



On \tilde{J} -tangent Affine Hyperspheres

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Abstract. In this paper we study \tilde{J} -tangent affine hyperspheres, where \tilde{J} is the canonical para-complex structure on \mathbb{R}^{2n+2} . The main purpose of this paper is to give a classification of \tilde{J} -tangent affine hyperspheres of an arbitrary dimension with an involutive distribution \mathcal{D} . In particular, we classify all such hyperspheres in the 3-dimensional case. We also show that there is a direct relation between \tilde{J} -tangent affine hyperspheres and Calabi products. As an application we obtain certain classification results. In particular, we show that, with one exception, all odd dimensional proper flat affine hyperspheres are, after a suitable affine transformation, \tilde{J} -tangent. Some examples of \tilde{J} -tangent affine hyperspheres are also given.

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1. Introduction

Para-complex and paracontact geometry plays an important role in mathematical physics. On the other hand affine differential geometry and in particular affine hyperspheres have been extensively studied over past decades. Some relations between para-complex and affine differential geometry can be found in [1–3].

In [4] the author studied J -tangent affine hypersurfaces and gave a local classification of J -tangent affine hyperspheres with an involutive contact distribution.

In this paper we study real affine hyperspheres $f: M^{2n+1} \rightarrow \mathbb{R}^{2n+2} \cong \tilde{\mathbb{C}}^{n+1}$ of the para-complex space $\tilde{\mathbb{C}}^{n+1}$ with a \tilde{J} -tangent transversal vector field C and an induced almost paracontact structure (φ, ξ, η) . First we show

that when C is centro-affine (not necessarily Blaschke) then f can be locally expressed in the form:

$$f(x_1, \dots, x_{2n}, z) = \tilde{J}g(x_1, \dots, x_{2n}) \cosh z - g(x_1, \dots, x_{2n}) \sinh z, \quad (1.1)$$

where g is some smooth immersion defined on an open subset of \mathbb{R}^{2n} . Basing on the above result we provide a local classification of all \tilde{J} -tangent affine hyperspheres with an involutive distribution \mathcal{D} . We also show that there are no improper \tilde{J} -tangent affine hyperspheres. In particular, using results from [1], we find all 3-dimensional \tilde{J} -tangent affine hyperspheres with the involutive distribution \mathcal{D} . We also give an example of a \tilde{J} -tangent affine hypersphere with non-involutive distribution \mathcal{D} .

In Sect. 2 we briefly recall the basic formulas of affine differential geometry, we recall the notion of an affine hypersphere and some basic results from para-complex geometry. We also recall the notion of a para-complex affine hypersphere (for details we refer to [1]).

In Sect. 3 we recall the definitions of an almost paracontact structure introduced for the first time in [5]. We also recall some elementary results for induced almost paracontact structures that will be used later in this paper.

Sections 4 and 5 contain the main results of this paper. In the Sect. 4 we introduce the notion of a \tilde{J} -tangent affine hypersphere and prove classification results. In particular, we show that \tilde{J} -tangent affine hyperspheres must be proper and there is a strict relation between \tilde{J} -tangent affine hyperspheres with the involutive distribution \mathcal{D} and proper para-complex affine hyperspheres. Finally we show that \tilde{J} -tangent affine hyperspheres can be constructed using lower dimensional proper affine hyperspheres. As an application, we classify all 3-dimensional proper \tilde{J} -tangent affine hyperspheres with the involutive distribution \mathcal{D} .

In the Sect. 5 we show some applications of the results obtained in Sect. 4. We show that \tilde{J} -tangent affine hyperspheres can be classified in terms of Calabi products. Among other we show that (with one exception) all odd dimensional proper flat affine hyperspheres are \tilde{J} -tangent affine hyperspheres with the involutive distribution \mathcal{D} . Moreover, we show that the above mentioned exceptional affine hypersphere is J -tangent, where J is the standard complex structure on \mathbb{R}^{2n+2} .

2. Preliminaries

We briefly recall the basic formulas of affine differential geometry. For more details, we refer to [6].

Let $f: M \rightarrow \mathbb{R}^{n+1}$ be an orientable connected differentiable n -dimensional hypersurface immersed in the affine space \mathbb{R}^{n+1} equipped with its usual flat connection D . Then for any transversal vector field C we have

$$D_X f_*Y = f_*(\nabla_X Y) + h(X, Y)C \quad (2.1)$$

and

$$D_X C = -f_*(SX) + \tau(X)C, \tag{2.2}$$

where X, Y are vector fields tangent to M . It is known that ∇ is a torsion-free connection, h is a symmetric bilinear form on M , called *the second fundamental form*, S is a tensor of type $(1, 1)$, called *the shape operator*, and τ is a 1-form, called *the transversal connection form*. Recall that the formula (2.1) is known as the formula of Gauss and the formula (2.2) is known as the formula of Weingarten.

For a hypersurface immersion $f: M \rightarrow \mathbb{R}^{n+1}$ a transversal vector field C is said to be *equiaffine* (resp. *locally equiaffine*) if $\tau = 0$ (resp. $d\tau = 0$). For an affine hypersurface $f: M \rightarrow \mathbb{R}^{n+1}$ with the transversal vector field C we consider the following volume element on M :

$$\Theta(X_1, \dots, X_n) := \det[f_*X_1, \dots, f_*X_n, C]$$

for all $X_1, \dots, X_n \in \mathcal{X}(M)$. We call Θ *the induced volume element* on M . Immersion $f: M \rightarrow \mathbb{R}^{n+1}$ is said to be a *centro-affine hypersurface* if the position vector x (from origin o) for each point $x \in M$ is transversal to the tangent plane of M at x . In this case $S = I$ and $\tau = 0$. If h is nondegenerate (that is h defines a semi-Riemannian metric on M), we say that the hypersurface or the hypersurface immersion is *nondegenerate*. In this paper we assume that f is always nondegenerate. We have the following

Theorem 2.1 [6, Fundamental equations]. *For an arbitrary transversal vector field C the induced connection ∇ , the second fundamental form h , the shape operator S and the 1-form τ satisfy the following equations:*

$$R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY, \tag{2.3}$$

$$(\nabla_X h)(Y, Z) + \tau(X)h(Y, Z) = (\nabla_Y h)(X, Z) + \tau(Y)h(X, Z), \tag{2.4}$$

$$(\nabla_X S)(Y) - \tau(X)SY = (\nabla_Y S)(X) - \tau(Y)SX, \tag{2.5}$$

$$h(X, SY) - h(SX, Y) = 2d\tau(X, Y). \tag{2.6}$$

The Eqs. (2.3), (2.4), (2.5) and (2.6) are called the equations of Gauss, Codazzi for h , Codazzi for S and Ricci, respectively.

When f is nondegenerate, there exists a canonical transversal vector field C called *the affine normal field* (or *the Blaschke field*). The affine normal field is uniquely determined up to sign by the following conditions:

- (1) the metric volume form ω_h of h is ∇ -parallel,
- (2) ω_h coincides with the induced volume form Θ .

Recall that ω_h is defined by

$$\omega_h(X_1, \dots, X_n) = |\det[h(X_i, X_j)]|^{1/2},$$

where $\{X_1, \dots, X_n\}$ is any positively oriented basis relative to the induced volume form Θ . The affine immersion f with a Blaschke field C is called a *Blaschke hypersurface*. In this case fundamental equations can be rewritten as follows

Theorem 2.2 [6, Fundamental equations]. *For a Blaschke hypersurface f , we have the following fundamental equations:*

$$\begin{aligned} R(X, Y)Z &= h(Y, Z)SX - h(X, Z)SY, \\ (\nabla_X h)(Y, Z) &= (\nabla_Y h)(X, Z), \\ (\nabla_X S)(Y) &= (\nabla_Y S)(X), \\ h(X, SY) &= h(SX, Y). \end{aligned}$$

A Blaschke hypersurface is called an *affine hypersphere* if $S = \lambda I$, where $\lambda = \text{const}$. If $\lambda = 0$ f is called an *improper affine hypersphere*, if $\lambda \neq 0$ a hypersurface f is called a *proper affine hypersphere*.

Now, we will recall a notion of para-complex affine hypersurfaces, for details we refer to [1]. More information on para-complex geometry one may find for example in [7, 8].

Let $g: M^{2n} \rightarrow \mathbb{R}^{2n+2}$ be an immersion and let \tilde{J} be the standard para-complex structure on \mathbb{R}^{2n+2} . That is

$$\tilde{J}(x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1}) := (y_1, \dots, y_{n+1}, x_1, \dots, x_{n+1}).$$

We always identify $(\mathbb{R}^{2n+2}, \tilde{J})$ with $\tilde{\mathbb{C}}^{n+1}$.

