Results in Mathematics

Classification and Construction of Minimal Translation Surfaces in Euclidean Space

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Abstract. A translation surface of Euclidean space $\mathbb{R}^3$ is the sum of two regular curves α and β, called the generating curves. In this paper we classify the minimal translation surfaces of $\mathbb{R}^3$ and we give a method of construction of explicit examples. Besides the plane and the minimal surfaces of Scherk type, it is proved that up to reparameterizations of the generating curves, any minimal translation surface is described as $\Psi(s,t) = \alpha(s) + \alpha(t)$, where α is a curve parameterized by arc length s, its curvature κ is a positive solution of the autonomous ODE $(y')^2 + y^4 + c_3 y^2 + c_1^2 y^{-2} + c_1 c_2 = 0$ and its torsion is $\tau(s) = c_1/\kappa(s)^2$. Here $c_1 \neq 0$, c_2 and c_3 are constants such that the cubic equation $-\lambda^3 + c_2\lambda^2 - c_3\lambda + c_1 = 0$ has three real roots λ_1, λ_2 and λ_3.

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1. Introduction and Statement of the Result

The surfaces of our study have its origin in the classical text of Darboux [1, Livre I] where the so-called "surfaces définies pour des propertiétés cinématiques" are considered, and later known as Darboux surfaces in the literature. A *Darboux surface* is defined kinematically as the movement of a curve by a uniparametric family of rigid motions of $\mathbb{R}^3$. Hence a parameterization of a such surface is $\Psi(s,t) = A(t) \cdot \alpha(s) + \beta(t)$, where α and β are two space curves and $A(t)$ is an orthogonal matrix. The case that we are investigating in this paper is when $A(t)$ is the identity. More precisely, we give the following definition.

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Definition 1.1. A surface $S \subset \mathbb{R}^3$ is called a translation surface if S can be locally written as the sum $\Psi(s,t) = \alpha(s) + \beta(t)$ of two space curves $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ and $\beta : J \subset \mathbb{R} \to \mathbb{R}^3$. The curves α and β are called the generating curves of S. If α and β are planar curves, the surface is called a translation surface of planar type.

Darboux dealt with translation surfaces in Secs. 81-84 [1, pp. 137–142] and its name is due to the fact that the surface S is obtained by the translation of α along β (or *vice versa* because the roles of α and β are interchanged). In particular, all parametric curves $s = ct$ are congruent by translations (similarly for parametric curves $t = ct$). It is natural to ask for the classification of the translation surfaces of $\mathbb{R}^3$ under some condition on its curvature. Recently, the authors of the present paper succeeded with the complete classification of all translation surfaces with constant Gaussian curvature K, proving that the only ones are cylindrical surfaces and thus, K must be 0 [3].

In this paper we are concerned with the following

Problem: Classify all minimal translation surfaces in $\mathbb{R}^3$.

Recall that a minimal surface in $\mathbb{R}^3$ is a surface with zero mean curvature everywhere. Of course, the plane is a trivial example of a minimal translation surface. A first approach to the posed problem is to assume that the generating curves are planar curves contained in orthogonal planes. In such a case, after an appropriate choice of coordinate system, the surface S is locally parameterized by

$$\Psi(s,t) = (s, 0, f(s)) + (0, t, g(t)) = (s, t, f(s) + g(t))$$

for some two smooth functions f and g. Thus the problem transforms into finding surfaces represented by the graph of a function $z = f(x) + g(y)$ with zero mean curvature. It is not difficult to see that, besides the plane and a rigid motion, the solution is the known as Scherk surface

$$z = \frac{1}{c} \log \frac{\cos(cy)}{\cos(cx)}, \quad x, y \in \left(\frac{-\pi}{2}, \frac{\pi}{2} \right),$$

where c is a positive constant. This surface was obtained by Scherk solving the minimal surface equation by separation of variables, namely, $z = f(x) + g(y)$ [12]. In fact, this surface belongs to a uniparametric family of minimal surfaces discovered by Scherk and given by

$$\mathcal{S}_\theta(x,y) = \left(x + y\cos\theta, y\sin\theta, \frac{1}{c} \log \frac{\cos(cy)}{\cos(cx)} \right),$$

where $\theta \in \mathbb{R}$ [12]; see also [11, § 81]. For $\theta = 0$, $\mathcal{S}_0$ is the plane and if $\theta = \pi/2$, $\mathcal{S}_{\pi/2}$ is the Scherk surface. Let us observe that $\mathcal{S}_\theta$ is a translation surface where the generating curves are planar but now not necessarily contained in orthogonal planes. Indeed, $\mathcal{S}_\theta$ can be expressed as

$$\mathcal{S}_\theta(s,t) = \left(s, 0, -\frac{1}{c}\log(\cos(cs))\right) + \left(t\cos\theta, t\sin\theta, \frac{1}{c}\log(\cos(ct))\right)$$
$$= \alpha(s) + \beta(t).$$

Then α is contained in the xz-plane and β lies in the plane of equation $\sin\theta x - \cos\theta y = 0$, which makes an angle θ with the xz-plane.

Other minimal translation surface, and already known by Lie, is the helicoid $X(u,v) = (\cos u\cos v, \sin u\cos v, u)$ [11, § 77]. This surface is obtained by the sum of a circular helix with itself. Indeed, let α be the circular helix parameterized as $\alpha(s) = (\cos s, \sin s, s)/2$. Making the change of coordinates $s = u + v$, $t = u - v$, we find $\Psi(s,t) = \alpha(s) + \alpha(t) = X(u,v)$. In fact, the helicoid can be generated in an infinite number of ways as a translation surface of helices for suitable choices of pitches.

In the literature, other works have appeared on the study of translation surfaces with constant mean curvature, also in other ambient spaces: we refer [2,4–6,9,10], without to be a complete list. However, in all these works, the translation surface is of planar type, so the problem of finding such surfaces reduces into a problem of solving an ODE by separation of variables. It deserves to point out that it was proved in [2] that if one generating curve is planar, then the other is also planar and the surface belongs to the family of Scherk surfaces.

Definitively, besides the plane, the Scherk surfaces and the helicoid, the motivating question for the present paper is as follows:

Question: Are there any other minimal translation surfaces in $\mathbb{R}^3$?

Recently the second author and O. Perdomo have undertaken the problem of classification in all its generality assuming that the generating curves are space curves [7]. It was proved that one generating curve is the rigid motion of the other one, hence we can write $\Psi(s,t) = \alpha(s) + \alpha(t)$ and if κ and τ are the curvature and the torsion of α respectively, then $\kappa^2\tau$ is a non-zero constant. Furthermore, the velocity vectors $\alpha'(s)$ must lie in a cone of the form $\{\mathbf{x} \in \mathbb{R}^3 : \langle A\mathbf{x}, \mathbf{x}\rangle = 0\}$ for a particular symmetric matrix A: see [7] for details.

In this paper we offer an alternative approach to the study of the minimal translation surfaces. Besides to obtain similar results than the ones of [7], we give a new method of construction of minimal translation surfaces based on the resolution of an ODE which seems to us simpler than the methods used in [7]. The advantage lies in the fact that provides a technique by means of a recipe that gives a plethora of examples. The results may be summarized as follows.

Theorem 1.2 (Classification and construction). *Let $\Psi(s,t) = \alpha(s) + \beta(t)$ be a minimal translation surface where α and β parameterized by arc length. Suppose that the curvatures κ_α and κ_β are positive everywhere and the torsions τ_α, τ_β are non-zero everywhere. Then, up to a rigid motion, the curve β coincides*

with α and

$$\kappa_\alpha^2 \tau_\alpha = c_1, \quad \frac{1}{\tau_\alpha}\left(\frac{\kappa_\alpha'}{\kappa_\alpha}\right)' + \frac{\kappa_\alpha^2}{\tau_\alpha} - 2\tau_\alpha = c_2, \tag{1.1}$$

where $c_1 \neq 0$ and c_2 are two constants. Furthermore, κ_α is a positive solution of the autonomous ODE

$$y'^2 + y^4 + c_3 y^2 + \frac{c_1^2}{y^2} + c_1 c_2 = 0 \tag{1.2}$$

for some constant c_3 and the curve α can be expressed as

$$\alpha(s) = \left(A\int^s \cos w(s), B\int^s \sin w(s), \int^s \sqrt{1 - A^2\cos^2 w(s) - B^2\sin^2 w(s)}\right).$$

Here $A = \sqrt{\lambda_3/(\lambda_3 - \lambda_1)}$ and $B = \sqrt{\lambda_3/(\lambda_3 - \lambda_2)}$ where λ_i, $1 \leq i \leq 3$, are the three real roots of the cubic equation

$$-\lambda^3 + c_2\lambda^2 - c_3\lambda + c_1 = 0 \tag{1.3}$$

and the function w is $w(s) = \int^s \sqrt{\kappa_\alpha(s)^2 + \lambda_1\lambda_2}$.

