separated* by a hyperplane $H \subset \mathrm{R}^n$ provided K_1 and K_2 lie in the opposite closed halfspaces determined by H. Furthermore, K_1 and K_2 are *properly* separated if $K_1 \cup K_2 \not\subset H$, and they are *nontrivially* separated if $K_1 \not\subset H$ and $K_2 \not\subset H$. Also, K_1 and K_2 are *strictly* separated by H if $K_1 \cap H = K_2 \cap H = \emptyset$. Finally, K_1 and K_2 are *strongly* separated by H if suitable open ρ-neighborhoods $U_\rho(K_1)$ and $U_\rho(K_2)$ of these sets are separated by H (see [9, p. 95]). The following lemma provides well-known criteria for proper and strong separation.

Lemma 2. ([9], §11) *If K_1 and K_2 are convex sets in R^n, then*

1) *K_1 and K_2 are properly separated if and only if* $\operatorname{rint} K_1 \cap \operatorname{rint} K_2 = \emptyset$,
2) *K_1 and K_2 are strongly separated if and only if* $\delta(K_1, K_2) > 0$.

□

Theorem 3. *Let K_1 and K_2 be convex sets in R^n, and $H \subset \mathrm{R}^n$ be a hyperplane. The assertions below hold.*

1) *H separates K_1 and K_2 if and only if H separates $P(K_1, K_2)$ and $P(K_2, K_1)$.*
2) *H properly separates K_1 and K_2 if and only if H nontrivially separates $P(K_1, K_2)$ and $P(K_2, K_1)$.*
3) *H strictly separates K_1 and K_2 if and only if H strictly separates $P(K_1, K_2)$ and $P(K_2, K_1)$.*
4) *H strongly separates K_1 and K_2 if and only if H strongly separates $P(K_1, K_2)$ and $P(K_2, K_1)$.*

Proof. 1) Let H separate K_1 and K_2. Denote by V_1 and V_2 the closed halfspaces determined by H and containing K_1 and K_2, respectively. Without loss of generality, we may suppose that V_1 and V_2 are given by (2).

We assert that $P(K_1, K_2) \subset V_1$. Indeed, choose any point $u \in P(K_1, K_2)$. Then $u = \mu x_1 + (1 - \mu) x_2$ for a suitable scalar $\mu \ge 1$ and points $x_1 \in K_1$ and $x_2 \in K_2$. Since $x_1 \in V_1$ and $x_2 \in V_2$, one has $x_1 \cdot e \le \gamma$ and $x_2 \cdot e \ge \gamma$. Therefore,

$$u \cdot e = \mu x_1 \cdot e + (1 - \mu) x_2 \cdot e \le \mu\gamma + (1 - \mu)\gamma = \gamma. \tag{8}$$

Hence $u \in V_1$, and thus $P(K_1, K_2) \subset V_1$. Similarly, $P(K_2, K_1) \subset V_2$, which implies that H separates $P(K_1, K_2)$ and $P(K_2, K_1)$.

The proof of the "if" part immediately follows from the inclusions $K_1 \subset P(K_1, K_2)$ and $K_2 \subset P(K_2, K_1)$.

2) Assume that H properly separates K_1 and K_2 such that $K_1 \subset V_1$ and $K_2 \subset V_2$. Let, for instance, $K_1 \not\subset H$. Then $P(K_1, K_2) \not\subset H$ due to the inclusion $K_1 \subset P(K_1, K_2)$. Now, choose points $x_1 \in K_1 \setminus H$ and $x_2 \in K_2$. Then $x_1 \cdot e < \gamma$, and, with $\mu > 1$, the point $v = \mu x_2 + (1 - \mu)x_1 \in P(K_2, K_1)$ satisfies the inequality

$$v \cdot e = \mu x_2 \cdot e + (1 - \mu)x_1 \cdot e > \mu\gamma + (1 - \mu)\gamma = \gamma.$$

Hence $P(K_2, K_1) \not\subset H$, which shows that H nontrivially separates $P(K_1, K_2)$ and $P(K_2, K_1)$. The proof of the "if" part is similar.

The proofs of assertions 3)–4) are similar to that of 1) and use the following refinements of (8):

a) if $x_1 \cdot e < \gamma$ and $x_2 \cdot e > \gamma$, then $u \cdot e < \gamma$,
b) if $x_1 \cdot e \le \gamma - \varepsilon$ and $x_2 \cdot e \ge \gamma + \varepsilon$, where $\varepsilon > 0$, then $u \cdot e \le \gamma - \varepsilon$.

□

Remark 2. Part 2) of Theorem 3 implies that $P(K_1, K_2)$ and $P(K_2, K_1)$ are properly separated by a hyperplane H if and only if they are nontrivially separated by H.

We recall (see [9, p. 100]) that a hyperplane $H \subset \mathrm{R}^n$ *nontrivially* supports a convex set $K \subset \mathrm{R}^n$ if H supports K such that $K \not\subset H$.

Theorem 4. *Let K_1 and K_2 be convex sets in* R^n. *If a hyperplane $H \subset \mathrm{R}^n$ supports* (*nontrivially supports*) $\operatorname{cl} P(K_1, K_2)$, *then H separates* (*properly separates*) *K_1 and K_2. If, additionally, K_1 is bounded, then H supports* $\operatorname{cl} K_1$.