Assume now that $g_*(TM)$ is \tilde{J} -invariant and $\tilde{J}|_{g_*(T_x M)}$ is a para-complex structure on $g_*(T_x M)$ for every $x \in M$. Then \tilde{J} induces an almost para-complex structure on M , which we will also denote by \tilde{J} . Moreover, since $(\mathbb{R}^{2n+2}, \tilde{J})$ is para-complex then (M, \tilde{J}) is para-complex as well. By assumption we have that $dg \circ \tilde{J} = \tilde{J} \circ dg$ that is $g: M^{2n} \rightarrow \mathbb{R}^{2n+2} \cong \tilde{\mathbb{C}}^{n+1}$ is a para-holomorphic immersion. Since para-complex dimension of M is n , immersion g is called a *para-holomorphic hypersurface*.

Let $g: M^{2n} \rightarrow \mathbb{R}^{2n+2}$ be an affine hypersurface of codimension 2 with a transversal bundle \mathcal{N} . If g is para-holomorphic then it is called *affine para-holomorphic hypersurface*. If additionally the transversal bundle \mathcal{N} is \tilde{J} -invariant then g is called a *para-complex affine hypersurface*.

Let $g: M^{2n} \rightarrow \mathbb{R}^{2n+2}$ be a para-holomorphic hypersurface. We say that g is *para-complex centro-affine hypersurface* if $\{g, \tilde{J}g\}$ is a transversal bundle for g .

Theorem 2.3 [1]. *Let $g: M^{2n} \rightarrow \mathbb{R}^{2n+2}$ be a para-holomorphic hypersurface. Then for every $x \in M$ there exists a neighborhood U of x and a transversal vector field $\zeta: U \rightarrow \mathbb{R}^{2n+2}$ such that $\{\zeta, \tilde{J}\zeta\}$ is a transversal bundle for $g|_U$. That is $g|_U$ considered with $\{\zeta, \tilde{J}\zeta\}$ is a para-complex affine hypersurface.*

Now let $g: M^{2n} \rightarrow \mathbb{R}^{2n+2}$ be a para-holomorphic hypersurface and let $\zeta: U \rightarrow \mathbb{R}^{2n+2}$ be a local transversal vector field on $U \subset M$ such that $\{\zeta, \tilde{J}\zeta\}$ is a transversal bundle to g . For all tangent vector fields $X, Y \in \mathcal{X}(U)$ we can

decompose $D_X Y$ and $D_X \zeta$ into tangent and transversal part. Namely, we have

$$D_X g_* Y = g_*(\nabla_X Y) + h_1(X, Y)\zeta + h_2(X, Y)\tilde{J}\zeta \quad (\text{formula of Gauss}),$$

$$D_X \zeta = -g_*(SX) + \tau_1(X)\zeta + \tau_2(X)\tilde{J}\zeta \quad (\text{formula of Weingarten}),$$

where ∇ is a torsion free affine connection on U , h_1 and h_2 are symmetric bilinear forms on U , S is a $(1, 1)$ -tensor field on U and τ_1 and τ_2 are 1-forms on U . We have the following relations between h_1 and h_2 .

Lemma 2.4 [1, 3].

$$h_1(X, \tilde{J}Y) = h_1(\tilde{J}X, Y) = h_2(X, Y), \tag{2.7}$$

$$h_2(X, \tilde{J}Y) = h_1(X, Y). \tag{2.8}$$

On U we define the volume form θ_ζ by the formula

$$\theta_\zeta(X_1, \dots, X_{2n}) := \det(g_* X_1, \dots, g_* X_{2n}, \zeta, \tilde{J}\zeta)$$

for tangent vectors $X_i, i = 1, \dots, 2n$. Let us consider the function H_ζ on U defined by

$$H_\zeta := \det[h_1(X_i, X_j)]_{i,j=1..2n},$$

where X_1, \dots, X_{2n} is a local basis on TU such that $\theta_\zeta(X_1, \dots, X_{2n}) = 1$. This definition is independent of the choice of basis. We say that a hypersurface is *nondegenerate* if h_1 (and in consequence h_2) is nondegenerate. When g is nondegenerate there exist transversal vector fields ζ satisfying the following two conditions:

$$\begin{aligned} |H_\zeta| &= 1, \\ \tau_1 &= 0. \end{aligned}$$

Such vector fields are called *affine normal vector fields*. In [1] we showed that on every para-holomorphic hypersurface we may find (at least locally) an affine normal vector field.

A nondegenerate para-complex hypersurface is said to be a *proper para-complex affine hypersphere* if there exists an affine normal vector field ζ such that $S = \alpha I$, where $\alpha \in \mathbb{R} \setminus \{0\}$ and $\tau_2 = 0$. If there exists an affine normal vector field ζ such that $S = 0$ and $\tau_2 = 0$ we say about an *improper para-complex affine hypersphere*. Note that the above definition is very analogous to the definition of complex affine hypersphere introduced by Dillen et al. [9].

3. Almost Paracontact Structures

Let $\dim M = 2n + 1$ and $f: M \rightarrow \mathbb{R}^{2n+2}$ be a nondegenerate (relative to the second fundamental form) affine hypersurface. We always assume that \mathbb{R}^{2n+2} is endowed with the standard para-complex structure \tilde{J}

$$\tilde{J}(x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1}) = (y_1, \dots, y_{n+1}, x_1, \dots, x_{n+1}).$$

Let C be a transversal vector field on M . We say that C is \tilde{J} -tangent if $\tilde{J}C_x \in f_*(T_xM)$ for every $x \in M$. We also define a distribution \mathcal{D} on M as the biggest \tilde{J} -invariant distribution on M , that is

$$\mathcal{D}_x = f_*^{-1}(f_*(T_xM) \cap \tilde{J}(f_*(T_xM)))$$

for every $x \in M$. We have that $\dim \mathcal{D}_x \geq 2n$. If for some x the $\dim \mathcal{D}_x = 2n + 1$ then $\mathcal{D}_x = T_xM$ and it is not possible to find a \tilde{J} -tangent transversal vector field in a neighbourhood of x . Since we only study hypersurfaces with a \tilde{J} -tangent transversal vector field, then we always have $\dim \mathcal{D} = 2n$. The distribution \mathcal{D} is smooth as an intersection of two smooth distributions and because $\dim \mathcal{D}$ is constant. A vector field X is called a \mathcal{D} -field if $X_x \in \mathcal{D}_x$ for every $x \in M$. We use the notation $X \in \mathcal{D}$ for vectors as well as for \mathcal{D} -fields. We say that the distribution \mathcal{D} is nondegenerate if h is nondegenerate on \mathcal{D} .

A $(2n + 1)$ -dimensional manifold M is said to have an *almost paracontact structure* if there exist on M a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η which satisfy

$$\varphi^2(X) = X - \eta(X)\xi, \tag{3.1}$$

$$\eta(\xi) = 1 \tag{3.2}$$

for every $X \in TM$ and the tensor field φ induces an almost para-complex structure on the distribution $\mathcal{D} = \ker \eta$. That is the eigendistributions $\mathcal{D}^+, \mathcal{D}^-$ corresponding to the eigenvalues $1, -1$ of φ have equal dimension n .

Let $f: M \rightarrow \mathbb{R}^{2n+2}$ be a nondegenerate affine hypersurface with a \tilde{J} -tangent transversal vector field C . Then we can define a vector field ξ , a 1-form η and a tensor field φ of type $(1,1)$ as follows:

$$\xi := \tilde{J}C; \tag{3.3}$$

$$\eta|_{\mathcal{D}} = 0 \text{ and } \eta(\xi) = 1; \tag{3.4}$$

$$\varphi|_{\mathcal{D}} = \tilde{J}|_{\mathcal{D}} \text{ and } \varphi(\xi) = 0. \tag{3.5}$$

It is easy to see that (φ, ξ, η) is an almost paracontact structure on M . This structure is called the *induced almost paracontact structure*. For an induced almost paracontact structure we have the following theorem

Theorem 3.1 [10]. *Let $f: M \rightarrow \mathbb{R}^{2n+2}$ be an affine hypersurface with a \tilde{J} -tangent transversal vector field C . If (φ, ξ, η) is an induced almost paracontact*

structure on M then the following equations hold:

$$\eta(\nabla_X Y) = h(X, \varphi Y) + X(\eta(Y)) + \eta(Y)\tau(X), \tag{3.6}$$

$$\varphi(\nabla_X Y) = \nabla_X \varphi Y - \eta(Y)SX - h(X, Y)\xi, \tag{3.7}$$

$$\begin{aligned} \eta([X, Y]) &= h(X, \varphi Y) - h(Y, \varphi X) + X(\eta(Y)) - Y(\eta(X)) \\ &\quad + \eta(Y)\tau(X) - \eta(X)\tau(Y), \end{aligned} \tag{3.8}$$

$$\varphi([X, Y]) = \nabla_X \varphi Y - \nabla_Y \varphi X + \eta(X)SY - \eta(Y)SX, \tag{3.9}$$

$$\eta(\nabla_X \xi) = \tau(X), \tag{3.10}$$

$$\eta(SX) = -h(X, \xi) \tag{3.11}$$

for every $X, Y \in \mathcal{X}(M)$.