    Conversely, any minimal translation surface of $\mathbb{R}^3$ of non-planar type is constructed by this process. Exactly let $c_1 \neq 0$, c_2 and c_3 be three constants such that the cubic Eq. (1.3) has three real roots λ_1, λ_2, λ_3. Let $\kappa(s)$ be a positive and non-constant solution of (1.2). If $\alpha(s)$ is a curve parameterized by arc length s with curvature $\kappa(s)$ and torsion $c_1/\kappa(s)^2$, then the translation surface $\Psi(s,t) = \alpha(s) + \alpha(t)$ is minimal.

    This paper is organized as follows. In Sect. 2 we recall some known formulae of the local theory of curves and surfaces in $\mathbb{R}^3$. In Sect. 3 we prove, for the sake of completeness, some known results with alternative proofs. So, we prove the result of [2] and we obtain the helicoid when one generating curve is a circular helix (Theorem 3.2). We also characterize any minimal translation surface by the two relations (1.1) between the curvature and the torsion of the generating curves. In Sect. 4, we show the main results of this paper. Here it will be essential the definition of a set of self-adjoint linear operators on $\mathbb{R}^3$ associated to each point $\alpha(s)$ and $\beta(t)$, which it will be proved later that, indeed, they coincide for all s and t. The two results of this section (Theorems 4.3 and 4.4) classify and describe the construction of all minimal translation surfaces in $\mathbb{R}^3$. The section finishes showing explicit examples of translation minimal surfaces by the procedure previously proved (see also Theorem 1.2).

2. Preliminaries

For a general reference on curves and surfaces, we refer to [8]. All the curves and surfaces considered in this paper will be assumed to be of class C^∞. Let $\alpha(s)$, $s \in I$, and $\beta(t)$, $t \in J$, be two curves in $\mathbb{R}^3$ parameterized by arc length with oriented Frenet trihedrons $\{\mathbf{t}_\alpha(s), \mathbf{n}_\alpha(s), \mathbf{b}_\alpha(s)\}$, $\{\mathbf{t}_\beta(t), \mathbf{n}_\beta(t), \mathbf{b}_\beta(t)\}$, for

every $s \in I$, $t \in J$, respectively. Throughout this paper let $\kappa_\alpha(s) > 0$ and $\kappa_\beta(t) > 0$ denote the curvatures of α and β respectively, as well as, $\tau_\alpha(s)$ and $\tau_\beta(t)$ the torsions. Let $\{\alpha(s) + \beta(t) : s \in I, t \in J\} \subset \mathbb{R}^3$ be the set obtained by the sum of the curves α and β. Then S is a regular (translation) surface, and $\Psi(s,t) = \alpha(s) + \beta(t)$ is a parameterization of S, if $\mathbf{t}_\alpha(s) \times \mathbf{t}_\beta(t) \neq 0$ for all $(s,t) \in I \times J$, where $\times$ represents the vector product of $\mathbb{R}^3$: throughout the paper, we will make this assumption. Recall that the parametric curves $t = ct$ are congruent and translations of $\alpha(s)$. Hence, they have the same curvature and torsion at corresponding points (similarly for the parametric curves $s = ct$).

We calculate the Gauss curvature and the mean curvature of S. For notational convenience we omit the dependence on s and t of the function which are implicitly understood. The derivatives of order 1 of Ψ are $\Psi_s = \mathbf{t}_\alpha$, $\Psi_t = \mathbf{t}_\beta$, with $\Psi_s \times \Psi_t \neq 0$. Let $\phi(s,t)$, $0 < \phi(s,t) < \pi$, be the angle that $\mathbf{t}_\alpha(s)$ makes with $\mathbf{t}_\beta(t)$ at point $\Psi(z,t)$, that is,

$$\cos\phi(s,t) = \langle \mathbf{t}_\alpha(s), \mathbf{t}_\beta(t) \rangle,$$

where $\langle,\rangle$ stands for the usual scalar product of $\mathbb{R}^3$. Then the coefficients of the first fundamental form in the basis $\{\Psi_s, \Psi_t\}$ are

$$E = 1, \ F = \cos\phi, \ G = 1,$$

and the unit normal vector $N(s,t)$ at $\Psi(s,t)$ is

$$N(s,t) = \frac{\mathbf{t}_\alpha(s) \times \mathbf{t}_\beta(t)}{\sin\phi(s,t)}.$$

The derivatives of Ψ of order 2 are $\Psi_{ss} = \kappa_\alpha \mathbf{n}_\alpha$, $\Psi_{st} = 0$ and $\Psi_{tt} = \kappa_\beta \mathbf{n}_\beta$, hence the coefficients of the second fundamental form are

$$l = -\frac{\kappa_\alpha}{\sin\phi}\langle \mathbf{b}_\alpha, \mathbf{t}_\beta \rangle, \ m = 0, \ n = \frac{\kappa_\beta}{\sin\phi}\langle \mathbf{t}_\alpha, \mathbf{b}_\beta \rangle.$$

The Gaussian curvature K and the mean curvature H of S are

$$K = -\frac{\kappa_\alpha \kappa_\beta}{\sin^4\phi}\langle \mathbf{b}_\alpha, \mathbf{t}_\beta \rangle \langle \mathbf{t}_\alpha, \mathbf{b}_\beta \rangle, \quad H = \frac{-\kappa_\alpha \langle \mathbf{b}_\alpha, \mathbf{t}_\beta \rangle + \kappa_\beta \langle \mathbf{t}_\alpha, \mathbf{b}_\beta \rangle}{2\sin^3\phi}. \quad (2.1)$$

Consequently, S is a minimal surface ($H = 0$) if and only if

$$\kappa_\alpha \langle \mathbf{b}_\alpha, \mathbf{t}_\beta \rangle = \kappa_\beta \langle \mathbf{t}_\alpha, \mathbf{b}_\beta \rangle \quad \text{for all } s \in I, \ t \in J. \quad (2.2)$$

Remark 2.1. If the generating curves α and β are not parameterized by arc length, then the minimality condition $H = 0$ is equivalent to

$$|\beta'(t)|^2 \langle \alpha'(s) \times \alpha''(s), \beta'(t) \rangle = |\alpha'(s)|^2 \langle \alpha'(s), \beta'(t) \times \beta''(t) \rangle \quad (2.3)$$

for all $s \in I$, $t \in J$.

The following curve will be useful later,

$$\alpha(u) = \left(u, 0, -\frac{1}{c}\log(\cos(cu)) \right), \quad u \in \left(\frac{-\pi}{2c}, \frac{\pi}{2c} \right), \quad (2.4)$$

where c is a positive constant, which is nothing more than the generating curve of the Scherk surface $\mathcal{S}_{\pi/2}$. Its curvature, with parameter the arc length s and origin $u = 0$, is

$$\kappa_\alpha(s) = \frac{2ce^{cs}}{1 + e^{2cs}}. \tag{2.5}$$

3. Auxiliary Results

In this section we characterize any minimal translation surface by the two relations between the curvature and the torsion of the generating curves. and we prove some known results with alternative proofs. Let $S \subset \mathbb{R}^3$ be a minimal translation surface with parameterization $\Psi(s, t) = \alpha(s) + \beta(t)$ where the generating curves α and β are parameterized by arc length s and t. Having in mind the Frenet equations, we take the derivative with respect to s of (2.2) and then dividing by κ_α, we arrive at

$$\left\langle -\tau_\alpha \mathbf{n}_\alpha + \frac{\kappa'_\alpha}{\kappa_\alpha} \mathbf{b}_\alpha, \mathbf{t}_\beta \right\rangle = \langle \mathbf{n}_\alpha, \kappa_\beta \mathbf{b}_\beta \rangle. \tag{3.1}$$

Differentiating (3.1) with respect to s again and taking into account (2.2) and (3.1), we obtain

$$\left\langle \kappa_\alpha \tau_\alpha \mathbf{t}_\alpha - \left(\frac{\kappa'_\alpha}{\kappa_\alpha} \tau_\alpha + \tau'_\alpha \right) \mathbf{n}_\alpha + \left(\left(\frac{\kappa'_\alpha}{\kappa_\alpha} \right)' + \kappa_\alpha^2 - \tau_\alpha^2 \right) \mathbf{b}_\alpha, \mathbf{t}_\beta \right\rangle$$
$$= \tau_\alpha \langle \mathbf{b}_\alpha, \kappa_\beta \mathbf{b}_\beta \rangle. \tag{3.2}$$

In the same way, for the curve β we have

$$\tau_\beta \langle \kappa_\alpha \mathbf{b}_\alpha, \mathbf{b}_\beta \rangle = \left\langle \mathbf{t}_\alpha, \kappa_\beta \tau_\beta \mathbf{t}_\beta - \left(\frac{\kappa'_\beta}{\kappa_\beta} \tau_\beta + \tau'_\beta \right) \mathbf{n}_\beta + \left(\left(\frac{\kappa'_\beta}{\kappa_\beta} \right)' + \kappa_\beta^2 - \tau_\beta^2 \right) \mathbf{b}_\beta \right\rangle. \tag{3.3}$$

Once obtained the above formulae, and for the completeness of this work, we insert in this section the result proved in [2] with an alternative proof.