Proof. Let H support $\operatorname{cl} P(K_1, K_2)$. Expressing H in the form (1), we may suppose that the opposite closed halfspaces V_1 and V_2 determined by H are given by (2). Assume that $\operatorname{cl} P(K_1, K_2) \subset V_1$. By Theorem 1, $K_1 \subset P(K_1, K_2) \subset V_1$. Consider the hyperplane $H' = \{x \in \mathrm{R}^n : x \cdot e = \gamma'\}$, where

$$\gamma' = \sup\{x_1 \cdot e : x_1 \in K_1\}.$$

Clearly, K_1 is contained in the closed halfspace $V_1' = \{x \in \mathrm{R}^n : x \cdot e \le \gamma'\}$. We assert that K_2 is contained in the opposite closed halfspace $V_2' = \{x \in \mathrm{R}^n : x \cdot e \ge \gamma'\}$. Indeed, assume for a moment the existence of a point $x_2 \in K_2 \setminus V_2'$. Then $x_2 \cdot e < \gamma'$. Choose a point $x_1 \in K_1$ so close to H' that $x_2 \cdot e < x_1 \cdot e \le \gamma'$. For a scalar $\mu \ge 1$, let $z_\mu = \mu x_1 + (1 - \mu)x_2$. Then $z_\mu \in P(K_1, K_2) \subset V_1$. On the other hand,

$$z_\mu \cdot e = (\mu x_1 + (1 - \mu)x_2) \cdot e = x_2 \cdot e + \mu(x_1 - x_2) \cdot e > \gamma$$

for a sufficiently large μ. Consequently, $z_\mu \notin V_1$, in contradiction with the choice of V_1. Summing up, $K_2 \subset V_2'$.

So, H' separates K_1 and K_2. Theorem 3 implies that $P(K_1, K_2) \subset V_1' \subset V_1$. Since H supports $\mathrm{cl}\, P(K_1, K_2)$, it follows that $V_1 = V_1'$ and $H = H'$, as desired.

Assume now that H nontrivially supports $\mathrm{cl}\, P(K_1, K_2)$. If $K_1 \cup K_2$ contained in H, then, by Theorem 1, we would have $P(K_1, K_2) = \mathrm{aff}\,(K_1, K_2) \subset H$, contrary to the assumption. Hence H properly separates K_1 and K_2.

The second assertion of the theorem follows from the choice of H' and compactness of $\mathrm{cl}\, K_1$. □

Remark 3. The hyperplane H in Theorem 4 may not support $\mathrm{cl}\, K_1$ if K_1 is unbounded. For instance, let the closed convex sets K_1 and K_2 in R^2 be given by

$$K_1 = \{(x, y) : y \geq 2^x + 1\} \quad \text{and} \quad K_2 = \{(x, y) : y \leq 0\}.$$

Then $\mathrm{cl}\, P(K_1, K_2) = \{(x, y) : y \geq 1\}$ and $y = 1$ is the only line that supports $\mathrm{cl}\, P(K_1, K_2)$. On the other hand, this line is asymptotic to K_1 (that is, $H \cap K_1 = \emptyset$ and $\delta(H, K_1) = 0$).

Theorem 5. *If convex sets K_1 and K_2 in R^n satisfy the condition $\mathrm{rint}\, K_1 \cap \mathrm{rint}\, K_2 = \emptyset$, then $\mathrm{cl}\, P(K_1, K_2)$ is the intersection of all closed halfspaces containing $P(K_1, K_2)$ such that their boundary hyperplanes properly separate K_1 and K_2. If, additionally, K_1 is bounded, then the above halfspaces can be chosen such that their boundary hyperplanes support $\mathrm{cl}\, K_1$.*

Proof. First, we assert that $\mathrm{cl}\, P(K_1, K_2)$ is not a plane. For this, let $L = \mathrm{aff}\,(K_1 \cup K_2)$ and $r = \dim L$. Lemma 2 shows the existence of a hyperplane $H \subset \mathrm{R}^n$ properly separating K_1 and K_2. Denote by V_1 and V_2 the closed halfspaces determined by H and containing K_1 and K_2, respectively. Since $K_1 \cup K_2 \not\subset H$, the affine span $\mathrm{aff}\,(K_1 \cup K_2 0$ is not included in H. Then the plane $L = H \cap \mathrm{aff}\,(K_1 \cup K_2)$ has dimension $r - 1$ and determines two closed halfplanes $M_1 = L \cap V_1$ and $M_2 = L \cap V_2$ (see [10], Lemma 2.72 and Theorem 2.73). By Theorem 3, $P(K_1, K_2) \subset M_1$, while $\mathrm{aff}\, P(K_1, K_2) = L$ (see Theorem 1). Hence $P(K_1, K_2) \neq \mathrm{aff}\, P(K_1, K_2)$, which implies that $P(K_1, K_2)$ is not a plane.

Since $P(K_1, K_2)$ is not a plane, $\mathrm{cl}\, P(K_1, K_2)$ is the intersection of all closed halfspaces nontrivially supporting $\mathrm{cl}\, P(K_1, K_2)$ (see [10], Theorem 9.39). By Theorem 4, the boundary hyperplanes of these halfspaces properly separate K_1 and K_2, as desired. If, additionally, K_1 is bounded, then the boundary hyperplanes of these halfspaces support $\mathrm{cl}\, K_1$. □

Remark 4. As it shown in the proof of Theorem 5, the penumbra $P(K_1, K_2)$ is not a plane if $\mathrm{rint}\, K_1 \cap \mathrm{rint}\, K_2 = \emptyset$. In this regard, it would be interesting to know whether $P(K_1, K_2) = \mathrm{aff}\,(K_1 \cup K_2)$ provided $\mathrm{rint}\, K_1 \cap \mathrm{rint}\, K_2 \neq \emptyset$.

