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Results in Mathematics

An Isoperimetric Inequality for Isoptic Curves of Convex Bodies

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Abstract. Given a convex body in the plane, we define the isoptic curve of angle α, for an arbitrary but fixed angle α, as the curve from which K is always seen under an angle α In this article we prove an inequality between the perimeter of any convex body in the plane and its isotopic curve. Moreover, we also prove some characterizations of the Euclidean disc by means of the constancy of some elements of the isoptic curve.

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Keywords. Isoptic curve, Isoperimetric inequality, Fourier series, Euclidean disc.

1. Introduction

Given a convex body K in the plane, i.e., a compact, convex set with non-empty interior, we are interested in a special curve associated to K known as *isoptic curve* (see, for instance, [2] and p. 271 in [8]). For a fixed number $\alpha \in (0, \pi)$, the *isoptic of angle* α or α*-isoptic*, denoted by K_α, is defined as the set of points in the plane from which K is seen under the constant angle α. We are interested in the geometric relations between a given convex body and its isoptic curves, for instance: how are the perimeters of K and K_α related? What can be said about the areas?

In this work we denote the *area* of K by $A(K)$ and its *perimeter* by $L(\gamma)$, where γ is a curve that parametrizes ∂K. Also, γ_α will denote a curve that parametrizes ∂K_α.

The following isoperimetric type inequality was proved in [5]:

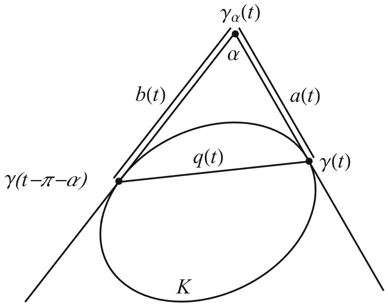


FIGURE 1. Parameters of the isoptic curve

Theorem GGJ *Let K be a convex body in the plane and let $\alpha \in (0, \pi)$. Then*

$$A(K_\alpha) \geq \frac{1}{\left(\sin \frac{\alpha}{2}\right)^2} \cdot A(K),$$

with equality if and only if K is a Euclidean disc.

In this paper we prove the following statement for the case of perimeters.

Theorem 1. *Let $K \subset \mathbb{R}^2$ be a strictly convex body and let $\alpha \in (0, \pi)$ be a fixed angle. Then we have*

$$\frac{1}{\sin \frac{\alpha}{2}} \cdot L(\gamma) \leq L(\gamma_\alpha) < \frac{1}{(\sin \frac{\alpha}{2})^2} \cdot L(\gamma),$$

with equality in the left side if and only if K is the constant angle caustic for the curve K_α, with respect to the angle $\frac{\pi}{2} - \frac{\alpha}{2}$.

Suppose a support line of K intersects K_α at the point $\gamma_\alpha(t)$ (as shown in Fig. 1). The other support line of K through $\gamma_\alpha(t)$ intersects ∂K at the point $\gamma(t + \pi - \alpha)$. Denote as $a(t) = |\gamma_\alpha(t) - \gamma(t)|$ and $b(t) = |\gamma_\alpha(t) - \gamma(t + \pi - \alpha)|$.

Using this notation we state the following results, which are also proved in this paper.

Theorem 2. *Let K be a strictly convex body in the plane and let $\alpha \in (0, \pi)$ be a fixed angle. Suppose that $\frac{\alpha}{\pi}$ is a rational number and for every $t \in [0, 2\pi]$ it holds that $a(t) = b(t)$. Then K is a Euclidean disc.*

Theorem 3. *Let K be a strictly convex body in the plane and let $\alpha \in (0, \pi)$ be a fixed angle. Suppose that $\frac{\alpha}{\pi}$ is a rational number and for every $t \in [0, 2\pi]$ it holds that $a(t)+b(t) = \lambda$, for a positive fixed number λ. Then K is a Euclidean disc.*

Moreover, we give examples of curves different to the Euclidean disc which have the properties mentioned in Theorems 2 and 3 for some suitable angles α.