Proposition 3.1. *Let S be a non planar minimal translation surface. Assume that one, say α, of the generating curves is a plane curve. Then:*

1. *The curvature κ_α of α satisfies the autonomous ODE*

$$\left(\frac{y'}{y} \right)' + y^2 = 0. \tag{3.4}$$

2. *The curve α is a rigid motion of the curve (2.4).*
3. *The other generating curve β is also a plane curve and S is a surface of Scherk type.*

Proof.   1. From (3.2), because of $\tau_\alpha = 0$, we have

$$\left(\left(\frac{\kappa'_\alpha}{\kappa_\alpha}\right)' + \kappa^2_\alpha\right)\langle \mathbf{b}_\alpha, \mathbf{t}_\beta\rangle = 0$$

for all s and t. If $\langle \mathbf{b}_\alpha, \mathbf{t}_\beta\rangle = 0$ on an open set, then from (2.1) we have $K = 0$. Since $H = K = 0$, then S is a plane, a contradiction. So, we must have

$$\left(\frac{\kappa'_\alpha}{\kappa_\alpha}\right)' + \kappa^2_\alpha = 0.$$

2. The general solution of the autonomous ODE (3.4) is

$$\kappa_\alpha(t) = \frac{2ce^{\pm(ct+c_1)}}{1 + e^{\pm 2(ct+c_1)}},$$

where $c > 0$, c_1 are constants. After the change $s = \pm(ct + c_1)$, we see that the curvature of α is the same than (2.5). From the fundamental theorem of plane curves, the curve α is a rigid motion of the curve (2.4).

3. After a rigid motion, we may suppose that α is as in (2.4). Let $\beta(v) = (\beta_1(v), \beta_2(v), \beta_3(v))$ denote the other generating curve parameterized by arc length v. Then the minimality condition (2.3) gives

$$(\beta'_1\beta''_2 - \beta''_1\beta'_2)\sin(cu) + (c\beta'_2 + \beta'_2\beta''_3 - \beta''_2\beta'_3)\cos(cu) = 0.$$

Since the functions $\sin(cu)$ and $\cos(cu)$ are linearly independent, we deduce

$$\beta'_1\beta''_2 - \beta''_1\beta'_2 = 0, \quad c\beta'_2 + \beta'_2\beta''_3 - \beta''_2\beta'_3 = 0. \tag{3.5}$$

Combining both equations, we have $\beta'_2(\beta''_1\beta'_3 - c\beta'_1 - \beta'_1\beta''_3) = 0$. If β_2 is a constant function, then β is a plane curve. On the contrary, $\beta''_1\beta'_3 = c\beta'_1 + \beta'_1\beta''_3$. From the first equation in (3.5), we obtain $\beta'_1\beta'''_2 = \beta'''_1\beta'_2$. Then the mixed product $(\beta', \beta'', \beta''')$ is

$$(\beta', \beta'', \beta''') = -c\beta'''_1\beta'_2 - \beta'''_2(\beta'_1\beta''_3 - \beta''_1\beta'_3)$$
$$= \beta'''_2(c\beta'_1 + \beta'_1\beta''_3 - \beta''_1\beta'_3) = 0.$$

This implies that $\tau_\beta = 0$ and β is planar. Now, according to the item 2 of the proposition, β is, up to a rigid motion, the curve parameterized in (2.4). Set $\beta(v) = A\sigma(v)$, where A is an orthogonal matrix, $\sigma(v) = (v, 0, -\frac{1}{d}\log\cos(dv))$ and $d > 0$ is a positive constant. Applying the minimality condition (2.3) again, we have

$$c\left(a_{21} + a_{23}\frac{\sin(dv)}{\cos(dv)}\right)$$
$$= d\left(a_{23}a_{31} - a_{21}a_{33} + (a_{13}a_{21} - a_{11}a_{23})\frac{\sin(cu)}{\cos(cu)}\right).$$

Due to the linear independence, in the first step, of $\cos(cu)$, $\sin(cu)$, and then of $\cos(dv)$, $\sin(dv)$, we deduce $a_{23} = 0$ and

$$a_{13}a_{21} = 0 \quad ca_{21} + da_{21}a_{33} = 0.$$

In case $a_{21} = 0$, and using that A is orthogonal, it follows that $a_{22} = \pm 1$ and it is not difficult to see that β is a curve contained in the xz-plane, the same that α: this implies that S is a plane, a contradiction. Thus $a_{21} \neq 0$ and $a_{13} = 0$ and $c = -da_{33}$. Using that A is orthogonal, then $a_{31} = a_{32} = 0$ and $a_{33} = \pm 1$. In particular, because c and d are positive, we find $a_{33} = -1$. Definitively, we have two possibilities for the matrix A, namely,

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In both cases, the parameterization $\Psi(s,t)$ is

$$\Psi(u,v) = \alpha(u) + A\sigma(v) = \left(u + v\cos\theta, v\sin\theta, \frac{1}{c}\log\frac{\cos(cv)}{\cos(cu)} \right),$$

and S is the surface $\mathcal{S}_\theta$ belonging to the Scherk family. $\square$

Recall that the helicoid is a minimal translation surface obtained as the sum of a circular helix with itself. We prove that this is consequence of the following result (see also [7, Cor. 3.4]).

Theorem 3.2. *Let S be a minimal translation surface. If one of the generating curves is a circular helix, then the other curve is a congruent circular helix and S is the helicoid.*

Proof. Assume that the generating curve α of S is the circular helix

$$\alpha(s) = (a\cos\varphi(s), a\sin\varphi(s), b\varphi(s)),$$

where $\varphi(s) = s/\sqrt{a^2 + b^2}$, $a > 0$, $b \neq 0$ are two constants. Then $\kappa_\alpha = a/(a^2 + b^2)$, $\tau_\alpha = b/(a^2 + b^2)$ and

$$\mathbf{t}_\alpha(s) = \frac{1}{\sqrt{a^2 + b^2}} \left(-a\sin\varphi(s), a\cos\varphi(s), b \right)$$

$$\mathbf{b}_\alpha(s) = \frac{1}{\sqrt{a^2 + b^2}} \left(b\sin\varphi(s), -b\cos\varphi(s), a \right).$$

If $\beta(t) = (\beta_1(t), \beta_2(t), \beta_3(t))$ is the other generating curve parameterized by arc length t, then $\kappa_\beta = |\beta''| = |\beta' \times \beta''|$, $\mathbf{t}_\beta = (\beta_1', \beta_2', \beta_3')$ and

$$\mathbf{b}_\beta = \frac{1}{|\beta' \times \beta''|} ((\beta' \times \beta'')_1, (\beta' \times \beta'')_2, (\beta' \times \beta'')_3).$$

Applying the minimality condition (2.2), it follows or equivalently,

$$\frac{a}{a^2+b^2}\left(\beta_1'b\sin\varphi(s)-\beta_2'b\cos\varphi(s)+a\beta_3'\right)$$
$$=\left(-a(\beta'\times\beta'')_1\sin\varphi(s)+a(\beta'\times\beta'')_2\cos\varphi(s)+b(\beta'\times\beta'')_3\right).$$

Since the functions $\{1,\cos\varphi(s),\sin\varphi(s)\}$ are linearly independent, we derive

$$\frac{b}{a^2+b^2}\beta_1'=-(\beta'\times\beta'')_1=-\beta_2'\beta_3''+\beta_2''\beta_3'$$
$$\frac{b}{a^2+b^2}\beta_2'=-(\beta'\times\beta'')_2=-\beta_3'\beta_1''+\beta_3''\beta_1' \qquad (3.6)$$
$$\frac{a^2}{a^2+b^2}\beta_3'=b(\beta'\times\beta'')_3=b(\beta_1'\beta_2''-\beta_1''\beta_2').$$

Multiplying the first and second of (3.6) by β_1' and β_2' respectively, we deduce

$$\frac{b}{a^2+b^2}(\beta_1'^2+\beta_2'^2)=\beta_3'(\beta_1'\beta_2''-\beta_1''\beta_2')=\frac{a^2\beta_3'^2}{b(a^2+b^2)},$$

where in the last identity we have used the third of (3.6). Since $\beta_1'^2+\beta_2'^2=1-\beta_3'^2$, then $\beta_3'^2=b^2/(a^2+b^2)$, hence

$$\beta_1'^2+\beta_2'^2=\frac{a^2}{a^2+b^2}.$$

Without loss of generality, we take $\beta_3'=b/\sqrt{a^2+b^2}$. Then the images of tangents of α and β will be lying to the same hemisphere. Then we must have

$$\beta_1'(t)=-\frac{a}{\sqrt{a^2+b^2}}\sin\varphi(t),\quad \beta_2'=\frac{a}{\sqrt{a^2+b^2}}\cos\varphi(t)$$

and thus, up to a translation, $\beta(t)=(a\cos\varphi(t),a\sin\varphi(t),b\varphi(t))$ and β coincides with α. $\square$

As a consequence of Proposition 3.1, from now on we may suppose that the generating curves α and β are non planar. We need to introduce the following notation. For a non plane curve parameterized by arc length with curvature κ and torsion τ, we set

$$R=\frac{\kappa'}{\kappa}+\frac{\tau'}{\tau},\quad \Sigma=\left(\frac{\kappa'}{\kappa}\right)'+\kappa^2-\tau^2. \qquad (3.7)$$

The subscript α or β in R and Σ indicates that we are working in the corresponding curve α or β.

