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Results in Mathematics

Spacelike Hypersurfaces Immersed in Weighted Standard Static Spacetimes: Uniqueness, Nonexistence and Stability

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Abstract. A spacetime endowed with a globally defined timelike Killing vector field admits a certain model of warped product, called the standard static spacetime, and, when the volume element is modified by a factor that depends on a smooth function (which is called density function), we say that this ambient is a *weighted standard static spacetime*. In such spacetimes, we study some aspects of the geometry of spacelike hypersurfaces through of drift Laplacian of two functions support naturally related to them. For such hypersurfaces, with some restrictions on density function and the geometry of the ambient spacetime, we begin by stating and showing some results of uniqueness and nonexistence, several of them not assuming that the hypersurface to be of constant weighted mean curvature. Versions of these results are given for entire Killing graphs, that is, graphs constructed over an integral leaf of the distribution of smooth vector fields orthogonal to timelike Killing vector field. Finally, for closed spacelike hypersurface immersed in a weighted standard static spacetime with constant weighted mean curvature, we study a notion of stability via the first eigenvalue of the drift Laplacian.

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1. Introduction

Standard static spacetimes are part of the so called stationary spacetimes. Let us recall here that a *stationary spacetime* is a time-orientable Lorentzian manifold $(\overline{M}^{n+1}, \overline{g})$ where there exists an infinitesimal symmetry given by a timelike Killing vector field Y (cf. [37]). The existence of Y enables us to define around each point a coordinate system $(t, x_1, \ldots, x_n)$ such that Y coincides with the coordinate vector field $\partial/\partial t$ on its domain of definition and such that the components of the metric tensor in these coordinates are independent of t. When we normalize Y we obtain an observers vector field $Z = Y/\sqrt{-\overline{g}(Y,Y)}$. These observers measure a metric tensor that does not change with time. Furthermore, if this timelike Killing vector field is also irrotational (i.e., the distribution $Y^{\perp}$ of all smooth vector fields on $\overline{M}^{n+1}$ that are orthogonal to Y is involutive), then a local warped product structure appears and the spacetime is called *static* (for more details see, for instance, [2]). In fact, when this structure is global this spacetime is known as a *standard static spacetime*. More precisely, a standard static spacetime $(\overline{M}^{n+1}, \overline{g})$ endowed with a globally defined timelike Killing vector field Y is isometric to the warped product

$$\left(\mathbb{P}^n \times_\rho \mathbb{R}_1\,,\, \pi^*_{\mathbb{P}^n}(\widetilde{g}) + (\rho \circ \pi_{\mathbb{P}^n})^2 \pi^*_{\mathbb{R}}(-dt^2)\right)$$

where $\pi_{\mathbb{P}^n}$ and $\pi_{\mathbb{R}}$ denote the canonical projections from $\mathbb{P}^n \times \mathbb{R}_1$ onto each factor, $\widetilde{g}$ is the Riemannian metric on the base $\mathbb{P}^n$, $\mathbb{R}_1$ is the manifold $\mathbb{R}$ endowed with the metric $-dt^2$ and $\rho = \sqrt{-\overline{g}(Y,Y)}$ is the warping function. In this context, it is known that any static spacetime is locally isometric to a standard static one (cf. [31, Proposition 12.38]). Conversely, Sánchez in [39] and more recently Aledo, Romero and Rubio in [2] obtained some sufficient conditions for a static spacetime to be standard. Other properties on the geometry of standard static spacetimes were studied by Sánchez in [38–40].

The importance of standard static spacetimes also comes from the fact that they include some classical spacetimes, such as the $(n+1)$-dimensional *Lorentz–Minkowski space* $\mathbb{L}^{n+1}$, *Einstein static universe* as well as models that describe an universe where there is only a spherically symmetric non-rotating mass, as a star or a black hole, like *exterior Schwarzschild spacetime* and some regions of *Reissner–Nordström spacetime* (see Remark 1).

The study of spacelike hypersurfaces immersed with constant mean curvature in a spacetime has attracted the interest of a considerable group of geometers as evidenced by the amount of works that it has generated in the last decades. This is due not only to its mathematical interest, but also to its relevance in General Relativity. For example, constant mean curvature spacelike hypersurfaces are particularly suitable for studying the propagation of gravitational radiation. See, for instance, [26, 41] for a summary of several reasons justifying this interest. From the mathematical point of view, the study of the geometry of constant mean curvature spacelike hypersurfaces is mostly due to the fact that they exhibit nice *Calabi–Bernstein type properties*. More precisely, this study had its beginnings when S. Bernstein [8] proved that the only entire minimal graphs in the 3-dimensional Euclidean space $\mathbb{R}^3$ are planes. In the Lorentzian setting, there is a result analogue to the Riemannian one that states that the only entire maximal graphs in the 3-dimensional Lorentz–Minkowski space $\mathbb{L}^3$ are spacelike planes. This result was first proved by Calabi [11], and extended to the general n-dimensional case by Cheng and Yau [15].

A natural extension to the Calabi–Bernstein problem, as it became known these days, is to determine a reasonable set of sufficient conditions which guarantee the uniqueness (or nonexistence) of complete noncompact spacelike hypersurfaces immersed into a certain spacetime. When this spacetime is standard static $\mathbb{P}^n \times_\rho \mathbb{R}_1$, there is a remarkable family of spacelike hypersurfaces, namely, it spacelike slices $\mathbb{P}^n \times \{t_0\}$, with $t_0 \in \mathbb{R}$, which are totally geodesics and constitute a foliation in any standard static spacetime. Therefore, it is natural to approach Calabi–Bernstein problems in a standard static spacetime. Recently in [17] the first and second authors of this article together with Lima Jr. and Medeiros extended a technique due to Romero et al. [34] to establish sufficient conditions to guarantee the parabolicity of complete spacelike hypersurfaces in $\mathbb{P}^n \times_\rho \mathbb{R}_1$ whose Riemannian base $\mathbb{P}^n$ has parabolic universal Riemannian covering and, as applications, they obtain uniqueness results concerning these hypersurfaces.

On the other hand, the study of variational questions associated to the area functional in (Riemannian or Lorentzian) manifolds with density, also called weighted manifolds, has been a focus of attention in the last years. We recall that a *weighted manifold* $\mathfrak{M}_f^{n+1}$ is a (Riemannian or Lorentzian) manifold $(\mathfrak{M}^{n+1}, \mathfrak{g})$ endowed with a weighted volume form $d\mu = e^{-f} d\mathfrak{M}$, where the *weighted function* f is a real-valued smooth function on $\mathfrak{M}^{n+1}$ and $d\mathfrak{M}$ is the volume element induced by the metric $\mathfrak{g}$ (for details see, for instance, [9] and [29]). In this scenario, Rosales, Cañete, Bayle and Morgan [35] investigated the isoperimetric problem for Euclidean space endowed with a continuous density, showing that, for a radial log-convex density, balls about the origin are isoperimetric regions. Afterwards, Cañete and Rosales [10] studied smooth Euclidean solid cones endowed with a smooth homogeneous weighted function. They proved that the unique compact, orientable, second order minima of the weighted area under variations preserving the weighted volume and with free boundary in the boundary of the cone are intersections with the cone of round spheres centered at the vertex. In [25], Impera and Rimoldi established stability properties concerning f-minimal hypersurfaces (that is, with f-mean curvature identically zero) isometrically immersed in a weighted Riemannian manifold $\mathfrak{M}_f^{n+1}$ with non-negative Bakry–Émery Ricci curvature under volume growth conditions. Meanwhile, Castro and Rosales [13] obtained variational characterizations of critical points and second order minima of the weighted area with or without volume constraint in a weighted Riemannian manifolds with boundary $\mathfrak{M}_f^{n+1}$. In the Lorentzian context, the second and fourth authors in collaboration with Oliveira and Santos established in [18] a notion of strong f-stability concerning closed spacelike hypersurfaces immersed with constant f-mean curvature in a certain weighted spacetime $\mathfrak{M}_f^{n+1}$ and they obtained sufficient conditions which assure that a strongly f-stable closed spacelike hypersurface must be either f-maximal or totally umbilical.

Also in the branch of manifolds with density, Batista, Cavalcante and Pyo [6] showed some general inequalities involving the weighted mean curvature of compact submanifolds immersed in a weighted Riemannian manifold $\mathfrak{M}_f^{n+1}$. As application, they obtained an isoperimetric inequality for such submanifolds. Concerning the weighted product space $\mathbb{G}^n \times \mathbb{R}$, where $\mathbb{G}^n$ stands for the so-called *Gaussian space* which is nothing but that the Euclidian space $\mathbb{R}^n$ endowed with the Gaussian probability density $e^{-f(x)} = (2\pi)^{-\frac{n+1}{2}} e^{-\frac{|x|^2}{2}}$, $x \in \mathbb{R}^n$, Hieu and Nam [24] extended the classical Bernstein's theorem showing that the only weighted minimal graphs $\Sigma(z)$ of smooth functions $z(x) = t$ over $\mathbb{G}^n$ are the affine hyperplanes $t =$ constant. Afterwards, McGonagle and Ross [27] showed that the hyperplane is the only stable, smooth solution to the isoperimetric problem in the $\mathbb{G}^{n+1}$. In the Lorentzian context, in the works [1,14,30] were applied suitable generalized maximum principles in order to obtain new Calabi–Bernstein type results concerning complete spacelike hypersurfaces immersed in a certain class weighted spacetimes.

Motivated by the works described above, in this work our objective is to carry out a study on the uniqueness, nonexistence and stability of spacelike hypersurfaces immersed into a weighted standard static spacetime $(\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$ endowed with a weighted function f does not depend on the parameter $t \in \mathbb{R}$, that is, $\langle \overline{\nabla} f, \partial/\partial t\rangle = 0$. For simplicity, we denote such ambient spacetime as

$$\mathbb{P}_f^n \times_\rho \mathbb{R}_1.$$

The restriction that we adopt in the ambient space is motivated by a splitting theorem due to Case (cf. [12]), which states that: *if $\mathfrak{M}_f^{n+1}$ is a weighted timelike geodesically complete spacetime that contains a timelike line $\mathfrak{L}$, admits a bounded weighted function f and has Bakry–Émery–Ricci tensor nonnegative for all timelike vector fields then f z must be constant along timelike line* $\mathfrak{L}$. Consequently, in any weighted standard static spacetime $(\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$ having nonnegative Bakry–Émery–Ricci tensor for timelike vector fields and with bounded weighted function f, we have that f does not depend on the parameter of the flow associated to the Killing vector field $\partial/\partial t$.

We start our study by obtaining explicit formulas for the Laplacian of the height function h (see Proposition 1) and the drift Laplacian of the angle function Θ (see Proposition 2), both functions naturally related to a spacelike hypersurface Σ^n immersed into $\mathbb{P}_f^n \times_\rho \mathbb{R}_1$. Then, applying some analytical results to subharmonic smooth functions on complete Riemannian manifolds (for example: some parabolocity criteria, a weak form of the Omori–Yau maximum principle and an extension of the Hopf's Theorem due to Yau) and considering suitable constraints on the f-mean curvature of Σ^n, height function h, sometimes on angle function Θ and the Bakry–Émery–Ricci tensor of $\mathbb{P}^n$, we establish some uniqueness results (see Theorems 1, 2, 3, 4 and 5, and Corollary 1) and some nonexistence results (see Corollaries 2, 4 and 6). In Remark 4 we exhibit a large family of standard static spacetimes that verify the hypotheses adopted in Theorem 4 and also in their corollary. Next, in Corollaries 3, 5 and 7 we make a particular study on the Calabi–Bernstein type properties of entire Killing graphs $\Sigma^n(z)$ constructed from a smooth function z defined on the base $\mathbb{P}^n$ of $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$.