2. Auxiliary Results

For every $t \in [0, 2\pi]$ denote by $\ell(t)$ the support line of K with outward normal vector $u(t) = (\cos t, \sin t)$, and let $p(t)$ denote the distance with the sign from the origin O to $\ell(t)$. Using the support function, ∂K can be parametrized by (see, for instance, [10])

$$\gamma(t) = p(t)u(t) + p'(t)u'(t), \text{ for } t \in [0, 2\pi]. \tag{1}$$

The support function of a compact convex set is a periodic function, with period 2π, and is also absolutely continuous, hence we can consider its expansion in terms of the Fourier series (see [6]), i.e.,

$$p(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt).$$

Moreover, the first and second derivatives of p are expressed as

$$p'(t) = -\sum_{n=1}^{\infty} (na_n \sin nt - nb_n \cos nt),$$

$$p''(t) = -\sum_{n=1}^{\infty} n^2 (a_n \cos nt + b_n \sin nt).$$

The perimeter of K can be computed in terms of $p(t)$ by Cauchy's formula as

$$L(\gamma) = \int_0^{2\pi} p(t)dt. \tag{2}$$

Hence we have that the perimeter of K is $2\pi a_0$.

With respect to the elements of the isoptic curves we have (see, for instance, [2]):

$$a(t) = \frac{1}{\sin\alpha}\left[p(t+\pi-\alpha) + p(t)\cos\alpha - p'(t)\sin\alpha\right], \tag{3}$$

$$b(t) = \frac{1}{\sin\alpha}\left[p(t+\pi-\alpha)\cos\alpha + p'(t+\pi-\alpha)\sin\alpha + p(t)\right]. \tag{4}$$

Using the expression for $a(t)$ it is simple to show that the parameterization of K_α is

$$\gamma_\alpha(t) = p(t)u(t) + \left[p(t)\cot\alpha + \frac{1}{\sin\alpha}p(t+\pi-\alpha)\right]u'(t). \tag{5}$$

Since we are going to talk about caustics of convex bodies, we need to introduce some notions of mathematical billiards (see, for instance, [9] and Section 17.2 and p. 423 in [8]). A mathematical *billiard* consists of a domain, say in the plane (a billiard table), and a point-mass (a billiard ball) that moves inside the domain freely. This means that the point moves along a straight line with a constant speed until it hits the boundary. The reflection on the

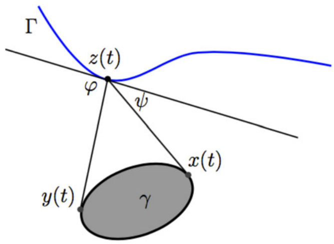


FIGURE 2. Reflective property

boundary is elastic and subject to a familiar law: the angle of incidence equals the angle of reflection.

A *caustic* is a curve inside a plane billiard table such that if a segment of a billiard trajectory is tangent to this curve, then so is each reflected segment.

To every convex curve we can associate a curve with similar properties of the ellipse. Such a curve can be obtained by the so called *Gardener's construction* which consists to wrap a closed non-stretchable string around γ, pull it tight at a point and move this point around γ to obtain a curve Γ. Then the billiard inside Γ has γ as its caustic and the length of this string meets the following.

Lemma 1. *Let γ be a strictly convex, differentiable and closed curve and let $z(t)$ be a point that moves along a differentiable curve Γ in the exterior of γ. Suppose the tangent lines to γ from the point $z(t)$ touch γ at the points $y(t)$ and $x(t)$, as shown in Fig. 2. Let $f(t) = |x(t) - z(t)| + |y(t) - z(t)| + \widehat{x(t)y(t)}$ be the perimeter of the convex hull of $\{z(t)\} \cup \gamma$. Then*

$$f'(t) = |z'(t)|(\cos\varphi - \cos\psi), \tag{6}$$

where φ is the angle between the vectors $z'(t)$ and $y'(t)$, and ψ is the angle between $z'(t)$ and $x'(t)$.

Proof. This result is well known. For a proof, the interested reader may consult [9]. □

3. Proof of Theorem 1

For the proof of the main result in this section we will need the following lemma from elementary geometry. Its proof is straightforward.

Lemma 2. *Let $\triangle xyz$ be a triangle such that $|x - y| + |x - z| = \lambda$, for a fixed value λ, and such that the angle $\measuredangle yxz = \alpha$, for a fixed value α. Then*

$$\frac{\lambda}{2}\sin\frac{\alpha}{2} \le |y - z| < \lambda,$$

and the minimum is reached when $|x - y| = |x - z|$.