We now have the following key result.

Proposition 3.3. *If* $\Psi(s,t)=\alpha(s)+\beta(t)$ *is a minimal translation surface, then*

$$\kappa_\alpha^2\tau_\alpha=c_1\neq 0,\ \kappa_\beta^2\tau_\beta=\bar{c}_1\neq 0,\ \frac{\Sigma_\alpha}{\tau_\alpha}-\tau_\alpha=c_2,\ \frac{\Sigma_\beta}{\tau_\beta}-\tau_\beta=\bar{c}_2, \qquad (3.8)$$

where c_1, c_2, $\bar{c}_1$ *and* $\bar{c}_2$ *are constants.*

Proof. Dividing (3.2) by τ_α, we have

$$\left\langle \kappa_\alpha \mathbf{t}_\alpha - R_\alpha \mathbf{n}_\alpha + \frac{\Sigma_\alpha}{\tau_\alpha} \mathbf{b}_\alpha, \mathbf{t}_\beta \right\rangle = \langle \mathbf{b}_\alpha, \kappa_\beta \mathbf{b}_\beta \rangle.$$

By differentiation this equation with respect to s and taking into account the Frenet equations and (3.1), we arrive at

$$\langle u, \mathbf{t}_\beta \rangle = 0, \tag{3.9}$$

where we have set

$$u = (\kappa_\alpha R_\alpha + \kappa_\alpha')\mathbf{t}_\alpha + (\kappa_\alpha^2 - \tau_\alpha^2 - \Sigma_\alpha - R_\alpha')\mathbf{n}_\alpha$$
$$+ \left(\left(\frac{\Sigma_\alpha}{\tau_\alpha} \right)' - \tau_\alpha R_\alpha + \frac{\tau_\alpha}{\kappa_\alpha} \kappa_\alpha' \right) \mathbf{b}_\alpha. \tag{3.10}$$

Differentiating (3.9) with respect to t and because $\kappa_\beta > 0$, we have

$$\langle u, \mathbf{n}_\beta \rangle = 0. \tag{3.11}$$

Finally, we differentiate again (3.11) with respect to t and, taking into account (3.9) and $\tau_\beta \neq 0$, we have

$$\langle u, \mathbf{b}_\beta \rangle = 0. \tag{3.12}$$

From Eqs. (3.9), (3.11) and (3.12), we find $u = 0$. Hence we deduce from (3.10)

$$\left. \begin{aligned} \kappa_\alpha R_\alpha + \kappa_\alpha' &= 0 \\ \Sigma_\alpha + R_\alpha' - \kappa_\alpha^2 + \tau_\alpha^2 &= 0 \\ \left(\frac{\Sigma_\alpha}{\tau_\alpha} \right)' - \tau_\alpha R_\alpha + \frac{\kappa_\alpha'}{\kappa_\alpha} \tau_\alpha &= 0. \end{aligned} \right\} \tag{3.13}$$

The first of (3.13) implies

$$R_\alpha = -\frac{\kappa_\alpha'}{\kappa_\alpha}. \tag{3.14}$$

By the definition of R_α in (3.7), we derive $2\kappa_\alpha' \tau_\alpha + \kappa_\alpha \tau_\alpha' = 0$. Then there exists a constant $c_1 \neq 0$ such that $\kappa_\alpha^2 \tau_\alpha = c_1$. By the definition of Σ_α in (3.7), the second relation of (3.13) is $(\kappa_\alpha'/\kappa_\alpha)' + R_\alpha' = 0$, which is valid because of (3.14). The third of (3.13) and the definition of R_α give $(\Sigma_\alpha/\tau_\alpha)' - \tau_\alpha' = 0$, hence

$$\frac{\Sigma_\alpha}{\tau_\alpha} - \tau_\alpha = c_2$$

for some constant c_2. In a similar way, we deduce the corresponding results for the curve β by using (3.3). $\square$

Remark 3.4. In conclusion, with successive differentiations of (2.2) with respect to s, ss, t, tt, ts, sst, tts and $ttss$ we, respectively, find

$$\left.\begin{array}{r}
\langle -\tau_\alpha \mathbf{n}_\alpha - R_\alpha \mathbf{b}_\alpha, \mathbf{t}_\beta \rangle = \langle \mathbf{n}_\alpha, \kappa_\beta \mathbf{t}_\beta \rangle \\[2mm]
\left\langle \kappa_\alpha \mathbf{t}_\alpha - R_\alpha \mathbf{n}_\alpha + \dfrac{\Sigma_\alpha}{\tau_\alpha} \mathbf{b}_\alpha, \mathbf{t}_\beta \right\rangle = \langle \mathbf{b}_\alpha, \kappa_\beta \mathbf{b}_\beta \rangle \\[2mm]
\langle \kappa_\alpha \mathbf{b}_\alpha, \mathbf{n}_\beta \rangle = \langle \mathbf{t}_\alpha, -\tau_\beta \mathbf{n}_\beta - R_\beta \mathbf{b}_\beta \rangle \\[2mm]
\langle \kappa_\alpha \mathbf{b}_\alpha, \mathbf{b}_\beta \rangle = \left\langle \mathbf{t}_\alpha, \kappa_\beta \mathbf{t}_\beta - R_\beta \mathbf{n}_\beta + \dfrac{\Sigma_\beta}{\tau_\beta} \mathbf{b}_\beta \right\rangle \\[2mm]
\langle -\tau_\alpha \mathbf{n}_\alpha - R_\alpha \mathbf{b}_\alpha, \mathbf{n}_\beta \rangle = \langle \mathbf{n}_\alpha, -\tau_\beta \mathbf{n}_\beta - R_\beta \mathbf{b}_\beta \rangle \\[2mm]
\left\langle \kappa_\alpha \mathbf{t}_\alpha - R_\alpha \mathbf{n}_\alpha + \dfrac{\Sigma_\alpha}{\tau_\alpha} \mathbf{b}_\alpha, \mathbf{n}_\beta \right\rangle = \langle \mathbf{b}_\alpha, -\tau_\beta \mathbf{n}_\beta - R_\beta \mathbf{b}_\beta \rangle \\[2mm]
\langle -\tau_\alpha \mathbf{n}_\alpha - R_\alpha \mathbf{b}_\alpha, \mathbf{b}_\beta \rangle = \left\langle \mathbf{n}_\alpha, \kappa_\beta \mathbf{t}_\beta - R_\beta \mathbf{n}_\beta + \dfrac{\Sigma_\beta}{\tau_\beta} \mathbf{b}_\beta \right\rangle \\[2mm]
\left\langle \kappa_\alpha \mathbf{t}_\alpha - R_\alpha \mathbf{n}_\alpha + \dfrac{\Sigma_\alpha}{\tau_\alpha} \mathbf{b}_\alpha, \mathbf{b}_\beta \right\rangle = \left\langle \mathbf{b}_\alpha, \kappa_\beta \mathbf{t}_\beta - R_\beta \mathbf{n}_\beta + \dfrac{\Sigma_\beta}{\tau_\beta} \mathbf{b}_\beta \right\rangle.
\end{array}\right\} \qquad (3.15)$$

Another useful result is the following.

Proposition 3.5. *Let α be a curve in $\mathbb{R}^3$ parameterized by arc length with curvature $\kappa_\alpha > 0$ and torsion $\tau_\alpha \neq 0$. If $\sigma_1 \neq 0$ and σ_2 are two constants such that*

$$\kappa_\alpha^2 \tau_\alpha = \sigma_1 \neq 0 \quad and \quad \frac{\Sigma_\alpha}{\tau_\alpha} - \tau_\alpha = \sigma_2, \qquad (3.16)$$

then κ_α is a positive solution of the autonomous ODE

$$y'^2 + y^4 + \sigma_3 y^2 + \frac{\sigma_1^2}{y^2} + \sigma_1 \sigma_2 = 0 \qquad (3.17)$$

for some constant σ_3.

Conversely, let $c_1 \neq 0$, c_2 be two constant. Then for any positive and non-constant solution $\kappa(s)$ of (3.16) and choosing $\tau(s) = \sigma_1/\kappa(s)^2$, a curve α parameterized by the arc length s with curvature and torsion κ and τ, respectively, satisfies

$$\frac{\Sigma_\alpha}{\tau_\alpha} - \tau_\alpha = \sigma_2, \quad \Sigma_\alpha + R_\alpha^2 + \kappa_\alpha^2 = -\sigma_3 \qquad (3.18)$$

for some constant σ_3.