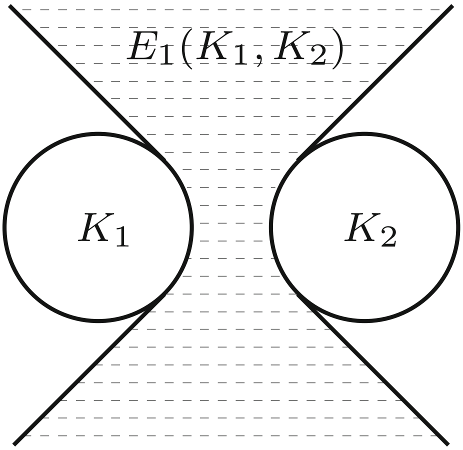

FIGURE 2. The set $E_1(K_1, K_2)$

4. Properly Separating Hyperplanes

Definition 2. Given convex sets K_1 and K_2 in R^n, denote by $\mathcal{H}_1(K_1, K_2)$ (respectively, by $\mathcal{H}_2(K_1, K_2)$ and $\mathcal{H}_3(K_1, K_2)$) the family of all hyperplanes properly (respectively, strictly and strongly) separating K_1 and K_2. Also, let

$$E_i(K_1, K_2) = \cup\,(H : H \in \mathcal{H}_i), \quad i = 1, 2, 3.$$

A combination of Theorem 5.3 from [13] and the above Theorem 2 implies the following assertion.

Theorem 6. ([13]) *If convex sets K_1 and K_2 in R^n satisfy the condition* $\operatorname{rint} K_1 \cap \operatorname{rint} K_2 = \emptyset$*, then*

$$\begin{aligned} E_1(K_1, K_2) &= \mathrm{R}^n \setminus (P_0(\operatorname{rint} K_1, \operatorname{rint} K_2) \cup P_0(\operatorname{rint} K_2, \operatorname{rint} K_1)) \\ &= \mathrm{R}^n \setminus (\operatorname{rint} P(K_1, K_2) \cup \operatorname{rint} P(K_2, K_1)). \end{aligned}$$

Theorem 7. *Let K_1 and K_2 be convex sets in R^n satisfying the condition $\operatorname{rint} K_1 \cap \operatorname{rint} K_2 = \emptyset$. A hyperplane $H \subset \mathrm{R}^n$ properly separates K_1 and K_2 if and only if $H \subset E_1(K_1, K_2)$ and $H \cap \operatorname{aff}(K_1 \cup K_2) \neq \emptyset$.*

Proof. Let a hyperplane H properly separate K_1 and K_2. Then the inclusion $H \subset E_1(K_1, K_2)$ immediately follows from Definition 2. Denote by V_1 and V_2 the closed halfspaces determined by H and containing K_1 and K_2, respectively. Choose distinct points $x_1 \in K_1$ and $x_2 \in K_2$ (this is possible due to the assumption $\operatorname{rint} K_1 \cap \operatorname{rint} K_2 = \emptyset$). Then the line through x_1 and x_2 lies in $\operatorname{aff}(K_1 \cup K_2)$ and meets H (see [13, Theorem 2.35]). Hence $H \cap \operatorname{aff}(K_1 \cup K_2) \neq \emptyset$.

Conversely, let a hyperplane H satisfy the conditions $H \subset E_1(K_1, K_2)$ and $H \cap \operatorname{aff}(K_1 \cup K_2) \neq \emptyset$. Theorem 6 shows that H is disjoint from the set

$$Q = P_0(\operatorname{rint} K_1, \operatorname{rint} K_2) \cup P_0(\operatorname{rint} K_2, \operatorname{rint} K_1).$$

Hence $Q \subset \mathrm{R}^n \setminus H$. We assert that $P_0(\operatorname{rint} K_1, \operatorname{rint} K_2)$ and $P_0(\operatorname{rint} K_2, \operatorname{rint} K_1)$ are contained, respectively, in the opposite open halfspaces determined by H. Indeed, assume for a moment that both sets

$$P_0(\operatorname{rint} K_1, \operatorname{rint} K_2) \quad \text{and} \quad P_0(\operatorname{rint} K_2, \operatorname{rint} K_1)$$

are contained in the same open halfspace W determined by H. By Theorem 1, both sets $\operatorname{rint} K_1$ and $\operatorname{rint} K_2$ are contained in W. Expressing H in the form (1), we may suppose that W is given by

$$W = \{x \in \mathrm{R}^n : x \cdot e < \gamma\}.$$

Choose any points $x_1 \in \operatorname{rint} K_1$ and $x_2 \in \operatorname{rint} K_2$. For any scalar $\mu > 1$, one has

$$u := \mu x_1 + (1 - \mu)x_2 \in P_0(\operatorname{rint} K_1, \operatorname{rint} K_2) \subset W.$$

Therefore,

$$x_2 \cdot e + \mu(x_1 - x_2) \cdot e = (\mu x_1 + (1 - \mu)x_2) \cdot e = u \cdot e < \gamma.$$

These inequalities hold for all $\mu > 1$ only if $x_1 \cdot e \leq x_2 \cdot e$. In a similar way, considering the points

$$v := \mu x_2 + (1 - \mu)x_1 \in P_0(\operatorname{rint} K_2, \operatorname{rint} K_1) \subset W,$$

we obtain that $x_2 \cdot e \leq x_1 \cdot e$. Hence $x_1 \cdot e = x_2 \cdot e$ for all $x_1 \in \operatorname{rint} K_1$ and $x_2 \in \operatorname{rint} K_2$. This argument shows that the set $\operatorname{rint} K_1 \cup \operatorname{rint} K_2$ lies in a hyperplane $H' \subset W$ of the form $H' = \{x \in \mathrm{R}^n : x \cdot e = \gamma'\}$, where $\gamma' < \gamma$. Consequently, by Lemma 1,

$$K_1 \cup K_2 \subset \operatorname{cl}(\operatorname{rint} K_1) \cup \operatorname{cl}(\operatorname{rint} K_2) \subset H',$$

which gives the inclusion $\operatorname{aff}(K_1 \cup K_2) \subset H'$. Because $H \cap H' = \emptyset$, the latter inclusion contradicts the assumption $H \cap \operatorname{aff}(K_1 \cup K_2) \neq \emptyset$.

