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Results in Mathematics

On the Periodicity of Entire Functions

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Abstract. The purpose of this paper is mainly to prove that if f is a transcendental entire function of hyper-order strictly less than 1 and $f(z)^n + a_1 f'(z) + \cdots + a_k f^{(k)}(z)$ is a periodic function, then $f(z)$ is also a periodic function, where n, k are positive integers, and $a_1, \cdots, a_k$ are constants. Meanwhile, we offer a partial answer to Yang's Conjecture, theses results extend some previous related theorems.

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1. Introduction and Main Results

Herein let f denote a non-constant meromorphic function and we assume that the reader is familiar with the fundamental results of Nevanlinna theory and its standard notation such as $m(r, f)$, $N(r, f)$, $T(r, f)$, etc (see e.g., [4] and [11]). In the sequel, $S(r, f)$ will be used to denote a quantity that satisfies $S(r, f) = o\big(T(r, f)\big)$ as $r \to \infty$, outside possibly an exceptional set of r values of finite linear measure, and a meromorphic function a is said to be a small function of f if $T(r, a) = S(r, f)$. We use $\rho(f)$ and $\rho_2(f)$ to denote the order and hyper-order of f respectively.

The convergence exponent of zeros of f is defined as

$$\tau(f) = \limsup_{r\to\infty} \frac{\log N(r, \frac{1}{f})}{\log r} = \limsup_{r\to\infty} \frac{\log n(r, \frac{1}{f})}{\log r}.$$

In addition, a complex number a is said to be a Borel exceptional value of f if

$$\limsup_{r\to\infty} \frac{\log^+ n\left(r, \frac{1}{f-a}\right)}{\log r} < \rho(f).$$

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In this note, we mainly consider the periodicity of entire functions, namely, if $f(z)^n + a_1 f'(z) + \cdots + a_k f^{(k)}(z)$ is a periodic function, then $f(z)$ is also a periodic function.

The motivation of this paper arises from the study of the real transcendental entire solutions of the differential equation

$$f(z)f^{(k)}(z) = p(z)\sin^2 z,$$

where $p(z)$ is a non-zero polynomial. It seems to us that Titchmarsh [9] firstly proved that the differential equation $f(z)f''(z) = -\sin^2 z$ has no real entire solutions of finite order other than $f(z) = \pm \sin z$. The follow-up works were due to Li, Lü and Yang in [8], where they considered the similar problem when $f(z)$ is real and of finite order. They obtained $f(z)f''(z) = -\sin^2 z$ has entire solutions $f(z) = \pm \sin z$ and no other solutions. Recently, Yang proposed the following interesting conjecture, see e.g., [8] and [10].

Yang's Conjecture. Let f be a transcendental entire function and $k\,(\geq 1)$ be an integer. If $f(z)f^{(k)}(z)$ is a periodic function, then $f(z)$ is also a periodic function.

From then on, a number of papers have focused on Yang's Conjecture, see e.g., [6,7] and references therein.

Recently, regarding Yang's Conjecture, Liu et al. [5] obtained the following result.

Theorem A. *Let f be a transcendental entire function and n, k be positive integers. If $f(z)^n f^{(k)}(z)$ is a periodic function and one of the following conditions is satisfied*

(i) $k = 1$;
(ii) $f(z) = \mathrm{e}^{h(z)}$*, where h is a non-constant polynomial;*
(iii) f has a non-zero Picard exceptional value and f is of finite order,

then $f(z)$ is also a periodic function.

A natural question would arise: what will happen if we drop the condition " finite order " in Theorem A. In this note, by considering a different proofs, we obtain the following result, which offers a partial answer to Yang's Conjecture, and improves Theorem A and references therein.

Theorem 1.1. *Let f be a transcendental entire function of hyper-order strictly less than* 1*, and n, k be positive integers. Suppose that $f(z)$ has a finite Borel exceptional value b, and $f(z)^n f^{(k)}(z)$ is a periodic function, then $f(z)$ is also a periodic function.*

Remark 1.1. If b is a Picard exceptional of f, then b is a Borel exceptional of f.

In addition, Liu et al. [5] also obtained the following result.