Proof. The second identity of (3.16), by taking into account the first one, becomes

$$\left(\frac{\kappa'}{\kappa} \right)' + \kappa^2 - \frac{2\sigma_1^2}{\kappa^4} - \frac{\sigma_1 \sigma_2}{\kappa^2} = 0.$$

We now obtain a first integral of this equation. Set $w = \log \kappa$, that is, $\kappa = e^w$. Since $w' = \kappa'/\kappa$, we find

$$w'' + e^{2w} - 2\sigma_1^2 e^{-4w} - \sigma_1 \sigma_2 e^{-2w} = 0.$$

In order to solve this ODE, put $z = w'$ and consider $z = z(w)$. Because of $w'' = \frac{1}{2}\frac{d(z^2)}{dw}$, we have

$$\frac{d(z^2)}{dw} + 2e^{2w} - 4\sigma_1^2 e^{-4w} - 2\sigma_1\sigma_2 e^{-2w} = 0.$$

An integration of this ODE leads to

$$(w')^2 = z^2 = -e^{2w} - \sigma_1^2 e^{-4w} - \sigma_1\sigma_2 e^{-2w} - \sigma_3$$

for some constant σ_3. Since $w = \log\kappa$,

$$\left(\frac{\kappa'}{\kappa}\right)^2 + \kappa^2 + \sigma_1^2\kappa^{-4} + \sigma_1\sigma_2\kappa^{-2} + \sigma_3 = 0,$$

or equivalently,

$$\kappa'^2 + \kappa^4 + \sigma_3\kappa^2 + \sigma_1^2\kappa^{-2} + \sigma_1\sigma_2 = 0.$$

This proves (3.17).

For the converse of the proposition, let $\kappa(s)$ be a positive and non-constant solution of (3.17) and put $\tau(s) = \sigma_1/\kappa^2(s)$. Consider α a curve parameterized by arc length with curvature κ and torsion τ. From (3.17), it follows

$$\left(\frac{\kappa'_\alpha}{\kappa_\alpha}\right)^2 + \kappa_\alpha^2 + \sigma_1^2\kappa_\alpha^{-4} + \sigma_1\sigma_2\kappa_\alpha^{-2} + \sigma_3 = 0. \tag{3.19}$$

Differentiating with respect to s and since $\kappa'_\alpha \neq 0$, we obtain

$$\left(\frac{\kappa'_\alpha}{\kappa_\alpha}\right)' + \kappa_\alpha^2 - 2\sigma_1^2\kappa_\alpha^{-4} - \sigma_1\sigma_2\kappa_\alpha^{-2} = 0.$$

Since $\tau_\alpha = \sigma_1/\kappa_\alpha^2$, then

$$\left(\frac{\kappa'_\alpha}{\kappa_\alpha}\right)' + \kappa_\alpha^2 - 2\tau_\alpha^2 - \sigma_2\tau_\alpha = 0.$$

This equation implies the first of (3.18). Since $\sigma_1^2\kappa_\alpha^{-4} + \sigma_1\sigma_2\kappa_\alpha^{-2} = \tau_\alpha^2 + \sigma_2\tau_\alpha = \Sigma_\alpha$, we derive from (3.19) the second identity of (3.18). $\qquad\square$

Remark 3.6. With the notation of Proposition 3.3, the generating curves α, β of a minimal translation surface $\Psi(s,t) = \alpha(s) + \beta(t)$ satisfy the conditions of Proposition 3.5 with $\sigma_1 = c_1$, $\sigma_2 = c_2$ and $\sigma_1 = \bar{c}_1$, $\sigma_2 = \bar{c}_2$, respectively. So we find

$$R_\alpha^2 + \kappa_\alpha^2 + \tau_\alpha^2 + c_2\tau_\alpha + c_3 = 0, \quad R_\beta^2 + \kappa_\beta^2 + \tau_\beta^2 + \bar{c}_2\tau_\beta + \bar{c}_3 = 0,$$

for some constants c_3 and $\bar{c}_3$.

Motivated by Eq. (2.2) and the set of identities (3.15), we define the functions $V_i = V_i(s)$, $W_i = W_i(t)$, $1 \leq i \leq 3$ by

$$\left. \begin{aligned} V_1 &= \kappa_\alpha \mathbf{b}_\alpha \\ V_2 &= -\tau_\alpha \mathbf{n}_\alpha - R_\alpha \mathbf{b}_\alpha \\ V_3 &= \kappa_\alpha \mathbf{t}_\alpha - R_\alpha \mathbf{n}_\alpha + \frac{\Sigma_\alpha}{\tau_\alpha} \mathbf{b}_\alpha \end{aligned} \right\}, \qquad \left. \begin{aligned} W_1 &= \kappa_\beta \mathbf{b}_\beta \\ W_2 &= -\tau_\beta \mathbf{n}_\beta - R_\beta \mathbf{b}_\beta \\ W_3 &= \kappa_\beta \mathbf{t}_\beta - R_\beta \mathbf{n}_\beta + \frac{\Sigma_\beta}{\tau_\beta} \mathbf{b}_\beta. \end{aligned} \right\} \quad (3.20)$$

It is not difficult to see that V_i an W_i satisfy the following equations of Frenet type:

$$\left. \begin{aligned} V_1' &= \kappa_\alpha V_2 \\ V_2' &= -\kappa_\alpha V_1 + \tau_\alpha V_3 \\ V_3' &= -\tau_\alpha V_2 \end{aligned} \right\} \qquad \left. \begin{aligned} W_1' &= \kappa_\beta W_2 \\ W_2' &= -\kappa_\beta W_1 + \tau_\beta W_3 \\ W_3' &= -\tau_\beta W_2, \end{aligned} \right\}$$

respectively.

Also it is immediate from (3.20) that their mixed products are

$$(V_1, V_2, V_3) = \kappa_\alpha^2 \tau_\alpha = c_1, \quad (W_1, W_2, W_3) = \kappa_\beta^2 \tau_\beta = \bar{c}_1.$$

With the above notation, the identity (2.2) and the eight relations (3.15) are written, respectively, as

$$\left. \begin{aligned} \langle V_1, \mathbf{t}_\beta \rangle &= \langle \mathbf{t}_\alpha, W_1 \rangle, \\ \langle V_2, \mathbf{t}_\beta \rangle &= \langle \mathbf{n}_\alpha, W_1 \rangle \\ \langle V_3, \mathbf{t}_\beta \rangle &= \langle \mathbf{b}_\alpha, W_1 \rangle, \\ \langle V_1, \mathbf{n}_\beta \rangle &= \langle \mathbf{t}_\alpha, W_2 \rangle \\ \langle V_1, \mathbf{b}_\beta \rangle &= \langle \mathbf{t}_\alpha, W_3 \rangle, \\ \langle V_2, \mathbf{n}_\beta \rangle &= \langle \mathbf{n}_\alpha, W_2 \rangle \\ \langle V_3, \mathbf{n}_\beta \rangle &= \langle \mathbf{b}_\alpha, W_2 \rangle, \\ \langle V_2, \mathbf{b}_\beta \rangle &= \langle \mathbf{n}_\alpha, W_3 \rangle \\ \langle V_3, \mathbf{b}_\beta \rangle &= \langle \mathbf{b}_\alpha, W_3 \rangle. \end{aligned} \right\} \qquad (3.21)$$

All above facts and formulas are needed in order to prove the main results in the next section.

4. Classification and Construction Results

Let $S \subset \mathbb{R}^3$ be a minimal translation surface with parameterization $\Psi(s,t) = \alpha(s) + \beta(t)$, where we suppose α and β parameterized by arc length. Motivated by the relations (3.20), for each point $\alpha(s)$ and $\beta(t)$, we define a set of linear transformations $L_{\alpha(s)}, L_{\beta(t)} : \mathbb{R}^3 \to \mathbb{R}^3$ with matrices

$$L_{\alpha(s)} = \begin{pmatrix} 0 & 0 & \kappa_\alpha(s) \\ 0 & -\tau_\alpha(s) & -R_\alpha(s) \\ \kappa_\alpha(s) & -R_\alpha(s) & \frac{\Sigma_\alpha}{\tau_\alpha}(s) \end{pmatrix}, \quad L_{\beta(t)} = \begin{pmatrix} 0 & 0 & \kappa_\beta(t) \\ 0 & -\tau_\beta(t) & -R_\beta(t) \\ \kappa_\beta(t) & -R_\beta(t) & \frac{\Sigma_\beta}{\tau_\beta}(t) \end{pmatrix}$$

with respect to the basis $\{\mathbf{t}_\alpha(s), \mathbf{n}_\alpha(s), \mathbf{b}_\alpha(s)\}$ and $\{\mathbf{t}_\beta(t), \mathbf{n}_\beta(t), \mathbf{b}_\beta(t)\}$, respectively.

Since the matrix $L_{\alpha(s)}$ is symmetric with respect to an orthonormal basis, the linear map $L_{\alpha(s)}$ is self-adjoint for all s. Its characteristic equation is

$$-\lambda^3 + \left(\frac{\Sigma_\alpha}{\tau_\alpha} - \tau_\alpha\right)\lambda^2 + (\Sigma_\alpha + R_\alpha^2 + \kappa_\alpha^2)\lambda + \kappa_\alpha^2 \tau_\alpha = 0,$$

or, because of (3.8) and (3.18), $-\lambda^3 + c_2\lambda^2 - c_3\lambda + c_1 = 0$. Thus the real eigenvalues λ_1, λ_2 and λ_3 are three constants independent of s and satisfy

$$\left.\begin{array}{r} \lambda_1 + \lambda_2 + \lambda_3 = c_2 \\ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = c_3 \\ \lambda_1\lambda_2\lambda_3 = c_1 \end{array}\right\}. \tag{4.1}$$

Analogously the real eigenvalues μ_1, μ_2 and μ_3 of $L_{\beta(t)}$ are constants and satisfy

$$\left.\begin{array}{r} \mu_1 + \mu_2 + \mu_3 = \bar{c}_2 \\ \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 = \bar{c}_3 \\ \mu_1\mu_2\mu_3 = \bar{c}_1 \end{array}\right\}. \tag{4.2}$$

Remark 4.1. Since the cubic equation

$$-\lambda^3 + c_2\lambda^2 - c_3\lambda + c_1 = 0$$

has three real roots, its discriminant

$$\Delta = 18c_1c_2c_3 - 4c_1c_2^3 + c_2^2c_3^2 - 4c_3^3 - 27c_2^2$$

is non-negative. In the case where $\Delta = 0$, the cubic equation has a multiple root.