Proof of Theorem 1. Denote the length of the chord $[\gamma(t+\pi-\alpha),\gamma(t)]$ by $q(t)$ for every $t\in[0,2\pi]$ (see Fig. 1). The perimeter of the isoptic curve K_α, parameterized by γ_α, is calculated by:

$$L(\gamma_\alpha)=\int_0^{2\pi}|\gamma_\alpha'(t)|dt.$$

It was proved in [2] by W. Cieślak, A. Miernowski, and W. Mozgawa that $|\gamma_\alpha'(t)|=\dfrac{q(t)}{\sin\alpha}$, hence

$$L(\gamma_\alpha)=\frac{1}{\sin\alpha}\int_0^{2\pi}q(t)dt.$$

Then, we can prove the theorem if we find the appropriate lower and upper bounds for $\int_0^{2\pi}q(t)dt$.

We will first prove the right side of the inequality. In order to do this set $c(t)=a(t)+b(t)$. By Lemma 2 we know that for every t it holds that $q(t)<c(t)$, hence

$$\begin{aligned}
L(\gamma_\alpha)&<\frac{1}{\sin\alpha}\int_0^{2\pi}c(t)dt,\\
&=\frac{1}{\sin^2\alpha}\int_0^{2\pi}[p(t)+p(t+\pi+\alpha)](1+\cos\alpha)dt\\
&\quad+\frac{1}{\sin\alpha}\int_0^{2\pi}[p'(t+\pi+\alpha)-p'(t)]dt,\\
&=\frac{1+\cos\alpha}{\sin^2\alpha}\int_0^{2\pi}[p(t)+p(t+\pi+\alpha)]dt,\\
&=\frac{1}{2\sin^2\frac{\alpha}{2}}\int_0^{2\pi}[p(t)+p(t+\pi+\alpha)]dt,\\
&=\frac{1}{(\sin\frac{\alpha}{2})^2}\cdot L(\gamma).
\end{aligned}$$

Now we will prove the left side of the inequality. By Lemma 2 we know that $q(t)\geq c(t)\sin\dfrac{\alpha}{2}$, for every t, hence

$$\begin{aligned}
L(\gamma_\alpha)&=\frac{1}{\sin\alpha}\int_0^{2\pi}q(t)dt,\\
&\geq\frac{1}{\sin\alpha}\int_0^{2\pi}c(t)\sin\frac{\alpha}{2}dt,\\
&=\frac{1}{2\cos\frac{\alpha}{2}}\int_0^{2\pi}c(t)dt,\\
&=\left(\frac{1}{2\cos\frac{\alpha}{2}}\right)\left(\frac{1+\cos\alpha}{\sin\alpha}\right)(2L(\gamma)),\\
&=\frac{1}{\sin\frac{\alpha}{2}}\cdot L(\gamma).
\end{aligned}$$

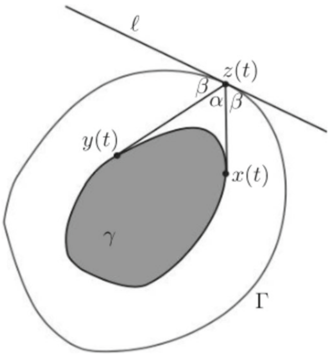


FIGURE 3. Constant angle caustic

Now, for the equality in the left side it is need that $q(t) = c(t) \sin \frac{\alpha}{2}$ for every t. By Lemma 2 we know that $a(t) = b(t)$ for every t if and only if $q(t) = c(t) \sin \frac{\alpha}{2}$. By a known result on isoptic curves we have that the angle between the line ℓ and the segment $[\gamma_\alpha(t), \gamma(t)]$ is equal to the angle between ℓ and the segment $[\gamma_\alpha(t), \gamma(t+\pi-\alpha)]$. In other words, with respect to Fig. 2 we have that $\varphi = \psi$ if and only if $a(t) = b(t)$. By Lemma 1 we have that $a(t) = b(t)$, for any t, if and only if the perimeter of the convex hull of $K \cup \{\gamma_\alpha(t)\}$ has a constant value λ. This last condition means that K_α is obtained by the Gardener's construction applied to K with a string of length λ. It follows that K_α is the boundary of a convex set (see [9]) that has K as a caustic of constant angle (see [7]). □

Remark 1. As far as we know, the fact that $|z(t) - x(t)| = |z(t) - y(t)|$ if and only if γ is a caustic of constant angle for the billiard table Γ (also proved in [1]), is not very known (see Fig. 3).