Proceeding, in Sect. 5, we show that closed spacelike hypersurfaces immersed with constant f-mean curvature in a weighted standard static spacetime $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$ are solutions of the variational problem of maximizing the weighted area functional for all variations that keeps the balance of weighted volume equal to zero (see Proposition 3). As a consequence, we establish the notion of f-stability for such hypersurfaces (Definition 1) and provide an appropriate stability criterion (Proposition 4). Finally, in Theorem 6 we obtain a characterization of f-stable closed spacelike hypersurfaces of $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$ through the first nonzero eigenvalue of the drift Laplacian.

2. Preliminaries

Along this paper, we will consider an $(n+1)$-dimensional Lorentzian manifold $\overline{M}^{n+1}$ with Lorentzian metric $g = g(\cdot,\cdot)$ and endowed with a timelike Killing vector field Y. Here timelike referred to a vector field means that $Y_p \in T_p\overline{M}$ is a timelike vector (and so nonzero) for each $p \in \overline{M}^{n+1}$. On the other hand, Killing mean that the $\mathcal{L}_Y g = 0$, where $\mathcal{L}_Y$ stands for the Lie derivative of g in the direction of Y.

We observe that the distribution $\mathcal{D}$ of all smooth vector fields of $\overline{M}^{n+1}$ that are orthogonal to Y, defined at each point by

$$\overline{M}^{n+1} \ni p \ \longmapsto \ \mathcal{D}(p) = \left\{ v \in T_p\overline{M} \ : \ g(v, Y_p) = 0 \right\},$$

is of constant rank and integrable. Given a Riemannian integral leaf $\mathbb{P}^n$ of that distribution $\mathcal{D}$, let $\Psi : \mathbb{I} \times \mathbb{P}^n \to \overline{M}^{n+1}$ be the flow generated by Y with initial values in $\mathbb{P}^n$, where $\mathbb{I}$ is a maximal interval of definition. Without loss of generality, in what follows we will consider $\mathbb{I} = \mathbb{R}$. In this setting, our space $\overline{M}^{n+1}$ can be regarded as the *standard static spacetime* $\mathbb{P}^n \times_\rho \mathbb{R}_1$ (cf. Proposition 12.38 of [31]), that is, the Lorentzian product manifold $\mathbb{P}^n \times \mathbb{R}_1$ endowed with the warping metric

$$\langle \cdot,\cdot \rangle = \pi^*_{\mathbb{P}^n}(\langle \cdot,\cdot \rangle_{\mathbb{P}^n}) + (\rho \circ \pi_{\mathbb{P}^n})^2 \pi^*_{\mathbb{R}}(-dt^2), \tag{2.1}$$

where $\pi_{\mathbb{P}^n}$ and $\pi_{\mathbb{R}}$ denote the canonical projections from $\mathbb{P}^n \times \mathbb{R}_1$ onto each factor, $\langle \cdot,\cdot \rangle_{\mathbb{P}^n}$ is the induced Riemannian metric on the base $\mathbb{P}^n$, $\mathbb{R}_1$ is the manifold $\mathbb{R}$ endowed with the metric $-dt^2$ and

$$\rho = |Y| = \sqrt{-\langle Y, Y\rangle} > 0$$

is the warping function. We mean by $C^\infty(\mathbb{P}^n \times_\rho \mathbb{R}_1)$ the ring of real functions of class C^∞ on $\mathbb{P}^n \times_\rho \mathbb{R}_1$ and by $\mathfrak{X}(\mathbb{P}^n \times_\rho \mathbb{R}_1)$ the $C^\infty(\mathbb{P}^n \times_\rho \mathbb{R}_1)$-module of vector fields of class C^∞ on $\mathbb{P}^n \times_\rho \mathbb{R}_1$. Let $\overline{\nabla}$ and $\widetilde{\nabla}$ be the Levi–Civita connections of $\mathbb{P}^n \times_\rho \mathbb{R}_1$ and $\mathbb{P}^n$, respectively.

Remark 1. The importance of standard static spacetimes comes from the fact that they include some classical spacetimes. In what follows we list some of them:

(a) A simple example is given by the Lorentz–Minkowski space $\mathbb{L}^{n+1}$, which is isometric to the warped product $\big(\mathbb{R}^n \times \mathbb{R}_1\,,\, \pi^*_{\mathbb{R}^n}(g_{\mathbb{R}^n}) + \pi^*_{\mathbb{R}}(-dt^2)\big)$.

(b) The Einstein static universe $\big(\mathbb{S}^n \times \mathbb{R}_1\,,\, \pi^*_{\mathbb{S}^n}(g_{\mathbb{S}^n}) + \pi^*_{\mathbb{R}}(-dt^2)\big)$ is also a standard static space (cf. Example 5.11 of [7]).

(c) Another example is given by the exterior Schwarzschild spacetime, which is defined as follows. Let $\mathbb{R}^4$ be given coordinates (t, r, θ, φ), where (r, θ, φ) are the usual spherical coordinates on $\mathbb{R}^3$. Given a positive constant m, the exterior Schwarzschild spacetime is defined on the subset $r > 2m$ of $\mathbb{R}^4$, a subset which is topologically $\mathbb{R}^2 \times \mathbb{S}^2$. The Schwarzschild metric for the region $r > 2m$ is given in (t, r, θ, φ) coordinates by

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right).$$

Since the metric for this spacetime is invariant under time translations $t \to t + a$, the coordinate vector field $\partial/\partial t$ is a (globally defined) timelike Killing vector field (cf. Section 5.2 of [7] or Chapter 13 of [31]). Consequently, the exterior Schwarzschild spacetime is a standard static spacetime.

(d) A model that also presents static regions (which appeared shortly after the Schwarzschild spacetime) is the Reissner–Nordström spacetime, whose metric in (t, r, θ, φ) coordinates admits the representation

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right).$$

This metric has singularities in $r = 0$, $r = r_+$ and $r = r_-$, where $r_\pm = m \pm (m^2 - e^2)^{1/2}$, and in regions corresponding to $+\infty > r > r_+$ and $r_- > r > 0$ we have that the Reissner–Nordström spacetime is static (cf. Section 5.5 of [23]).

Now, in the configuration described above, let $(\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$ be a *weighted standard static spacetime*, namely, a standard static spacetime $\mathbb{P}^n \times_\rho \mathbb{R}_1$ endowed with a weighted volume form $d\overline{\sigma} = e^{-f} d\overline{v}$, where $f \in C^\infty(\mathbb{P}^n \times_\rho \mathbb{R}_1)$

is a real-valued function, called *weighted function* (or *density function*), and $d\overline{v}$ is the volume element induced by the warping metric $\langle \cdot, \cdot \rangle$ defined in (2.1). For $(\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$, the Bakry–Émery–Ricci tensor $\overline{\mathrm{Ric}}_f$ (cf. [9]) is defined by

$$\overline{\mathrm{Ric}}_f = \overline{\mathrm{Ric}} + \overline{\mathrm{Hess}} f, \tag{2.2}$$

where $\overline{\mathrm{Ric}}$ and $\overline{\mathrm{Hess}}$ are the Ricci tensor and the Hessian operator in $\mathbb{P}^n \times_\rho \mathbb{R}_1$, respectively.

Throughout this work, we will deal with complete spacelike hypersurfaces

$$x : \Sigma^n \hookrightarrow (\mathbb{P}^n \times_\rho \mathbb{R}_1)_f\,,$$

namely, isometric immersions from a (connected) n-dimensional Riemannian manifold Σ^n into weighted standard static spacetime $(\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$. In this setting, let ∇ denote the Levi–Civita connection of Σ^n. As $(\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$ is time-orientable by timelike vector field Y and $x : \Sigma^n \hookrightarrow (\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$ is a spacelike hypersurface, then Σ^n is orientable (cf. Proposition 5.26 of [31]) and one can choose a globally defined unit normal vector field N on Σ^n having the same time-orientation of $(\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$ (cf. Proposition 5.29 of [31]), that is,

$$\langle Y, N \rangle < 0. \tag{2.3}$$

Such N is said the *future-pointing Gauss map* of $x : \Sigma^n \hookrightarrow (\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$. Let A denote the shape operator of $x : \Sigma^n \hookrightarrow (\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$ with respect to N, so that at each $p \in \Sigma^n$, A restricts to a self-adjoint linear map

$$\begin{aligned} A_p : T_p\Sigma &\to T_p\Sigma \\ v &\mapsto A_p v = -\overline{\nabla}_v N. \end{aligned}$$

According to Gromov [22], the *weighted mean curvature* (or simply the *f-mean curvature*) H_f of the spacelike hypersurfaces $x : \Sigma^n \hookrightarrow (\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$ is given by

$$nH_f = nH - \langle \overline{\nabla} f, N \rangle, \tag{2.4}$$

where $H = -\frac{1}{n}\mathrm{tr}(A)$ denotes the standard mean curvature of $x : \Sigma^n \hookrightarrow (\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$ with respect to its orientation N. Moreover, we say that $x : \Sigma^n \hookrightarrow (\mathbb{P}^n \times_\rho \mathbb{R})_f$ is *f-maximal* when its f-mean curvature vanishes identically.

The *f-divergence* on Σ^n (cf. [9]) is defined by

$$\begin{aligned} \mathrm{div}_f : C^\infty(\Sigma^n) &\to C^\infty(\Sigma^n) \\ X &\mapsto \mathrm{div}_f(X) = \mathrm{div} X - \langle \nabla f, X \rangle, \end{aligned}$$

where $\mathrm{div}(\,\cdot\,)$ denotes the standard divergence on Σ^n. We define the *f-Laplacian* (also called the *drift Laplacian*) of Σ^n (cf. [9]) by

$$\begin{aligned} \Delta_f : C^\infty(\Sigma^n) &\to C^\infty(\Sigma^n) \\ u &\mapsto \Delta_f(u) = \mathrm{div}_f(\nabla u) = \Delta u - \langle \nabla f, \nabla u \rangle \end{aligned} \tag{2.5}$$

where Δ is the standard Laplacian on Σ^n.

Remark 2. Since the timelike Killing vector field Y has identically zero conformal factor ϕ (more precisely, $\phi = \frac{1}{n+1}\mathrm{Div}Y \equiv 0$, where Div stands for the divergence on $\mathbb{P}^n \times_\rho \mathbb{R}_1$), it follows from Proposition 1 of [28] that Y determines in $\mathbb{P}^n \times_\rho \mathbb{R}_1$ a codimension one Riemannian foliation by totally geodesic slices $\Sigma^n_{t_0} = \mathbb{P}^n \times \{t_0\}$, $t_0 \in \mathbb{R}$, with respect to the orientation determined by $\frac{\partial}{\partial t} \equiv Y$. Moreover, assuming that the weighted function $f \in C^\infty(\mathbb{P}^n \times_\rho \mathbb{R})$ is invariant along the flow determinate by Y, that is, $\langle \overline{\nabla} f, Y\rangle = 0$, from (2.4) we get that each slice $\Sigma^n_{t_0}$ is f-maximal.

Remark 3. We observe that the following result is a consequence of a splitting theorem due to Case (see Theorem 1.2 of [12]):

"Let $\overline{M}^{n+1}_f$ be a weighted timelike geodesically complete spacetime that contains a timelike line with $\overline{\mathrm{Ric}}_f(X,X) \geq 0$ for all timelike vector fields X, and whose weighted function f is bounded. Then f must be constant along timelike line of $\overline{M}^{n+1}_f$."