Now we prove the key property that all transformations $L_{\alpha(s)}$ and $L_{\beta(t)}$ coincide for any s and t.

Proposition 4.2. *Let $\Psi(s,t) = \alpha(s) + \beta(t)$ be a minimal translation surface. Then $L_{\alpha(s)} = L_{\beta(t)}$ for all $s \in I$ and $t \in J$.*

Proof. We shall prove that $L_{\beta(t)}$ is the adjoint of $L_{\alpha(s)}$ for any s and t, and since $L_{\alpha(s)}$ is self-adjoint, we conclude $L_{\alpha(s)} = L_{\beta(t)}$, proving the result.

Thus we need to show that $\langle L_{\alpha(s)}(v), w\rangle = \langle v, L_{\beta(t)}(w)\rangle$ for all $v, w \in \mathbb{R}^3$. Let

$$v = a_1\mathbf{t}_\alpha(s) + a_2\mathbf{n}_\alpha(s) + a_3\mathbf{b}_\alpha(s), \quad w = b_1\mathbf{t}_\beta(t) + b_2\mathbf{n}_\beta(t) + b_3\mathbf{b}_\beta(t),$$

where $a_i = a_i(s), b_i = b_i(t) \in \mathbb{R}$. Then

$$\langle L_\alpha(v), w\rangle = \langle a_1 L_\alpha(\mathbf{t}_\alpha) + a_2 L_\alpha(\mathbf{n}_\alpha) + a_3 L_\alpha(\mathbf{b}_\alpha), b_1\mathbf{t}_\beta + b_2\mathbf{n}_\beta + b_3\mathbf{b}_\beta\rangle,$$

where we omit the dependence on s and t. In the right hand side of this identity appears nine summands. For each on them, we use the definition of L_α and the relations (3.21). For instance, we find for the first summand that

$$\langle L_\alpha(\mathbf{t}_\alpha), \mathbf{t}_\beta \rangle = \langle \kappa_\alpha \mathbf{b}_\alpha, \mathbf{t}_\beta \rangle = \langle \mathbf{t}_\alpha, \kappa_\beta \mathbf{b}_\beta \rangle = \langle \mathbf{t}_\alpha, L_\beta(\mathbf{t}_\beta) \rangle.$$

On applying this argument summand-by-summand, we see that

$$\langle L_\alpha(v), w \rangle = \langle v, b_1 L_\beta(\mathbf{t}_\beta) + b_2 L_\beta(\mathbf{n}_\beta) + b_3 L_\beta(\mathbf{b}_\beta) \rangle = \langle v, L_\beta(w) \rangle,$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Because of $L_{\alpha(s)} = L_{\beta(t)}$ for all s, t, we conclude that the eigenvalues of $L_{\alpha(s)}$ and $L_{\beta(t)}$ coincide. Let λ_1, λ_2 and λ_3 denote the three eigenvalues. It follows from (4.1) and (4.2) that $c_i = \bar{c}_i$, $1 \leq i \leq 3$. Moreover, $L_{\alpha(s)}$, $L_{\beta(t)}$ have a common eigensystem independent of s and t, for all $s \in I$, $t \in J$.

Consider now the common eigensystem of L_α and L_β as an orthonormal reference system. With respect to this system, we write α in coordinates, say, $\alpha(s) = (\alpha_1(s), \alpha_2(s), \alpha_3(s))$, being s the arc length. Then

$$\mathbf{t}_\alpha(s) = \alpha'(s) = (\alpha_1'(s), \alpha_2'(s), \alpha_3'(s)),$$

and

$$\kappa_\alpha(s)\mathbf{b}_\alpha(s) = L_\alpha(\mathbf{t}_\alpha(s)) = (\lambda_1 \alpha_1'(s), \lambda_2 \alpha_2'(s), \lambda_3 \alpha_3'(s)).$$

The identities $\langle \mathbf{t}_\alpha, \mathbf{t}_\alpha \rangle = 1$ and $\langle \mathbf{t}_\alpha, \kappa_\alpha \mathbf{b}_\alpha \rangle = 0$ write as

$$\alpha_1'(s)^2 + \alpha_2'(s)^2 + \alpha_3'(s)^2 = 1 \qquad\qquad\qquad (4.3)$$

and

$$\lambda_1 \alpha_1'(s)^2 + \lambda_2 \alpha_2'(s)^2 + \lambda_3 \alpha_3'(s)^2 = 0, \qquad\qquad\qquad (4.4)$$

respectively. Because of the third of (4.1) and (4.4), we conclude that all λ_i, $1 \leq i \leq 3$, are non-zero and without the same sign. In the case where $c_1 > 0$, by renumbering the axis, we may choose $\lambda_1 \leq \lambda_2 < 0 < \lambda_3$. Analogously, if $c_1 < 0$, we may choose $\lambda_1 \geq \lambda_2 > 0 > \lambda_3$. Set

$$A = \sqrt{\frac{\lambda_3}{\lambda_3 - \lambda_1}}, \; B = \sqrt{\frac{\lambda_3}{\lambda_3 - \lambda_2}}.$$

Taking into account (4.3) and (4.4), we may assume that

$$\alpha_1'(s) = A \cos w(s)$$
$$\alpha_2'(s) = B \sin w(s)$$
$$\alpha_3'(s) = \sqrt{1 - A^2 \cos^2 w(s) - B^2 \sin^2 w(s)}.$$

Obviously, we have $\alpha_3'(s) \neq 0$ everywhere. We briefly write

$$\alpha'(s) = (A \cos w(s), B \sin w(s), \alpha_3'(s)).$$

In order to calculate κ_α and τ_α, we need the computations of α'' and α'''. For α'' we have

$$\alpha'' = w'\left(-A\sin w, B\cos w, \frac{(A^2 - B^2)\cos w \sin w}{\alpha'_3}\right).$$

The expression of α'' is longer, namely,

$$w''\left(-A\sin w, B\cos w, \frac{(A^2 - B^2)\cos w \sin w}{\alpha'_3}\right)$$

$$+ w'^2\left(\begin{array}{c} -A\cos w \\ -B\sin w \\ \frac{(A^2 - B^2)\left((1 - A^2\cos^2 w)\cos^2 w - (1 - B^2\sin^2 w)\sin^2 w\right)}{\alpha'^3_3} \end{array}\right).$$

Then

$$\alpha'(s) \times \alpha''(s) = \frac{w'(s)}{\alpha'_3(s)}\left(B(A^2 - 1)\cos w(s), A(B^2 - 1)\sin w(s), AB\alpha'_3(s)\right) \quad (4.5)$$

and

$$(\alpha'(s), \alpha''(s), \alpha'''(s)) = AB(1 + A^2 B^2 - A^2 - B^2)\left(\frac{w'(s)}{\alpha'_3(s)}\right)^3. \quad (4.6)$$

Since $\tau_\alpha = (\alpha'(s), \alpha''(s), \alpha'''(s))/\kappa_\alpha^2$, it follows from (3.16) that $(\alpha'(s), \alpha''(s), \alpha'''(s)) = c_1$. The computation of the right-hand side of (4.6) using the third of (4.1) yields

$$\left(\frac{w'(s)}{\alpha'_3(s)}\right)^3 = ((\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2))^{3/2},$$

or, equivalently,

$$\frac{w'(s)}{\alpha'_3(s)} = \sqrt{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}. \quad (4.7)$$

By a direct calculation from (4.5), we obtain

$$\kappa_\alpha(s)^2 = \left(\frac{w'(s)}{\alpha'_3(s)}\right)^2 (A^2 + B^2 - A^2 B^2 - 1 + \alpha'_3(s))^2,$$

which can be written as

$$\kappa_\alpha(s)^2 - w'(s)^2 = (A^2 + B^2 - A^2 B^2 - 1)\left(\frac{w'(s)}{\alpha'_3(s)}\right)^2.$$

Using the value of A and B and (4.7), the above equation reduces into $\kappa_\alpha(s)^2 - w'(s)^2 = -\lambda_1\lambda_2$, hence

$$w'(s) = \sqrt{\kappa_\alpha(s)^2 + \lambda_1\lambda_2}.$$

Similarly, the above argument applies to the curve β.

