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Irreducible Twisted Heisenberg–Virasoro Modules from Tensor Products

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Abstract. In this paper, we realize polynomial $\mathcal{H}$-modules $\Omega(\lambda,\alpha,\beta)$ from irreducible twisted Heisenberg–Virasoro modules $\mathcal{A}_{\alpha,\beta}$. It follows from $\mathcal{H}$-modules $\Omega(\lambda,\alpha,\beta)$ and $\mathrm{Ind}(M)$ that we obtain a class of tensor product modules $\left(\bigotimes_{i=1}^{m}\Omega(\lambda_i,\alpha_i,\beta_i)\right)\otimes\mathrm{Ind}(M)$. We give the necessary and sufficient conditions under which these modules are irreducible and isomorphic, and also give that the irreducible modules in this class are new.

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1. Introduction

It is well-known that the *twisted Heisenberg–Virasoro algebra* $\mathcal{H}$ [1] is the universal central extension of the Lie algebra $\bar{\mathcal{H}}$ of differential operators of order at most one on the Laurent polynomial algebra $\mathbb{C}[t,t^{-1}]$, where

$$\bar{\mathcal{H}} := \left\{ f(t)\frac{d}{dt} + g(t) \,\middle|\, f(t), g(t) \in \mathbb{C}[t,t^{-1}] \right\}.$$

More precisely, $\mathcal{H}$ is an infinite-dimensional Lie algebra with $\mathbb{C}$-basis $\{L_m = t^{m+1}\frac{d}{dt}, I_m = t^m, C_i \mid m\in\mathbb{Z},\ i=1,2,3\}$ subject to the Lie bracket as follows:

$$\begin{aligned}
&[L_m,L_n] = (n-m)L_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}C_1,\\
&[L_m,I_n] = nI_{m+n} + \delta_{m+n,0}(m^2+m)C_2,\\
&[I_m,I_n] = n\delta_{m+n,0}C_3,\\
&[\mathcal{H},C_1] = [\mathcal{H},C_2] = [\mathcal{H},C_3] = 0.
\end{aligned}$$

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It is clear that the subspaces spanned by $\{I_m, C_3 \mid 0 \neq m \in \mathbb{Z}\}$ and by $\{L_m, C_1 \mid m \in \mathbb{Z}\}$ are respectively the Heisenberg algebra and the Virasoro algebra. Notice that the center of $\mathcal{H}$ is spanned by $\{I_0, C_i \mid i = 1, 2, 3\}$.

The twisted Heisenberg–Virasoro algebra $\mathcal{H}$ has been studied by Arbarello et al. (see [1]), where a connection is established between the second cohomology of certain moduli spaces of curves and the second cohomology of the Lie algebra of differential operators of order at most one. Furthermore, when the central element of the Heisenberg subalgebra acts in a non-zero way, an irreducible highest weight module for $\mathcal{H}$ is isomorphic to the tensor product of an irreducible module for the Virasoro algebra and an irreducible module for the infinite-dimensional Heisenberg algebra was proved. A more general result for this was given in [14].

The representation theory on the twisted Heisenberg–Virasoro algebra has attracted a lot of attention from mathematicians and physicists. The theory of weight twisted Heisenberg–Virasoro modules with finite-dimensional weight spaces is fairly well-developed (see [9,12,16]). While weight modules with an infinite dimensional weight spaces were also studied (see [4,15]). In the last few years, various families of non-weight irreducible twisted Heisenberg–Virasoro modules were investigated (see, e.g., [2–6,10]). These are basically various versions of Whittaker modules and $\mathcal{U}(\mathbb{C}L_0)$-free modules constructed using different tricks. Whittaker modules for $\mathcal{H}$ were defined in [10], but the correct classification of irreducible modules appeared in [2]. A more general setting for Whittaker modules over $\mathcal{H}$ was given in [14]. However, the theory of representation over the twisted Heisenberg–Virasoro algebra is far more from being well-developed.

In the present paper, we construct a class of non-weight $\mathcal{H}$-modules by taking tensor products of a finite number of irreducible $\mathcal{H}$-modules $\Omega(\lambda, \alpha, \beta)$ with irreducible $\mathcal{H}$-modules $\mathrm{Ind}(M)$. We present the necessary and sufficient conditions under which these modules are irreducible and also determine all the equivalent irreducible modules in this class. The irreducibility and isomorphism problem of the $\mathcal{H}$-modules $\big(\bigotimes_{i=1}^m \Omega(\lambda_i, \alpha_i, \beta_i)\big) \otimes \mathrm{Ind}(M)$ were also given in Examples 8, 10 and Theorem 35 in [14]. At the same time, inspired by [11], a class of $\mathcal{H}$-modules $\mathcal{A}_{\alpha,\beta}$ are given. Then many interesting examples for such irreducible twisted Heisenberg–Virasoro modules with different features are provided. In particular, a class of irreducible polynomial modules $\Omega(\lambda, \alpha, \beta)$ over the twisted Heisenberg–Virasoro algebra are defined.