4. Proof of Theorem 2

Proof. Since $a(t) = b(t)$, from equations (3) and (4), after some simplifications, we obtain the differential equation

$$[p(t+\pi-\alpha) - p(t)] \tan \frac{\alpha}{2} = p'(t) + p'(t+\pi-\alpha).$$

By the substitution of the Fourier series representation of p and p' into the above equation we have

$$\tan \frac{\alpha}{2} \sum_{n=1}^{\infty} [a_n \cos n(t+\pi-\alpha) - a_n \cos nt + b_n \sin n(t+\pi-\alpha) - b_n \sin nt]$$
$$= \sum_{n=1}^{\infty} n[-a_n \sin nt - a_n \sin n(t+\pi-\alpha) + b_n \cos nt + b_n \cos n(t+\pi-\alpha)].$$

Using the trigonometric identities for cosine and sine of sums of angles and comparing the coefficients of $\cos nt$ and $\sin nt$ we obtain the following system of equations written in matrix structure as

$$M \cdot \begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where M is the following 2×2 matrix:

$$\begin{bmatrix} (\cos n(\pi-\alpha)-1)\tan\frac{\alpha}{2} + n\sin n(\pi-\alpha) & \sin n(\pi-\alpha)\tan\frac{\alpha}{2} - n(1+\cos n(\pi-\alpha)) \\ n(1+\cos n(\pi-\alpha)) - \sin n(\pi-\alpha)\tan\frac{\alpha}{2} & (\cos n(\pi-\alpha)-1)\tan\frac{\alpha}{2} + n\sin n(\pi-\alpha) \end{bmatrix}.$$

Since

$$\det M = [\cos n(\pi-\alpha)\tan\frac{\alpha}{2} - \tan\frac{\alpha}{2} + n\sin n(\pi-\alpha)]^2 + [\sin n(\pi-\alpha)\tan\frac{\alpha}{2} - n - n\cos n(\pi-\alpha)]^2,$$

we see that $\det M = 0$ if and only if

$$\tan\frac{\alpha}{2}(1-\cos n(\pi-\alpha)) = n\sin n(\pi-\alpha).$$

This last condition reduces to

$$\tan\frac{\alpha}{2} = n\cot n\left(\frac{\pi-\alpha}{2}\right).$$

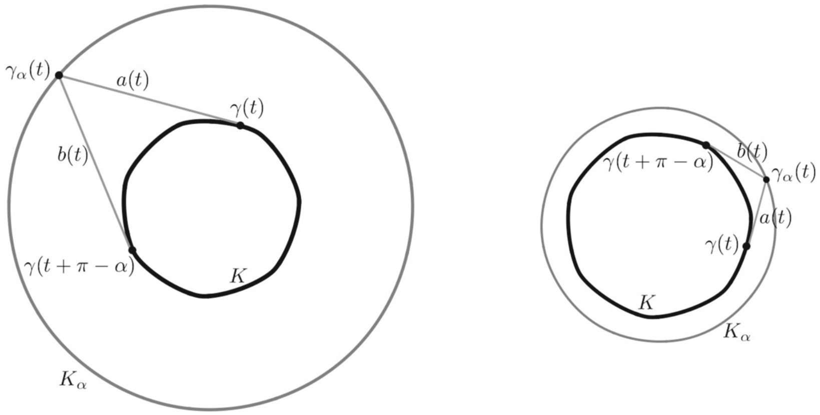
If we consider $\beta = \frac{\pi-\alpha}{2}$, we see that $\det M = 0$ if and only if

$$\tan n\beta = n\tan\beta.$$

For $n=1$ the matrix M is the zero matrix, which implies that a_1 and b_1 can be arbitrarily chosen. However, by Cyr's Theorem (see [4]) we know that for any $n>1$ this condition is never satisfied if $\frac{\beta}{\pi}$ is a rational number. Since $\frac{\alpha}{\pi}$ is rational if and only if $\frac{\beta}{\pi}$ is rational, we conclude that p is of the form $p(t) = a_0 + a_1\sin t + b_1\cos t$, with a_0, a_1 and b_1 real constants. Therefore, K is a Euclidean disc. □

Now we give an example of a convex body K, which is not a Euclidean disc, with the property that $a(t) = b(t)$ for every $t \in [0, 2\pi]$ and for two different angles α. The problem of isoptic curves, with the property here established, is also solved in [3]. A convex body K and its two corresponding isoptic curves are shown in Fig. 4.