In the meantime, the curvatures $\kappa_\alpha(s)$, $\kappa_\beta(t)$ are positive solutions of the autonomous ODE

$$y'^2 + y^4 + c_3 y^2 + \frac{c_1^2}{y^2} + c_1 c_2 = 0.$$

Using the value of c_i from (4.1), this equation is equivalent to

$$y'^2 + \frac{1}{y^2}(y^2 + \lambda_1\lambda_2)(y^2 + \lambda_1\lambda_3)(y^2 + \lambda_2\lambda_3) = 0. \qquad (4.8)$$

The positive equilibrium solutions are $y_1 = \sqrt{-\lambda_1\lambda_3}$, $y_2 = \sqrt{-\lambda_2\lambda_3}$, which give stationary solutions of (4.8). So the positive solutions $\kappa_\alpha(s)$ and $\kappa_\beta(t)$ are included in the strip bounded by the values y_1, y_2 and $\kappa_\beta(t)$ is a horizontal translate of $\kappa_\alpha(s)$. That is, $\kappa_\beta(t) = \kappa_\alpha(\pm t + c_0)$, $c_0 \in \mathbb{R}$. By a reparameterization of β, we conclude that $\kappa_\beta(t) = \kappa_\alpha(t)$ and thus $\tau_\beta(t) = \tau_\alpha(t)$ from (3.8). Hence the generating curves α and β are congruent.

Summarizing, we have proved the following classification result.

Theorem 4.3. *Let $\Psi(s,t) = \alpha(s) + \beta(t)$ be a minimal translation surface with α and β parameterized by arc length. Suppose $\kappa_\alpha, \kappa_\beta > 0$ and $\tau_\alpha \neq 0, \tau_\beta \neq 0$ everywhere. Then:*

1. *There are two constants $c_1, c_2 \in \mathbb{R}$, $c_1 \neq 0$, such that*

$$\kappa_\alpha^2 \tau_\alpha = \kappa_\beta^2 \tau_\beta = c_1, \qquad \frac{\Sigma_\alpha}{\tau_\alpha} - \tau_\alpha = \frac{\Sigma_\beta}{\tau_\beta} - \tau_\beta = c_2.$$

2. *The curvature $\kappa_\alpha, \kappa_\beta$ are positive solutions of the autonomous ODE*

$$y'^2 + y^4 + c_3 y^2 + \frac{c_1^2}{y^2} + c_1 c_2 = 0$$

   for some constant c_3, and the curves α and β have the same orbit.

3. *Up to a rigid motion, we have*

$$\alpha(s) = \left(A \int^s \cos w(s), B \int^s \sin w(s), \int^s \sqrt{1 - A^2 \cos^2 w(s) - B^2 \sin^2 w(s)} \right),$$

$$\beta(s) = \left(A \int^t \cos w(t), B \int^t \sin w(t), \int^t \sqrt{1 - A^2 \cos^2 w(t) - B^2 \sin^2 w(t)} \right),$$

   where

$$A = \sqrt{\frac{\lambda_3}{\lambda_3 - \lambda_1}}, \quad B = \sqrt{\frac{\lambda_3}{\lambda_3 - \lambda_2}}$$

   and $\lambda_1 \leq \lambda_2 < 0 < \lambda_3$ (resp. $\lambda_1 \geq \lambda_2 > 0 > \lambda_3$) are the real roots of the cubic equation $-\lambda^3 + c_2\lambda^2 - c_3\lambda + c_1 = 0$ and

$$w(s) = \int^s \sqrt{\kappa_\alpha(s)^2 + \lambda_1\lambda_2}. \qquad (4.9)$$

In the sequel, we will prove one more result which is the converse of Theorem 4.3 and, by the way, it provides an useful tool for constructing minimal translation surfaces.

Theorem 4.4. *Suppose that $c_1 \neq 0$, c_2 and c_3 are constants such that the cubic equation*

$$- \lambda^3 + c_2 \lambda^2 - c_3 \lambda + c_1 = 0 \tag{4.10}$$

has three real roots λ_1, λ_2, λ_3. Consider the autonomous ODE

$$y'^2 + y^4 + c_3 y^2 + \frac{c_1^2}{y^2} + c_1 c_2 = 0 \tag{4.11}$$

and let $\kappa_\alpha(s) = \kappa_\alpha(s; c_1, c_2, c_3)$ be a positive and non-constant solution of (4.11). Denote by $\alpha(s)$ the curve parameterized by arc length s with curvature $\kappa_\alpha(s)$ and torsion $\tau_\alpha(s) = c_1/\kappa_\alpha(s)^2$. Then the translation surface $\Psi(s,t) = \alpha(s) + \alpha(t)$ is minimal.

Proof. The autonomous ODE (4.11) takes the form (4.8). Hence $\lambda_i \neq 0$ and without the same sign. In the case where $c_1 > 0$, we may choose $\lambda_1 \leq \lambda_2 < 0 < \lambda_3$ (analogously, if $c_1 < 0$, we choose $\lambda_1 \geq \lambda_2 > 0 > \lambda_3$). By the converse of Proposition 3.5, we deduce for the curve α that

$$\kappa_\alpha^2 \tau_\alpha = c_1, \quad \frac{\Sigma_\alpha}{\tau_\alpha} - \tau_\alpha = c_2, \quad \Sigma_\alpha + R_\alpha^2 + \kappa_\alpha^2 = -c_3.$$

At the point $\alpha(s)$, we define the linear transformation $L_{\alpha(s)}$ by the relations

$$\left.\begin{aligned}
L_{\alpha(s)}(\mathbf{t}_\alpha(s)) &= \kappa_\alpha(s)\mathbf{b}_\alpha(s) \\
L_{\alpha(s)}(\mathbf{n}_\alpha(s)) &= -\tau_\alpha(s)\mathbf{n}_\alpha(s) - R_\alpha(s)\mathbf{b}_\alpha(s) \\
L_{\alpha(s)}(\mathbf{b}_\alpha(s)) &= \kappa_\alpha(s)\mathbf{t}_\alpha(s) - R_\alpha(s)\mathbf{n}_\alpha(s) + \frac{\Sigma_\alpha}{\tau_\alpha}(s)\mathbf{b}_\alpha(s)
\end{aligned}\right\}. \tag{4.12}$$

The matrix of this transformation with respect to the basis $\{\mathbf{t}_\alpha(s), \mathbf{n}_\alpha(s), \mathbf{b}_\alpha(s)\}$ is symmetric and thus any $L_{\alpha(s)}$ is self-adjoint. The characteristic equation of $L_{\alpha(s)}$ is (4.10) for any s. Moreover, by differentiation of (4.12) with respect to s, and taking into account the Frenet equations, we find

$$L'_{\alpha(s)}(\mathbf{t}_\alpha(s)) = 0, \ L'_{\alpha(s)}(\mathbf{n}_\alpha(s)) = 0, \ L'_{\alpha(s)}(\mathbf{b}_\alpha(s)) = 0.$$

Thus, $L_\alpha = L_{\alpha(s)}$ is a constant transformation and has a constant eigensystem for any s. Taking the eigensystem as the reference system as in Theorem 4.3, we obtain $\alpha'(s) = (A\cos w(s), B\sin w(s), \alpha_3'(s))$, where

$$\alpha_3'(s) = \sqrt{1 - A^2 \cos^2 w(s) - B^2 \sin^2 w(s)},$$

and $A = \sqrt{\lambda_3/(\lambda_3 - \lambda_1)}$ and $B = \sqrt{\lambda_3/(\lambda_3 - \lambda_2)}$. Moreover,

$$(\alpha'(s), \alpha''(s), \alpha'''(s)) = \kappa_\alpha^2 \tau_\alpha = c_1 = \lambda_1 \lambda_2 \lambda_3$$

$$= AB(1 + A^2 B^2 - A^2 - B^2)\left(\frac{w'(s)}{\alpha_3'(s)}\right)^3,$$

hence

$$\kappa_\alpha(s)^2 - w'(s)^2 = -\lambda_1 \lambda_2.$$

We now prove that the surface $\Psi(s,t) = \alpha(s) + \alpha(t)$ is minimal. The condition $H = 0$ in (2.3) is now $\langle \alpha'(s) \times \alpha''(s), \alpha'(t) \rangle = \langle \alpha'(s), \alpha'(t) \times \alpha''(t) \rangle$. The computation of both Euclidean products give

$$
\begin{aligned}
&\langle \alpha'(s) \times \alpha''(s), \alpha'(t) \rangle \\
&= \frac{ABw'(s)}{\alpha'_3(s)} \big((A^2 - 1) \cos w(s) \cos w(t) \\
&\quad + (B^2 - 1) \sin w(s) \sin w(t) + \alpha'_3(s)\alpha'_3(t) \big). \\
&\langle \alpha'(s), \alpha'(t) \times \alpha''(t) \rangle \\
&= \frac{ABw'(t)}{\alpha'_3(t)} \big((A^2 - 1) \cos w(s) \cos w(t) \\
&\quad + (B^2 - 1) \sin w(s) \sin w(t) + \alpha'_3(s)\alpha'_3(t) \big).
\end{aligned}
$$

Thus the surface is minimal if and only if we prove that $w'(s)/\alpha'_3(s) = w'(t)/\alpha'_3(t)$ for all s and t. However this holds because of (4.7), we deduce

$$
\frac{w'(s)}{\alpha'_3(s)} = \sqrt{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} = \frac{w'(t)}{\alpha'_3(t)}.
$$

$\square$

Remark 4.5. If the characteristic Eq. (4.10) has a double root, that is, $\lambda_1 = \lambda_2$, then $A = B = \sqrt{\lambda_3/(\lambda_3 - \lambda_1)}$, $\alpha'(s) = (A \cos w(s), A \sin w(s), \sqrt{1 - A^2})$ and $\kappa_\alpha = A^2 w'^2$. So, because of $w'^2 = \kappa_\alpha^2 + \lambda_1\lambda_2$, we see that $\kappa_\alpha^2 = -\lambda_1\lambda_3$ and $\tau_\alpha = -\lambda_1$. Since the curvature and torsion are constant, the curve α is a circular helix. On the other hand, the autonomous ODE (4.11) becomes

$$
y'^2 + \frac{1}{y^2}(y^2 + \lambda_1^2)(y^2 + \lambda_1\lambda_3)^2 = 0,
$$

we conclude that there are no non-constant solutions of (4.11).