We briefly give a summary of the paper below. In Sects. 2 and 3, we recall some known results and construct a class of modules $\mathcal{A}_{\alpha,\beta}$ over the twisted Heisenberg–Virasoro algebra. In Sect. 4, we determine the necessary and sufficient conditions for $\big(\bigotimes_{i=1}^m \Omega(\lambda_i, \alpha_i, \beta_i)\big) \otimes \mathrm{Ind}(M)$ to be irreducible. In Sect. 5, we present the necessary and sufficient conditions for $\mathcal{H}$-modules $\big(\bigotimes_{i=1}^m \Omega(\lambda_i, \alpha_i, \beta_i)\big) \otimes \mathrm{Ind}(M)$ to be isomorphic. At last, we present that $\mathcal{H}$-modules $\big(\bigotimes_{i=1}^m \Omega(\lambda_i, \alpha_i, \beta_i)\big) \otimes \mathrm{Ind}(M)$ are new.

Throughout this paper, we respectively denote by $\mathbb{C}, \mathbb{C}^*, \mathbb{Z}, \mathbb{Z}_+$ and $\mathbb{N}$ the sets of complex numbers, nonzero complex numbers, integers, nonnegative integers and positive integers. All vector spaces are assumed to be over $\mathbb{C}$.

2. Some Known Results

In this section, we recall some definitions and known results.

Let $\mathbb{C}[t]$ be the (associative) polynomial algebra. For convenience, we denote $\partial := t\frac{d}{dt}$. We see that $\partial t^n = t^n(\partial + n)$ for $n \in \mathbb{Z}$. It is obvious that associative algebra $\widetilde{\mathcal{A}} = \mathbb{C}[t, \partial]$ is a proper subalgebra of the rank 1 Weyl algebra $\mathbb{C}[t, \frac{d}{dt}]$. Then $\widetilde{\mathcal{A}}$ is the universal enveloping algebra of the 2-dimensional solvable Lie algebra $\mathfrak{b}_1 = \mathbb{C}L_0 \oplus \mathbb{C}L_1$, which subjects to $[L_0, L_1] = L_1$. Now, assume that $\mathcal{K} = \mathbb{C}[t, t^{-1}, \partial]$ is the Laurent polynomial differential operator algebra.

Definition 2.1. Assume that $\mathcal{P}$ is an associative or Lie algebra over $\mathbb{C}$ and $\mathcal{Q}$ is a subspace of $\mathcal{P}$. If there exists $0 \neq q \in \mathcal{Q}$ such that $qv = 0$ for some $0 \neq v \in V$, the $\mathcal{P}$-module V is called *$\mathcal{Q}$-torsion*; otherwise V is called *$\mathcal{Q}$-torsion-free.*

Let us recall some results about irreducible modules over the associative algebra $\mathcal{K}$.

Lemma 2.2. [11] *Assume that V is a $\mathbb{C}[t]$-torsion-free irreducible module over the associative algebra $\widetilde{\mathcal{A}}$. Then V can be extended into a $\mathcal{K} = \mathbb{C}[t, t^{-1}, \frac{d}{dt}]$-module, i.e., the action of $\widetilde{\mathcal{A}}$ on V is a restriction of an irreducible $\mathbb{C}[t, t^{-1}]$-torsion-free $\mathcal{K}$-module.*

Lemma 2.3. [11] *Assume that μ is an irreducible element in the associative algebra $\mathbb{C}(t)[\partial]$. Then $\mathcal{K}/(\mathcal{K} \cap (\mathbb{C}(t)[\partial]\mu))$ is a $\mathbb{C}[t, t^{-1}]$-torsion-free irreducible $\mathcal{K}$-module. In this way any $\mathbb{C}[t, t^{-1}]$-torsion-free irreducible $\mathcal{K}$-module can be realized.*

For any $\lambda \in \mathbb{C}^*$, it follows from $t^m\partial^n = \lambda^m(\partial - m)^n, \partial\partial^n = \partial^{n+1}$ for all $n \in \mathbb{Z}_+, m \in \mathbb{Z}$ that we can define a $\mathcal{K}$-module structure on the space $\Omega(\lambda) = \mathbb{C}[\partial]$. Then $\Omega(\lambda)$ is an irreducible module over the associative algebra $\mathcal{K}$ for any $\lambda \in \mathbb{C}^*$ (see [11]).

Lemma 2.4. [11] *Assume that V is an irreducible module over the associative algebra $\mathcal{K}$ on which $\mathbb{C}[t, t^{-1}]$ is torsion. Then $V \cong \Omega(\lambda)$ for some $\lambda \in \mathbb{C}^*$.*

Combining Lemmas 2.3 and 2.4 , a classification for all irreducible modules over the associative algebra $\mathcal{K}$ are obtained.

Now let us recall a large class of irreducible $\mathcal{H}$-modules, which includes the known irreducible modules such as highest weight modules and Whittaker modules. For any $e \in \mathbb{Z}_+$, denote by $\mathcal{H}_e$ the subalgebra

$$\sum_{m \in \mathbb{Z}_+} (\mathbb{C}L_m \oplus \mathbb{C}I_{m-e}) \oplus \mathbb{C}C_1 \oplus \mathbb{C}C_2 \oplus \mathbb{C}C_3.$$

Take $M(c_0, c_1, c_2, c_3)$ to be an irreducible $\mathcal{H}_e$-module such that I_0, C_1, C_2 and C_3 act on it as scalars c_0, c_1, c_2, c_3 respectively. For convenience, we denote $M(c_0, c_1, c_2, c_3)$ by M and form the induced $\mathcal{H}$-module

$$\mathrm{Ind}(M) := \mathcal{U}(\mathcal{H}) \otimes_{\mathcal{U}(\mathcal{H}_e)} M. \tag{2.1}$$

Theorem 2.5. [2] *Let $e \in \mathbb{Z}_+$ and M be a simple $\mathcal{H}_e$-module with $c_3 = 0$. Assume there exists $k \in \mathbb{Z}_+$ such that*

(1) $\begin{cases} \textit{the action of } I_k \textit{ on } M \textit{ is injective} & \textit{if } \ k \neq 0, \\ c_0 + (n-1)c_2 \neq 0 \quad \text{for all } n \in \mathbb{Z}\backslash\{0\} & \textit{if } \ k = 0, \end{cases}$

(2) $I_n M = L_m M = 0$ *for all* $n > k$ *and* $m > k + e$.