Summing up, $P_0(\operatorname{rint} K_1, \operatorname{rint} K_2)$ and $P_0(\operatorname{rint} K_2, \operatorname{rint} K_1)$ are contained in the opposite open halfspaces determined by H. By Theorem 2, $P(K_1, K_2)$ and $P(K_2, K_1)$ are contained in the opposite closed halfspaces determined by H such that neither of these set lies in H. Theorem 3 shows that H properly separates K_1 and K_2. □

The following example illustrates Theorem 7.

Example 1. In the plane R^2, consider the closed convex sets

$$K_1 = \{(x, 0) : x \leq 0\} \quad \text{and} \quad K_2 = \{(x, 0), x \geq 1\}.$$

Then $P(K_1, K_2) = K_1$ and $P(K_2, K_1) = K_2$. Any horizontal line $y = b$, where $b \neq 0$, lies in $E_1(K_1, K_2) = \mathrm{R}^2 \setminus (K_1 \cup K_2)$, while only slant lines which meet the x-axis at the segment $\{(x, 0) : 0 \leq x \leq 1\}$ properly separate K_1 and K_2.

Corollary 1. *Let K_1 and K_2 be convex sets in* R^n *satisfying the condition* $\mathrm{cl}\, K_1 \cap \mathrm{cl}\, K_2 = \emptyset$. *A hyperplane $H \subset \mathrm{R}^n$ strictly separates* $\mathrm{cl}\, K_1$ *and* $\mathrm{cl}\, K_2$ *if and only if H is contained in the set*

$$F_2(K_1, K_2) := \mathrm{R}^n \setminus (P(\mathrm{cl}\, K_1, \mathrm{cl}\, K_2) \cup P(\mathrm{cl}\, K_2, \mathrm{cl}\, K_1))$$

and $H \cap \mathrm{aff}\,(K_1 \cup K_2) \neq \emptyset$.

Proof. Let a hyperplane H strictly separate $\mathrm{cl}\, K_1$ and $\mathrm{cl}\, K_2$. By Theorem 3, H strictly separates $P(\mathrm{cl}\, K_1, \mathrm{cl}\, K_2)$ and $P(\mathrm{cl}\, K_2, \mathrm{cl}\, K_1)$. Hence $H \subset F_2(K_1, K_2)$. Since H properly separates K_1 and K_2, Theorem 7 shows that $H \cap \mathrm{aff}\,(K_1 \cup K_2) \neq \emptyset$.

Conversely, suppose that $H \subset F_2(K_1, K_2)$ and $H \cap \mathrm{aff}\,(K_1 \cup K_2) \neq \emptyset$. Then H is disjoint from both $P(\mathrm{cl}\, K_1, \mathrm{cl}\, K_2)$ and $P(\mathrm{cl}\, K_2, \mathrm{cl}\, K_1)$, and Theorem 1 shows that H is disjoint from both $\mathrm{cl}\, K_1$ and $\mathrm{cl}\, K_2$. By Theorem 7, H properly separates K_1 and K_2. Hence H strictly separates $\mathrm{cl}\, K_1$ and $\mathrm{cl}\, K_2$. □

Remark 5. Corollary 1 shows that $E_2(K_1, K_2)$ is contained in the set $F_2(K_1, K_2)$. This inclusion may be proper, as illustrated by the convex sets K_1 and K_2 from Remark 1. Indeed, for these sets, $E_2(K_1, K_2) = \{(x, y) : 0 < y < 1\}$, while

$$F_2(K_1, K_2) = E_2(K_1, K_2) \cup \{(x, 1) : x < 0\} \cup \{(x, 1) : x > 1\}.$$

5. Strongly Separating Hyperplanes

Theorem 8. *If K_1 and K_2 are strongly disjoint convex sets in* R^n, *then*

$$\mathrm{cl}\, P(K_1, K_2) = \cap\, (P_0(U_\rho(K_1), U_\rho(K_2)) : \rho > 0).$$

Proof. For any scalar $\rho > 0$, both sets $U_\rho(K_i)$, $i = 1, 2$, are open. So, part 1) of Theorem 2 gives

$$\begin{aligned} \mathrm{int}\, P_0(U_\rho(K_1), U_\rho(K_2)) &= P_0(\mathrm{int}\, U_\rho(K_1), \mathrm{int}\, U_\rho(K_2)) \\ &= P_0(U_\rho(K_1), U_\rho(K_2)). \end{aligned}$$

Hence the set $P_0(U_\rho(K_1), U_\rho(K_2))$ also is open. Therefore, the obvious inclusion

$$P_0(\mathrm{rint}\, K_1, \mathrm{rint}\, K_2) \subset P_0(U_\rho(K_1), U_\rho(K_2))$$

and the same part 1) give

$$\mathrm{cl}\, P(K_1, K_2) = \mathrm{cl}\, P_0(\mathrm{rint}\, K_1, \mathrm{rint}\, K_2) \subset P_0(U_\rho(K_1), U_\rho(K_2)).$$