4. \tilde{J} -Tangent Affine Hyperspheres

An affine hypersphere with a transversal \tilde{J} -tangent Blaschke field we call a \tilde{J} -tangent affine hypersphere. We start this section with the following useful lemma related to differential equations

Lemma 4.1. *Let $F: I \rightarrow \mathbb{R}^{2n}$ be a smooth function on an interval I . If F satisfies the differential equation*

$$F'(z) = -\tilde{J}F(z), \tag{4.1}$$

then F is of the form

$$F(z) = \tilde{J}v \cosh z - v \sinh z, \tag{4.2}$$

where $v \in \mathbb{R}^{2n}$.

Proof. It is not difficult to check that functions of the form (4.2) satisfy the differential equation (4.1). On the other hand, since (4.1) is a first-order ordinary differential equation, the Picard-Lindelöf theorem implies that any solution of (4.1) must be of the form (4.2). \square

Using the above lemma, we can prove the following theorem.

Theorem 4.2. *Let $f: M \rightarrow \mathbb{R}^{2n+2}$ be a centro-affine hypersurface with a \tilde{J} -tangent centro-affine vector field. Then f can be locally expressed in the form*

$$f(x_1, \dots, x_{2n}, z) = \tilde{J}g(x_1, \dots, x_{2n}) \cosh z - g(x_1, \dots, x_{2n}) \sinh z \tag{4.3}$$

for all $(x_1, \dots, x_{2n}, z) \in U \times I$, where $U \subset \mathbb{R}^{2n}$ is an open subset, $I \subset \mathbb{R}$ is an open interval and $g: U \rightarrow \mathbb{R}^{2n+2}$ is an immersion.

Proof. Denote $C := -f$. Since f is a centro-affine hypersurface with a \tilde{J} -tangent transversal vector field then we have $\tilde{J}C = -\tilde{J}f \in f_*(TM)$. Therefore,

for every $x \in M$, there exists a neighborhood V of x and a map $\psi(x_1, \dots, x_{2n}, z)$ on V such that

$$f_* \frac{\partial}{\partial z} = \tilde{J}C.$$

That is f can be locally expressed in the form $f(x_1, \dots, x_{2n}, z)$, where $f_z = -\tilde{J}f$. Now using Lemma 4.1 we obtain the thesis. \square

When the distribution \mathcal{D} is involutive we have

Theorem 4.3. *Let $f: M \rightarrow \mathbb{R}^{2n+2}$ be an affine hypersurface with a centro-affine \tilde{J} -tangent vector field $C = -\overrightarrow{of}$. If the distribution \mathcal{D} is involutive then for every $x \in M$ there exists a para-complex centro-affine immersion $g: V \rightarrow \mathbb{R}^{2n+2}$ defined on an open subset $V \subset \mathbb{R}^{2n}$ such that f can be expressed in the neighborhood of x in the form*

$$f(x_1, \dots, x_{2n}, z) = \tilde{J}g(x_1, \dots, x_{2n}) \cosh z - g(x_1, \dots, x_{2n}) \sinh z. \tag{4.4}$$

Moreover, if $g: V \rightarrow \mathbb{R}^{2n+2}$ is a para-complex centro-affine immersion then f given by the formula (4.4) is an affine hypersurface with a centro-affine \tilde{J} -tangent vector field and an involutive distribution \mathcal{D} .

Proof. Let (φ, ξ, η) be an induced almost paracontact structure on M induced by C . The Frobenius theorem implies that for every $x \in M$ there exist an open neighborhood $U \subset M$ of x and linearly independent vector fields $X_1, \dots, X_{2n}, X_{2n+1} = \xi \in \mathcal{X}(U)$ such that $[X_i, X_j] = 0$ for $i, j = 1, \dots, 2n + 1$. For every $i = 1, \dots, 2n$ we have $X_i = D_i + \alpha_i \xi$ where $D_i \in \mathcal{D}$ and $\alpha_i \in C^\infty(U)$. Thus we have

$$0 = [X_i, \xi] = [D_i, \xi] - \xi(\alpha_i)\xi.$$

Now (3.8) and (3.11) imply that $[D_i, \xi]$ and $\xi(\alpha_i) = 0$. We also have

$$0 = [X_i, X_j] = [D_i, D_j] - D_j(\alpha_i)\xi + D_i(\alpha_j)\xi$$

for $i = 1, \dots, 2n$. Since \mathcal{D} is involutive the above equality implies that $[D_i, D_j] = 0$ for $i, j = 1, \dots, 2n$. Of course the vector fields D_1, \dots, D_{2n}, ξ are linearly independent, so there exists a map $\psi(x_1, \dots, x_{2n}, z)$ on U such that

$$\frac{\partial}{\partial z} = \xi, \quad \frac{\partial}{\partial x_i} = D_i, \quad i = 1, \dots, 2n.$$

Now applying Lemma 4.1 we find that f can be locally expressed in the form

$$f(x_1, \dots, x_{2n}, z) = \tilde{J}g(x_1, \dots, x_{2n}) \cosh z - g(x_1, \dots, x_{2n}) \sinh z,$$

where $g: V \rightarrow \mathbb{R}^{2n+2}$ is an immersion defined on an open subset $V \subset \mathbb{R}^{2n}$. Moreover, since $\frac{\partial}{\partial x_i} \in \mathcal{D}$ we have that

$$f_{x_i} = \tilde{J}g_{x_i} \cosh z - g_{x_i} \sinh z \in f_*(D).$$

Since $f_*(D)$ is \tilde{J} -invariant we also have

$$\tilde{J}f_{x_i} = g_{x_i} \cosh z - \tilde{J}g_{x_i} \sinh z \in f_*(D).$$

The above implies that $g_{x_i} \in f_*(D)$ for $i = 1, \dots, 2n$. Since $\{g_{x_i}\}$ are linearly independent, they form a basis of $f_*(D)$ (note that $\dim f_*(D) = 2n$) i. e.

$$f_*(D) = \text{span}\{g_{x_1}, \dots, g_{x_{2n}}\}.$$

Since $f_*(D)$ is \tilde{J} -invariant we also have that

$$\tilde{J}g_{x_i} \in f_*(D) = \text{span}\{g_{x_1}, \dots, g_{x_{2n}}\}.$$

That is, $\tilde{J}g_{x_i} = \sum \alpha_i g_{x_i}$, where $\alpha_i \in C^\infty(U)$. Since g does not depend on variable z , the functions α_i also do not, thus $\alpha_i \in C^\infty(V)$.

In this way we have shown that for $g: V \rightarrow \mathbb{R}^{2n+2}$ the tangent space TV is \tilde{J} -invariant (we can transfer \tilde{J} from $g_*(TV)$ to TV). Since $\tilde{J}|_{f_*(D)}$ is para-complex and $f_*(D) = \text{span}_{C^\infty(U)}\{g_{x_1}, \dots, g_{x_{2n}}\}$, \tilde{J} is a para-complex structure on TV . Finally g is para-holomorphic. Since f is an immersion, $\{g_{x_1}, \dots, g_{x_{2n}}, \tilde{J}g\}$ are linearly independent. Moreover, because f is centro-affine, we also have that g is linearly independent with $\{g_{x_1}, \dots, g_{x_{2n}}, \tilde{J}g\}$. That is $\{g, \tilde{J}g\}$ is a \tilde{J} -invariant transversal bundle for $g_*(TV)$ and, in consequence, g is a para-complex affine immersion.