Then

(i) $\mathrm{Ind}(M)$ *is a simple* $\mathcal{H}$*-module*;

(ii) *the actions of* I_n, L_m *on* $\mathrm{Ind}(M)$ *for all* $n > k$ *and* $m > k + e$ *are locally nilpotent.*

The following result will be used in the following (see [13]).

Proposition 2.6. *Let P be a vector space over $\mathbb{C}$ and P_1 a subspace of P. Suppose that $\mu_1, \mu_2, \ldots, \mu_s \in \mathbb{C}^*$ are pairwise distinct, $v_{i,j} \in P$ and $f_{i,j}(t) \in \mathbb{C}[t]$ with $\deg f_{i,j}(t) = j$ for $i = 1, 2, \ldots, s; j = 0, 1, 2, \ldots, k$. If*

$$\sum_{i=1}^{s} \sum_{j=0}^{k} \mu_i^m f_{i,j}(m) v_{i,j} \in P_1$$
$$\textit{for } K < m \in \mathbb{Z} \ (K \textit{ any fixed element in } \mathbb{Z} \cup \{-\infty\})$$

then $v_{i,j} \in P_1$ for all i, j.

3. Realize $\mathcal{H}$-Module $\Omega(\lambda, \alpha, \beta)$

Let $\mathcal{A}$ be an irreducible module over the associative algebra $\mathcal{K}$. For any $\alpha, \beta \in \mathbb{C}$, we define the action of $\mathcal{H}$ on $\mathcal{A}$ as follows

$$L_m v = \big(t^m \partial + m\alpha t^m\big) v, \ I_m v = \beta t^m v, \ C_i v = 0 \tag{3.1}$$

for $i \in \{1, 2, 3\}, m \in \mathbb{Z}, v \in \mathcal{A}$. Denote the above action by $\mathcal{A}_{\alpha,\beta}$.

Proposition 3.1. *For any $\alpha, \beta \in \mathbb{C}$, we obtain that $\mathcal{A}_{\alpha,\beta}$ is an $\mathcal{H}$-module under the action given in* (3.1).

Proof. It follows from (3.1) that we have

$$(L_m I_n - I_n L_m) v = \beta\big(t^m \partial + m\alpha t^m\big) t^n v - \beta t^n \big(t^m \partial + m\alpha t^m\big) v = n I_{m+n} v.$$

That is, $L_m I_n - I_n L_m = n I_{n+m}$ holds on $\mathcal{A}_{\alpha,\beta}$. By [11], $L_m L_n - L_n L_m = (n-m) L_{n+m}$ holds on $\mathcal{A}_{\alpha,\beta}$. Finally, the relation $I_m I_n - I_n I_m = 0$ on $\mathcal{A}_{\alpha,\beta}$ is trivial. Thus, we obtain that $\mathcal{A}_{\alpha,\beta}$ is an $\mathcal{H}$-module. □

Now we recall the necessary and sufficient conditions for $\mathcal{A}_{\alpha,\beta}$ to be irreducible (see [11]).

Theorem 3.2. *Let $\alpha, \beta \in \mathbb{C}$ and $\mathcal{A}$ be an irreducible module over the association algebra $\mathcal{K}$. Then $\mathcal{A}_{\alpha,\beta}$ as an irreducible $\mathcal{H}$-modules if and only if one of the following holds*

(1) $\alpha \notin \{0, 1\}$ *or* $\beta \neq 0$.
(2) $\alpha = 1, \beta = 0$ *and* $\partial\mathcal{A} = \mathcal{A}$.
(3) $\alpha = \beta = 0$ *and* $\mathcal{A}$ *is not isomorphic to the natural* $\mathcal{K}$ *module* $\mathbb{C}[t, t^{-1}]$.

The isomorphism results for modules $\mathcal{A}_{\alpha,\beta}$ as follows.

Theorem 3.3. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$ and $\mathcal{A}, \mathcal{B}$ be irreducible modules over the associative algebra $\mathcal{K}$. Then $\mathcal{A}_{\alpha_1,\beta_1} \cong \mathcal{B}_{\alpha_2,\beta_2}$ as $\mathcal{H}$-modules if and only if one of the following holds*

(1) $\mathcal{A} \cong \mathcal{B}$ *as $\mathcal{K}$-modules,* $\alpha_1 = \alpha_2$ *and* $\beta_1 = \beta_2$.
(2) $\mathcal{A} \cong \mathcal{B}$ *as $\mathcal{K}$-modules,* $\alpha_1 = 1, \alpha_2 = 0, \beta_1 = \beta_2 = 0$ *and* $\partial\mathcal{A} = \mathcal{A}$.
(3) $\mathcal{A} \cong \mathcal{B}$ *as $\mathcal{K}$-modules,* $\alpha_1 = 0, \alpha_2 = 1, \beta_1 = \beta_2 = 0$ *and* $\partial\mathcal{B} = \mathcal{B}$.