Remark 4.6. The Eq. (4.11) gives the curvature of κ of the generating curve α of the translation minimal surface $\Psi(s,t) = \alpha(s) + \alpha(t)$. With the change $u = -(\kappa^2 + c_3/3)$, then (4.11) can be rewritten as

$$
u'^2 - 4u^3 + g_2 u + g_3 = 0,
$$

where

$$
g_2 = \frac{4}{3}c_3^2 - 4c_1c_2, \quad g_3 = 4c_1^2 - \frac{4}{3}c_1c_2c_3 + \frac{8}{27}c_3^3.
$$

Hence, u is a real form of the Weierstrass $\mathcal{P}$-function with invariants g_2 and g_3. Thus κ can be explicitly written in terms of real forms of elliptic equations.

We finish this paper showing explicit examples of the procedure for constructing translation minimal surfaces with non-planar generating curves according Theorem 4.4. Recall that by the item 1 of Theorem 4.3, if the generating curve α has constant curvature (resp. constant torsion), then its torsion
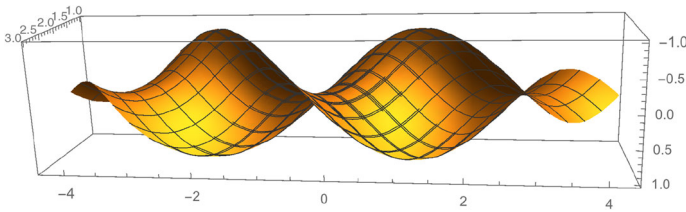
FIGURE 1. The helicoid

(resp. curvature) is constant as well, hence the curve is a circular helix and the resulting surface is a helicoid by Theorem 3.2.

Fixing the constants c_i is equivalent to fix the roots λ_i of the cubic polynomial (4.10).

Remark 4.7. The family of minimal translation surfaces is constructed in terms of the roots of the cubic polynomial equation $-\lambda^3 + c_2\lambda^2 - c_3\lambda + c_1 = 0$. After a homothety of the ambient space $\mathbb{R}^3$, which preserves the minimality of the surface and the property to be a translation surface, we can fix one of the roots of this equation. As a consequence, the minimal translation surfaces are parameterized by two parameters.

Following Theorems 4.3 and 4.4, we present here the scheme through a series of steps for constructing examples of minimal translation surfaces in Euclidean space.

1. Fix the roots λ_i of (4.10). By simplicity, we may consider $\lambda_1 \leq \lambda_2 < 0 < \lambda_3$. The root λ_3 will be fixed to be $\lambda_3 = 1$. Compute A, B.
2. Compute c_i and the polynomial Eq. (4.10).
3. Compute the equilibrium points $y_1 = \sqrt{-\lambda_2\lambda_3}$ and $y_2 = \sqrt{-\lambda_1\lambda_3}$ of (4.11).
4. Fix the initial value y_0 of (4.11), where $y_1 < y_0 < y_2$.
5. Solve numerically the Eqs. (4.11). Fix a initial value w_0 to solve numerically (4.9) and the function w.
6. Solve the curve α.

Example 1. Case of helicoid. Choose a double root $\lambda_1 = \lambda_2 = -1$. Then (4.10) is $-\lambda^3 - \lambda^2 + \lambda + 1 = 0$ and $A = B = 1/\sqrt{2}$. The equilibrium points as $y_1 = y_2 = 1$. Thus take $y_0 = 1$ as initial condition in (4.11). Then the solution is $\kappa(s) = 1$, so $\tau = 1$ (Fig. 1).

Example 2. Take $\lambda_1 = -4$ and $\lambda_2 = -1$. Then (4.10) is $-\lambda^3 - 4\lambda^2 + \lambda + 4 = 0$ and $A = 0.447$ and $B = 0.707$. The equilibrium points as $y_1 = 1$ and $y_2 = 2$. Choose $y_0 = 1.3$ as initial condition in (4.11) (Fig. 2).
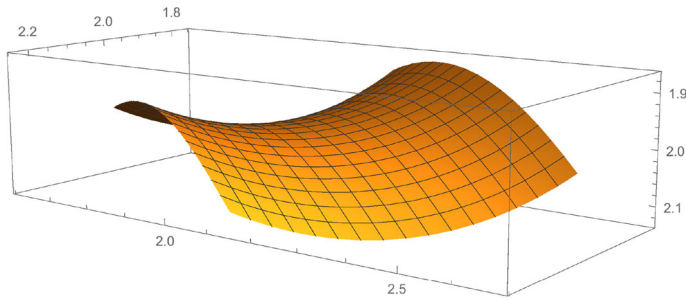
FIGURE 2. Case $\lambda_1 = -4$, $\lambda_2 = -1$ and $\lambda_3 = 1$
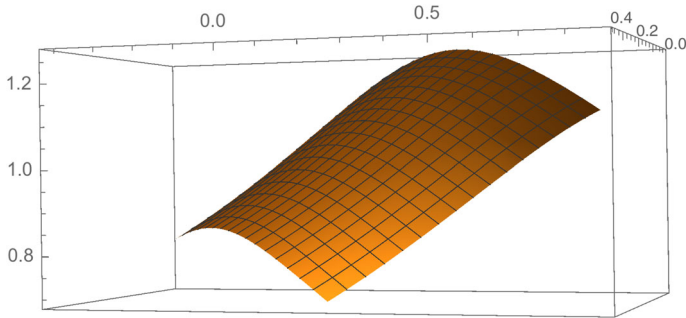
FIGURE 3. Case $\lambda_1 = -2$, $\lambda_2 = -1$ and $\lambda_3 = 1$

Example 3. Consider $\lambda_1 = -2$ and $\lambda_2 = -1$. Then the polynomial is $p[\lambda] = -\lambda^3 - 2\lambda^2 + \lambda + 2$. The equilibrium points as $y_1 = 1.412$ and $y_2 = 1$. Also $A = 0.577$ and $B = 0.707$. The initial value is $y_0 = 1.1$ (Fig. 3).

References

[1] Darboux, G.: Leçons sur la Théorie Générale des Surfaces et ses Applications Géométriques du Calcul Infinitésimal, vol. 1–4. Chelsea Publ. Co, reprint (1972)

[2] Dillen, F., Van de Woestyne, I., Verstraelen, L., Walrave, J.T.: The surface of Scherck in E^3: a special case in the class of minimal surfaces defined as the sum of two curves. Bull. Inst. Math. Acad. Sin. **26**, 257–267 (1998)

[3] Hasanis, T., López, R.: Translation surfaces in Euclidean space with constant Gaussian curvature. Commun. Anal. Geom. to appear

[4] Liu, H.: Translation surfaces with constant mean curvature in 3-dimensional spaces. J. Geom. **64**, 141–149 (1999)

[5] López, R.: Minimal translation surfaces in hyperbolic space. Beitr. Algebra Geom. **52**, 105–112 (2011)

[6] López, R., Munteanu, M.I.: Surfaces with constant mean curvature in Sol geometry. Differ. Geom. Appl. **29**(suppl. 1), S238–S245 (2011)

[7] López, R., Perdomo, O.: Minimal translation surfaces in Euclidean space. J. Geom. Anal. **27**, 2926–2937 (2017)

[8] Montiel, S., Ros, A.: Curves and Surfaces. Graduate Studies in Mathematics, vol. 69. American Mathematical Society, Providence (2009)

[9] Moruz, M., Munteanu, M.I.: Minimal translation hypersurfaces in E^4. J. Math. Anal. Appl. **439**, 798–812 (2016)

[10] Munteanu, M.I., Palmas, O., Ruiz-Hernández, G.: Minimal translation hypersurfaces in Euclidean space. Mediterr. J. Math. **13**, 2659–2676 (2016)

[11] Nitsche, J.C.C.: Lectures on Minimal Surfaces. Cambridge University Press, Cambridge (1989)

[12] Scherk, H.F.: Bemerkungen über die kleinste Fläche innerhalb gegebener Grenzen. J. Reine Angew. Math. **13**, 185–208 (1835)

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