Thus

$$\mathrm{cl}\, P(K_1, K_2) \subset \cap\, (P_0(U_\rho(K_1), U_\rho(K_2)) : \rho > 0).$$

To prove the opposite inclusion, let

$$Q = \big(\cap\, (P_0(U_\rho(K_1), U_\rho(K_2)) : \rho > 0)\big).$$

Clearly, Q is a convex set as the intersection of convex sets $P_0(U_\rho(K_1), U_\rho(K_2))$, $\rho > 0$. Assume for a moment the existence of a point $u \in Q \setminus \mathrm{cl}\, P(K_1, K_2)$ and denote by z the metric projection of u on $\mathrm{cl}\, P(K_1, K_2)$. It is known that z is uniquely determined by u due to the convexity of $\mathrm{cl}\, P(K_1, K_2)$.

Since K_1 and K_2 are strongly disjoint, Lemma 2 implies the existence of a hyperplane $H \subset \mathrm{R}^n$ strongly separating K_1 and K_2. Denote by W_1 and W_2 the opposite open halfspaces determined by H containing K_1 and K_2, respectively. Choose a pair of hyperplanes $H_1 \subset W_1$ and $H_2 \subset W_2$ which are parallel to H and each of them strongly separates K_1 and K_2. Denote by W_i' the open halfspace determined by H_i and contained in W_i, $i = 1, 2$. Choose a scalar $\rho > 0$ such that $U_\rho(K_i)$ is contained in W_i' and is strongly disjoint from H_i, $i = 1, 2$.

Expressing H in the form (1), we may suppose that

$$H_i = \{x \in \mathrm{R}^n : x \cdot e = \gamma_i\}, \quad i = 1, 2,$$

with $\gamma_1 < \gamma < \gamma_2$ and

$$W_1' = \{x \in \mathrm{R}^n : x \cdot e < \gamma_1\}, \;\; W_2' = \{x \in \mathrm{R}^n : x \cdot e > \gamma_2\}.$$

By the above argument, each of the hyperplanes H_1 and H_2 strongly separates $U_\rho(K_1)$ and $U_\rho(K_2)$. Consequently, Theorem 3 implies that each of H_1 and H_2 strongly separates $P(U_\rho(K_1), U_\rho(K_2))$ and $P(U_\rho(K_2), U_\rho(K_1))$. Hence

$$P(U_\rho(K_1), U_\rho(K_2)) \subset W_1', \quad P(U_\rho(K_2), U_\rho(K_1)) \subset W_2'.$$

Since the open segment (u, z) lies in Q and is disjoint from $\mathrm{cl}\, P(K_1, K_2)$, we can choose a point $v \in (u, z) \subset Q$ so close to z that $v \in W_1'$. Denote by v_1 and v_2 the orthogonal projections of v on H_1 and H_2, respectively.

For any integer $r \geq 1$, we can write

$$v = \mu_r x_r + (1 - \mu_r) y_r, \text{ where } \mu_r > 1, \; x_r \in U_{1/r}(K_1), \; y_r \in U_{1/r}(K_2).$$

Choose an integer r_0 such that $1/r_0 < \rho$. Then $x_r \in W_1'$ and $y_r \in W_2'$ for all $r \geq r_0$. In particular, $x_r \neq y_r$ and the line segment $[x_r, y_r]$ meets H_1 and H_2 at distinct points x_r' and y_r', respectively, $r \geq r_0$. Since the hyperplanes H_1 and H_2 are parallel, one has

$$\|v - x_r'\| / \|x_r' - y_r'\| = \|v - v_1\| / \|v_1 - v_2\|.$$

The equalities

$$v - y_r = \mu_r (x_r - y_r), \quad r \geq r_0,$$

give

$$\mu_r = \frac{\|v - y_r\|}{\|x_r - y_r\|} = \frac{\|v - x_r\| + \|x_r - y_r\|}{\|x_r - y_r\|} = \frac{\|v - x_r\|}{\|x_r - y_r\|} + 1 \tag{9}$$

$$\leq \frac{\|v - x_r'\|}{\|x_r' - y_r'\|} + 1 = \frac{\|v - v_1\|}{\|v_1 - v_2\|} + 1, \quad r \geq r_0. \tag{10}$$

Let G be the hyperplane through z orthogonal to the segment $[v, z]$. It is well known that G supports $\operatorname{cl} P(K_1, K_2)$ at z and separates v from $\operatorname{cl} P(K_1, K_2)$. By Theorem 4, G separates K_1 and K_2. Denote by T_1 and T_2 the opposite closed halfspaces determined by G and containing K_1 and K_2, respectively. Expressing G as

$$G = \{x \in \mathrm{R}^n : x \cdot c = \eta\}, \quad c \neq o,$$

we may suppose that T_1 and T_2 are given by

$$T_1 = \{x \in \mathrm{R}^n : x \cdot c \leq \eta\} \ \text{ and } \ T_2 = \{x \in \mathrm{R}^n : x \cdot c \geq \eta\}.$$

Then $v \cdot c > \eta$ due to the inclusion $v \in \operatorname{int} T_2$. Furthermore, dividing both c and η by $\|c\|$, we may assume that c is a unit vector.