Theorem B. *Let f be a transcendental entire function and $n \geq 2$, $k \geq 1$ be integers. If $f(z)^n + f^{(k)}(z)$ is a periodic function with period c and one of the following conditions is satisfied*

(i) $k = 1$;
(ii) $f(z+c) - f(z)$ *has no zeros;*
(iii) *the zeros multiplicity of* $f(z+c) - f(z)$ *is great than or equal to* k; *then* $f(z)$ *is also a periodic function with period* c *or* $2c$.

In this paper, we will prove the following result.

Theorem 1.2. *Let f be a transcendental entire function of hyper-order strictly less than 1, and $n\,(\geq 2)$, $k\,(\geq 1)$ be integers. If $f(z)^n + a_1 f'(z) + \cdots + a_k f^{(k)}(z)$ is a periodic function, where $a_1, \cdots, a_k$ are constants, then $f(z)$ is also a periodic function.*

Remark 1.2. (i) The condition " $n \geq 2$ " in Theorem 1.2 is necessary. For example, let $f(z) = z\mathrm{e}^{-z}$. Then

$$f(z) + f'(z) + f''(z) + f'''(z) = 2\mathrm{e}^{-z}$$

is a periodic function, however $f(z) = z\mathrm{e}^{-z}$ is not a periodic function.
(ii) Carefully checking the proof of Theorem 1.2, we may find when $n = 2$ or $n \geq 4$, the hypothesis " $\rho_2(f) < 1$ " can be removed from Theorem 1.2.

2. Lemmas

In order to prove our results, we need the following lemmas.

Lemma 2.1 (see, e.g., [3]). *Let f be a non-constant meromorphic function with $\rho_2(f) < 1$, $c \in \mathbb{C}$. Then*

$$m\Big(r, \frac{f(z+c)}{f(z)}\Big) = S(r,f),$$

outside of a possible exceptional set with finite logarithmic measure.

It is pointed out that if f is of finite order, we have

Lemma 2.1' (see, e.g., [2]). Let f be a meromorphic function with $\rho = \rho(f) < +\infty$, $c\,(\neq 0) \in \mathbb{C}$. Then for each $\varepsilon > 0$, we have

$$m\Big(r, \frac{f(z+c)}{f(z)}\Big) = O(r^{\rho-1+\varepsilon}).$$

By applying Lemma 2.1 and the Logarithmic Derivative Lemma, we have the following result.

Lemma 2.2. *Let f be a non-constant meromorphic function with $\rho_2(f) < 1$. Then for $c \in \mathbb{C}$ and any positive integer k, we have*

$$m\Big(r, \frac{f^{(k)}(z+c)}{f(z)}\Big) = S(r,f),$$

outside of a possible exceptional set with finite logarithmic measure.

Lemma 2.3 ([11], Lemma 5.1). *Let f denote a non-constant periodic function. Then $\rho(f) \geq 1$.*

Lemma 2.4 ([1]). *Let g be a function transcendental and meromorphic in the plane of order less than 1, and $h > 0$. Then there exists an ε-set E such that*

$$\frac{g'(z+c)}{g(z+c)} \to 0, \quad \frac{g(z+c)}{g(z)} \to 1 \quad as \quad z \to \infty \ in \ \mathbb{C}\backslash E,$$

uniformly in c for $|c| \leq h$. Further, E may be chosen so that for large z not in E the function g has no zeros or poles in $|\zeta - z| \leq h$.

Remark 2.1. According to the works of Hayman (see, e.g., [4]), an ε set E is defined to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. Suppose that E is an ε set, then the set of $r \geq 1$ for which the circle $S(0,r)$ meets E has finite logarithmic measure and for almost all real θ the intersection of E with the ray $\arg z = \theta$ is bounded.

Lemma 2.5. ([11], Theorem 1.62) *Suppose that $f_j (j = 1, 2, \cdots, n)$ $(n \geq 3)$ are meromorphic functions which are not constants except for f_n. Furthermore, let*

$$\sum_{j=1}^{n} f_j = 1.$$

If $f_n \not\equiv 0$ and

$$\sum_{j=1}^{n} N(r, \frac{1}{f_j}) + (n-1)\sum_{j=1}^{n} \overline{N}(r, f_j) < (\lambda + o(1))T(r, f_k),$$

where $r \in I$, I is a set whose linear measure is infinite, $k \in \{1, 2, \cdots, n-1\}$ and $\lambda < 1$, then $f_n \equiv 1$.