Proof. (1) The sufficiency of the conditions is clear. Now suppose that $\varphi : \mathcal{A}_{\alpha_1,\beta_1} \to \mathcal{A}_{\alpha_2,\beta_2}$ is an $\mathcal{H}$-module isomorphism. For any $v \in \mathcal{A}$, we have $\varphi(I_0 v) = I_0\varphi(v)$, which gives $\beta_1 = \beta_2$. In particular, $\beta_1 \neq 0$. We note that $\varphi(I_m v) = I_m\varphi(v)$, which implies

$$\varphi(t^m v) = t^m\varphi(v) \tag{3.2}$$

for any $m \in \mathbb{Z}$. From $\varphi(L_0^m v) = L_0^m\varphi(v)$, we have

$$\varphi(\partial^m v) = \partial^m\varphi(v) \tag{3.3}$$

for $m \in \mathbb{Z}$. Combining (3.2) and (3.3), we obtain $\mathcal{A} \cong \mathcal{B}$ as $\mathcal{K}$-modules. From (3.2) and (3.3), it is easy to get

$$\begin{aligned} 0 &= \varphi(L_m v) - L_m\varphi(v) = \varphi\big((t^m\partial + m\alpha_1 t^m)v\big) - (t^m\partial + m\alpha_2 t^m)\partial\varphi(v) \\ &= m(\alpha_1 - \alpha_2)t^m\varphi(v). \end{aligned}$$

Then $\alpha_1 = \alpha_2$. If $\beta_1 = 0$, these modules reduce to the Virasoro modules (see [11]). This is (1).

By the [11, Theorem 12], we get (2) and (3). □

Now we realize $\mathcal{H}$-modules $\Omega(\lambda, \alpha, \beta)$ from $\mathcal{A}_{\alpha,\beta}$. Let $\lambda \in \mathbb{C}^*$ and $\alpha, \beta \in \mathbb{C}$. Then we get the irreducible $\mathcal{K}$-module $\Omega(\lambda)$, which has a basis $\{\partial^k : k \in \mathbb{Z}_+\}$, and the $\mathcal{K}$-actions are given by

$$t^m \cdot \partial^n = \lambda^m(\partial - m)^n, \ \partial \cdot \partial^m = \partial^{m+1} \quad \text{for } m \in \mathbb{Z}, n \in \mathbb{Z}_+.$$

According to (3.1) we have $\mathcal{H}$-modules $\Omega(\lambda, \alpha, \beta) = \mathbb{C}[\partial]$ with the action:

$$\begin{aligned} &L_m\partial^n = \lambda^m\big(\partial + m(\alpha - 1)\big)(\partial - m)^n, \ I_m\partial^n = \lambda^m\beta(\partial - m)^n \\ &\quad \text{for } m \in \mathbb{Z}, n \in \mathbb{Z}_+. \end{aligned}$$

Then $\Omega(\lambda, \alpha, \beta)$ is irreducible if and only if $\alpha \neq 1$ or $\beta \neq 0$ (see [3]). In the following sections, we will consider a class of tensor product $\mathcal{H}$-modules related to $\Omega(\lambda, \alpha, \beta)$.

Now we describe some other examples about irreducible $\mathcal{H}$-modules $\mathcal{A}_{\alpha,\beta}$, such as intermediate series modules, degree two modules and degree n modules.

Example 3.4. Let $\gamma \in \mathbb{C}[t,t^{-1}], \beta \in \mathbb{C}$ and $\mu = \partial - \gamma$ in Lemma 2.3. Then we obtain the irreducible $\mathcal{K}$-module

$$\mathcal{A} = \mathcal{K}/\big(\mathcal{K} \cap (\mathbb{C}(t)[\partial]\mu)\big) = \mathcal{K}/(\mathcal{K}\mu)$$

with a basis $\{t^k : k \in \mathbb{Z}\}$. We see that the $\mathcal{K}$-actions on $\mathcal{A}$ are given by

$$\partial \cdot t^n = t^n(\gamma + n),\ t^m \cdot t^n = t^{m+n} \quad \text{for } m, n \in \mathbb{Z}.$$

It follows from (3.1) that we get $\mathcal{H}$-modules $\mathcal{A}_{\gamma,\alpha,\beta} = \mathbb{C}[t,t^{-1}]$ with the action:

$$L_m t^n = (\gamma + n + m\alpha)t^{m+n},\ I_m t^n = \beta t^{m+n} \quad \text{for } m, n \in \mathbb{Z}.$$

If $\gamma \in \mathbb{C}\backslash\mathbb{Z}$ or $\alpha \notin \{0,1\}$ or $\beta \neq 0$, then $\mathcal{A}_{\gamma,\alpha,\beta}$ is an irreducible $\mathcal{H}$-module (see [8,9]). In particular, $\gamma \in \mathbb{C}$ these modules $\mathcal{A}_{\gamma,\alpha,\beta}$ are the intermediate series modules of $\mathcal{H}$ (see [7,9]).

Some degree two irreducible elements in $\mathbb{C}(t)[\partial]$ were first constructed in [11].