Since $x_r \in U_{1/r}(K_1)$, there is a point $b_r \in K_1$ such that $\|x_r - b_r\| < 1/r$. Consequently,

$$(x_r - b_r) \cdot c \leq \|x_r - b_r\| < 1/r.$$

Because $b_r \cdot c \leq \eta$ due to the inclusion $b_r \in K_1 \subset T_1$, we obtain that

$$x_r \cdot c \leq b_r \cdot c + 1/r < \eta + 1/r.$$

Similarly, $y_r \cdot c > \eta - 1/r$. Hence

$$\begin{aligned} v \cdot c &= (\mu_r x_r + (1 - \mu_r) y_r) \cdot c = \mu_r x_r \cdot c + (1 - \mu_r) y_r \cdot c \\ &< \mu_r(\eta + 1/r) + (1 - \mu_r)(\eta - 1/r) = \eta + (2\mu_r - 1)/r. \end{aligned}$$

Since $v \cdot c > \eta$, one has

$$\mu_r > \frac{r(v \cdot c - \eta) + 1}{2} \to \infty \ \text{ as } \ r \to \infty. \tag{11}$$

Because (10) and (11) contradict each other, our assumption on the existence of a point $u \in Q \setminus \operatorname{cl} P(K_1, K_2)$ is false, as desired. □

The following lemma will be used in the proof of Theorem 9.

Lemma 3. ([13]) *Let M_1 and M_2 be disjoint convex sets in R^n. If a point z belongs to the set $\mathrm{R}^n \setminus (P_0(M_1, M_2) \cup P_0(M_2, M_1))$, then the generated cones $C_z(M_1)$ and $C_z(M_2)$ satisfy the conditions*

$$C_z(M_1) \cap C_z(M_2) = \{z\} \quad \text{and} \quad \operatorname{rint} C_z(M_1) \cap \operatorname{rint} C_z(M_2) = \emptyset.$$

Theorem 9. *If convex sets K_1 and K_2 in R^n are strongly disjoint, then*

$$E_3(K_1, K_2) = \mathrm{R}^n \setminus (\operatorname{cl} P(K_1, K_2) \cup \operatorname{cl} P(K_2, K_1)).$$

Proof. First, we assert that

$$E_3(K_1, K_2) \cap \operatorname{cl} P(K_1, K_2) = \emptyset. \tag{12}$$

Indeed, H be a hyperplane strongly separating K_1 and K_2. By Theorem 3, H strongly separates $P(K_1, K_2)$ and $P(K_2, K_1)$. So, there is a scalar $\rho > 0$ such that H separates $U_\rho(P(K_1, K_2))$ and $U_\rho(P(K_2, K_1))$. Since the set

$U_\rho(P(K_1,K_2))$ is open, one has $H \cap U_\rho(P(K_1,K_2)) = \emptyset$. Finally, the inclusion $\mathrm{cl}\, P(K_1,K_2) \subset U_\rho(P(K_1,K_2))$ shows that $H \cap \mathrm{cl}\, P(K_1,K_2) = \emptyset$. Consequently, (12) holds.

In a similar way, $E_3(K_1,K_2) \cap \mathrm{cl}\, P(K_2,K_1) = \emptyset$. Therefore,

$$E_3(K_1,K_2) \subset \mathrm{R}^n \setminus (\mathrm{cl}\, P(K_1,K_2) \cup \mathrm{cl}\, P(K_2,K_1)).$$

For the opposite inclusion, choose any point

$$z \in \mathrm{R}^n \setminus (\mathrm{cl}\, P(K_1,K_2) \cup \mathrm{cl}\, P(K_2,K_1)).$$

Since K_1 and K_2 are strongly disjoint, there is a scalar $\rho > 0$ such that the sets $U_\rho(K_1)$ and $U_\rho(K_2)$ are strongly disjoint. By Theorem 8, the scalar ρ can be chosen so small that

$$z \in \mathrm{R}^n \setminus \big(P_0(U_\rho(K_1), U_\rho(K_2)) \cup P_0(U_\rho(K_2), U_\rho(K_1))\big).$$

Lemma 3 implies that

$$\mathrm{rint}\, C_z(U_\rho(K_1)) \cap \mathrm{rint}\, C_z(U_\rho(K_2)) = \emptyset.$$

Hence Lemma 2 shows the existence of a hyperplane $H \subset \mathrm{R}^n$ that properly separates the cones $C_z(U_\rho(K_1))$ and $C_z(U_\rho(K_2))$. Consequently, H separates the open sets $U_\rho(K_1)$ and $U_\rho(K_2)$. In other words, H strongly separates K_1 and K_2. So, $H \in \mathcal{H}_3(K_1,K_2)$ and $z \in H \subset E_3(K_1,K_2)$, as desired. □

Remark 6. An assertion similar to Theorem 7 and Corollary 1 does not hold for the case of strong separation. For instance, let the closed convex sets K_1 and K_2 in R^2 be given by

$$K_1 = \{(x,y) : x > 0, xy \geq 1\} \quad \text{and} \quad K_2 = \{(x,y) : x < 0, xy \geq 1\}.$$

Then $P(K_1,K_2) = K_1$ and $P(K_2,K_1) = K_2$. Furthermore, the coordinate axes of R^2 are subsets of $E_3(K_1,K_2)$ and meet $\mathrm{aff}\,(K_1 \cup K_2) = \mathrm{R}^2$, while each of these axes separates K_1 and K_2 strictly, but not strongly.

6. Further Properties of Penumbra

Theorem 10. *If K_1 and K_2 are convex sets in R^n, then*

$$P(K_1,K_2) = K_1 + C_o(K_1 - K_2).$$

Furthermore, $\dim C_o(K_1 - K_2) = \dim(\mathrm{aff}\,(K_1 \cup K_2))$.