3. Proof of Theorem 1.1

Note that b is a finite Borel exceptional value of f. Next, two cases will be considered.

Case 1. If $b = 0$, by the Hadamard factorization theorem, we get

$$f(z) = Q(z)e^{p(z)},$$

where Q is the canonical product of f formed with its zeros, and p is a non-constant entire function satisfying $\rho(p) < 1$. Using the facts (see., e.g. [11], Theorem 2.2 and Theorem 2.3), it is easy to deduce that

$$\rho(Q) = \tau(Q) = \tau(f) < \rho(f).$$

Thus, $\rho(f) = \rho(\mathrm{e}^p)$. Besides, since $f(z)^n f^{(k)}(z)$ is a periodic function with period c, then

$$f(z)^n f^{(k)}(z) = f(z+c)^n f^{(k)}(z+c). \tag{3.1}$$

Substituting $f(z) = Q(z)\mathrm{e}^{p(z)}$ into (3.1), it follows without difficulty that

$$[Q(z)\mathrm{e}^{p(z)}]^n \mathrm{e}^{p(z)} H_1(z) = [Q(z+c)\mathrm{e}^{p(z+c)}]^n \mathrm{e}^{p(z+c)} H_1(z+c), \tag{3.2}$$

where H_1 is a differential polynomial of Q and p, namely,

$$H_1(z) = Q^{(k)}(z) + A_1 Q^{(k-1)}(z)p'(z) + A_2 Q^{(k-2)}(z)[p''(z)]^2 + \cdots + Q(z)p^{(k)}(z)$$

with constants $A_i\,(i = 1, 2, \cdots)$.

Thereby, $\rho(H_1) \leq \max\{\rho(Q), \rho(p)\} < \rho(f)$.

Now, we can rewrite (3.2) as

$$\mathrm{e}^{(n+1)[p(z)-p(z+c)]} = \frac{H_1(z+c)}{H_1(z)} \frac{Q(z+c)^n}{Q(z)^n}. \tag{3.3}$$

In addition, (3.3) shows that $\rho(\mathrm{e}^{p(z)-p(z+c)}) < +\infty$ since $\rho(H_1) < +\infty$, $\rho(Q) < +\infty$. This implies $p(z) - p(z+c)$ is a polynomial, say $p(z) - p(z+c) = q_0 z^m + \cdots + q_m$, where m is a natural number and $q_0, \cdots, q_m$ are constants.

If $m > 1$, then $p^{(m+1)}(z) - p^{(m+1)}(z+c) \equiv 0$, which implies $p^{(m+1)}$ is a periodic function. Therefore, Lemma 2.3 and $\rho(p^{(m+1)}) = \rho(p) < 1$ show that $p^{(m+1)}(z)$ is a constant, this leads to p is a polynomial, say $p(z) = a_0 z^{m+1} + \cdots + a_{m+1}$. In this case, it is easy to see $\rho(f) = m+1$, and $\rho(p) = 0$.

Set $\rho(Q) = \sigma$. Then $\rho(H_1) \leq \sigma$, and $\sigma < m+1$.

Again, applying Lemma 2.1' to (3.3), we obtain

$$m(r, \mathrm{e}^{(n+1)[p(z)-p(z+c)]}) = m\Big(r, \frac{H_1(z+c)}{H_1(z)} \frac{Q(z+c)^n}{Q(z)^n}\Big),$$

which implies $r^m \leq O(r^{\sigma-1+\varepsilon})$. This is impossible since we can choose $\varepsilon > 0$ small enough such that $\sigma - 1 + \varepsilon < m$.

Thus, $p(z) = a_0 z + a_1$, where a_0, a_1 are constants. Furthermore, if we set $\mathrm{e}^{(n+1)a_0 c} = A$, then

$$A = \frac{H_1(z+c)}{H_1(z)} \frac{Q(z+c)^n}{Q(z)^n}.$$

On the other hand, by Lemma 2.4, there exists a $\varepsilon-$set E such that

$$\frac{H_1(z+c)}{H_1(z)} \to 1, \quad \frac{Q(z+c)}{Q(z)} \to 1, \quad \text{as} \;\; z \to \infty \;\text{in}\;\; \mathbb{C}\backslash E.$$

Trivially, $A=1$, and

$$\frac{H_1(z+c)}{H_1(z)}\frac{Q(z+c)^n}{Q(z)^n}=1.$$

It means that $H_1(z)Q(z)^n$ is a periodic function. Hence $\rho(H_1(z)Q(z)^n)\geq 1$ if $H_1(z)Q(z)^n$ is not a constant. It follows by $\rho(H_1(z)Q(z)^n)<1$ that $H_1(z)Q(z)^n$ is a constant. Therefore, Q must be a constant. Thus, we conclude that $f(z)$ must be a periodic function with period $\frac{2\pi \mathrm{i}}{a_0}$.