Consequently, in any weighted standard static spacetime $(\mathbb{P}^n \times_\rho \mathbb{R}_1)_f$ having nonnegative Bakry–Émery–Ricci tensor for timelike vector fields and with bounded weighted function f, we have that f does not depend on the parameter of the flow associated to the Killing vector field $\dfrac{\partial}{\partial t} \equiv Y$.

Motivated by Remarks 2 and 3, along this work we will consider static spacetimes $\mathbb{P}^n \times_\rho \mathbb{R}_1$ endowed with a weighted function f does not depend on the parameter $t \in \mathbb{R}$, that is, $\langle \overline{\nabla} f, Y\rangle = 0$. For sake of simplicity, we will denote such an ambient space by $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$.

In what follows, associated with a spacelike hypersurface $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$, we will consider two particular smooth functions, namely, the (vertical) *height function*

$$h = (\pi_{\mathbb{R}})\Big|_{\Sigma^n} : \Sigma^n \to \mathbb{R} \tag{2.6}$$

and the *angle function*

$$\begin{aligned} \Theta : \Sigma^n &\to \quad \mathbb{R} \\ p \ &\mapsto \Theta(p) = \langle N(p), Y(p)\rangle, \end{aligned} \tag{2.7}$$

where N is the future-pointing Gauss map of Σ^n and Y is the Killing vector field on $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$. From (2.3), we note that Θ will be always a negative function on Σ^n.

We have that

$$\nabla h = -\frac{1}{\rho^2} Y^\top, \tag{2.8}$$

where $(\,\cdot\,)^{\top}$ denote the projection of a smooth vector field in $\mathfrak{X}(\mathbb{P}^n \times_\rho \mathbb{R}_1)$ on $\mathfrak{X}(\Sigma^n)$. Moreover, it holds that

$$N^* = N + \frac{1}{\rho^2}\Theta Y, \tag{2.9}$$

where $(\,\cdot\,)^*$ denote the projection of a smooth vector field in $\mathfrak{X}(\mathbb{P}^n \times_\rho \mathbb{R}_1)$ on $\mathfrak{X}(\mathbb{P}^n)$. Hence, from (2.8) and (2.9) it is not difficult to verify that the following relation holds

$$|\nabla h|^2 = \frac{1}{\rho^2}|N^*|^2_{\mathbb{P}^n}. \tag{2.10}$$

Indeed, we have that

$$\begin{aligned}\langle \nabla h, \nabla h\rangle &= \frac{1}{\rho^4}\langle Y^{\top}, Y^{\top}\rangle = \frac{1}{\rho^4}\langle Y + \Theta N, Y + \Theta N\rangle \\ &= \frac{1}{\rho^2}\left(\frac{\Theta^2}{\rho^2} - 1\right) = \frac{1}{\rho^2}\langle N + \frac{\Theta}{\rho^2}Y, N + \frac{\Theta}{\rho^2}Y\rangle \\ &= \frac{1}{\rho^2}\langle N^*, N^*\rangle = \frac{1}{\rho^2}\langle N^*, N^*\rangle_{\mathbb{P}^n}.\end{aligned}$$

3. Uniqueness and Nonexistence Results in Standard Static Spacetimes

We begin this section by providing a formula for the classical Laplacian of the height function of a spacelike hypersurface immersed in a standard static space $\mathbb{P}^n \times_\rho \mathbb{R}$ in terms of a certain weighted mean curvature. More precisely, we have the following

Proposition 1. *Let $x : \Sigma^n \hookrightarrow \mathbb{P}^n \times_\rho \mathbb{R}_1$ be a spacelike hypersurface and let $h \in C^\infty(\Sigma^n)$ be the height function defined in (2.6). Then*

$$\Delta h = -n\rho^{-2}\Theta\, H_{\log \rho^2}, \tag{3.1}$$

where Θ is the angle function defined in (2.7) and $H_{\log\rho^2}$ is the $\log\rho^2$-mean curvature of Σ^n.

Proof. Let $\{E_1, \ldots, E_n\}$ be an orthonormal frame defined in a neighborhood of some point of Σ^n. From (2.8) we note that

$$\begin{aligned}\rho^{-2}\mathrm{div}\,(\nabla h) &= \rho^{-2}\mathrm{div}\left(-\rho^{-2}\,Y^{\top}\right) \\ &= -\rho^{-2}\Big\langle \nabla\rho^{-2}, Y^{\top}\Big\rangle - \rho^{-4}\mathrm{div}\left(Y^{\top}\right) \\ &= \Big\langle \nabla\rho^{-2}, \nabla h\Big\rangle - \rho^{-4}\mathrm{div}\,(Y + \Theta N) \\ &= \Big\langle \nabla\rho^{-2}, \nabla h\Big\rangle - \rho^{-4}\sum_{i=1}^{n}\langle \nabla_{E_i}(Y + \Theta N), E_i\rangle\end{aligned}$$

$$
\begin{aligned}
&= \left\langle \nabla \rho^{-2}, \nabla h \right\rangle - \rho^{-4} \sum_{i=1}^{n} \left\langle \overline{\nabla}_{E_i} \left(Y + \Theta N \right), E_i \right\rangle \\
&= \left\langle \nabla \rho^{-2}, \nabla h \right\rangle - \rho^{-4} \sum_{i=1}^{n} \underbrace{\left\langle \overline{\nabla}_{E_i} Y, E_i \right\rangle}_{0} - \rho^{-4} \sum_{i=1}^{n} \left\langle \overline{\nabla}_{E_i} \left(\Theta N \right), E_i \right\rangle \\
&= \left\langle \nabla \rho^{-2}, \nabla h \right\rangle - \rho^{-4} \sum_{i=1}^{n} \left[E_i(\Theta) \underbrace{\langle N, E_i \rangle}_{0} + \Theta \langle \overline{\nabla}_{E_i} N, E_i \rangle \right] \\
&= \left\langle \nabla \rho^{-2}, \nabla h \right\rangle + \rho^{-4} \Theta \operatorname{tr}(A) \; = \; \left\langle \nabla \rho^{-2}, \nabla h \right\rangle - n \rho^{-4} H \Theta.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Delta h = \operatorname{div}\left(\nabla h\right) \; &= \; \rho^2 \left\langle \nabla \rho^{-2}, \nabla h \right\rangle - n \rho^{-2} H \Theta \\
&= \left\langle \nabla \log \rho^{-2}, -\rho^2 Y^{\top} \right\rangle - n \rho^{-2} H \Theta \\
&= -\rho^{-2} \left\langle \overline{\nabla} \log \rho^{-2}, Y^{\top} \right\rangle - n \rho^{-2} H \Theta \\
&= -\rho^{-2} \left\langle \overline{\nabla} \log \rho^{-2}, Y + \Theta N \right\rangle - n \rho^{-2} H \Theta \\
&= -\rho^{-2} \underbrace{\left\langle \overline{\nabla} \log \rho^{-2}, Y \right\rangle}_{0} - \rho^{-2} \left\langle \overline{\nabla} \log \rho^{-2}, N \right\rangle \Theta - n \rho^{-2} H \Theta \\
&= -\rho^{-2} \Theta \left\{ nH + \left\langle \overline{\nabla} (\log \rho^{-2}), N \right\rangle \right\} \; = \; -n \rho^{-2} \Theta \, H_{\log \rho^2},
\end{aligned}
$$

where in the last equality we use (2.4). □

In order to obtain our first result, we will need another key lemma. The next one corresponds to Theorem 3 of [43]. In what follows, given a n-dimensional Riemannian manifold Σ^n, we use the notation

$$
\mathcal{L}^q(\Sigma^n) = \left\{ u : \Sigma^n \to \mathbb{R} \; : \; \int_{\Sigma^n} |u|^q d\Sigma \; \ll +\infty \right\},
$$

where $d\Sigma$ denotes the standard volume element of Σ^n.

Lemma 1. *Let u be a nonnegative smooth subharmonic function on a complete Riemannian manifold Σ^n. If $u \in \mathcal{L}^q\left(\Sigma^n\right)$, for some $q > 1$, then u is constant on Σ^n.*

We will apply the previous lemma to get the following result

Theorem 1. *The only complete spacelike hypersurfaces immersed into standard static spacetime $\mathbb{P}^n \times_\rho \mathbb{R}_1$ with nonnegative $\log \rho^2$-mean curvature and whose height function h is nonnegative and satisfies the condition $h \in \mathcal{L}^q\left(\Sigma^n\right)$, for same $q > 1$, are the slices $\mathbb{P}^n \times \{t\}$, $t \in \mathbb{R}$.*

Proof. In fact, let $x : \Sigma^n \hookrightarrow \mathbb{P}^n \times_\rho \mathbb{R}_1$ be such a spacelike hypersurface. Since $\Theta < 0$ and $H_{\log\rho^2} \geq 0$ on Σ^n, from (3.1) we have that $\Delta h \geq 0$ on Σ^n. From Lemma 1, we conclude that h is constant on Σ^n and, hence, there is $t_0 \in \mathbb{R}$ such that $x(\Sigma^n) = \mathbb{P}^n \times \{t_0\}$. □

Let's remember that a Riemannian manifold Σ^n is said to be *parabolic* if every bounded solution of $\Delta u \geq 0$ must be identically constant. Let us also remember that a *slab* of a weighted standard static spacetime $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$ is a region of the type

$$\mathbb{P}^n_f \times_\rho [t_1, t_2] = \left\{ (q,t) \in \mathbb{P}^n_f \times_\rho \mathbb{R}_1 \;:\; t_1 \leq t \leq t_2 \right\}.$$

Note that if a spacelike hypersurface $x : \Sigma^n \hookrightarrow \mathbb{P}^n \times_\rho \mathbb{R}_1$ is contained in a slab then its height function h will be limited. In this context, from the proof of Theorem 1, we get the following

Corollary 1. *The only parabolic complete spacelike hypersurfaces immersed into standard static spacetime $\mathbb{P}^n \times_\rho \mathbb{R}_1$ with nonnegative $\log\rho^2$-mean curvature and lying in a slab of $\mathbb{P}^n \times_\rho \mathbb{R}_1$ are the slices $\mathbb{P}^n \times \{t\}$, $t \in \mathbb{R}$.*

A Riemannian manifold Σ^n is said to be *stochastically complete* if, for some (and, hence, for any) $(x,t) \in \Sigma^n \times (0, +\infty)$, the heat kernel $p(x,y,t)$ of the Laplace–Beltrami operator Δ (that is, the minimal, positive fundamental solution of the heat operator $\Delta - \partial/\partial_t$; for more details concerning the heat kernel of the Laplace–Beltrami operator, see [21]) satisfies the conservation property

$$\int_{\Sigma^n} p(x,y,t)d\mu(y) = 1. \tag{3.2}$$

From the probabilistic viewpoint, stochastically completeness is the property of a stochastic process to have infinite life time. For the Brownian motion on a manifold, the conservation property (3.2) means that the total probability of the particle to be found in the state space is constantly equal to one (cf. [19,20,36]).

Any parabolic manifold is stochastically complete but the opposite implication is not true. For example, all Euclidean spaces $\mathbb{R}^n$ (with Euclidean measure) are stochastically complete, whereas $\mathbb{R}^n$ is parabolic if and only if $n \in \{1,2\}$. On the other hand, Pigola, Rigoli and Setti showed that stochastic completeness turns out to be equivalent to the validity of a weak form of the Omori–Yau maximum principle (see Theorem 1.1 of [32] or Theorem 3.1 of [33]), as can be expressed below

Lemma 2. *A Riemannian manifold Σ^n is stochastically complete if and only if, for every $u \in C^2(\Sigma^n)$ satisfying $\sup_{\Sigma^n} u \ll +\infty$, there exists a sequence of points $\{p_j\}_{j=1}^{+\infty} \subset \Sigma^n$ such that*

$$\lim_{j\to+\infty} u(p_j) = \sup_{\Sigma^n} u \qquad \textit{and} \qquad \limsup_{j\to+\infty} \Delta u(p_j) \leq 0.$$

We will also need of the next lemma, which is just a consequence of a more general extension of Liouville's theorem due to Yau in [42].