Example 3.5. Let $f(t) \in \mathbb{C}[t,t^{-1}]$ be such that $\partial^2 - f(t)$ is irreducible in $\mathbb{C}(t)[\partial]$. Take $\mu = \partial^2 - f(t)$ in Lemma 2.3. Then one obtain the irreducible $\mathcal{K}$-module

$$\mathcal{A} = \mathcal{K}/\big(\mathcal{K} \cap (\mathbb{C}(t)[\partial]\mu)\big) = \mathcal{K}/(\mathcal{K}\mu),$$

which has a basis $\{t^k, t^k\partial : k \in \mathbb{Z}\}$. The $\mathcal{K}$-actions on $\mathcal{A}$ are given by

$$\begin{aligned} &t^m \cdot t^n = t^{m+n},\ t^m \cdot (t^n\partial) = t^{m+n}\partial, \\ &\partial \cdot t^n = t^n(\partial + n),\ \partial \cdot (t^n\partial) = t^n(f(t) + n\partial), \end{aligned}$$

where $m, n \in \mathbb{Z}$. From (3.1), for $\alpha \neq 1$ or $\beta \neq 0$, we have irreducible $\mathcal{H}$-modules $\mathcal{A}_{\alpha,\beta} = \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}[t,t^{-1}]\partial$ with the action:

$$\begin{aligned} &L_m \cdot t^n = t^{m+n}(n + m\alpha + \partial),\ L_m \cdot (t^n\partial) = t^{m+n}(f(t) + m\alpha + n\partial), \\ &I_m \cdot t^n = \beta t^{m+n},\ I_m \cdot (t^n\partial) = \beta t^{m+n}\partial. \end{aligned}$$

Some degree n irreducible elements in $\mathbb{C}(t)[\partial]$ were first constructed in [11].

Example 3.6. For any $n \in \mathbb{N}$, let $\mu = (\frac{d}{dt})^n - t$ in Lemma 2.3. Then we have the irreducible $\mathcal{K}$-module

$$\mathcal{A} = \mathcal{K}/(\mathcal{K} \cap (\mathbb{C}(t)[\partial]\mu)) = \mathcal{K}/(\mathcal{K}\mu),$$

which has a basis $\{t^k(\frac{d}{dt})^m : k \in \mathbb{Z}, m = 0, 1, \ldots, n-1\}$. The actions of $\mathcal{K} = \mathbb{C}[t, t^{-1}][\frac{d}{dt}]$ are given by

$$t^k \cdot \left(t^r \left(\frac{d}{dt}\right)^m\right) = t^{k+r}(\frac{d}{dt})^m \quad \text{for } k, r \in \mathbb{Z},\ 0 \leq m \leq n-1,$$

$$\frac{d}{dt} \cdot \left(t^r \left(\frac{d}{dt}\right)^m\right) = rt^{r-1}\left(\frac{d}{dt}\right)^m + t^r \left(\frac{d}{dt}\right)^{m+1} \quad \text{for } r \in \mathbb{Z},\ 0 \leq m < n-1,$$

$$\frac{d}{dt} \cdot \left(t^r \left(\frac{d}{dt}\right)^{n-1}\right) = rt^{r-1}\left(\frac{d}{dt}\right)^{n-1} + t^{r+1} \quad \text{for } r \in \mathbb{Z}.$$

Using (3.1), for $\alpha \neq 1$ or $\beta \neq 0$, we obtain irreducible $\mathcal{H}$-modules $\mathcal{A}_{\alpha,\beta} = \mathbb{C}[t, t^{-1}] \times (\Sigma_{i=0}^{n-1}\mathbb{C}(\frac{d}{dt})^i)$ with the action:

$$L_k \cdot \left(t^r \left(\frac{d}{dt}\right)^m\right) = (rt^{k+r} + \alpha k t^{k+r+1})\left(\frac{d}{dt}\right)^m + t^{k+r+1}\left(\frac{d}{dt}\right)^{m+1},$$

$$L_k \cdot \left(t^r \left(\frac{d}{dt}\right)^{n-1}\right) = (rt^{k+r} + \alpha k t^{k+r+1})\left(\frac{d}{dt}\right)^{n-1} + t^{k+r+2},$$

$$I_k \cdot \left(t^r \left(\frac{d}{dt}\right)^m\right) = \beta t^{k+r}\left(\frac{d}{dt}\right)^m,$$

where $k, r \in \mathbb{Z}, 0 \leq m < n-1$.

4. Irreducibilities

In this section, we will determine the irreducibility of $\left(\bigotimes_{i=1}^m \Omega(\lambda_i, \alpha_i, \beta_i)\right) \otimes \mathrm{Ind}(M)$.

Now we introduce some notations. Let $m \in \mathbb{N}, \lambda_i, \alpha_i, \beta_i \in \mathbb{C}$ for $i = 1, 2, \ldots, m$. Denote $\Omega(\lambda_i, \alpha_i, \beta_i) = \mathbb{C}[\partial_i]$. The actions of $\mathcal{H}$ on $\Omega(\lambda_i, \alpha_i, \beta_i)$ are

$$L_k \partial_i^n = \lambda_i^k(\partial_i + k\alpha_i)(\partial_i - k)^n,\ I_k \partial_i^n = \lambda_i^k \beta_i (\partial_i - k)^n,\ C_j \partial_i^n = 0$$

for $k \in \mathbb{Z}, n \in \mathbb{Z}_+, i = 1, 2, \ldots, m, j = 1, 2, 3$. Then $\Omega(\lambda_i, \alpha_i, \beta_i)$ is irreducible if and only if $\alpha_i \neq 0$ or $\beta_i \neq 0$ for $i = 1, \ldots, m$. For convenience, we write $\bigotimes_{i=1}^m \Omega(\lambda_i, \alpha_i, \beta_i) = \mathbb{C}[\partial_1, \partial_2, \ldots, \partial_m]$ for $m \in \mathbb{N}$.