Case 2. If $b\neq 0$, then by the Hadamard factorization theorem, we get

$$f(z)=Q(z)\mathrm{e}^{p(z)}+b,$$

where Q is the canonical product of $f-b$ formed with its zeros, and p is a non-constant entire function satisfying $\rho(p)<1$. Using the same methods as the proof in **Case 1**, $\rho(Q)=\tau(Q)=\tau(f-b)<\rho(f-b)=\rho(f)$ follows. Thus, $\rho(f)=\rho(\mathrm{e}^p)$.

Since $f(z)^nf^{(k)}(z)$ is a periodic function with period c, then

$$f(z)^nf^{(k)}(z)=f(z+c)^nf^{(k)}(z+c). \tag{3.4}$$

Substituting $f(z)=Q(z)\mathrm{e}^{p(z)}+b$ into (3.4), we have

$$[Q(z)\mathrm{e}^{p(z)}+b]^n\mathrm{e}^{p(z)}H_1(z)=[Q(z+c)\mathrm{e}^{p(z+c)}+b]^n\mathrm{e}^{p(z+c)}H_1(z+c),$$

where H_1 is a differential polynomial of Q and p, namely,

$$H_1(z)=Q^{(k)}(z)+A_1Q^{(k-1)}(z)p'(z)+A_2Q^{(k-2)}(z)[p''(z)]^2+\cdots+Q(z)p^{(k)}(z)$$

with constants $A_i(i=1,2,\cdots)$. In this case, we conclude

$$\rho(H_1)\leq\max\{\rho(Q),\rho(p)\}<\rho(f).$$

Besides, we find

$$\begin{aligned}&Q(z)^n\mathrm{e}^{(n+1)p(z)}+C_n^1bQ(z)^{n-1}\mathrm{e}^{np(z)}+\cdots+C_n^{n-1}b^{n-1}Q(z)\mathrm{e}^{2p(z)}+b^n\mathrm{e}^{p(z)}=\\&\quad H(z)[Q(z+c)^n\mathrm{e}^{(n+1)p(z+c)}+C_n^1bQ(z+c)^{n-1}\mathrm{e}^{np(z+c)}\\&\quad+\cdots+C_n^{n-1}b^{n-1}Q(z+c)\mathrm{e}^{2p(z+c)}+b^n\mathrm{e}^{p(z+c)}],\end{aligned}$$

where $H(z)=\frac{H_1(z+c)}{H_1(z)}$, and $\rho(H)<\rho(f)$.

Dividing both sides of the above equation by $b^n\mathrm{e}^{p(z)}$ gives

$$\begin{aligned}&\frac{H(z)}{b^n}\mathrm{e}^{(n+1)p(z+c)-p(z)}Q(z+c)^n+\cdots+H(z)\mathrm{e}^{p(z+c)-p(z)}\\&\quad-\frac{Q(z)^n}{b^n}\mathrm{e}^{np(z)}-\cdots-C_n^{n-1}\frac{Q(z)}{b}\mathrm{e}^{p(z)}=1.\end{aligned} \tag{3.5}$$

Next, we will prove $mp(z+c)-p(z)\,(m=2,\cdots,n+1)$ are not constants. In fact, if p is a non-constant polynomial, it is obvious. Now, we assume that p is a transcendental entire function. In this case, if $mp(z+c)-p(z)=q$, here q is a constant, then $mp'(z+c)=p'(z)$. Noting $\rho(p)=\rho(p')<1$, we apply Lemma 2.4 to p' and obtain $m=1$, a contradiction. Thereby, $mp(z+c)-p(z)(m=2,\cdots,n+1)$ can not be constants. To complete the proof, we now employ Lemma 2.5 to (3.5) and have $H(z)\mathrm{e}^{p(z+c)-p(z)} \equiv 1$. It means that $H_1(z+c)\mathrm{e}^{p(z+c)} = H_1(z)\mathrm{e}^{p(z)}$, and

$$[b + Q(z)\mathrm{e}^{p(z)}]^n = [b + Q(z+c)\mathrm{e}^{p(z+c)}]^n$$

follows. Thus $f(z)^n = f(z+c)^n$ shows that f is a periodic function with period c or nc.