Example 1. We solve the equation $n\tan\beta = \tan n\beta$, with $n=7$, and we get two solutions $\beta_1 = \arctan\left(\sqrt{\frac{7+\sqrt{7}}{5-\sqrt{7}}}\right)$ and $\beta_2 = \arctan\left(\sqrt{\frac{7-\sqrt{7}}{5+\sqrt{7}}}\right)$. This leads to two angles: $\alpha_i = \pi - 2\beta_i$, with $i = 1, 2$. Note that for these α's, the matrix



(a) Case α_1 (b) Case α_2

FIGURE 4. A convex body K with two α-isoptic such that $a(t) = b(t)$

M is the zero matrix, so we can choose a_7 and b_7 arbitrarily. We consider the following support function for the convex body K:

$$p(t) = a_0 + a_1 \cos t + b_1 \sin t + a_7 \cos 7t + b_7 \sin 7t,$$

where $a_0 = 100$ is chosen to ensure that K is a convex body and $a_1 = b_1 = a_7 = b_7 = 1$.

5. Proof of Theorem 3

Before proving Theorem 3 we shall confirm a set of useful lemmas.

Lemma 3. *Suppose $\theta \in (0, \pi)\setminus\{\frac{\pi}{2}\}$. If there exists a natural number $n > 1$ such that*

$$\tan\theta = n \tan n\theta,$$

then

$$\frac{\sin(n-1)\theta}{\sin(n+1)\theta} = -\frac{n-1}{n+1}.$$

Proof. It is well known that for any complex number $z \in \mathbb{C}\setminus\{(k+\frac{1}{2})\pi : k \in \mathbb{Z}\}$ one has

$$\tan z = i\frac{e^{-iz} - e^{iz}}{e^{iz} + e^{-iz}}.$$

For the number θ we know that $|\tan\theta| < \infty$, and by the condition of the lemma we also have that $|\tan n\theta| < \infty$. Hence the condition of the lemma can be rewritten as

$$i\frac{e^{-i\theta}-e^{i\theta}}{e^{i\theta}+e^{-i\theta}}=n\left(i\frac{e^{-in\theta}-e^{in\theta}}{e^{in\theta}+e^{-in\theta}}\right).$$

From this equality we obtain

$$(n+1)(e^{i(n-1)\theta}-e^{-i(n-1)\theta})=-(n-1)(e^{i(n+1)\theta}-e^{-i(n+1)\theta}).$$

It follows that

$$\frac{\dfrac{e^{i(n-1)\theta}-e^{-i(n-1)\theta}}{2i}}{\dfrac{e^{i(n+1)\theta}-e^{-i(n+1)\theta}}{2i}}=-\frac{(n-1)}{(n+1)},$$

i.e.,

$$\frac{\sin(n-1)\theta}{\sin(n+1)\theta}=-\frac{(n-1)}{(n+1)}.$$

□

The following lemma is due to V. Cyr, and it was proved in [4].

Lemma 4. *If* $\theta \in (0,\pi)\backslash\{\frac{\pi}{2}\}$ *is such that* $\frac{\theta}{\pi}$ *is a rational number, and* k *and* m *are integer numbers such that* $\sin m\theta \neq 0$, *then*

$$\frac{\sin k\theta}{\sin m\theta}$$

is either -1, 0, 1 or irrational.

Using the two above lemmas we can prove the following.

Lemma 5. *If* $\theta \in (0,\pi)\backslash\{\frac{\pi}{2}\}$ *is such that* $\frac{\theta}{\pi}$ *is a rational number, then there is no integer number* $n > 1$ *such that*

$$\tan\theta = n\tan n\theta.$$

Proof. Suppose $\frac{\theta}{\pi}$ is a rational number and there is an integer number $n > 1$ such that $\tan\theta = n\tan n\theta$. By Lemma 3 we have that

$$\frac{\sin(n-1)\theta}{\sin(n+1)\theta}=-\frac{(n-1)}{(n+1)}.$$

Since $n > 1$, we have that

$$-\frac{(n-1)}{(n+1)}\neq -1,0,1,$$

and so by Lemma 4

$$\frac{\sin(n-1)\theta}{\sin(n+1)\theta}$$

must be an irrational number. However, $-\frac{(n-1)}{(n+1)}$ cannot be an irrational number. This contradiction shows that there is no integer number $n > 1$ such that $\tan\theta = n\tan n\theta$ if $\frac{\theta}{\pi}$ is rational. □

Lemma 6. *Let α be a rational number, modulus π, in the interval $(0,\pi)$, i.e., $\frac{\alpha}{\pi} \in \mathbb{Q}$, and let $p : \mathbb{R} \to \mathbb{R}$ be a differentiable periodic function, with period 2π, that satisfies the differential equation*

$$\cot\frac{\alpha}{2}\left(p'(t) + p'(t+\pi-\alpha)\right) = p''(t) - p''(t+\pi-\alpha). \tag{7}$$

Then p is of the form $p(t) = c + a\sin t + b\cos t$, where a, b and c are real constants.