In order to prove the second part of the theorem, note that since g is a centro-affine para-complex affine immersion, then $\{f_{x_1}, \dots, f_{x_{2n}}, f_z, f\}$ are linearly independent. It means that f is an immersion and is centro-affine. Moreover, f is \tilde{J} -tangent since $\tilde{J}(-\overrightarrow{of}) = -g \cosh z + \tilde{J}g \sinh z = f_z$. In particular, g is para-holomorphic. That is, we have $\tilde{J}g_{x_i} = \sum_{j=1}^{2n} \alpha_{ij} g_{x_j}$ for $i = 1, \dots, 2n$. Now, by straightforward computations we get $\sum_{j=1}^{2n} \alpha_{ij} f_{x_j} = \tilde{J}f_{x_i}$ for $i = 1, \dots, 2n$. That is, $\tilde{J}f_{x_i} \in \text{span}\{f_{x_1}, \dots, f_{x_{2n}}\}$. In this way we have shown that $\text{span}\{f_{x_1}, \dots, f_{x_{2n}}\}$ is \tilde{J} -invariant and since its dimension is $2n$ it must be equal to $f_*(D)$. Now it is easy to see that $\mathcal{D} = \{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{2n}}\}$ is involutive as generated by the canonical vector fields. \square

For \tilde{J} -tangent affine hyperspheres we have the following classification theorems:

Theorem 4.4. *There are no improper \tilde{J} -tangent affine hyperspheres.*

Proof. By (3.11) we have $\eta(SX) = -h(X, \xi)$ for all $X \in \mathcal{X}(M)$. Since $S = 0$ we have $h(X, \xi) = 0$ for every $X \in \mathcal{X}(M)$, which contradicts nondegeneracy of h . \square

Theorem 4.5. *Let $f: M \rightarrow \mathbb{R}^{2n+2}$ be a \tilde{J} -tangent affine hypersphere with an involutive distribution \mathcal{D} . Then f can be locally expressed in the form:*

$$f(x_1, \dots, x_{2n}, z) = \tilde{J}g(x_1, \dots, x_{2n}) \cosh z - g(x_1, \dots, x_{2n}) \sinh z, \quad (4.5)$$

where g is a proper para-complex affine hypersphere. Moreover, the converse is also true in the sense that if g is a proper para-complex affine hypersphere then f given by the formula (4.5) is a \tilde{J} -tangent affine hypersphere with an involutive distribution \mathcal{D} .

Proof. (\Rightarrow) First note that due to Theorem 4.4 f must be a proper affine hypersphere. Let C be a \tilde{J} -tangent affine normal field. There exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $C = -\lambda f$. Since C is \tilde{J} -tangent and transversal the same is $\frac{1}{\lambda}C = -f$. That is, f satisfies assumptions of Theorem 4.3. By Theorem 4.3 there exists a para-complex centro-affine immersion $g: V \rightarrow \mathbb{R}^{2n+2}$ defined on an open subset $V \subset \mathbb{R}^{2n}$ and there exists an open interval I such that f can be locally expressed in the form

$$f(x_1, \dots, x_{2n}, z) = \tilde{J}g(x_1, \dots, x_{2n}) \cosh z - g(x_1, \dots, x_{2n}) \sinh z \quad (4.6)$$

for $(x_1, \dots, x_{2n}) \in V$ and $z \in I$.

Let $\zeta := -|\lambda|^{\frac{2n+3}{2n+4}}g$. Bundle $\{\zeta, \tilde{J}\zeta\}$ is transversal to g , because g is para-complex centro-affine. Let $\nabla, h_1, h_2, S, \tau_1, \tau_2$ be induced objects on V by ζ . Using the Weingarten formula for g and ζ we get

$$D_{\partial_{x_i}} \zeta = -g_*(S\partial_{x_i}) + \tau_1(\partial_{x_i})\zeta + \tau_2(\partial_{x_i})J\zeta.$$

On the other hand, by straightforward computations we have

$$D_{\partial_{x_i}} \zeta = \partial_{x_i}(\zeta) = -|\lambda|^{\frac{2n+3}{2n+4}}g_*(\partial_{x_i}).$$

Thus, we obtain

$$S = |\lambda|^{\frac{2n+3}{2n+4}}I, \quad \tau_1 = 0, \quad \tau_2 = 0. \quad (4.7)$$

Now, it is enough to show that ζ is an affine normal vector field that is $|H_\zeta| = 1$. Since g is para-holomorphic, without loss of generality, we may assume that

$$\partial_{x_{n+i}} = \tilde{J}\partial_{x_i}$$

for $i = 1, \dots, n$. Let h be the second fundamental form for f .

Using similar methods like in the proof of Theorem 4.1 from [4] one may compute

$$-\lambda h(\partial_{x_i}, \partial_{x_j}) = -|\lambda|^{\frac{2n+3}{2n+4}}h_1(\partial_{x_i}, \partial_{x_j}), \quad h(\partial_z, \partial_z) = -\frac{1}{\lambda}$$

and

$$h(\partial_z, \partial_{x_i}) = h(\partial_{x_i}, \partial_z) = 0$$

for $i, j = 1, \dots, 2n$. Let us denote

$$a := \theta_\zeta(\partial_{x_1}, \dots, \partial_{x_n}, \tilde{J}\partial_{x_1}, \dots, \tilde{J}\partial_{x_n}).$$

Then we have

$$\begin{aligned} \det h &:= \det \begin{bmatrix} h(\partial_{x_1}, \partial_{x_1}) & h(\partial_{x_1}, \partial_{x_2}) & \cdots & h(\partial_{x_1}, \partial_{x_{2n}}) & 0 \\ h(\partial_{x_2}, \partial_{x_1}) & h(\partial_{x_2}, \partial_{x_2}) & \cdots & h(\partial_{x_2}, \partial_{x_{2n}}) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h(\partial_{x_{2n}}, \partial_{x_1}) & h(\partial_{x_{2n}}, \partial_{x_2}) & \cdots & h(\partial_{x_{2n}}, \partial_{x_{2n}}) & 0 \\ 0 & 0 & \cdots & 0 & -\frac{1}{\lambda} \end{bmatrix} \\ &= -\frac{1}{\lambda} \det[h(\partial_{x_i}, \partial_{x_j})] = -\frac{1}{\lambda} \cdot \left(\frac{1}{\lambda}\right) \cdot |\lambda|^{\frac{2n+3}{2n+4}} a^{2n} \det[h_1(\partial_{x_i}, \partial_{x_j})] \\ &= -\frac{1}{\lambda} \cdot |\lambda|^{-\frac{2n}{2n+4}} a^2 H_\zeta \end{aligned}$$

and

$$(\omega_h)^2 = |\det h| = |\lambda|^{\frac{-2n-2}{n+2}} a^2 |H_\zeta|. \tag{4.8}$$

Again by similar computation like in [4] we get

$$\omega_h = -\lambda \cdot (|\lambda|^{-\frac{2n+3}{n+2}}) \cdot (-1)^{n+1} \theta_\zeta(\partial_{x_1}, \dots, \partial_{x_{2n}}) = (-1)^{n+2} \cdot \lambda \cdot (|\lambda|^{-\frac{2n+3}{n+2}}) a.$$

Using the above formula in (4.8) we easily obtain

$$|H_\zeta| = a^{-2} |\lambda|^{\frac{2n+2}{n+2}} \cdot \lambda^2 \cdot |\lambda|^{-\frac{4n+6}{n+2}} a^2 = 1.$$

(\Leftrightarrow) Let $g: U \rightarrow \mathbb{R}^{2n+2}$ be a proper para-complex affine hypersphere. Since g is a proper para-complex affine hypersphere there exists $\alpha \neq 0$ such that $\zeta = -\alpha g$ is an affine normal vector field. Without loss of generality we may assume that $\alpha > 0$. Since both, g and $\tilde{J}g$ are transversal, we see that $\{g_{x_1}, \dots, g_{x_{2n}}, g, \tilde{J}g\}$ forms the basis of \mathbb{R}^{2n+2} . The above implies that

$$f: U \times I \ni (x_1, \dots, x_{2n}, z) \mapsto f(x_1, \dots, x_{2n}, z) \in \mathbb{R}^{2n+2}$$

given by the formula:

$$f(x_1, \dots, x_{2n}, z) := \tilde{J}g(x_1, \dots, x_{2n}) \cosh z - g(x_1, \dots, x_{2n}) \sinh z$$

is an immersion and $C := -\alpha \frac{2n+4}{2n+3} \cdot f$ is a transversal vector field. The field C is \tilde{J} -tangent because $\tilde{J}C = \alpha \frac{2n+4}{2n+3} f_z$. Since C is equiaffine and $S = \alpha \frac{2n+4}{2n+3} I$ it is enough to show that $\omega_h = \theta$ for some positively oriented (relative to θ) basis on $U \times I$. Let $\partial_{x_1}, \dots, \partial_{x_{2n}}, \partial_z$ be a local coordinate system on $U \times I$. Since g is para-holomorphic we may assume that $\partial_{x_{n+i}} = \tilde{J}\partial_{x_i}$ for $i = 1, \dots, n$.