This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

By assumption, $f(z)^n + a_1 f'(z) + \cdots + a_k f^{(k)}(z)$ is a periodic function with period c, then

$$f(z+c)^n + a_1 f'(z+c) + \cdots + a_k f^{(k)}(z+c) = f(z)^n + a_1 f'(z) + \cdots + a_k f^{(k)}(z),$$

and thus

$$f(z+c)^n - f(z)^n = a_1[f'(z) - f'(z+c)] + \cdots + a_k[f^{(k)}(z) - f^{(k)}(z+c)]. \tag{4.1}$$

Next, we consider three cases.

Case 1. $n = 2$. In this case, we can rewrite (4.1) as

$$\begin{aligned} &[f(z+c) + f(z)][f(z+c) - f(z)] \\ &\quad = a_1[f'(z) - f'(z+c)] + \cdots + a_k[f^{(k)}(z) - f^{(k)}(z+c)]. \end{aligned} \tag{4.2}$$

If $f(z+c) - f(z) \equiv 0$, then $f(z)$ is a periodic function with period c.

Next, we may assume that $f(z+c) - f(z) \not\equiv 0$. In this case, (4.2) can be rewritten as

$$\begin{aligned} f(z+c) + f(z) &= -a_1 \frac{f'(z) - f'(z+c)}{f(z) - f(z+c)} - \cdots - a_k \frac{f^{(k)}(z) - f^{(k)}(z+c)}{f(z) - f(z+c)} \\ &= -a_1 \frac{g'(z)}{g(z)} - \cdots - a_k \frac{g^{(k)}(z)}{g(z)}, \end{aligned} \tag{4.3}$$

where

$$g(z) = f(z) - f(z+c). \tag{4.4}$$

Define $p_i(z) = \frac{g^{(i)}(z)}{g(z)}$ $(i = 1, 2, \cdots, k)$, and

$$H(z) = -a_1 p_1(z) - \cdots - a_k p_k(z). \tag{4.5}$$

Then (4.3) becomes

$$f(z+c) + f(z) = H(z). \tag{4.6}$$

Besides, applying the Logarithmic Derivative Lemma to (4.3), we have

$$T(r,H) = m(r,H) \le m(r,p_1) + \cdots + m(r,p_k) + O(1) \le S(r,g). \tag{4.7}$$

Combining (4.4) and (4.6) yields that

$$f(z)=\frac{1}{2}[H(z)+g(z)],\ f(z+c)=\frac{1}{2}[H(z)-g(z)].$$

Thus, a routine computation leads to

$$g(z)+g(z+c)=H(z)-H(z+c). \tag{4.8}$$

Moreover, (4.8) results in

$$g^{(i)}(z)+g^{(i)}(z+c)=H^{(i)}(z)-H^{(i)}(z+c). \tag{4.9}$$

Thereby, it follows by $g^{(i)}(z)=p_i(z)g(z)$ and (4.9) that

$$\begin{cases} p_1(z)g(z)+p_1(z+c)g(z+c)=H'(z)-H'(z+c), \\ p_2(z)g(z)+p_2(z+c)g(z+c)=H''(z)-H''(z+c), \\ \cdots\cdots \\ p_k(z)g(z)+p_k(z+c)g(z+c)=H^{(k)}(z)-H^{(k)}(z+c). \end{cases} \tag{4.10}$$

Now, combining (4.5) and (4.10) yields

$$\begin{aligned} H(z)g(z)+H(z+c)g(z+c) &= -a_1[H'(z)-H'(z+c)] \\ &\quad -\cdots-a_k[H^{(k)}(z)-H^{(k)}(z+c)]. \end{aligned} \tag{4.11}$$