Now we consider the tensor product $\left(\bigotimes_{i=1}^m \Omega(\lambda_i, \alpha_i, \beta_i)\right) \otimes \mathrm{Ind}(M)$. Define a total order "$\prec$" on the subset

$$\{\partial_1^{p_1} \cdots \partial_m^{p_m} \otimes v \mid \mathbf{P} = (p_1, \ldots, p_m) \in \mathbb{Z}_+^m, m \in \mathbb{N}, 0 \neq v \in \mathrm{Ind}(M)\},$$

by

$$\partial_1^{p_1} \cdots \partial_m^{p_m} \otimes u \prec \partial_1^{q_1} \cdots \partial_m^{q_m} \otimes v$$
$$\iff \exists k \in \mathbb{N} \text{ such that } p_k < q_k \text{ and } p_n = q_n \text{ for } n < k.$$

Then each non-zero element w in $\left(\bigotimes_{i=1}^m \Omega(\lambda_i,\alpha_i,\beta_i)\right)\otimes \mathrm{Ind}(M)$ can be (uniquely) written as follows

$$w=\sum_{\mathbf{p}\in I}\partial_1^{p_1}\cdots\partial_m^{p_m}\otimes v_{\mathbf{p}},$$

where I is a finite subset of $\mathbb{Z}_+^m$ and the $v_{\mathbf{p}}$ are nonzero vectors of $\mathrm{Ind}(M)$. Now we define $\deg(w)=(p_1,\ldots,p_m)$, where $\partial_1^{p_1}\cdots\partial_m^{p_m}\otimes v_{\mathbf{p}}$ is the term with maximal order in the sum. Notice that $\deg(1\otimes v)=\mathbf{0}=(\underbrace{0,0,\ldots,0}_{m})$.

Lemma 4.1. *Let $S=\{1,2\}, S'\subseteq S, \lambda\in\mathbb{C}^*, \beta_i,\alpha_j\in\mathbb{C}^*, \beta_j=0$ for $i\in S', j\in S\backslash S'$ and $s\in\mathbb{Z}_+$. Denote W_s the vector subspace of $\Omega(\lambda,\alpha_1,\beta_1)\otimes\Omega(\lambda,\alpha_2,\beta_2)$ spanned by $\{f(\partial_1)(\partial_1+\partial_2)^n\mid n\in\mathbb{Z}_+, 0\le\deg(f)\le s\}$ or $\{(\partial_1+\partial_2)^n f(\partial_2)\mid n\in\mathbb{Z}_+, 0\le\deg(f)\le s\}$. Then W_s is a submodule of $\Omega(\lambda,\alpha_1,\beta_1)\otimes\Omega(\lambda,\alpha_2,\beta_2)$.*

Proof. Without loss of generality, we may assume $\lambda=1$. For any $n\in\mathbb{Z}_+, f(\partial_1)\in W_s, k\in\mathbb{Z}$, it is easy to get

$$\begin{aligned}I_k\big(f(\partial_1)(\partial_1+\partial_2)^n\big)&=I_k\Big(\sum_{i=0}^n\binom{n}{i}f(\partial_1)\partial_1^i\partial_2^{n-i}\Big)\\&=\sum_{i=0}^n\binom{n}{i}\Big(\beta_1 f(\partial_1-k)(\partial_1-k)^i\partial_2^{n-i}+\beta_2 f(\partial_1)\partial_1^i(\partial_2-k)^{n-i}\Big)\\&=\big(\beta_1 f(\partial_1-k)+\beta_2 f(\partial_1)\big)(\partial_1+\partial_2-k)^n\in W_s.\end{aligned}$$

By Theorem 9 of [17], we have $L_k\big(f(\partial_1)(\partial_1+\partial_2)^n\big)\in W_s$. By the similar method, we obtain $L_k\big((\partial_1+\partial_2)^n f(\partial_2)\big)\in W_s$ and $I_k\big((\partial_1+\partial_2)^n f(\partial_2)\big)\in W_s$. Thus, W_s is a submodule of $\Omega(\lambda,\alpha_1,\beta_1)\otimes\Omega(\lambda,\alpha_2,\beta_2)$, completing the proof. □

Corollary 4.2. *Let $S=\{1,2\}, S'\subseteq S, \lambda\in\mathbb{C}^*, \beta_i,\alpha_j\in\mathbb{C}^*, \beta_j=0$ for $i\in S', j\in S\backslash S'$ and $s\in\mathbb{Z}_+$. Assume that W_s is the subspace of $\Omega(\lambda,\alpha_1,\beta_1)\otimes\Omega(\lambda,\alpha_2,\beta_2)$, where W_s is spanned by $\{f(\partial_1)(\partial_1+\partial_2)^n\mid n\in\mathbb{Z}_+, 0\le\deg(f)\le s\}$ or $\{(\partial_1+\partial_2)^n f(\partial_2)\mid n\in\mathbb{Z}_+, 0\le\deg(f)\le s\}$. Then $\Omega(\lambda,\alpha_1,\beta_1)\otimes\Omega(\lambda,\alpha_2,\beta_2)$ has a series of $\mathcal{L}$-submodules*

$$W_1\subset W_2\subset\cdots\subset W_s\subset\cdots$$

such that $W_s/W_{s-1}\cong\Omega\big(\lambda,s+\alpha_1+\alpha_2,\beta_1+\beta_2\big)$ as $\mathcal{L}$-module for each $s\ge1$.