Then we have

$$\theta(\partial_{x_1}, \dots, \partial_{x_{2n}}, \partial_z) = -\alpha^{-\frac{2n+2}{2n+3}} \cdot (-1)^{n+1} \theta_\zeta(\partial_{x_1}, \dots, \partial_{x_{2n}}).$$

That is

$$\theta_\zeta(\partial_{x_1}, \dots, \partial_{x_{2n}}) = (-1)^n \alpha^{\frac{2n+2}{2n+3}} \cdot \theta(\partial_{x_1}, \dots, \partial_{x_{2n}}, \partial_z). \tag{4.9}$$

In a similar way as in the proof of the first implication we compute

$$\begin{aligned} \det h &= -\alpha^{-\frac{2n+4}{2n+3}} \cdot \left(\frac{\alpha}{\alpha^{\frac{2n+4}{2n+3}}}\right)^{2n} \det h_1 \\ &= -\alpha^{-\frac{2n+4}{2n+3}} \cdot \alpha^{-\frac{2n}{2n+3}} \det h_1 \\ &= -\alpha^{-\frac{4n-4}{2n+3}} \det h_1. \end{aligned}$$

The above implies that

$$(\omega_h)^2 = |\det h| = \alpha^{-\frac{4n-4}{2n+3}} |\det h_1|.$$

Since

$$|\det h_1| = |H_\zeta| |\theta_\zeta(\partial_{x_1}, \dots, \partial_{x_{2n}})|^2,$$

we obtain

$$(\omega_h)^2 = \alpha^{-\frac{4n-4}{2n+3}} |H_\zeta| |\theta_\zeta(\partial_{x_1}, \dots, \partial_{x_{2n}})|^2.$$

Finally, using the fact that $|H_\zeta| = 1$ and (4.9), we get

$$\omega_h = |\theta(\partial_{x_1}, \dots, \partial_{x_{2n}}, \partial_z)|.$$

The proof is completed. □

Immediately from the proof of the above theorem we get

Corollary 4.6. *If f is a \tilde{J} -tangent affine hypersphere with the shape operator $S = \lambda \text{id}$ and g is a para-complex affine hypersphere (related to f) with the shape operator $\tilde{S} = \alpha \text{id}$ then λ and α are related by the following formula:*

$$|\lambda| = |\alpha|^{\frac{2n+4}{2n+3}}.$$

Now, we shall recall a classification theorem for para-complex affine hyperspheres.

Theorem 4.7 [1]. *Let $g: M \rightarrow \mathbb{R}^{2n+2}$ be a para-complex affine hypersphere with a transversal bundle $\{\zeta, \tilde{J}\zeta\}$. Then g can be locally expressed in the form*

$$g = f_1 \times f_2 + \tilde{J} \circ (f_1 \times (-f_2)), \tag{4.10}$$

where $U_1 \subset \mathbb{R}^n, U_2 \subset \mathbb{R}^n$ are open subsets and

$$f_1: U_1 \rightarrow \mathbb{R}^{n+1}, \quad f_2: U_2 \rightarrow \mathbb{R}^{n+1}$$

are (real) affine hyperspheres.

Moreover, if g is proper (respectively improper) then both f_1 and f_2 are proper (respectively improper) as well. The converse is also true, in the sense, that for every two proper (respectively improper) n -dimensional affine hyperspheres f_1 and f_2 the formula (4.10) defines a proper (respectively improper) para-complex affine hypersphere.

The following theorem allows us to construct \tilde{J} -tangent affine hyperspheres using standard proper affine hyperspheres. Namely we have

Theorem 4.8. *Let $f: M \rightarrow \mathbb{R}^{2n+2}$ be a \tilde{J} -tangent affine hypersphere with an involutive distribution \mathcal{D} . Then f can be locally expressed in the form:*

$$\begin{aligned}
 & f(x_1, \dots, x_n, y_1, \dots, y_n, z) \\
 &= \left(\tilde{J} \circ (f_1 \times f_2) + f_1 \times (-f_2) \right) (x_1, \dots, x_n, y_1, \dots, y_n) \cosh z \\
 &\quad - \left((f_1 \times f_2) + \tilde{J} \circ (f_1 \times (-f_2)) \right) (x_1, \dots, x_n, y_1, \dots, y_n) \sinh z, \quad (4.11)
 \end{aligned}$$

where f_1 and f_2 are proper n -dimensional affine hyperspheres. Moreover, the converse is also true in the sense that if f_1 and f_2 are proper n -dimensional affine hyperspheres then f given by the above formula is a proper \tilde{J} -tangent affine hypersphere with an involutive distribution \mathcal{D} .

Proof. The proof is an immediate consequence of Theorems 4.5 and 4.7. \square

Since the only 1-dimensional proper affine spheres are the ellipse and hyperbola we can obtain the complete local classification of 3-dimensional \tilde{J} -tangent affine hyperspheres with an involutive distribution \mathcal{D} . Namely we have

Theorem 4.9. *Let $f: M^3 \rightarrow \mathbb{R}^4$ be a \tilde{J} -tangent affine hypersphere with an involutive distribution \mathcal{D} . Then up to \tilde{J} -invariant affine transformation f is locally equivalent to one of the following hypersurfaces:*

$$f_1(x, y, z) = \begin{pmatrix} \cos x + \cos y \\ \sin x + \sin y \\ \cos x - \cos y \\ \sin x - \sin y \end{pmatrix} \cosh z - \begin{pmatrix} \cos x - \cos y \\ \sin x - \sin y \\ \cos x + \cos y \\ \sin x + \sin y \end{pmatrix} \sinh z, \quad (4.12)$$

$$f_2(x, y, z) = \begin{pmatrix} \cosh x + \cosh y \\ \sinh x + \sinh y \\ \cosh x - \cosh y \\ \sinh x - \sinh y \end{pmatrix} \cosh z - \begin{pmatrix} \cosh x - \cosh y \\ \sinh x - \sinh y \\ \cosh x + \cosh y \\ \sinh x + \sinh y \end{pmatrix} \sinh z, \quad (4.13)$$

$$f_3(x, y, z) = \begin{pmatrix} \cos x + \cosh y \\ \sin x + \sinh y \\ \cos x - \cosh y \\ \sin x - \sinh y \end{pmatrix} \cosh z - \begin{pmatrix} \cos x - \cosh y \\ \sin x - \sinh y \\ \cos x + \cosh y \\ \sin x + \sinh y \end{pmatrix} \sinh z, \quad (4.14)$$

$$f_4(x, y, z) = \begin{pmatrix} \cosh x + \cos y \\ \sinh x + \sin y \\ \cosh x - \cos y \\ \sinh x - \sin y \end{pmatrix} \cosh z - \begin{pmatrix} \cosh x - \cos y \\ \sinh x - \sin y \\ \cosh x + \cos y \\ \sinh x + \sin y \end{pmatrix} \sinh z. \quad (4.15)$$

Proof. By Theorem 4.8 f can be locally obtained from 1-dimensional proper affine spheres f_1 and f_2 . Since the only 1-dimensional proper affine spheres are the ellipse and hyperbola then f_i is affinely equivalent to either $\gamma_1(t) = (\cos t, \sin t)$ or $\gamma_2(t) = (\cosh t, \sinh t)$. Now one may find affine transformations P, Q of \mathbb{R}^2 such that

$$f_1 = P \circ \gamma_{i_0} \quad \text{and} \quad f_2 = Q \circ \gamma_{j_0} \quad (4.16)$$

for some $i_0, j_0 \in \{1, 2\}$. Applying (4.16) to (4.11) we get

$$\begin{aligned}
 f(x, y, z) &= A \circ \left[\left(\tilde{J} \circ (\gamma_{i_0} \times \gamma_{j_0}) + \gamma_{i_0} \times (-\gamma_{j_0}) \right) (x, y) \cosh z \right. \\
 &\quad \left. - \left((\gamma_{i_0} \times \gamma_{j_0}) + \tilde{J} \circ (\gamma_{i_0} \times (-\gamma_{j_0})) \right) (x, y) \sinh z \right] \\
 &= A \circ \left[\begin{pmatrix} \gamma_{i_0}(x) + \gamma_{j_0}(y) \\ \gamma_{i_0}(x) - \gamma_{j_0}(y) \end{pmatrix} \cosh z - \begin{pmatrix} \gamma_{i_0}(x) - \gamma_{j_0}(y) \\ \gamma_{i_0}(x) + \gamma_{j_0}(y) \end{pmatrix} \sinh z \right]
 \end{aligned}$$

where

$$A = \begin{bmatrix} \frac{1}{2}(P + Q) & \frac{1}{2}(P - Q) \\ \frac{1}{2}(P - Q) & \frac{1}{2}(P + Q) \end{bmatrix}.$$

Hence f is (up to \tilde{J} -invariant affine transformation A) equivalent to

$$f_0(x, y, z) = \begin{pmatrix} \gamma_{i_0}(x) + \gamma_{j_0}(y) \\ \gamma_{i_0}(x) - \gamma_{j_0}(y) \end{pmatrix} \cosh z - \begin{pmatrix} \gamma_{i_0}(x) - \gamma_{j_0}(y) \\ \gamma_{i_0}(x) + \gamma_{j_0}(y) \end{pmatrix} \sinh z.$$

Now taking different combinations of i_0, j_0 we easily obtain (4.12)–(4.15). \square

Remark 4.10. Note that (4.14) and (4.15) are affinely equivalent but the affine transformation mapping (4.14) onto (4.15) is not \tilde{J} -invariant.