Furthermore, substituting $g(z+c)=H(z)-H(z+c)-g(z)$ in (4.11), we find

$$\begin{aligned} &H(z)g(z)+H(z+c)[H(z)-H(z+c)-g(z)] \\ &\quad =-a_1[H'(z)-H'(z+c)]-\cdots-a_k[H^{(k)}(z)-H^{(k)}(z+c)]. \end{aligned}$$

If $H(z+c)\not\equiv H(z)$, we obtain

$$g(z)=\frac{-a_1[H'(z)-H'(z+c)]-\cdots-a_k[H^{(k)}(z)-H^{(k)}(z+c)]+H(z+c)^2-H(z+c)H(z)}{H(z)-H(z+c)}. \tag{4.12}$$

It follows from (4.7), (4.12) and Lemma 2.2 that

$$T(r,g)\leq S(r,g),$$

which is impossible. Hence, $H(z+c)=H(z)$, and (4.8) gives $g(z)+g(z+c)=0$, this implies that f is a periodic function with period $2c$.

Case 2. $n=3$. Now, we can rewrite (4.1) as

$$\begin{aligned} &[f(z+c)-f(z)][f(z+c)-\eta f(z)][f(z+c)-\eta^2 f(z)] \\ &\quad =a_1[f'(z)-f'(z+c)]+\cdots+a_k[f^{(k)}(z)-f^{(k)}(z+c)], \end{aligned} \tag{4.13}$$

where $\eta(\neq 1)$ is a cube-root of the unity.

If $f(z+c)-f(z)\equiv 0$, then f is a periodic function with period c.

If $f(z+c)-f(z)\not\equiv 0$, (4.13) can be rewritten as

$$\begin{aligned}&[f(z+c)-\eta f(z)][f(z+c)-\eta^2 f(z)]\\&\quad=-a_1\frac{f'(z)-f'(z+c)}{f(z)-f(z+c)}-\cdots-a_k\frac{f^{(k)}(z)-f^{(k)}(z+c)}{f(z)-f(z+c)}\\&\quad=-a_1\frac{g'(z)}{g(z)}-\cdots-a_k\frac{g^{(k)}(z)}{g(z)},\end{aligned}\tag{4.14}$$

where

$$g(z)=f(z)-f(z+c).\tag{4.15}$$

Define $p_i(z)=\frac{g^{(i)}(z)}{g(z)}$, $i=1,2,\cdots,k$, and $H(z)=-a_1p_1(z)-\cdots-a_kp_k(z)$. Then (4.14) becomes

$$[f(z+c)-\eta f(z)][f(z+c)-\eta^2 f(z)]=H(z).\tag{4.16}$$

Besides, the Logarithmic Derivative Lemma gives

$$\begin{aligned}T(r,H)=m(r,H)&\le m(r,p_1)+\cdots+m(r,p_k)+O(1)\\&=O\Big(\log(rT(r,g))\Big)\,(r\to\infty, r\notin E_0),\end{aligned}$$

where E_0 is a set whose linear measure is not greater than 2.

Note that $\rho_2(f)<1$, $T(r,f(z+c))=T(r,f(z))+S(r,f)$ (see, e.g., [2] and [3]). By making use of (4.15), it is easy to see that $T(r,g)\le O(T(r,f))$. It clearly follows by $\rho_2(f)<1$ that $\log T(r,f)\le r^{\lambda}$, where $\lambda\,(<1)$ is a positive number. Hence, $T(r,H)\le O(\log rT(r,g))\le O(r^{\lambda})$, which implies that $\rho(H)<1$. In addition, by the Hadamard factorization theorem, (4.16) can be changed as

$$f(z+c)-\eta f(z)=\Pi_1(z)\mathrm{e}^{\alpha(z)}\tag{4.17}$$

and

$$f(z+c)-\eta^2 f(z)=\Pi_2(z)\mathrm{e}^{-\alpha(z)},\tag{4.18}$$

where α is a non-constant entire function satisfying $\rho(\alpha)<1$, $\Pi_1(z)$ is the canonical product of $f(z+c)-\eta f(z)$ formed with its zeros, $\Pi_2(z)$ is the canonical product of $f(z+c)-\eta^2 f(z)$ formed with its zeros, and $\Pi_1(z)$, $\Pi_2(z)$ satisfy