Proof. For $s,n\in\mathbb{Z}_+, k\in\mathbb{Z}$, it follows from Lemma 4.1 that we check

$$\begin{aligned}I_k\big(\partial_1^s(\partial_1+\partial_2)^n\big)&=I_k\left(\sum_{i=0}^n\binom{n}{i}\partial_1^{i+s}\partial_2^{n-i}\right)\\&\equiv\lambda^k(\beta_1+\beta_2)\partial_1^s(\partial_1+\partial_2-k)^n\pmod{W_{s-1}}.\end{aligned}$$

From Corollary 10 of [17], we get

$$L_k\big(\partial_1^s(\partial_1+\partial_2)^n\big) \equiv \lambda^k\partial_1^s\big(\partial_1+\partial_2-k(s+\alpha_0+\alpha_1)\big)(\partial_1+\partial_2-k)^n \pmod{W_{s-1}}.$$

By the similar method, we have $I_k\big(\partial_1^s(\partial_1+\partial_2)^n\big) \equiv \lambda^k(\beta_1+\beta_2)\partial_1^s(\partial_1+\partial_2-k)^n (\mathrm{mod}\ W_{s-1})$. and $L_k\big(\partial_1^s(\partial_1+\partial_2)^n\big) \equiv \lambda^k\partial_1^s\big(\partial_1+\partial_2-k(s+\alpha_0+\alpha_1)\big)(\partial_1+\partial_2-k)^n (\mathrm{mod}\ W_{s-1})$.

These show that the $\mathcal{L}$-module isomorphism $W_s/W_{s-1} \cong \Omega(\lambda, s+\alpha_1+\alpha_2, \beta_1+\beta_2)$. ☐

By the similar method in Lemma 3 of [17], we get the following results.

Lemma 4.3. *Let $m \in \mathbb{N}, \lambda_i \in \mathbb{C}^*, \alpha_i, \beta_i \in \mathbb{C}$ for $i = 1, 2, \ldots, m$ with the λ_i pairwise distinct. Then $\underbrace{1\otimes\cdots\otimes 1}_{m}\otimes v$ generates the $\mathcal{H}$-module $\big(\bigotimes_{i=1}^m \Omega(\lambda_i, \alpha_i, \beta_i)\big) \otimes \mathrm{Ind}(M)$.*

Now we are ready to prove the irreducibility of $\mathcal{H}$-module $\big(\bigotimes_{i=1}^m \Omega(\lambda_i, \alpha_i, \beta_i)\big) \otimes \mathrm{Ind}(M)$.

Theorem 4.4. *Let $m \in \mathbb{N}, S = \{1, \ldots, m\}, S' \subseteq S$ and $\lambda_i \in \mathbb{C}^*$ for $i \in S$ with the λ_i pairwise distinct. Let $\beta_i, \alpha_j \in \mathbb{C}^*, \beta_j = 0$ for $i \in S', j \in S\backslash S'$. Assume $\mathrm{Ind}(M)$ is an $\mathcal{H}$-module defined by (2.1) for which M satisfies the conditions in Theorem 2.5. Then the tensor product $\big(\bigotimes_{i=1}^m \Omega(\lambda_i, \alpha_i, \beta_i)\big) \otimes \mathrm{Ind}(M)$ is an irreducible $\mathcal{H}$-module.*

Proof. For any $w \in \mathrm{Ind}(M)$, there exists $K(w) \in \mathbb{Z}_+$ such that $L_k\cdot w = I_k\cdot w = 0$ for all $k \geq K(w)$ by Theorem 2.5. Suppose W is a nonzero submodule of $\big(\bigotimes_{i=1}^m \Omega(\lambda_i, \alpha_i, \beta_i)\big) \otimes \mathrm{Ind}(M)$. Choose a nonzero element $u \in W$ with minimal degree. We claim that $\deg(u) = 0$. If not, we assume

$$u = \sum_{\mathbf{p}\in I} \partial_1^{p_1}\cdots\partial_m^{p_m} \otimes w_{\mathbf{p}} \in W,$$

where I is a finite subset of $\mathbb{Z}_+^m$ and $w_{\mathbf{p}}$ are nonzero vectors of $\mathrm{Ind}(M)$. Let $\partial_1^{p_1}\cdots\partial_m^{p_m}\otimes w_{\mathbf{p}}$ be maximal among the terms in the sum with respect to "$\prec$" and let i' be minimal such that $p_{i'} > 0$.