Proof. By the definition of $P(K_1,K_2)$ and $C_o(K_1 - K_2)$,

$$\begin{aligned} P(K_1,K_2) &= \{\mu x_1 + (1-\mu)x_2 : \mu \geq 1, x_1 \in K_1, x_2 \in K_2\} \\ &= \{x_1 + (\mu - 1)(x_1 - x_2) : \mu \geq 1, x_1 \in K_1, x_2 \in K_2\} \\ &\subset \{x + \lambda y : \lambda \geq 0, x \in K_1,\ y \in K_1 - K_2\} \\ &= K_1 + C_o(K_1 - K_2). \end{aligned}$$

For the opposite inclusion, choose any point $u \in K_1 + C_o(K_1 - K_2)$. Then $u = x_1 + \lambda(y_1 - y_2)$ for suitable points $x_1, y_1 \in K_1$, $y_2 \in K_2$, and a scalar $\lambda \geq 0$. Put

$$x_1' = \frac{1}{\lambda+1}x_1 + \frac{\lambda}{\lambda+1}y_1.$$

Clearly, $x_1' \in [x_1, y_1] \subset K_1$ due to the convexity of K_1. Furthermore,

$$\begin{aligned} u &= x_1 + \lambda(y_1 - y_2) = (\lambda+1)\big(\frac{1}{\lambda+1}\,x_1 + \frac{\lambda}{\lambda+1}\,y_1\big) - \lambda y_2 \\ &= (1+\lambda)x_1' - \lambda y_2 \in P(K_1, K_2). \end{aligned}$$

For the second assertion, we observe first that

$$K_1 - K_2 \subset C_o(K_1 - K_2) \subset \operatorname{span}(K_1 - K_2),$$

which implies the equality $\operatorname{span} C_o(K_1 - K_2) = \operatorname{span}(K_1 - K_2)$. Consequently,

$$\dim C_o(K_1 - K_2) = \dim(\operatorname{span}(K_1 - K_2)) = \dim(\operatorname{aff}(K_1 \cup K_2)),$$

where the second equality is proved in [13]. □

We recall that a convex set $K \subset \mathrm{R}^n$ is called M-*predecomposable* if it can be expressed as a sum $K = B + C$, where B is a compact convex set B and C is a convex cone. If, additionally, the cone C is closed, then K is called M-*decomposable* (see [2,5,11,12] for priority publications and further results).

Theorem 11. *For convex sets K_1 and K_2 in R^n, the following assertions hold.*

1) If K_1 is compact, then $P(K_1, K_2)$ is an M-predecomposable set.

2) If both K_1 and K_2 are compact and $K_1 \cap K_2 = \emptyset$, then $P(K_1, K_2)$ is an M-decomposable set.

Proof. 1) If K_1 is compact, then, according to Theorem 10, the set $P(K_1, K_2)$ is M-decomposable as the sum of K_1 and the convex cone $C_o(K_1 - K_2)$.

2) If both K_1 and K_2 are compact and $K_1 \cap K_2 = \emptyset$, then $o \notin K_1 - K_2$ and the convex set $K_1 - K_2$ is compact. Consequently, the generated cone $C_o(K_1 - K_2)$ is closed (see [10], Theorem 5.45). By Theorem 10, the set $P(K_1, K_2)$ is M-decomposable. □

Problem 1. Characterize those M-decomposable (M-predecomposable) sets in R^n which are penumbras of suitable convex sets. In particular, which M-decomposable sets K_1 and K_2 satisfy the conditions $K_1 = P(K_1, K_2)$ and $K_2 = P(K_2, K_1)$?

The concept of penumbra is routinely extendable to the case of arbitrary nonempty sets. In this regard, the following assertion holds.

Theorem 12. *If X and Y are nonempty sets in R^n, then*

$$\operatorname{conv} P(X, Y) = P(\operatorname{conv} X, \operatorname{conv} Y).$$

Proof. The obvious inclusion $P(X, Y) \subset P(\operatorname{conv} X, \operatorname{conv} Y)$ and the convexity of $P(\operatorname{conv} X, \operatorname{conv} Y)$ (see Theorem 1) give

$$\operatorname{conv} P(X, Y) \subset P(\operatorname{conv} X, \operatorname{conv} Y).$$

For the opposite inclusion, choose any point $u \in P(\operatorname{conv} X, \operatorname{conv} Y)$. Then $u = \mu x + (1-\mu)y$, where $\mu \geq 1$, $x \in \operatorname{conv} X$, and $y \in \operatorname{conv} Y$. We can express x and y as convex combinations of suitable points from X and Y, respectively:

$$x = \alpha_1 x_1 + \cdots + \alpha_p x_p, \quad y = \beta_1 y_1 + \cdots + \beta_q y_q,$$
$$x_i \in X,\ y_j \in Y,\ \alpha_i \geq 0,\ \beta_j \geq 0,\ \sum_{i=1}^{p} \alpha_i = \sum_{j=1}^{q} \beta_j = 1.$$

Since

$$\mu x_i + (1-\mu) y_j \in \mu X + (1-\mu) Y, \quad 1 \leq i \leq p,\ 1 \leq j \leq q,$$

the equalities

$$u = \sum_{i=1}^{p} \sum_{j=1}^{q} \alpha_i \beta_j (\mu x_i + (1-\mu) y_j), \quad \alpha_i \beta_j \geq 0, \quad \sum_{i=1}^{p} \sum_{j=1}^{q} \alpha_i \beta_j = 1,$$

show that u is a convex combination of points from $\mu X + (1-\mu)Y$. By the definition of $P(X, Y)$, the set $\mu X + (1-\mu)Y$ is contained in $P(X, Y)$ for any choice of $\mu \geq 1$. Hence

$$u \in \operatorname{conv}(\mu X + (1-\mu)Y) \subset \operatorname{conv} P(X, Y).$$

□

We recall that a *polyhedron* is the intersection of finitely many closed halfspaces, and a *polytope* is a bounded polyhedron.