Remark 4.11. It is worth to mention that hypersurfaces from Theorem 4.9 are flat and have parallel cubic form. Actually they are the only proper 3-dimensional affine hyperspheres with this property (see [11, 12] for details).

To conclude this section, we give an example of a \tilde{J} -tangent affine hypersphere with a non-involutive distribution \mathcal{D} .

Example 4.12. Let f be defined as follows:

$$f: \mathbb{R}^3 \ni (x, y, z) \mapsto \begin{pmatrix} xy + 1 \\ x + \frac{1}{2}y \\ xy \\ x - \frac{1}{2}y \end{pmatrix} \cosh z - \begin{pmatrix} xy \\ x - \frac{1}{2}y \\ xy + 1 \\ x + \frac{1}{2}y \end{pmatrix} \sinh z \in \mathbb{R}^4.$$

It is not difficult to check that f is an immersion and the vector field $C: \mathbb{R}^3 \ni (x, y, z) \mapsto -f(x, y, z) \in \mathbb{R}^4$ is transversal to $f_*(\mathbb{R}^3)$.

In the canonical basis $\left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\}$ the second fundamental form h is expressed as follows

$$h = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 2x \\ 0 & 2x & -1 \end{bmatrix}.$$

The above implies that f is nondegenerate. By straightforward computations we obtain that C is the affine normal field. Since $\tilde{J}C = -f_z \in f_*(TM)$ it

follows that f is a \tilde{J} -tangent affine hypersphere. Moreover, we have that $\tilde{J}f_x = f_x$, so $\frac{\partial}{\partial x} \in \mathcal{D}^+$. We also have

$$\tilde{J}(2x^2 f_x + f_y + 2x f_z) = -(2x^2 f_x + f_y + 2x f_z),$$

so the vector field $W := 2x^2 \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + 2x \frac{\partial}{\partial z}$ belongs to \mathcal{D}^- . Now, we compute that

$$h\left(\frac{\partial}{\partial x}, W\right) = 2x^2 h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) + h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) + 2x h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right) = -1.$$

Using the formula (3.8) and the above we get

$$\eta\left(\left[\frac{\partial}{\partial x}, W\right]\right) = h\left(\frac{\partial}{\partial x}, \varphi W\right) - h\left(W, \varphi \frac{\partial}{\partial x}\right) = -2h\left(\frac{\partial}{\partial x}, W\right) = 2.$$

Since $\ker \eta = \mathcal{D}$, the above implies that $[\frac{\partial}{\partial x}, W] \notin \mathcal{D}$ and in consequence the distribution \mathcal{D} is not involutive.

5. Some Applications

In this section we show some applications of results obtained in the previous section. In particular, we show that \tilde{J} -tangent affine hyperspheres can be classified in terms, of so called, Calabi products.

Recall that [13] the Calabi product of two proper affine hyperspheres

$$\psi_1: M_1 \rightarrow \mathbb{R}^{n_1+1} \quad \text{and} \quad \psi_2: M_2 \rightarrow \mathbb{R}^{n_2+1}$$

is an affine immersion

$$\psi: M_1 \times M_2 \times \mathbb{R} \rightarrow \mathbb{R}^{n_1+n_2+2}$$

defined by the formula

$$\psi(x, y, z) := (c_1 e^{\sqrt{\frac{n_2+1}{n_1+1}}az} \psi_1(x), c_2 e^{-\sqrt{\frac{n_1+1}{n_2+1}}az} \psi_2(y))$$

where c_1, c_2 and a are nonzero constants.

Let f_1 and f_2 and f be the affine hyperspheres from Theorem 4.8 with the affine normal fields $C_1 = -\alpha f_1$, $C_2 = -\beta f_2$ and $C = -\lambda f$, respectively ($\alpha, \beta, \lambda > 0$). Let us denote by $CP(f_1, f_2)$ the Calabi product of f_1 and f_2 with $c_1 = 2$, $c_2 = 1$ and $a = -1$. That is we have

$$CP(f_1, f_2)(x, y, z) := (2e^{-z} f_1(x), e^z f_2(y))$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. We shall always consider $CP(f_1, f_2)$ with a transversal vector field $C_{CP} := -\lambda \cdot CP(f_1, f_2)$.

For the affine hypersphere f_i ($i = 1, 2$) we shall denote by $\nabla_i, h_i, S_i, \Theta_i$ the Blaschke connection, the Blaschke metric, the shape operator and the induced volume form, respectively. Similarly for the affine hypersurface $CP(f_1, f_2)$ we denote the induced affine objects by $\nabla^{CP}, h_{CP}, S_{CP}$ and Θ_{CP} .

Using similar methods like in [14] one may obtain that the affine metric h_{CP} of $CP(f_1, f_2)$ is given by the product metric

$$h_{CP} = \frac{1}{2} \frac{\alpha}{\lambda} h_1 \otimes \frac{1}{2} \frac{\beta}{\lambda} h_2 \otimes \left(-\frac{1}{\lambda}\right) dz^2 \tag{5.1}$$

and the connection ∇^{CP} can be expressed in terms of ∇_1, ∇_2 and h_1, h_2 as follows:

$$\nabla^{CP}_{\partial_{x_i}} \partial_{x_j} = \nabla_{1\partial_{x_i}} \partial_{x_j} + \frac{1}{2} \alpha h_1(\partial_{x_i}, \partial_{x_j}) \partial_z \tag{5.2}$$

$$\nabla^{CP}_{\partial_{y_i}} \partial_{y_j} = \nabla_{2\partial_{y_i}} \partial_{y_j} - \frac{1}{2} \beta h_2(\partial_{y_i}, \partial_{y_j}) \partial_z \tag{5.3}$$

$$\nabla^{CP}_{\partial_{x_i}} \partial_{y_j} = \nabla^{CP}_{\partial_{y_j}} \partial_{x_i} = 0 \tag{5.4}$$

$$\nabla^{CP}_{\partial_{x_i}} \partial_z = \nabla^{CP}_{\partial_z} \partial_{x_i} = -\partial_{x_i} \tag{5.5}$$

$$\nabla^{CP}_{\partial_{y_i}} \partial_z = \nabla^{CP}_{\partial_z} \partial_{y_i} = \partial_{y_i} \tag{5.6}$$

$$\nabla^{CP}_{\partial_z} \partial_z = 0. \tag{5.7}$$

By straightforward computations we obtain

$$\omega_{h_{CP}} = \sqrt{\frac{\alpha^n \beta^n}{2^{2n} \lambda^{2n+1}}} \omega_{h_1} \otimes \omega_{h_2} \tag{5.8}$$

and

$$\Theta_{CP} = (-2)^{n+2} \cdot \frac{\lambda}{\alpha\beta} \Theta_1 \otimes \Theta_2. \tag{5.9}$$

Let us define a $(2n + 2) \times (2n + 2)$ matrix A by the formula

$$A := \begin{bmatrix} \frac{1}{2} I_{n+1} & I_{n+1} \\ \frac{1}{2} I_{n+1} & -I_{n+1} \end{bmatrix}$$

where by I_n we denote an identity matrix of dimension $n \times n$. It is easy to see that $\det A = (-1)^{n+1}$ that is A is an equiaffine transformation of \mathbb{R}^{2n+2} . By straightforward calculation one may check that

$$f = A \circ CP(f_1, f_2) \tag{5.10}$$

that is we have the following

Corollary 5.1. *Let f_1, f_2 and f be like in Theorem 4.8 then f is up to equiaffine transformation the Calabi product of f_1 and f_2 .*

Since A is an equiaffine transformation then both f and $CP(f_1, f_2)$ are affine hyperspheres. In particular, using (5.8) and (5.9) we obtain

Corollary 5.2. *Let f_1, f_2 and f be like in Theorem 4.8 and let $C_1 = -\alpha f_1, C_2 = -\beta f_2$ and $C = -\lambda f$ be their affine normals. Then α, β and λ are related by the formula:*

$$\lambda = \left[\frac{(\alpha\beta)^{n+2}}{2^{4n+4}} \right]^{\frac{1}{2n+3}}.$$

Calabi products have many interesting properties. In particular, they preserve parallel cubic form (see [14]). Moreover the affine metric of Calabi product is flat if and only if both components have a flat affine metric.