Lemma 3. *The only harmonic semi-bounded functions defined on an n-dimensional complete Riemannian manifold whose Ricci curvature is nonnegative are the constant ones.*

Applying these previous lemmas, we obtain the following result

Theorem 2. *Let $x : \Sigma^n \hookrightarrow \mathbb{P}^n \times_\rho \mathbb{R}_1$ be a stochastically complete spacelike hypersurface which lies in a slab of $\mathbb{P}^n \times_\rho \mathbb{R}_1$. If the $\log \rho^2$-mean curvature $H_{\log \rho^2}$ of $x : \Sigma^n \hookrightarrow \mathbb{P}^n \times_\rho \mathbb{R}_1$ is a nonnegative constant, then $x : \Sigma^n \hookrightarrow \mathbb{P}^n \times_\rho \mathbb{R}_1$ is $\log \rho^2$-maximal. Moreover, if $x : \Sigma^n \hookrightarrow \mathbb{P}^n \times_\rho \mathbb{R}_1$ is complete with nonnegative Ricci curvature, then $x(\Sigma^n)$ is a slice $\mathbb{P}^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$.*

Proof. From Proposition 1 we have that $\rho^2 \Delta h = -nH_{\log \rho^2} \Theta$ on Σ^n. So, taking into account that the height function h of Σ^n is bounded, from Lemma 2 we get a sequence $\{p_j\}_{j=1}^{+\infty} \subset \Sigma^n$ such that

$$\begin{aligned} 0 \ \geq \ \limsup_{j \to +\infty} \rho^2 \Delta h(p_j) &= n \limsup_{j \to +\infty} (-H_{\log \rho^2} \Theta(p_j)) \\ &= -nH_{\log \rho^2} \liminf_{j \to +\infty} \Theta(p_j) \ \geq \ 0. \end{aligned} \tag{3.3}$$

Then, we have that $H_{\log \rho^2} = 0$ on Σ^n and, hence, h is harmonic on Σ^n.

On the other hand, since Σ^n lies in a slab then there exists a constant β such that $h - \beta > 0$. Thus, if $\mathrm{Ric} \geq 0$, then from Lemma 3 we can conclude that h is constant on Σ^n. Therefore, we conclude that there is $t_0 \in \mathbb{R}$ such that $x(\Sigma^n) = \mathbb{P}^n \times \{t_0\}$. $\square$

In particular, from the analysis of signals realized in (3.3) we can established the following nonexistence result.

Corollary 2. *There do not exist stochastically complete spacelike hypersurface immersed into standard static spacetime $\mathbb{P}^n \times_\rho \mathbb{R}_1$ which lies in a slab of $\mathbb{P}^n \times_\rho \mathbb{R}_1$ and whose $\log \rho^2$-mean curvature is a positive constant.*

In order to establish our next result, we will need of an extension of Hopf's theorem on a complete noncompact Riemannian manifold due to Yau in [43].

Lemma 4. *Let u be a smooth function on a complete Riemannian manifold Σ^n, such that Δu does not change sign on Σ^n. If $|\nabla u| \in \mathcal{L}^1(\Sigma^n)$, then Δu vanishes identically on Σ^n.*

Now, we are in position to present the following result

Theorem 3. *Let $\mathbb{P}^n \times_\rho \mathbb{R}_1$ be a standard static spacetime and $x : \Sigma^n \hookrightarrow \mathbb{P}^n \times_\rho \mathbb{R}_1$ be a complete spacelike hypersurface whose $\log \rho^2$-mean curvature $H_{\log \rho^2}$ does not change sign. If the gradient ∇h of the height function h of $x : \Sigma^n \hookrightarrow$*

$\mathbb{P}^n \times_\rho \mathbb{R}_1$ has integrable norm on Σ^n then $x : \Sigma^n \hookrightarrow \mathbb{P}^n \times_\rho \mathbb{R}_1$ is $\log \rho^2$-maximal. Moreover, if $x\left(\Sigma^n\right)$ lies in a slab of $\mathbb{P}^n \times_\rho \mathbb{R}$ then $x\left(\Sigma^n\right)$ is a slice $\mathbb{P}^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$.

Proof. Taking into account our restrictions on $H_{\log \rho^2}$ and Θ, from (3.1) we get that Δh does not change sign on Σ^n. Moreover, since $|\nabla h| \in \mathcal{L}^1\left(\Sigma^n\right)$, from Lemma 4 we get that $\Delta h = 0$ and, returning again in (3.1) we have that Σ^n is $\log \rho^2$-maximal.

On the other hand, from (2.10) we also note that

$$\Delta h^2 \;=\; 2h\Delta h + 2|\nabla h|^2 \;=\; 2\rho^{-2}|N^*|^2 \;\geq\; 0. \tag{3.4}$$

If we assume that $x\left(\Sigma^n\right)$ lies in a slab of $\mathbb{P}^n \times_\rho \mathbb{R}$ then h is bounded on Σ^n. So, since h is bounded on Σ^n and using once more that $|\nabla h| \in \mathcal{L}^1\left(\Sigma^n\right)$, Lemma 4 guarantees also that $\Delta h^2 = 0$. Therefore, from (3.4) we obtain that N^* vanishes identically on Σ^n, which means that N and the Killing vector field Y are collinear. Since Y determines in $\mathbb{P}^n \times_\rho \mathbb{R}_1$ a codimension one Riemannian foliation by totally geodesic slices $\mathbb{P}^n \times \{t\}$, $t \in \mathbb{R}$, we conclude that there is $t_0 \in \mathbb{R}$ such that $x\left(\Sigma^n\right) = \mathbb{P}^n \times \{t_0\}$. □

According to [16], we define the *entire Killing graph* $\Sigma^n(z)$ associated to a smooth function $z \in C^\infty(\mathbb{P}^n)$ as been the hypersurface given by

$$\Sigma^n(z) = \left\{\, \Psi(x, z(x)) \;:\; x \in \mathbb{P}^n \,\right\} \subset \mathbb{P}^n \times_\rho \mathbb{R}_1.$$

where Ψ is the flow generated by the timelike Killing vector field Y. The metric induced on $\mathbb{P}^n$ from the Lorentzian metric (2.1) via $\Sigma^n(z)$ is given by

$$\langle\, \cdot, \cdot \rangle_z = \langle\, \cdot, \cdot \rangle_{\mathbb{P}^n} - \rho^2 dz^2. \tag{3.5}$$

We can observe that, $\Sigma^n(z)$ is spacelike if and only if $\rho^2|Dz|^2_{\mathbb{P}^n} < 1$, where Dz denotes the gradient of a function z with respect to the metric $\langle\, \cdot, \cdot \rangle_{\mathbb{P}^n}$ of $\mathbb{P}^n$. Indeed, if $\Sigma^n(z)$ is spacelike, then

$$0 \;<\; \langle Dz, Dz\rangle_z = \langle Dz, Dz\rangle_{\mathbb{P}^n} - \rho^2\langle Dz, Dz\rangle^2_{\mathbb{P}^n}$$

and, hence, we conclude that $\rho^2|Dz|^2_{\mathbb{P}^n} < 1$. Conversely, if $\rho^2|Dz|^2_{\mathbb{P}^n} < 1$ then for every $X \in \mathfrak{X}(\Sigma^n(z))$ we obtain from Cauchy–Schwarz inequality that

$$\langle X, X\rangle_z = \langle X^*, X^*\rangle_{\mathbb{P}^n} - \rho^2\langle Dz, X^*\rangle^2_{\mathbb{P}^n} \;\geq\; \langle X^*, X^*\rangle_{\mathbb{P}^n}\left(1 - \rho^2|Dz|^2_{\mathbb{P}^n}\right),$$

where X^* is the orthogonal projection of X onto $\mathfrak{X}(\mathbb{P}^n)$. Thus, $\langle X, X\rangle_z \geq 0$ and, $\langle X, X\rangle_z = 0$ if and only if $X = 0$.

On the other hand, the function

$$\begin{aligned} G : \mathbb{P}^n \times \mathbb{R}_1 \to &\quad \mathbb{R} \\ (x,t) \quad &\mapsto G(x,t) = z(x) - t, \end{aligned}$$

is such that $\Sigma^n(z) = \Psi(G^{-1}(0))$. Thus, for all vector field X tangent to $\mathbb{P}^n \times_\rho \mathbb{R}_1$, we have

$$X(G) = X^*(G) - \frac{1}{\rho^2}\Big\langle X, \frac{\partial}{\partial t}\Big\rangle \frac{\partial}{\partial t}(G) = \Big\langle \frac{1}{\rho^2}\,\frac{\partial}{\partial t} + Dz, X\Big\rangle.$$

So,

$$\overline{\nabla} G = \frac{1}{\rho^2}\frac{\partial}{\partial t} + Dz$$

is a normal vector field on $G^{-1}(0)$ and, consequently,

$$N_0 = \Psi_*(\overline{\nabla} G) = \frac{1}{\rho^2}\,Y + \Psi_*(Dz)$$

is a normal timelike vector field on $\Sigma^n(z)$. Since,

$$|N_0| = \frac{\left(1-\rho^2|Dz|^2_{\mathbb{P}^n}\right)^{1/2}}{\rho},$$

it follows that

$$N = \frac{N_0}{|N_0|} = \frac{1}{\rho\left(1-\rho^2|Dz|^2_{\mathbb{P}^n}\right)^{1/2}}\left(Y + \rho^2\Psi_*(Dz)\right) \tag{3.6}$$

defines the future-pointing Gauss map of $\Sigma^n(z)$ such that its angle function $\Theta = \langle N, Y\rangle$ is given by

$$\Theta = -\,\frac{\rho}{\left(1-\rho^2|Dz|^2_{\mathbb{P}^n}\right)^{1/2}} \,<\, 0.$$

As a consequence of Theorem 3, we will obtain the following non-parametric result concerning entire Killing graphs in $\mathbb{P}^n \times_\rho \mathbb{R}_1$.

Corollary 3. *Let $\Sigma^n(z)$ be an entire Killing graph which lies in a slab of the standard static spacetime $\mathbb{P}^n \times_\rho \mathbb{R}_1$ whose base $\mathbb{P}^n$ is complete. Suppose there is a positive constant $\alpha < 1$ such that the gradient Dz of the function $z \in C^\infty(\mathbb{P}^n)$ satisfies*

$$\sup_{\Sigma^n(z)} \rho^2|Dz|^2_{\mathbb{P}^n} \,\le\, \alpha. \tag{3.7}$$

If the $\log\rho^2$-mean curvature $H_{\log\rho^2}$ of $\Sigma^n(z)$ does not change sign and $|Dz| \in \mathcal{L}^1(\mathbb{P}^n)$, then $\Sigma^n(z)$ is a slice $\mathbb{P}^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$.