Theorem 13. *For convex sets K_1 and K_2 in R^n, the following assertions hold.*

1) *If both K_1 and K_2 in R^n are polyhedra, then* $\operatorname{cl} P(K_1, K_2)$ *is a polyhedron.*
2) *If both K_1 and K_2 are polytopes, then $P(K_1, K_2)$ is a polyhedron.*

Proof. 1) If both K_1 and K_2 are polyhedra, then (see, e. g., [10, Theorem 13.16]) they can be expressed as $K_i = \operatorname{conv}(X_i \cup Y_i)$, where $X_i = \{x_{i1}, \ldots, x_{ip_i}\}$ is a finite set of points and $Y_i = \{h_{i1}, \ldots, h_{iq_i}\}$ is a finite (possibly, empty) family of closed halflines, $i = 1, 2$. By Theorem 12,

$$P(K_1, K_2) = P(\operatorname{conv}(X_1 \cup Y_1), \operatorname{conv}(X_2 \cup Y_2)) \tag{13}$$
$$= \operatorname{conv} P((X_1 \cup Y_1), (X_2 \cup Y_2)). \tag{14}$$

Obviously, $P((X_1 \cup Y_1), (X_2 \cup Y_2))$ is the union of finitely many sets of the form

$$P(x_{1r}, x_{2s}), P(x_{1r}, h_{2s}), P(h_{1r}, x_{2s}), P(h_{1r}, h_{2s}), \tag{15}$$

where $x_{1r} \in X_1$, $x_{2s} \in X_2$, $h_{1r} \in Y_1$, and $h_{2s} \in Y_2$.

Elementary geometric arguments show that the sets from (15) can be described as follows (see illustration of cases c), d), and e) in Example 2 below).

a) $P(x_{1r}, x_{2s})$ is either a singleton or a closed halfline.
b) $P(x_{1r}, h_{2s})$ is either a closed halfline, a line, or the convex hull of the union of two closed halflines with common endpoint.
c) $P(h_{1r}, x_{2s})$ is either a closed halfline, a line, or the (nonclosed) convex hull of the union of two closed halflines with distinct endpoints.
d) If the halflines h_{1r} and h_{2s} are contained in a 2-dimensional plane, say L, then $P(h_{1r}, h_{2s})$ is either a closed halfline, a line, the convex hull of the union of two closed halflines, or the whole plane L.
e) If the halflines h_{1r} and h_{2s} are not contained in a 2-dimensional plane, then $P(h_{1r}, h_{2s})$ is a 3-dimensional convex set, which is the convex hull of the union of three closed halflines.

It it easy to show (see, e. g., [10, Theorem 4.2]) that, for any family $\{Z_\alpha\}$ of sets in R^n, one has

$$\operatorname{conv}(\cup Z_\alpha) = \operatorname{conv}(\cup \operatorname{conv} Z_\alpha).$$

This property of convex hulls and the above descriptions a)–e) of particular penumbras imply that $\operatorname{conv} P((X_1 \cup Y_1), (X_2 \cup Y_2))$ is the convex hull of the union of finitely many points and closed halflines. Consequently (see, e. g., [10, Theorem 13.18]), the closure of this set is a polyhedron. Equivalently, due to (14), $\operatorname{cl} P(K_1, K_2)$, is a polyhedron.

2) Suppose that both K_1 and K_2 are polytopes. Then the set $K_1 - K_2$ is a polytope, and $C_o(K_1 - K_2)$ is a polyhedral cone with apex o (see, e. g., [10, Theorem 5.46]). Theorem 10 shows that $P(K_1, K_2)$ is a polyhedron as the sum of the polytope K_1 and the polyhedral cone $C_o(K_1 - K_2)$ (see, e. g., [10, Theorem13.18]). □

The following example illustrates items c)–e) in the proof of Theorem 13.

Example 2. c) In R^2, if $h_{1r} = \{(x, 0) : x \geq 0\}$ and $x_{2s} = (0, 1)$, then the nonclosed set

$$P(h_{1r}, x_{2s}) = \{(0, 1)\} \cup \{(x, y) : x \leq 0, y > 1\}$$

is the convex hull of the union of closed halflines

$$g_1 = \{(0, y) : y \geq 1\}, \quad g_2 = \{(x, 2) : x \leq 0\}.$$

d) In R^2, if $h_{1r} = \{(x, 0) : x \geq 0\}$ and $h_{2s} = \{(x, 1) : x \leq 1\}$, then the set $P(h_{1r}, h_{2s})$ is R^2.

e) In R^3, if $h_{1r} = \{(x, 0, 0) : x \geq 0\}$ and $h_{2s} = \{(0, y, 1) : y \geq 0\}$, then the nonclosed set

$$P(h_{1r}, h_{2s}) = h_{2s} \cup \{(x, y, z) : x \leq 0, y \geq 0, z > 1\}$$

is the convex hull of the union of closed halflines

$$g_1 = \{(0, 0, z) : z \geq 1\}, \quad g_2 = h_{2s}, \quad g_3 = \{(x, 0, 2) : x \leq 0\}.$$

Problem 2. Characterize those polyhedra in R^n which are penumbras of suitable polyhedra. In particular, which polyhedra K_1 and K_2 satisfy the conditions $K_1 = P(K_1, K_2)$ and $K_2 = P(K_2, K_1)$?

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