Now using (5.10) and results from [12, 15, 16] one can obtain some classification results for \tilde{J} -tangent affine hyperspheres with the parallel cubic form. For example, when $\dim M = 5$, we have the following

Corollary 5.3. *Let $f: M \rightarrow \mathbb{R}^6$ be a \tilde{J} -tangent affine hypersphere with an involutive distribution \mathcal{D} and a parallel cubic form. Then f is locally affine equivalent to Calabi product $CP(f_1, f_2)$ where f_i ($i = 1, 2$) is one of the following surfaces:*

$$x^2 + y^2 + z^2 = 1 \tag{5.11}$$

$$x^2 + y^2 - z^2 = 1 \tag{5.12}$$

$$x^2 + y^2 - z^2 = -1 \tag{5.13}$$

$$xyz = 1 \tag{5.14}$$

$$(x^2 + y^2)z = 1 \tag{5.15}$$

As it was already mentioned in previous section (see Remark 4.11) all 3-dimensional \tilde{J} -tangent affine hyperspheres with involutive distribution \mathcal{D} are flat. Of course this is not the case in higher dimensions. In particular, since the only 2-dimensional proper flat affine spheres are (5.14) and (5.15) (see [17]), taking Calabi products, we get the following classification result in 5-dimensional case:

Corollary 5.4. *If $f: M \rightarrow \mathbb{R}^6$ is a flat \tilde{J} -tangent affine hypersphere with involutive distribution \mathcal{D} then f is locally affine equivalent to one of the following hypersurfaces:*

$$x_1x_2x_3x_4x_5x_6 = 1 \tag{5.16}$$

$$(x_1^2 + x_2^2)(x_3^2 + x_4^2)x_5x_6 = 1 \tag{5.17}$$

$$(x_1^2 + x_2^2)x_3x_4x_5x_6 = 1 \tag{5.18}$$

Note, that contrary to 3-dimensional case, not all 5-dimensional flat proper affine hyperspheres are (after suitable affine transformation) \tilde{J} -tangent affine hyperspheres with the involutive distribution \mathcal{D} . Indeed, it is well known that $(x_1^2 + x_2^2)(x_3^2 + x_4^2)(x_5^2 + x_6^2) = 1$ is a proper flat affine hypersphere but it is not affinely equivalent to any of (5.16)–(5.18). Actually we will show (see proof of Prop. 5.7) that this hypersphere cannot be transformed into \tilde{J} -tangent affine hypersphere.

Recall that we have the following general classification result [18].

Theorem 5.5 [18]. *Let M be an affine hypersphere in \mathbb{R}^{n+1} with constant sectional curvature c and with nonzero Pick invariant J . Then $c = 0$ and M is equivalent to*

$$(x_1^2 \pm x_2^2)(x_3^2 \pm x_4^2) \cdots (x_{2m-1}^2 \pm x_{2m}^2) = 1, \tag{5.19}$$

if $n = 2m - 1$ or with

$$(x_1^2 \pm x_2^2)(x_3^2 \pm x_4^2) \cdots (x_{2m-1}^2 \pm x_{2m}^2)x_{2m+1} = 1, \tag{5.20}$$

if $n = 2m$.

Before we proceed with classification result for flat \tilde{J} -tangent affine hyperspheres we shall show the following lemma

Lemma 5.6. *Let $f: M \rightarrow \mathbb{R}^{2n+2}$ be an affine hypersphere with the Blaschke field $C: M \rightarrow \mathbb{R}^{2n+2}$. f is affine equivalent to \tilde{J} -tangent affine hypersphere if and only if there exists an affine transformation $B: \mathbb{R}^{2n+2} \rightarrow \mathbb{R}^{2n+2}$ such that $B \circ C$ is tangent to f and $B \sim \tilde{J}$ (i.e. matrices for B and \tilde{J} are similar)*

Proof. If f is affine equivalent to \tilde{J} -tangent affine hypersphere then there exists an affine transformation $A: \mathbb{R}^{2n+2} \rightarrow \mathbb{R}^{2n+2}$ such that $A \circ f$ considered with the transversal vector field $A \circ C$ is \tilde{J} -tangent. That is if X_1, \dots, X_{2n+1} is a basis of vector fields on M then

$$\begin{aligned} 0 &= \det[A \circ f_*X_1, \dots, A \circ f_*X_{2n+1}, \tilde{J} \circ A \circ C] \\ &= \det A \det[f_*X_1, \dots, f_*X_{2n+1}, A^{-1} \circ \tilde{J} \circ A \circ C]. \end{aligned}$$

Now it is enough to take $B := A^{-1} \circ \tilde{J} \circ A$. On the other hand, when $B \sim \tilde{J}$, there exists an invertible matrix P such that $B := P^{-1} \circ \tilde{J} \circ P$. Now taking $A := P$ we prove the converse. \square

Now we obtain

Proposition 5.7. *Let $f: M \rightarrow \mathbb{R}^{2n+2}$ be a flat \tilde{J} -tangent affine hypersphere then distribution \mathcal{D} is involutive and f is affine equivalent to either*

$$(x_1^2 \pm x_2^2)(x_3^2 \pm x_4^2) \cdots (x_{2n+1}^2 \pm x_{2n+2}^2) = 1, \tag{5.21}$$

if n is odd or

$$(x_1^2 \pm x_2^2)(x_3^2 \pm x_4^2) \cdots (x_{2n-1}^2 \pm x_{2n}^2)x_{2n+1}x_{2n+2} = 1, \tag{5.22}$$

if n is even.

Proof. Since proper flat affine hyperspheres have nonzero Pick invariant, by Theorem 5.5, they are affine equivalent to (5.19) or (5.20). In particular, $2n + 1$ dimensional flat affine hyperspheres are equivalent to

$$(x_1^2 \pm x_2^2)(x_3^2 \pm x_4^2) \cdots (x_{2n+1}^2 \pm x_{2n+2}^2) = 1. \tag{5.23}$$

We shall show that all the above affine hyperspheres (with one exception) are, after suitable affine transformation, \tilde{J} -tangent. If n is odd (5.23) can be obtained as the Calabi product of two flat n -dimensional affine hyperspheres

$$(x_1^2 \pm x_2^2) \cdots (x_n^2 \pm x_{n+1}^2) = 1$$

and

$$(x_{n+2}^2 \pm x_{n+3}^2) \cdots (x_{2n+1}^2 \pm x_{2n+2}^2) = 1$$

and as such is affine equivalent to \tilde{J} -tangent affine hypersphere with involutive distribution \mathcal{D} . If n is even and at least one of “ \pm ” in (5.23) is “ $-$ ”, without loss of generality we may assume that (5.23) contains $(x_{2n+1}^2 - x_{2n+2}^2)$ term. Now applying affine transformation changing $(x_{2n+1}^2 - x_{2n+2}^2)$ into $x_{2n+1}x_{2n+2}$ we can transform (5.23) into (5.22). Since (5.22) is the Calabi product of two flat n -dimensional affine hyperspheres of form (5.20) it is affine equivalent to \tilde{J} -tangent affine hypersphere with involutive distribution \mathcal{D} .

Now it remained to show that for n even

$$(x_1^2 + x_2^2)(x_3^2 + x_4^2) \cdots (x_{2n+1}^2 + x_{2n+2}^2) = 1 \tag{5.24}$$

cannot be transformed by affine transformation into \tilde{J} -tangent affine hypersphere. First note that (5.24) can be parameterized as follows:

$$f(v_1, \dots, v_{n+1}, u_1, \dots, u_n) = \begin{pmatrix} e^{u_1} \cos v_1 \\ e^{u_1} \sin v_1 \\ \dots \\ e^{u_n} \cos v_n \\ e^{u_n} \sin v_n \\ e^{-u_1 - \dots - u_n} \cos v_{n+1} \\ e^{-u_1 - \dots - u_n} \sin v_{n+1} \end{pmatrix}$$

where $v_i, u_i \in \mathbb{R}$. Assume that f is affine equivalent to \tilde{J} -tangent affine hypersphere, then by Lemma 5.6 there exists a matrix $B = [b_{ij}] \in GL(2n + 2)$, $B \sim \tilde{J}$ such that $B \circ f$ is tangent to f . That is

$$W := \det[f_{v_1}, \dots, f_{v_{n+1}}, f_{u_1}, \dots, f_{u_n}, B \circ f] = 0.$$