Proof. First, from (3.7) we observe that $\Sigma^n(z)$ is spacelike. Now, we claim that $\Sigma^n(z)$ is complete. Indeed, let X be any vector field tangent to $\Sigma^n(z)$. From (3.5) and from the Cauchy–Schwarz inequality we get

$$\langle X, X\rangle_z = \langle X^*, X^*\rangle_{\mathbb{P}^n} - \rho^2\langle Dz, X^*\rangle_{\mathbb{P}^n} \,\ge\, (1-\rho^2|Dz|^2_{\mathbb{P}^n})\langle X^*, X^*\rangle_{\mathbb{P}^n}.$$

Then, from (3.7) we obtain

$$\ell_u(\gamma) \,\ge\, (1-\alpha)^{1/2}\ell_{\mathbb{P}^n}(\gamma^*),$$

where $\ell_z(\gamma)$ stands for the length of a curve γ on $\Sigma^n(z)$ with respect to the induced metric (3.5) and $\ell_{\mathbb{P}^n}(\gamma^*)$ denotes the length of the projection γ^* of γ onto $\mathbb{P}^n$ with respect to its metric $\langle\,\cdot,\cdot\rangle_{\mathbb{P}^n}$. Consequently, since projections onto $\mathbb{P}^n$ of divergent curves on $\Sigma^n(z)$ give divergent curves on $\mathbb{P}^n$ and as we are assume that the metric $\langle \cdot, \cdot \rangle_{\mathbb{P}^n}$ is complete, we can apply Hopf–Rinow theorem to conclude that the induced metric (3.5) is also complete.

On the other hand, from (3.6) we obtain

$$N^* = \frac{1}{\left(1 - \rho^2 |Dz|^2_{\mathbb{P}^n}\right)^{1/2}}\, \rho \Psi_*(Dz),$$

So, from (2.10) and (3.7) we have that the height function h of $\Sigma^n(z)$ satisfies

$$|\nabla h|^2 = \frac{1}{1 - \rho^2 |Dz|^2_{\mathbb{P}^n}}\, |Dz|^2_{\mathbb{P}^n} \leq \frac{1}{1-\alpha}\, |Dz|^2_{\mathbb{P}^n}. \tag{3.8}$$

Therefore, from Theorem 3 we get that $\Sigma^n(z)$ is a slice $\mathbb{P}^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$. $\square$

4. Uniqueness and Nonexistence Results in Weighted Standard Static Spacetimes

The following key proposition provides an explicit formula for the drift Laplacian of the angle function Θ defined in (2.7). As in the previous sections, let us denote by $\overline{\nabla}$, ∇ and $\widetilde{\nabla}$ the Levi–Civita connections of $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$, Σ^n and $\mathbb{P}^n$, respectively.

Proposition 2. *Let $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ be a spacelike hypersurface and let $\Theta \in C^\infty(\Sigma^n)$ be the angle function defined in (2.7). Then*

$$\Delta_f \Theta = nY^\top (H_f) + \left(\widetilde{\mathrm{Ric}}_f(N^*, N^*) - \frac{1}{\rho}\, \widetilde{\mathrm{Hess}}\, \rho(N^*, N^*) + \Theta^2 \frac{\widetilde{\Delta}_f(\rho)}{\rho^3} + |A|^2\right)\Theta. \tag{4.1}$$

Here, Y is the Killing vector field on $\mathbb{P}^n_f \times_\rho \mathbb{R}$, $\rho = |Y| > 0$, N is the unit normal vector field on Σ^n, Δ_f and $\widetilde{\Delta}_f$ represent the f-Laplacians on Σ^n and $\mathbb{P}^n$, respectively, $\widetilde{\mathrm{Ric}}_f$ and $\widetilde{\mathrm{Hess}}$ are the Bakry–Émery–Ricci tensor and the Hessian operator on $\mathbb{P}^n$, $|A|^2$ represent the square of the norm of the shape operator A of Σ^n with respect to the orientation given by N and N^* is the projection of N on the tangent bundle of $\mathbb{P}^n$.

Proof. Firstly, since Y is a Killing vector field for any $X \in \mathfrak{X}(\Sigma^n)$ we have

$$\begin{aligned}\left\langle \nabla\Theta, X \right\rangle = X(\Theta) &= X(\langle N, Y\rangle) = \left\langle \overline{\nabla}_X N, Y \right\rangle + \left\langle N, \overline{\nabla}_X Y \right\rangle \\ &= \left\langle -A(Y^\top) - \overline{\nabla}_N Y, X \right\rangle,\end{aligned}$$

which assures us that

$$\nabla\Theta = -A(Y^\top) - (\overline{\nabla}_N Y)^\top. \tag{4.2}$$

On the other hand, from (2.4) we note that

$$\begin{aligned} nY^\top(H) &= Y^\top\left(nH_f + \left\langle\overline{\nabla} f, N\right\rangle\right) \\ &= nY^\top\left(H_f\right) + Y^\top\left(\left\langle\overline{\nabla} f, N\right\rangle\right) \\ &= nY^\top\left(H_f\right) + \left\langle Y, \overline{\operatorname{Hess}} f(N)\right\rangle + \Theta\,\overline{\operatorname{Hess}} f(N,N) - \left\langle A(Y^\top), \overline{\nabla} f\right\rangle, \end{aligned} \tag{4.3}$$

where we used the decomposition $Y = Y^\top - \Theta N$.

Moreover, since f is supposed to be invariant along the flow determinate by Y, from (4.2) we get that

$$\begin{aligned} \left\langle\nabla\Theta, \overline{\nabla} f\right\rangle &= -\left\langle A(Y^\top) + (\overline{\nabla}_N Y)^\top, \overline{\nabla} f\right\rangle \\ &= -\left\langle A(Y^\top), \overline{\nabla} f\right\rangle - \left\langle\overline{\nabla}_N Y, \overline{\nabla} f\right\rangle \\ &= -\left\langle A(Y^\top), \overline{\nabla} f\right\rangle + \left\langle Y, \overline{\nabla}_N\overline{\nabla} f\right\rangle \\ &= -\left\langle A(Y^\top), \overline{\nabla} f\right\rangle + \left\langle Y, \overline{\operatorname{Hess}} f(N)\right\rangle. \end{aligned} \tag{4.4}$$

Substituting (4.4) into (4.3) we get

$$nY^\top(H) = nY^\top\left(H_f\right) + \Theta\,\overline{\operatorname{Hess}} f(N,N) + \left\langle\nabla\Theta, \overline{\nabla} f\right\rangle. \tag{4.5}$$

From Proposition 2.12 of [5] we have

$$\Delta\Theta = nY^\top\left(H\right) + \Theta\left(\operatorname{Ric}(N,N) + |A|^2\right), \tag{4.6}$$

Thus, from (2.5), (4.6) and (4.5) we obtain that

$$\Delta_f\,\Theta = nY^\top\left(H_f\right) + \left(\overline{\operatorname{Ric}}_f(N,N) + |A|^2\right)\Theta. \tag{4.7}$$

Now, if we consider the decomposition $N = N^* + N^\perp$ of N, where $(\,\cdot\,)^\perp$ denote the projection of a vector field in $\mathfrak{X}(\mathbb{P}^n \times_\rho \mathbb{R}_1)$ on $\mathfrak{X}(\mathbb{R}_1)$, we have

$$\begin{aligned} \overline{\operatorname{Hess}} f(N,N) &= \left\langle\overline{\nabla}_N\overline{\nabla} f, N\right\rangle \\ &= \left\langle\overline{\nabla}_N\widetilde{\nabla} f, N^* + N^\perp\right\rangle \\ &= \widetilde{\operatorname{Hess}} f(N^*,N^*) + \frac{1}{\rho}\left\langle\widetilde{\nabla} f, \widetilde{\nabla}\rho\right\rangle|N^\perp|^2 \\ &= \widetilde{\operatorname{Hess}} f(N^*,N^*) - \frac{1}{\rho^3}\left\langle\widetilde{\nabla} f, \widetilde{\nabla}\rho\right\rangle\Theta^2. \end{aligned} \tag{4.8}$$

From Corollary 7.43 of [31] we get that

$$\overline{\operatorname{Ric}}(N,N) = \widetilde{\operatorname{Ric}}(N^*,N^*) - \frac{1}{\rho}\,\widetilde{\operatorname{Hess}}\rho(N^*,N^*) + \Theta^2\frac{\widetilde{\Delta}(\rho)}{\rho^3} \tag{4.9}$$

Now, from Eqs. (2.2), (4.8) and (4.9), we have that

$$\overline{\mathrm{Ric}}_f(N,N) = \widetilde{\mathrm{Ric}}_f(N^*,N^*) - \frac{1}{\rho}\widetilde{\mathrm{Hess}}\rho(N^*,N^*) + \Theta^2\frac{\widetilde{\Delta}_f(\rho)}{\rho^3} \tag{4.10}$$

Therefore, from Eqs. (4.10) and (4.7) we obtain (4.1). $\square$

According to the classical terminology in linear potential theory, a weighted manifold Σ^n with weighted function f is said to be *f-parabolic* if every bounded solution of $\Delta_f(u) \geq 0$ must be identically constant. In this setting, we obtain the following result concerning f-parabolic spacelike hypersurfaces immersed in a weighted static spacetime. As usual, expressions that have $\widetilde{(\ \cdot\ )}$ correspond to objects defined on $\mathbb{P}^n$.

Theorem 4. *Let $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$ be a weighted standard static spacetime with $\widetilde{\mathrm{Ric}}_f \geq -\kappa$, for some constant $\kappa > 0$, and ρ being a convex warping function such that $\langle\widetilde{\nabla} f, \widetilde{\nabla}\rho\rangle \leq 0$. Let $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ be a f-parabolic spacelike hypersurface with constant f-mean curvature and angle function Θ bounded from below. If the height function h and the shape operator A of Σ^n satisfy*

$$|\nabla h|^2 \ \leq \ \frac{\alpha}{\kappa(n-1)\rho^2}\,|A|^2, \tag{4.11}$$

for some constant $\alpha \in (0,1)$, then $x(\Sigma^n)$ is contained in a slice $\mathbb{P}^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$.

Proof. Let us first observe that at points where N^* is different from zero we have

$$\frac{1}{\rho}\widetilde{\mathrm{Hess}}\,\rho(N^*,N^*) \ = \ \frac{|N^*|^2}{\rho}\widetilde{\mathrm{Hess}}\,\rho\left(\frac{N^*}{|N^*|},\frac{N^*}{|N^*|}\right) \ = \ \frac{\Theta^2-\rho^2}{\rho^3}\widetilde{\mathrm{Hess}}\,\rho\left(\frac{N^*}{|N^*|},\frac{N^*}{|N^*|}\right)$$

and, taking a local orthonormal frame $\left\{E_1 = \frac{N^*}{|N^*|}, E_2, \ldots, E_n\right\}$ tangent to $\mathbb{P}^n$, we also have

$$\frac{\Theta^2}{\rho^3}\,\widetilde{\Delta}(\rho) \ = \ \frac{\Theta^2}{\rho^3}\widetilde{\mathrm{Hess}}\,\rho\left(\frac{N^*}{|N^*|},\frac{N^*}{|N^*|}\right) + \frac{\Theta^2}{\rho^3}\sum_{i=2}^{n}\widetilde{\mathrm{Hess}}\,\rho(E_i,E_i).$$

Then,

$$-\frac{1}{\rho}\widetilde{\mathrm{Hess}}\,\rho(N^*,N^*) + \frac{\Theta^2}{\rho^3}\,\widetilde{\Delta}(\rho) \ = \ \frac{1}{\rho}\widetilde{\mathrm{Hess}}\,\rho\left(\frac{N^*}{|N^*|},\frac{N^*}{|N^*|}\right) + \frac{\Theta^2}{\rho^3}\sum_{i=2}^{n}\widetilde{\mathrm{Hess}}\,\rho(E_i,E_i)$$

and, from (2.5), we get

$$\begin{aligned} -\frac{1}{\rho}\widetilde{\mathrm{Hess}}\,\rho(N^*,N^*) + \frac{\Theta^2}{\rho^3}\,\widetilde{\Delta}_f(\rho) &= \frac{1}{\rho}\widetilde{\mathrm{Hess}}\,\rho\left(\frac{N^*}{|N^*|},\frac{N^*}{|N^*|}\right) \\ &\quad + \frac{\Theta^2}{\rho^3}\sum_{i=2}^{n}\widetilde{\mathrm{Hess}}\,\rho(E_i,E_i) - \frac{\Theta^2}{\rho^3}\langle\widetilde{\nabla} f, \widetilde{\nabla}\rho\rangle \ \geq \ 0, \end{aligned} \tag{4.12}$$

where in the last step we use the convexity of ρ and the hypothesis $\langle\widetilde{\nabla} f, \widetilde{\nabla}\rho\rangle \leq 0$.