$$\Pi_1(z)\Pi_2(z)=H(z).\tag{4.19}$$

Using Theorem 2.2 and Theorem 2.3 in [11] , it is easy to deduce that

$$\rho(\Pi_1)=\tau(\Pi_1)\le\tau(H)\le\rho(H)<1.$$

By applying the same analysis, we can easily conclude the following result

$$\rho(\Pi_2)<1.$$

Combining (4.17) and (4.18) yields

$$f(z) = \frac{\Pi_1(z)e^{\alpha(z)} - \Pi_2(z)e^{-\alpha(z)}}{\eta(\eta-1)}, \tag{4.20}$$

$$f(z+c) = \frac{\eta\Pi_1(z)e^{\alpha(z)} - \Pi_2(z)e^{-\alpha(z)}}{\eta-1}. \tag{4.21}$$

Thus, a routine computation leads to

$$\eta^2\Pi_1(z)e^{\alpha(z)} - \eta\Pi_2(z)e^{-\alpha(z)} = \Pi_1(z+c)e^{\alpha(z+c)} - \Pi_2(z+c)e^{-\alpha(z+c)}. \tag{4.22}$$

Now, dividing (4.22) by $\eta^2\Pi_1(z)e^{\alpha(z)}$, we obtain

$$\frac{1}{\eta}\frac{\Pi_2(z)}{\Pi_1(z)}e^{-2\alpha(z)} + \frac{1}{\eta^2}\frac{\Pi_1(z+c)}{\Pi_1(z)}e^{\alpha(z+c)-\alpha(z)} - \frac{1}{\eta^2}\frac{\Pi_2(z+c)}{\Pi_1(z)}e^{-\alpha(z+c)-\alpha(z)} = 1. \tag{4.23}$$

Since α is a non-constant entire function with $\rho(\alpha) < 1$, then $-\alpha(z+c) - \alpha(z)$ is not a constant. Otherwise, if $-\alpha(z+c) - \alpha(z)$ is a constant, then $\alpha'(z)$ is a periodic function, and $\rho(\alpha') = \rho(\alpha) \geq 1$, a contradiction. Now, applying Lemma 2.5 to (4.23) yields

$$\frac{1}{\eta^2}\frac{\Pi_1(z+c)}{\Pi_1(z)}e^{\alpha(z+c)-\alpha(z)} \equiv 1. \tag{4.24}$$

On the other hand, dividing (4.22) by $\eta\Pi_2(z)e^{-\alpha(z)}$ implies

$$\eta\frac{\Pi_1(z)}{\Pi_2(z)}e^{2\alpha(z)} - \frac{1}{\eta}\frac{\Pi_1(z+c)}{\Pi_2(z)}e^{\alpha(z+c)+\alpha(z)} + \frac{1}{\eta}\frac{\Pi_2(z+c)}{\Pi_2(z)}e^{-\alpha(z+c)+\alpha(z)} = 1. \tag{4.25}$$

Obviously, $2\alpha(z)$ and $\alpha(z+c)+\alpha(z)$ are not constants. Armed with Lemma 2.5 and (4.25), we deduce

$$\frac{1}{\eta}\frac{\Pi_2(z+c)}{\Pi_2(z)}e^{-\alpha(z+c)+\alpha(z)} \equiv 1. \tag{4.26}$$

Combining (4.24) and (4.26) yields

$$\Pi_1(z+c)\Pi_2(z+c) = \Pi_1(z)\Pi_2(z).$$

This suggests that $H(z+c) = H(z)$. We conclude from $\rho(H) < 1$ that H must be a constant. Thus, $f(z+c) - \eta f(z)$, $f(z+c) - \eta^2 f(z)$ have no zeros, which shows that $\Pi_1(z)$, $\Pi_2(z)$ are constants. It follows from (4.24) and (4.26) that

$$e^{\alpha(z+c)-\alpha(z)} = \eta^2 \quad \text{and} \quad e^{-\alpha(z+c)+\alpha(z)} = \eta,$$

this results in

$$\alpha(z+c) - \alpha(z) \equiv C, \tag{4.27}$$

where C is a constant. Differentiating (4.27) yields $\alpha'(z+c) - \alpha'(z) \equiv 0$, namely $\alpha'(z)$ is a periodic function. Noting $\rho(\alpha') = \rho(\alpha) < 1$, we know $\alpha'(z)$ must be a constant, say, A. Thus, $\alpha(z) = Az + B$ with a constant B. By (4.20), we see that f is a periodic function with period $\frac{2\pi \mathrm{i}}{A}$.