By straightforward (but quite long) computations one may obtain

$$\begin{aligned} W = & (-1)^{\frac{(n+1)(n+2)}{2}} \left(\sum_{k=1}^n \sum_{s=1}^n e^{-u_k + u_s} A_{k,s} \right. \\ & + e^{-(u_1 + \dots + u_n)} \sum_{k=1}^n e^{-u_k} A_{k,n+1} \\ & \left. + e^{u_1 + \dots + u_n} \sum_{k=1}^n e^{u_k} A_{n+1,k} + A_{n+1,n+1} \right), \tag{5.25} \end{aligned}$$

where

$$\begin{aligned} A_{i,j} := & (\cos v_i \cos v_j b_{2i-1,2j-1} + \cos v_i \sin v_j b_{2i-1,2j} \\ & + \sin v_i \cos v_j b_{2i,2j-1} + \sin v_i \sin v_j b_{2i,2j}) \end{aligned}$$

for $i, j = 1, \dots, n + 1$. Since $W = 0$ the above implies that $A_{k,s} = 0$ for $k, s = 1, \dots, n$, $k \neq s$ and $A_{k,n+1} = A_{n+1,k} = 0$ for $k = 1, \dots, n$. In consequence we obtain

$$b_{2k-1,2s-1} = b_{2k-1,2s} = b_{2k,2s-1} = b_{2k,2s} = 0$$

for $k, s = 1, \dots, n, k \neq s$ and

$$\begin{aligned} b_{2k-1,2n+1} &= b_{2k-1,2n+2} = b_{2k,2n+1} = b_{2k,2n+2} \\ &= b_{2n+1,2k-1} = b_{2n+2,2k-1} = b_{2n+1,2k} = b_{2n+2,2k} = 0 \end{aligned}$$

for $k = 1, \dots, n$. Moreover, from (5.25) we also have that $\sum_{k=1}^{n+1} A_{k,k} = 0$ that is

$$\begin{aligned} &\sum_{k=1}^{n+1} \left(\cos^2 v_k b_{2k-1,2k-1} + \sin v_k \cos v_k (b_{2k-1,2k} + b_{2k,2k-1}) + \sin^2 v_k b_{2k,2k} \right) \\ &= \sum_{k=1}^{n+1} \cos^2 v_k (b_{2k-1,2k-1} - b_{2k,2k}) + \sum_{k=1}^{n+1} \sin v_k \cos v_k (b_{2k-1,2k} + b_{2k,2k-1}) \\ &+ \sum_{k=1}^{n+1} b_{2k,2k} = 0. \end{aligned}$$

The above implies that $b_{2k-1,2k-1} = b_{2k,2k}, b_{2k-1,2k} = -b_{2k,2k-1}$ for $k = 1, \dots, n+1$ and $\sum_{k=1}^{n+1} b_{2k,2k} = 0$. Summarising, the matrix B can be expressed as a block diagonal matrix

$$B = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{n+1} \end{bmatrix}$$

where $B_k = \begin{bmatrix} b_{2k,2k} & b_{2k-1,2k} \\ -b_{2k-1,2k} & b_{2k,2k} \end{bmatrix}$ for $k = 1, \dots, n+1$. Note that $\det B_k > 0$ and in consequence $\det B = \det B_1 \cdot \dots \cdot \det B_{n+1} > 0$. On the other hand, since $B \sim \tilde{J}$, we have $\det B = \det \tilde{J} = (-1)^{n+1} = -1 < 0$, since n is even, what contradicts our assumption. \square

Let J be the standard complex structure on $\mathbb{R}^{2n+2} \cong \tilde{\mathbb{C}}^{n+1}$. Although (5.24) cannot be transformed into \tilde{J} -tangent affine hypersphere one may show that it is affine equivalent to J -tangent affine hypersphere (more details on J -tangent affine hyperspheres can be found in [4]). Actually we have the following general result

Proposition 5.8. *For every $n \geq 0$ the hypersurface*

$$(x_1^2 + x_2^2)(x_3^2 + x_4^2) \cdots (x_{2n+1}^2 + x_{2n+2}^2) = 1 \tag{5.26}$$

is (after suitable affine transformation) J -tangent affine hypersphere.

Proof. Applying $P: \mathbb{R}^{2n+2} \ni (x_1, \dots, x_{2n+2}) \mapsto (x_1, x_{n+2}, \dots, x_{n+1}, x_{2n+2}) \in \mathbb{R}^{2n+2}$ to (5.26) we obtain

$$(x_1^2 + x_{n+2}^2)(x_2^2 + x_{n+3}^2) \cdots (x_{n+1}^2 + x_{2n+2}^2) = 1. \tag{5.27}$$

Let us denote by G the gradient of (5.27). That is

$$G := \left[\frac{2x_1}{x_1^2 + x_{n+2}^2}, \dots, \frac{2x_{n+1}}{x_{n+1}^2 + x_{2n+2}^2}, \frac{2x_{n+2}}{x_1^2 + x_{n+2}^2}, \dots, \frac{2x_{2n+2}}{x_{n+1}^2 + x_{2n+2}^2} \right]^T.$$

Since $J(x_1, \dots, x_{2n+2}) = [-x_{n+2}, \dots, -x_{2n+2}, x_1, \dots, x_{n+1}]^T$ we see that G is orthogonal to $J(x_1, \dots, x_{2n+2})$ thus (5.27) is a J -tangent affine hypersphere. \square

Remark 5.9. The above results show that every proper $(2n + 1)$ -dimensional flat affine hypersphere is (after suitable affine transformation) either \tilde{J} -tangent or J -tangent. Moreover, when n is odd (5.26) is both \tilde{J} -tangent and J -tangent.

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References

- [1] Szancer, Z.: On para-complex affine hyperspheres. *Results Math.* **72**, 491–513 (2017)
- [2] Cortés, V., Lawn, M.A., Schäfer, L.: Affine hyperspheres associated to special para-Kähler manifolds. *Int. J. Geom. Methods Mod. Phys.* **3**, 995–1009 (2006)
- [3] Lawn, M.A., Schäfer, L.: Decompositions of para-complex vector bundles and para-complex affine immersions. *Results Math.* **48**, 246–274 (2005)
- [4] Szancer, Z.: J -tangent affine hyperspheres with an involutive contact distribution. *Publ. Math. Debrecen* **89**(4), 399–413 (2016)
- [5] Kaneyuki, S., Williams, F.L.: Almost paracontact and parahodge structures on manifolds. *Nagoya Math. J.* **99**, 173–187 (1985)
- [6] Nomizu, K., Sasaki, T.: *Affine Differential Geometry*. Cambridge University Press, Cambridge (1994)
- [7] Al-Aqeel, A., Bejancu, A.: On the geometry of paracomplex submanifolds. *Demonstr. Math.* **34**(4), 919–932 (2001)
- [8] Cruceanu, V., Fortuny, P., Gadea, P.M.: A survey on para-complex geometry. *Rocky Mt. J. Math.* **26**(1), 83–115 (1996)

- [9] Dillen, F., Vrancken, L., Verstraelen, L.: Complex affine differential geometry. *Atti Acc. Peloritana dei Pericolanti LXX I*, 232–260 (1988)
- [10] Szancer, Z.: \tilde{J} -tangent affine hypersurfaces with an induced almost paracontact structure. [arXiv:1710.10488](https://arxiv.org/abs/1710.10488)
- [11] Dillen, F., Vrancken, L.: 3-Dimensional affine hypersurfaces in \mathbb{R}^4 with parallel cubic form. *Nagoya Math. J.* **124**, 41–53 (1991)
- [12] Hu, Z., Li, C.: The classification of 3-dimensional Lorentzian affine hypersurfaces with parallel cubic form. *Differ. Geom. Appl.* **29**, 361–373 (2011)
- [13] Hu, Z., Li, C., Li, H., Vrancken, L.: Lorentzian affine hypersurfaces with parallel cubic form. *Results Math.* **59**(3), 577–620 (2011)
- [14] Dillen, F., Vrancken, L.: Calabi-type compositions of affine spheres. *Differ. Geom. Appl.* **4**(4), 303–328 (1994)
- [15] Hu, Z., Li, C., Vrancken, L.: Locally strongly convex affine hypersurfaces with parallel cubic form. *J. Differ. I Geom.* **87**(2), 239–308 (2011)
- [16] Magid, M.A., Nomizu, K.: On affine surfaces whose cubic forms are parallel relative to the affine metric. *Proc. Jpn. Acad. Ser. A Math. Sci.* **65**(7), 215–218 (1989)
- [17] Magid, M.A., Ryan, P.: Flat affine spheres. *Geometriae Dedicata* **33**, 277–288 (1990)
- [18] Vrancken, L.: The Magid–Ryan conjecture for equiaffine hyperspheres with constant sectional curvature. *J. Differ. Geom.* **54**, 99–138 (2000)

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