Now, noting that H_f is constant, $\Theta < 0$ on Σ^n and taking into account our constraint on $\widetilde{\mathrm{Ric}}$, from (2.10) and (4.12) jointly with Proposition 2 we obtain

$$\Delta_f \Theta \;\leq\; \left(-\kappa(n-1)\rho^2|\nabla h|^2 + |A|^2\right)\Theta. \tag{4.13}$$

Using hypothesis (4.11), from (4.13) we obtain that

$$\Delta_f(-\Theta) \;\geq\; (1-\alpha)|A|^2\,(-\Theta)\,. \tag{4.14}$$

Hence, from (4.14) we have that $-\Theta$ is a bounded positive subharmonic function on Σ^n and, since we are assuming that Σ^n is f-parabolic, $-\Theta$ must be constant on Σ^n. So, returning to (4.14), we see that Σ^n is totally geodesic. Therefore, hypothesis (4.11) assures that h is constant on Σ^n, that is, there exists $t_0 \in \mathbb{R}$ such that $\Sigma^n \subset \mathbb{P}^n \times \{t_0\}$. $\square$

As a direct consequence of Theorem 4, we get the following

Corollary 4. *Let $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$ be a weighted standard static spacetime with $\widetilde{\mathrm{Ric}}_f \geq -\kappa$, for some constant $\kappa > 0$, and ρ being a convex warping function such that $\langle\widetilde{\nabla} f, \widetilde{\nabla}\rho\rangle \leq 0$. There is not nonzero constant f-mean curvature f-parabolic spacelike hypersurface immersed into $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$ with angle function bounded from below and such that the height function and the its shape operator satisfy the condition (4.11), for some constant $\alpha \in (0,1)$.*

Remark 4. We note that there is a large family of weighted standard static spacetimes $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$ that satisfy the conditions of Theorem 4. For example, if we define on $\mathbb{P}^n$ the smooth function $f = a\rho + b$, with $a < 0$ and $b \in \mathbb{R}$, then we obtain that $\langle\widetilde{\nabla} f, \widetilde{\nabla}\rho\rangle = a|\widetilde{\nabla}\rho|^2 \leq 0$ and the Bakry–Émery–Ricci tensor $\widetilde{\mathrm{Ric}}_f$ of $\mathbb{P}^n$ is given by

$$\widetilde{\mathrm{Ric}}_f = \widetilde{\mathrm{Ric}} + a\,\widetilde{\mathrm{Hess}}\,\rho.$$

In addition, if $\widetilde{\mathrm{Ric}} \geq -\kappa$, for some positive constant κ, and ρ is chosen such that $0 \leq \widetilde{\mathrm{Hess}}\rho \leq \beta$ for some constant β, then $\widetilde{\mathrm{Ric}}_f \geq -(k + |a|\beta)$. Hence, $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$ verifies the requested conditions of Theorem 4.

Another situation happens when we define on $\mathbb{P}^n$ the smooth function $f = e^{a\rho} + b$, with $a < 0$ and $b \in \mathbb{R}$. In this other case, with the same constraints on ρ and $\widetilde{\mathrm{Ric}}$ assumed in the previous case, we have that $\langle\widetilde{\nabla} f, \widetilde{\nabla}\rho\rangle = ae^{a\rho}|\widetilde{\nabla}\rho|^2 \leq 0$ and

$$\widetilde{\mathrm{Ric}}_f \;=\; \widetilde{\mathrm{Ric}} + a^2 e^{a\rho}\langle\widetilde{\nabla}\rho, \cdot\,\rangle^2 + ae^{a\rho}\,\widetilde{\mathrm{Hess}}\,\rho \;\geq\; -(k + |a|\beta).$$

Therefore, this second ambient space also contemplates the hypothesis of Theorem 4.

In the context of Killing graphs, from Theorem 4 we obtain the following

Corollary 5. *Let $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$ be a weighted standard static spacetime with $\widetilde{\mathrm{Ric}}_f \geq -\kappa$, for some constant $\kappa > 0$, and ρ being a convex warping function such that*

$\langle \widetilde{\nabla} f, \widetilde{\nabla}\rho \rangle \leq 0$. *Let $\Sigma^n(z)$ be a f-parabolic entire Killing graph in $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$ with constant f-mean curvature, angle function Θ bounded from below and whose norm of its shape operator A satisfy*

$$|A|^2 \leq \frac{\kappa(n-1)}{1-\alpha} \tag{4.15}$$

for some constant $\alpha \in (0,1)$. If the gradient Dz of the function z satisfy

$$|Dz|^2_{\mathbb{P}^n} \leq \frac{(1-\alpha)\alpha}{\kappa(n-1)\rho^2}|A|^2 \tag{4.16}$$

then $\Sigma^n(z) = \mathbb{P}^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$.

Proof. From (4.15) and (4.16), we get $\sup_{\Sigma^n(z)} \rho^2 |Dz|^2_{\mathbb{P}^n} \leq \alpha$. So, from the first part of the proof of the Corollary 3 we obtain that $\Sigma^n(z)$ is spacelike and complete. Now, from (3.8) and (4.16) we obtain (4.11). Finally, the result is obtained as a direct application of the Theorem 4. □

In order to characterize slices of weighted standard static spacetimes $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$, we observe that one of the hypotheses of the Theorem 4 is exactly the inferior limitation of the Bakry–Émery–Ricci tensor $\widetilde{\mathrm{Ric}}_f$ of $\mathbb{P}^n$ by some constant. When this limitation is given by zero, we have the following result that establishes other sufficient conditions for an spacelike hypersurface to be a slice.

Theorem 5. *Let $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$ be a weighted standard static spacetime with $\widetilde{\mathrm{Ric}}_f \geq 0$ and ρ being a convex warping function such that $\langle \widetilde{\nabla} f, \widetilde{\nabla}\rho \rangle \leq 0$. Let $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ be a f-parabolic spacelike hypersurface with constant f-mean curvature and angle function Θ bounded from below. Then, Σ^n is totally geodesic. Moreover, if $\widetilde{\mathrm{Ric}}_f$ is strictly positive at some point p_0 of Σ^n, then $x(\Sigma^n)$ is contained in a slice $\mathbb{P}^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$.*

Proof. Since the f-mean curvature of $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ is constant, ρ is a concave function such that $\langle \widetilde{\nabla} f, \widetilde{\nabla}\rho \rangle \leq 0$ and $\widetilde{\mathrm{Ric}}_f$ is nonnegative, from Proposition 2 and (4.12) we obtain that

$$\Delta_f \Theta \leq \left(\widetilde{\mathrm{Ric}}_f(N^*, N^*) + |A|^2\right)\Theta \leq 0. \tag{4.17}$$

Thus, the weighted parabolicity of Σ^n assures that Θ is constant on it. So, returning to (4.17) we have that $|A| = 0$, that is, Σ^n is totally geodesic.

We claim that ρ is constant. Indeed, first we note that all $X \in \mathfrak{X}(\Sigma^n)$ can be written as

$$X = X^* - \frac{\langle X, Y\rangle}{\rho^2} Y,$$

where $(\,\cdot\,)^*$ denote the projection on $\mathfrak{X}(\mathbb{P}^n)$. Since Σ^n is totally geodesic, from Proposition 7.35 of [31], we have that

$$\begin{aligned}\langle\nabla\Theta, X\rangle &= X(\langle N, Y\rangle) = \langle N, \overline{\nabla}_X Y\rangle\\ &= \langle N, \rangle\overline{\nabla}_{X^*}Y - \frac{\langle X, Y\rangle}{\rho^2}\langle N, \overline{\nabla}_Y Y\rangle\\ &= \frac{1}{\rho}\langle X, \overline{\nabla}\rho\rangle\langle N, Y\rangle - \frac{1}{\rho}\langle X, Y\rangle\langle N, \overline{\nabla}\rho\rangle,\end{aligned}$$

which implies

$$\nabla\Theta = \frac{1}{\rho}\left(\Theta\overline{\nabla}\rho - \langle N, \overline{\nabla}\rho\rangle Y\right).$$

Given that Θ is constant on Σ^n, since the vector fields $\overline{\nabla}\rho$ and Y are linearly independent, from the last equation, we obtain that ρ is also constant on Σ^n. So, our affirmation stay showed.

On the other hand, from (2.9) it is not difficult to see that

$$|N^*|^2_{\mathbb{P}^n} = \left(\frac{\Theta^2}{\rho^2} - 1\right),$$

which implies that $|N^*|_{\mathbb{P}^n}$ is also constant. But, supposing that $\widetilde{\mathrm{Ric}}_f$ is strictly positive at some point p_0 of Σ^n, since (4.17) give us that $\widetilde{\mathrm{Ric}}_f(N^*, N^*)(p_0) = 0$, it follows that $N^*(p_0) = 0$. Therefore, N^* must be vanishes on Σ^n and, consequently, $x(\Sigma^n)$ must be contained in a slice $\mathbb{P}^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$. □

In particular, Theorem 5 gives us the following result of nonexistence.

Corollary 6. *Let $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$ be a weighted standard static spacetime with $\widetilde{\mathrm{Ric}}_f \geq 0$ and ρ being a convex warping function such that $\langle\widetilde{\nabla} f, \widetilde{\nabla}\rho\rangle \leq 0$. There is not nonzero constant f-mean curvature f-parabolic spacelike hypersurface immersed into $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$ with angle function bounded from below.*

From Theorem 5, we can reason as in the proof of Corollary 3 in order to obtain the following result:

Corollary 7. *Let $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$ be a weighted standard static spacetime with $\widetilde{\mathrm{Ric}}_f \geq 0$ and ρ being a convex warping function such that $\langle\widetilde{\nabla} f, \widetilde{\nabla}\rho\rangle \leq 0$. Let $\Sigma^n(z)$ be a f-parabolic entire graph with constant f-mean curvature and angle function Θ bounded from below. If the norm of the gradient Dz of the function $z \in C^\infty(\mathbb{P}^n)$ satisfies (3.7), then $\Sigma^n(z)$ is totally geodesic. Moreover, if $\widetilde{\mathrm{Ric}}_f$ is strictly positive at some point p_0 of $\Sigma^n(z)$, then $\Sigma^n(z)$ is a slice $\mathbb{P}^n \times \{t_0\}$, for some $t_0 \in \mathbb{R}$.*

5. A Notion of Stability in Weighted Standard Static Spacetimes

For a compact spacelike hypersurface $x : \Sigma^n \hookrightarrow \mathbb{P}_f \times_\rho \mathbb{R}_1$ with boundary $\partial\Sigma$ (possibly empty), we define a *variation* of it as being the smooth mapping

$$\begin{array}{rcl} X : (-\epsilon, \epsilon) \times \Sigma^n & \to & \mathbb{P}^n_f \times_\rho \mathbb{R}_1 \\ (s,p) & \mapsto & X(s,p) \end{array}$$

satisfying the following two conditions:

(i) for all $s \in (-\epsilon, \epsilon)$, the map $X_s : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ given by $X_s(p) = X(s,p)$ is a Riemannian immersion such that $X_0 = x$;

(ii) $X_s\Big|_{\partial\Sigma} = x\Big|_{\partial\Sigma}$, for all $s \in (-\epsilon, \epsilon)$.