Proof. Substituting the Fourier series representation of p and p' into (7) gives

$$\begin{aligned}\cot\frac{\alpha}{2}\sum_{n=1}^{\infty}[-a_n\sin n(t+\pi-\alpha) - a_n\sin nt + b_n\cos n(t+\pi-\alpha) + b_n\cos nt]\\ = \sum_{n=1}^{\infty} n[a_n\cos n(t+\pi-\alpha) + b_n\sin n(t+\pi-\alpha) - a_n\cos nt - b_n\sin nt].\end{aligned} \tag{8}$$

Using the trigonometric identities for cosine and sine of sums of angles and comparing the coefficients of $\cos nt$ and $\sin nt$, we obtain the following system of equations written in matrix structure as

$$M\cdot\begin{bmatrix}a_n\\ b_n\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix},$$

where M is the following 2×2 matrix:

$$\begin{bmatrix} n(1-\cos n(\pi-\alpha)) - \cot\frac{\alpha}{2}\sin n(\pi-\alpha) & \cot\frac{\alpha}{2}(1+\cos n(\pi-\alpha)) - n\sin n(\pi-\alpha) \\ n\sin n(\pi-\alpha) - (1+\cos n(\pi-\alpha))\cot\frac{\alpha}{2} & n(1-\cos n(\pi-\alpha)) - \cot\frac{\alpha}{2}\sin n(\pi-\alpha) \end{bmatrix}.$$

Since

$$\begin{aligned}\det M = {} & [n - \cot\frac{\alpha}{2}\sin n(\pi-\alpha) - n\cos n(\pi-\alpha)]^2\\ & +[\cot\frac{\alpha}{2}(1+\cos n(\pi-\alpha)) - n\sin n(\pi-\alpha)]^2,\end{aligned}$$

we see that $\det M = 0$ if and only if

$$n(1-\cos n(\pi-\alpha)) = \cot\frac{\alpha}{2}\sin n(\pi-\alpha).$$

This last condition reduces to

$$n\tan n\left(\frac{\pi-\alpha}{2}\right) = \cot\frac{\alpha}{2}.$$

If we consider $\beta = \frac{\pi-\alpha}{2}$, we see that $\det M = 0$ if and only if

$$n\tan n\beta = \tan\beta.$$

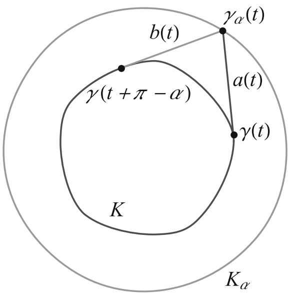


FIGURE 5. A curve with $a(t) + b(t)$ constant

For $n = 1$ this condition is always satisfied. However, by Lemma 5 we know that for any $n > 1$ this condition is never satisfied if $\frac{\beta}{\pi}$ is a rational number. Since $\frac{\alpha}{\pi}$ is rational if and only if $\frac{\beta}{\pi}$ is rational, we conclude that p is of the form $p(t) = a_0 + a_1 \sin t + b_1 \cos t$, with a_0, a_1 and b_1 real constants. □

Proof of Theorem 3. Since $a(t)+b(t)$ is constant, we have that $a'(t)+b'(t) = 0$. From deriving equations (3) and (4) we obtain the differential equation

$$[p'(t) + p''(t + \pi - \alpha)] \cot \frac{\alpha}{2} = p''(t) - p''(t + \pi - \alpha).$$

Now we apply Lemma 6 and obtain that

$$p(t) = c + a \cos t + b \sin t.$$

In other words, p is the support function of a Euclidean disc. □

Example 2. We consider the case $n = 5$. Solving the equation $n \tan n\beta = \tan \beta$ we find that $\beta = \arctan \sqrt{\frac{5}{3}}$. For this angle the support function must be of the form

$$p(t) = a_0 + a_1 \cos t + b_1 \sin t + a_5 \cos 5t + b_5 \sin 5t,$$

where a_0 must be chosen to ensure that K is a convex body. The obtained convex body K and its corresponding isoptic is shown in Fig. 5, for the support function

$$p(t) = 53 + \cos t + \sin t + \cos 5t + \sin 5t.$$

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