Case 3. $n \geq 4$. To complete the proof, we rewrite (4.1) as

$$\begin{aligned} &[f(z+c)^{n-1} + f(z+c)^{n-2}f(z) + \cdots + f(z)^{n-1}][f(z+c) - f(z)] \\ &\quad = a_1[f'(z) - f'(z+c)] + \cdots + a_k[f^{(k)}(z) - f^{(k)}(z+c)]. \end{aligned} \tag{4.28}$$

If $f(z+c) - f(z) \equiv 0$, then f is a periodic function with period c.

If $f(z+c) - f(z) \not\equiv 0$, we change (4.28) into

$$\begin{aligned} &f(z+c)^{n-1} + f(z+c)^{n-2}f(z) + \cdots + f(z)^{n-1} \\ &\quad = -a_1 \frac{f'(z) - f'(z+c)}{f(z) - f(z+c)} - \cdots - a_k \frac{f^{(k)}(z) - f^{(k)}(z+c)}{f(z) - f(z+c)} \\ &\quad = -a_1 \frac{g'(z)}{g(z)} - \cdots - a_k \frac{g^{(k)}(z)}{g(z)}, \end{aligned} \tag{4.29}$$

where

$$g(z) = f(z) - f(z+c). \tag{4.30}$$

Define $p_i(z) = \frac{g^{(i)}(z)}{g(z)}$, $i = 1, 2, \cdots, k$, and $H(z) = -a_1 p_1(z) - \cdots - a_k p_k(z)$. Then (4.29) becomes

$$f(z+c)^{n-1} + f(z+c)^{n-2}f(z) + \cdots + f(z)^{n-1} = H(z). \tag{4.31}$$

Now, using the Logarithmic Derivative Lemma, we have

$$T(r, H) = m(r, H) \leq m(r, p_1) + \cdots + m(r, p_k) + O(1) \leq S(r, g).$$

By (4.30), we conclude

$$f(z)\Big(1 - \frac{f(z+c)}{f(z)}\Big) = g(z). \tag{4.32}$$

Moreover, (4.31) gives

$$f(z)^{n-1}\left(\frac{f(z+c)^{n-1}}{f(z)^{n-1}} + \frac{f(z+c)^{n-2}}{f(z)^{n-2}} + \cdots + \frac{f(z+c)}{f(z)} + 1\right) = H(z). \tag{4.33}$$

Set $\omega(z) = \frac{f(z+c)}{f(z)}$. Obviously, $\omega \not\equiv 1$. Combining (4.32) and (4.33) yields

$$\frac{(1 - \omega(z))^{n-1}}{\omega(z)^{n-1} + \omega(z)^{n-2} + \cdots + \omega(z) + 1} = \frac{g(z)^{n-1}}{H(z)},$$

and so

$$(n-1)T(r, \omega) = (n-1)T(r, g) + S(r, g).$$

Therefore, equation (4.33) implies

$$\omega(z)^{n-1} + \omega(z)^{n-2} + \cdots + \omega(z) + 1 = H(z)\frac{1}{f(z)^{n-1}},$$

and, consequently

$$N\Big(r, \frac{1}{\omega^{n-1}+\omega^{n-2}+\cdots+\omega+1}\Big) = N\Big(r, \frac{1}{H}\Big) \le T(r, H) = S(r, g) = S(r, \omega).$$

Now, applying the second main theorem gives

$$\begin{aligned}(n-2)T(r,\omega) &\le N\Big(r, \frac{1}{\omega-1}\Big) + N\Big(r, \frac{1}{\omega^{n-1}+\omega^{n-2}+\cdots+\omega+1}\Big) + S(r,\omega)\\ &\le N\Big(r, \frac{1}{\omega-1}\Big) + S(r,\omega)\\ &\le T\Big(r, \frac{1}{\omega-1}\Big) + S(r,\omega).\end{aligned}$$

Thus, $\omega\ (\not\equiv 1)$ must be a constant. It follows by (4.32) that $T(r, f) = T(r, g) + S(r, f)$. A contradiction follows by (4.33) and $(n-1)T(r, f) = T(r, H) + S(r, f) = S(r, g) = S(r, f)$ since $n \ge 4$.

This finishes the proof of Theorem 1.2.

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