In all that follows, we let $d\Sigma_s$ for denote the volume element of the warping metric (2.1) induced on $\Sigma^n_s = X_s(\Sigma^n)$ and N_s will be the future-pointing Gauss map along of Σ^n_s. Moreover, we also consider in Σ^n_s the weighted volume form given by $d\sigma_s = e^{-f} d\Sigma_s$. When $s = 0$ all these objects coincide with ones defined in Σ^n, respectively. Moreover for any open subset Ω of $\mathbb{P}^n_f \times_\rho \mathbb{R}_1$ with compact closure, $\mathrm{Vol}_f(\Omega)$ and $\mathrm{Area}_f(\Omega)$ will denote the *weighted volume* and *weighted area* of Ω, respectively.

The *variational field* associated to the variation X is the smooth vector field $\frac{\partial X}{\partial s}\Big|_{s=0}$. Letting

$$u_s = -\Big\langle \frac{\partial X}{\partial s}, N_s \Big\rangle, \tag{5.1}$$

we get

$$\frac{\partial X}{\partial s}\Big|_{s=0} = u_0 N + \left(\frac{\partial X}{\partial s}\Big|_{s=0} \right)^{\top}.$$

The *balance of weighted volume* and the *weighted area functional* associated to the variation X are the functionals

$$\begin{array}{rl} \mathcal{V}_f : (-\epsilon, \epsilon) \to & \mathbb{R} \\ s \;\; \mapsto & \mathcal{V}_f(s) = \mathrm{Vol}_f\left(X\left([0,s] \times \Sigma^n \right) \right) = \displaystyle\int_{[0,s]\times\Sigma^n} X^*(d\overline{\sigma}) \end{array}$$

and

$$\begin{array}{rl} \mathcal{A}_f : (-\epsilon, \epsilon) \to & \mathbb{R} \\ s \;\; \mapsto & \mathcal{A}_f(s) = \mathrm{Area}_f\left(\Sigma^n_s \right) = \displaystyle\int_{\Sigma^n_s} d\sigma_s, \end{array}$$

respectively, where $d\overline{v}$ is the volume element on induced by the warping metric (2.1). We say that the variation X is *weighted volume-preserving* of Σ^n if $\mathcal{V}_f(s) = \mathcal{V}_f(0) = 0$, for all $s \in (-\epsilon, \epsilon)$.

The following result is well known and, in the context of weighted Lorentzian manifolds, it can be found in Lemmas 1 and 2 of [18].

Lemma 5. *Let $X : (-\epsilon, \epsilon) \times \Sigma^n \to \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ be a variation of the closed (that is, compact and without boundary) spacelike hypersurface $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$. If u_s is the smooth function given in (5.1) then*

$$\frac{d}{ds}\mathcal{V}_f(s) = \int_{\Sigma^n_s} u_s \, d\sigma_s \qquad \text{and} \qquad \frac{d}{ds}\mathcal{A}_f(s) = n\int_{\Sigma^n_s} (H_f)_s \, u_s \, d\sigma_s$$

where $(H_f)_s = H_f(s, \cdot)$ denotes the f-mean curvature of Σ^n_s. In particular, X is weighted volume-preserving of Σ^n if and only if $\int_{\Sigma^n_s} u_s \, d\sigma_s = 0$ for all $s \in (-\epsilon, \epsilon)$.

Remark 5. Applying the same topological arguments used to prove Proposition 3.2 of [3], we conclude that a closed spacelike hypersurface Σ^n immersed in a standard static spacetime $\mathbb{P}^n \times_\rho \mathbb{R}_1$ can only exist when the Riemannian base $\mathbb{P}^n$ is also compact. On the other hand, it is not difficult to verify that Lemma 2.2 of [5] still remains valid for the context of weighted standard static spacetimes. More specifically, given a closed spacelike hypersurface $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$, if $u \in C^\infty(\Sigma^n)$ is such that

$$\int_{\Sigma^n} u d\sigma = 0,$$

then there exists a weighted volume-preserving variation $X : (-\epsilon, \epsilon) \times \Sigma^n \to \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ of $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ whose variational field is $\left.\frac{\partial X}{\partial s}\right|_{s=0} = uN$.

In order to characterize closed spacelike hypersurfaces $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ with constant f-mean curvature, we consider the variational problem of maximizing the weighted area functional $\mathcal{A}_f$ for all variations $X : (-\epsilon, \epsilon) \times \Sigma^n \to \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ of $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ that keeps the balance of weighted volume $\mathcal{V}_f$ equal to zero. The Lagrange multiplier method leads us then to the associated weighted Jacobi functional

$$\begin{aligned} \mathcal{J}_f : (-\epsilon, \epsilon) &\to \quad \mathbb{R} \\ s \quad &\mapsto \mathcal{J}_f(s) = \mathcal{A}_f(s) - \lambda \mathcal{V}_f(s), \end{aligned} \tag{5.2}$$

where λ is a constant to be determined. As an immediate consequence of Lemma 5 we get that the first variation of $\mathcal{J}_f$ takes the following form

$$\frac{d}{ds}\mathcal{J}_f(s) = \int_{\Sigma^n_s} \left\{ n\,(H_f)_s - \lambda \right\} u_s \, d\sigma_s, \tag{5.3}$$

where u_s is the smooth function given in (5.1). Thinking about making the best possible choice of λ, let

$$\overline{\mathcal{H}} = \frac{1}{\mathrm{Area}_f(\Sigma^n)} \int_{\Sigma^n} H_f \, d\sigma \tag{5.4}$$

be an integral mean of the f-mean curvature H_f on Σ^n. We call the attention to the fact that, in case H_f is constant, we have

$$\overline{\mathcal{H}} = H_f, \tag{5.5}$$

and this notation will be used in what follows without further comments. Therefore, if we choose $\lambda = n\overline{\mathcal{H}}$, from (5.3) we arrive at

$$\frac{d}{ds}\,\mathcal{J}_f(s) = n\int_{\Sigma_s^n}\left\{(H_f)_s - \overline{\mathcal{H}}\right\} u_s\, d\sigma_s. \tag{5.6}$$

Reasoning as in the proof of Proposition 2.7 of [4] we get

Proposition 3. *The following statements are equivalent:*

(a) $x : \Sigma^n \hookrightarrow \mathbb{P}_f^n \times_\rho \mathbb{R}_1$ *is a closed spacelike hypersurface with constant f-mean curvature H_f;*

(b) $\dfrac{d}{ds}\,\mathcal{A}_f(0) = 0$ *for all weighted volume-preserving variation* $X : (-\epsilon, \epsilon) \times \Sigma^n \to \mathbb{P}_f^n \times_\rho \mathbb{R}_1$ *of* $x : \Sigma^n \hookrightarrow \mathbb{P}_f^n \times_\rho \mathbb{R}_1$*;*

(c) $\dfrac{d}{ds}\,\mathcal{J}_f(0) = 0$ *for every variation* $X : (-\epsilon, \epsilon) \times \Sigma^n \to \mathbb{P}_f^n \times_\rho \mathbb{R}_1$ *of* $x : \Sigma^n \hookrightarrow \mathbb{P}_f^n \times_\rho \mathbb{R}_1$.

Proof. We will show the result making the sequence $(a) \Rightarrow (c)$, $(c) \Rightarrow (b)$, $(b) \Rightarrow (a)$.

$(a) \Rightarrow (c)$: The result follows directly from (5.5) and (5.6).

$(c) \Rightarrow (b)$: We have

$$0 = \frac{d}{ds}\,\mathcal{J}_f(0) = \frac{d}{ds}\,\mathcal{A}_f(0) + n\overline{\mathcal{H}}\,\frac{d}{ds}\,\mathcal{V}_f(0)$$

or all variation $X : (-\epsilon, \epsilon) \times \Sigma^n \to \mathbb{P}_f^n \times_\rho \mathbb{R}_1$ of $x : \Sigma^n \hookrightarrow \mathbb{P}_f^n \times_\rho \mathbb{R}_1$. But if the variation preserves the volume of $x : \Sigma^n \hookrightarrow \mathbb{P}_f^n \times_\rho \mathbb{R}_1$ then $\dfrac{d}{ds}\,\mathcal{V}_f(0) = 0$. Hence, $\dfrac{d}{ds}\,\mathcal{A}_f(0) = 0$ for all weighted volume-preserving variation $X : (-\epsilon, \epsilon) \times \Sigma^n \to \mathbb{P}_f^n \times_\rho \mathbb{R}_1$ of $x : \Sigma^n \hookrightarrow \mathbb{P}_f^n \times_\rho \mathbb{R}_1$.

$(b) \Rightarrow (a)$: Suppose there is p_0 in Σ^n such that $(H_f - \overline{\mathcal{H}})(p_0) \neq 0$. We can assume that $(H_f - \overline{\mathcal{H}})(p_0) > 0$. From the definition of $\overline{\mathcal{H}}$ in (5.4) we can obtain another point $q_0 \in \Sigma^n$ such that $(H_f - \overline{\mathcal{H}})(q_0) < 0$. Indeed, from (5.5) we have

$$\begin{aligned}\int_{\Sigma^n}(H_f - \overline{\mathcal{H}})\, d\sigma &= \int_{\Sigma^n} H_f\, d\sigma - \overline{\mathcal{H}}\,\mathrm{Area}_f(\Sigma^n) \\ &= \int_{\Sigma^n} H_f\, d\sigma - \left(\frac{1}{\mathrm{Area}_f(\Sigma^n)}\int_{\Sigma^n} H_f\, d\sigma\right)\mathrm{Area}_f(\Sigma^n) = 0.\end{aligned} \tag{5.7}$$

So, if $(H_f - \overline{\mathcal{H}})(q) > 0$ for every $q \in \Sigma^n$, since there is $p_0 \in \Sigma^n$ such that $(H_f - \overline{\mathcal{H}})(p_0) > 0$, then

$$\int_{\Sigma^n}(H_f - \overline{\mathcal{H}})\, d\sigma > 0,$$

inequality that is in contradiction with (5.7).

Thus, the sets

$$\Sigma^+ = \left\{ q \in \Sigma^n \,:\, (H_f - \overline{\mathcal{H}})(q) > 0 \right\} \qquad \text{and}$$
$$Sigma^- = \left\{ q \in \Sigma^n \,:\, (H_f - \overline{\mathcal{H}})(q) < 0 \right\}$$

are well defined.

Now, consider nonnegative smooth functions φ and ψ such that $p_0 \in \operatorname{supp}\varphi \subset \Sigma^+$, $\operatorname{supp}\psi \subset \Sigma^-$ and

$$\int_{\Sigma^n} (\varphi + \psi)(H_f - \overline{\mathcal{H}})\, d\sigma = 0.$$

where $\operatorname{supp}\varphi$ and $\operatorname{supp}\psi$ denote the support of φ and the support of ψ, respectively. If we consider the smooth function $u = (\varphi + \psi)(H_f - \overline{\mathcal{H}})$ then, according to Remark 5, there is a weighted volume-preserving variation $X : (-\epsilon, \epsilon) \times \Sigma^n \to \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ of $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ whose variational field is $\frac{\partial X}{\partial s}\Big|_{s=0} = uN$. By hypothesis and Lemma 5,

$$0 = \frac{d}{ds}\, \mathcal{A}_f(0) = n \int_{\Sigma^n} H_f\, u\, d\sigma.$$

Since $\int_{\Sigma^n} u\, d\sigma = 0$, we obtain

$$\begin{aligned} 0 &= n \int_{\Sigma^n} H_f\, u\, d\sigma - n\overline{\mathcal{H}} \int_{\Sigma^n} u\, d\sigma \\ &= n \int_{\Sigma^n} (H_f - \overline{\mathcal{H}}) u\, d\sigma = n \int_{\Sigma^n} (\varphi + \psi)(H_f - \overline{\mathcal{H}})^2\, d\sigma \;>\; 0, \end{aligned}$$

which is a contradiction. Therefore, $H_f = \overline{\mathcal{H}}$ on Σ^n.

□

In particular, Proposition 3 guarantees that a closed spacelike hypersurface $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ is a critical point of the variational problem described above if and only if its f-mean curvature H_f is constant. Motivated by this fact, we establish the following

Definition 1. Let $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ be a closed spacelike hypersurface having constant f-mean curvature. We say that $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ is *f-stable* if $\dfrac{d^2}{ds^2}\, \mathcal{A}_f(0) \leq 0$, for all weighted volume-preserving variation $X : \Sigma^n \times (-\epsilon, \epsilon) \to \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ of $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$.

Let $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ be a closed spacelike hypersurface as described in Definition 1. We consider the set

$$\mathcal{G} = \left\{ \, u \in C^\infty(\Sigma^n) \;:\; \int_{\Sigma^n} u\, d\sigma = 0 \, \right\}. \tag{5.8}$$

Just as [4], we can establish the following criterion of f-stability.

Proposition 4. *With the notations considered above, $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ is f-stable if and only if $\frac{d^2}{ds^2}\mathcal{J}_f(0)(u) \le 0$ for all $u \in \mathcal{G}$.*

Proof. Suppose that $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ is f-stable and consider $u \in \mathcal{G}$. From Remark 5, there is a weighted volume-preserving variation $X : (-\epsilon, \epsilon) \times \Sigma^n \to \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ of $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ whose variational field is $\frac{\partial X}{\partial s}\Big|_{s=0} = uN$. Then, $\frac{d^2}{ds^2}\mathcal{V}_f(0)(u) = 0$. Hence, from (5.2) and Definition 1 we obtain

$$\frac{d^2}{ds^2}\mathcal{J}_f(0)(u) = \frac{d^2}{ds^2}\mathcal{A}_f(0)(u) - \lambda\frac{d^2}{ds^2}\mathcal{V}_f(0)(u) = \frac{d^2}{ds^2}\mathcal{A}_f(0)(u) \le 0$$

Conversely, suppose that $\frac{d^2}{ds^2}\mathcal{J}_f(0)(u) \le 0$ for all $u \in \mathcal{G}$. Let $X : (-\epsilon, \epsilon) \times \Sigma^n \to \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ be an weighted volume-preserving variation of $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$, and let uN be the normal component of the variation vector $\frac{\partial X}{\partial s}\Big|_{s=0}$. From Lemma 5,

$$\int_{\Sigma^n} u\, d\sigma = \frac{d}{ds}\mathcal{V}_f(0) = 0,$$

which implies that $u \in \mathcal{G}$. Therefore, from hypotheses,

$$0 \ge \frac{d^2}{ds^2}\mathcal{J}_f(0)(u) = \frac{d^2}{ds^2}\mathcal{A}_f(0)(u) - \lambda\underbrace{\frac{d^2}{ds^2}\mathcal{V}_f(0)(u)}_{0} = \frac{d^2}{ds^2}\mathcal{A}_f(0)(u),$$

which according to Definition 1 tells us that $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ is f-stable. □

The sought formula for the second variation of Jacobi functional $\mathcal{J}_f$ is given in the following

Proposition 5. *Let $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ be a closed spacelike hypersurface having constant f-mean curvature H_f. If $X : (-\epsilon, \epsilon) \times \Sigma^n \to \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ is a variation of $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ then the second variation $\frac{d^2}{ds^2}\mathcal{J}_f(0)$ of the weighted Jacobi functional $\mathcal{J}_f$ is given by*

$$\frac{d^2}{ds^2}\mathcal{J}_f(0)(u) = \int_{\Sigma^n} u\,\mathfrak{L}_f(u)\, d\sigma, \tag{5.9}$$

for any $u \in C^\infty(\partial\Omega)$, where $\mathfrak{L}_f : C^\infty(\Sigma^n) \to C^\infty(\Sigma^n)$ is the weighted Jacobi operator given by

$$\mathfrak{L}_f = \Delta_f - \left\{\widetilde{\mathrm{Ric}}_f(N^*, N^*) - \frac{1}{\rho}\widetilde{\mathrm{Hess}}\,\rho(N^*, N^*) + \Theta^2\frac{\widetilde{\Delta}_f(\rho)}{\rho^3} + |A|^2\right\}. \tag{5.10}$$

Here, Δ_f and $\widetilde{\Delta}_f$ represent the f-Laplacians on Σ^n and $\mathbb{P}^n$, respectively, Θ be the angle function defined in (2.7), N is the future-pointing Gauss map of $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$, $\widetilde{\mathrm{Ric}}_f$ and $\widetilde{\mathrm{Hess}}$ are the Bakry–Émery–Ricci tensor and *the Hessian operator on $\mathbb{P}^n$, $|A|^2$ represent the square of the norm of the shape operator A of $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ and N^* is the projection of N on the tangent bundle of $\mathbb{P}^n$.*

Proof. Since H_f is constant, from (5.6) and (5.5) we have that

$$\begin{aligned}\frac{d^2}{ds^2}\,\mathcal{J}_f(0)(u_0) &= n\int_{\Sigma^n}\left(\frac{\partial\,(H_f)_s}{\partial s}\Big|_{s=0}\right)u_0\,d\sigma\\ &\quad + n\int_{\Sigma^n}\left(\underbrace{H_f-\overline{\mathcal{H}}}_{0}\right)\frac{\partial}{\partial s}\,(u_s\,d\sigma_s)\Big|_{s=0},\end{aligned}$$

where u_s is the smooth function given in (5.1).

On the other hand, reasoning as in the proof of equation (3.5) of [13], we obtain

$$n\,\frac{\partial\,(H_f)_s}{\partial s}\Big|_{s=0} = \Delta_f\,(u_0) - \left\{\overline{\mathrm{Ric}}_f(N,N) + |A^2|\right\}u_0.$$

Hence,

$$\frac{d^2}{ds^2}\,\mathcal{J}_f(0)(u_0) = \int_{\Sigma^n}\left\{\Delta_f(u_0) - \left\{\overline{\mathrm{Ric}}_f(N,N) + |A|^2\right\}u_0\right\}u_0\,d\sigma. \quad (5.11)$$

Now, from Eqs. (4.10) and (5.11) we obtain

$$\frac{d^2}{ds^2}\,\mathcal{J}_f(0)(u_0) = \int_{\Sigma^n} u_0\,\mathfrak{L}_f(u_0)\,d\sigma, \quad (5.12)$$

where $\mathfrak{L}_f$ is given in (5.10). To finish the proof, we observe that the expression (5.12) depends only on the hypersurface Σ^n and on the function $u_0 \in C^\infty(\Sigma^n)$. □

To show our next result, let us remember that the *eigenvalue problem* for the drift Laplacian Δ_f on a closed Riemannianmanifold Σ^n is the determination of the existence or not of nontrivial solutions (that is, not identically zero) $u \in C^{\infty}(\Sigma)$ for the partial differential equation

$$\Delta_f(u) + \mu\, u = 0$$

on Σ^n. In this case, the corresponding function u is an *eigenfunction* associated with the *eigenvalue* μ. By the spectral theorem we know that all the eigenvalues of Δ_f are determined by a sequence of eigenvalues $\{\mu_j\}_{j=0}^{+\infty}$ satisfying

$$0 \;=\; \mu_0 \;<\; \mu_1 \;\leq <\; \mu_2 \;\leq\; \cdots \;\leq\; \mu_j \;\leq\; \mu_{j+1} \;\leq\; \cdots,$$

repeated according to their multiplicity, and

$$\lim_{j\to+\infty} \mu_j = +\infty$$

(see, for instance, Section 1 of [6]). Moreover, the variational characterization of μ_1 gives

$$\mu_1 = \min_{u \in \mathcal{G}\setminus\{0\}} \frac{-\int_{\Sigma^n} u\Delta_f(u)\, d\sigma}{\int_{\Sigma^n} u^2\, d\sigma}, \tag{5.13}$$

where $\mathcal{G}$ is defined in (5.8).

We can now present our characterization of f-stability concerning closed spacelike hypersurfaces immersed in a weighted standard static spacetime.

Theorem 6. *Let $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ be a closed spacelike hypersurface with constant f-mean curvature. Suppose that*

$$\mu = -\,\widetilde{\mathrm{Ric}}_f(N^*, N^*) + \frac{1}{\rho}\,\widetilde{\mathrm{Hess}}\,\rho(N^*, N^*) - \Theta^2\,\frac{\widetilde{\Delta}_f(\rho)}{\rho^3} - |A|^2$$

is a nonzero constant on Σ^n, where Δ_f and $\widetilde{\Delta}_f$ represent the f-Laplacians on Σ^n and $\mathbb{P}^n$, respectively, Θ be the angle function defined in (2.7), N is the future-pointing Gauss map of $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$, $\widetilde{\mathrm{Ric}}_f$ and $\widetilde{\mathrm{Hess}}$ are the Bakry–Émery–Ricci tensor and the Hessian operator on $\mathbb{P}^n$, $|A|^2$ represent the square of the norm of the shape operator A of $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ and N^* is the projection of N on the tangent bundle of $\mathbb{P}^n$. Then $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ is f-stable if and only if μ is the first nonzero eigenvalue of drift Laplacian Δ_f on Σ^n.

Proof. Initially, since the f-mean curvature of $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ and μ are constant on Σ^n, from Proposition 2 we can see that μ belongs to the sequence of eigenvalues $\{\mu_j\}_{j=0}^{+\infty}$ of the drift Laplacian Δ_f on Σ^n.

If $\mu = \mu_1$, then from (5.9), (5.10) and (5.13) we obtain

$$\frac{d^2}{ds^2}\,\mathcal{J}_f(0)(u) \;=\; \int_{\Sigma^n} \{u\Delta_f(u) + \mu\, u^2\}\, d\sigma \;\leq\; (-\mu + \mu) \int_{\Sigma^n} u^2\, d\sigma = 0$$

for any $u \in \mathcal{G}$ and, according to Proposition 4, $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ is f-stable.

Conversely, suppose that $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ is f-stable, so that $\frac{d^2}{ds^2}\mathcal{J}_f(0)(u) \leq 0$ for all $u \in \mathcal{G}$. Let u be an eigenfunction associated to the first nonzero eigenvalue μ_1 of Δ_f. Consequently, by (5.9) and (5.10) we get

$$0 \;\geq\; \frac{d^2}{ds^2}\mathcal{J}_f(0)(u) \;=\; (-\mu_1 + \mu)\int_{\Sigma^n} u^2\, d\sigma.$$

Therefore, since $\mu_1 \leq \mu$, we must have $\mu_1 = \mu$. □

Remark 6. From the proof of Theorem 6 it is immediate to verify that any constant f-mean curvature closed spacelike hypersurface $x : \Sigma^n \hookrightarrow \mathbb{P}^n_f \times_\rho \mathbb{R}_1$ for which

$$\mu_1 + \widetilde{\mathrm{Ric}}_f(N^*, N^*) - \frac{1}{\rho}\widetilde{\mathrm{Hess}}\,\rho(N^*, N^*) + \Theta^2\,\frac{\widetilde{\Delta}_f(\rho)}{\rho^3} + |A|^2 \geq 0$$

on Σ^n is stable, where μ_1 is the first nonzero eigenvalue of drift Laplacian on Σ^n. In particular, this happens when $\widetilde{\mathrm{Ric}}_f \geq 0$ and ρ is constant on $\mathbb{P}^n$.

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