First we consider $i' \in S'$. For enough large $k \in \mathbb{Z}$, we obtain

$$I_k\left(\sum_{\mathbf{p}\in I} \partial_1^{p_1}\cdots\partial_m^{p_m}\otimes w_{\mathbf{p}}\right) = \sum_{i=1}^m \sum_{\mathbf{p}\in I} \partial_1^{p_1}\cdots\lambda_i^k\beta_i(\partial_i-k)^{p_i}\cdots\partial_m^{p_m}\otimes w_{\mathbf{p}} \in W, \tag{4.1}$$

where I is a finite subset of $\mathbb{Z}_+^m$ and $w_{\mathbf{p}}$ are nonzero vectors of $\mathrm{Ind}(M)$. Now we consider $i' \in S\backslash S'$. For enough large $k \in \mathbb{Z}$, one can easily to get

$$L_k \left(\sum_{\mathbf{p}\in I} \partial_1^{p_1} \cdots \partial_m^{p_m} \otimes w_{\mathbf{p}} \right) = \sum_{i=1}^{m} \sum_{\mathbf{p}\in I} \partial_1^{p_1} \cdots \lambda_i^k (\partial_i + k\alpha_i)(\partial_i - k)^{p_i} \cdots \partial_m^{p_m} \otimes w_{\mathbf{p}} \in W, \qquad (4.2)$$

where I is a finite subset of $\mathbb{Z}_+^m$ and $w_{\mathbf{p}}$ are nonzero vectors of $\mathrm{Ind}(M)$.

By Proposition 2.6, we respectively consider the coefficient of $\lambda_{i'}^k k^{p_{i'}}$ and $\lambda_{i'}^k k^{p_{i'}+1}$ in (4.1) and (4.2), one has

$$\partial_1^{p_1} \cdots \partial_{i'-1}^{p_{i'-1}} \partial_{i'+1}^{p_{i'+1}} \cdots \partial_m^{p_m} \otimes w_{\mathbf{p}} \in W,$$

where $m \in \mathbb{N}, w_{\mathbf{p}}$ are nonzero vectors of $\mathrm{Ind}(M)$. Then

$$\partial_1^{p_1} \cdots \partial_{i'-1}^{p_{i'-1}} \partial_{i'+1}^{p_{i'+1}} \cdots \partial_m^{p_m} \otimes w_{\mathbf{p}} + \text{lower terms}$$

has lower degree than u, which is contrary to the choice of u. Hence, $\deg(u) = 0$.

By Lemma 4.3, we see that $\left(\bigotimes_{i=1}^m \Omega(\lambda_i, \alpha_i, \beta_i)\right) \otimes w_{\mathbf{0}}$ can be generated by $\underbrace{1 \otimes \cdots \otimes 1}_{m} \otimes w_{\mathbf{0}}$. It follows that $\left(\bigotimes_{i=1}^m \Omega(\lambda_i, \alpha_i, \beta_i)\right) \otimes \mathcal{U}(\mathcal{H}) w_{\mathbf{0}} \subseteq W$. Thus, $W = \left(\bigotimes_{i=1}^m \Omega(\lambda_i, \alpha_i, \beta_i)\right) \otimes \mathrm{Ind}(M)$, since the nonzero $\mathcal{H}$-submodule $\mathcal{U}(\mathcal{H}) w_{\mathbf{0}}$ of $\mathrm{Ind}(M)$ generated by $w_{\mathbf{0}}$ is equal to $\mathrm{Ind}(M)$ by the irreducibility of $\mathrm{Ind}(M)$. This completes the proof of Theorem 4.4. □

Remark 4.5. When $S' = \emptyset$ in Theorem 4.4, it was studied in [17].

It follows from Lemma 4.1 and Theorem 4.4 that we have the following remark.

Remark 4.6. Let $m \in \mathbb{N}, S = \{1, \ldots, m\}, S' \subseteq S$ and $\lambda_i \in \mathbb{C}^*$ for $i \in S$. Let $\beta_i, \alpha_j \in \mathbb{C}^*, \beta_j = 0$ for $i \in S', j \in S\backslash S'$. Assume $\mathrm{Ind}(M)$ is an $\mathcal{H}$-module defined by (2.1) for which M satisfies the conditions in Theorem 2.5. Then the tensor product $\left(\bigotimes_{i=1}^m \Omega(\lambda_i, \alpha_i, \beta_i)\right) \otimes \mathrm{Ind}(M)$ is an irreducible $\mathcal{H}$-module if and only if the λ_i pairwise distinct. The irreducibility of the $\mathcal{H}$-modules $\left(\bigotimes_{i=1}^m \Omega(\lambda_i, \alpha_i, \beta_i)\right) \otimes \mathrm{Ind}(M)$ was also given in Examples 8 and 10 in [14].

5. Isomorphism Classes

In this section, we will give isomorphism results for modules $\left(\bigotimes_{i=1}^m \Omega(\lambda_i, \alpha_i, \beta_i)\right) \otimes \mathrm{Ind}(M)$. We denote the number of elements in set A by $\mathrm{card}(A)$.

Theorem 5.1. *Let $m, n \in \mathbb{N}, S = \{1, \ldots, m\}, T = \{1, \ldots, n\}, S' \subseteq S, T' \subseteq T, \lambda_i, \mu_j \in \mathbb{C}^*$ with the λ_i pairwise distinct as well as μ_j pairwise distinct for $i \in S, j \in T$. Let $\beta_{i'}, \alpha_i \in \mathbb{C}^*, \beta_i = 0$ and $d_{j'}, c_j \in \mathbb{C}^*, d_j = 0$ for*

$i' \in S', i \in S\backslash S', j' \in T', j \in T\backslash T'$. Assume $\mathrm{Ind}(M_1)$ and $\mathrm{Ind}(M_2)$ are $\mathcal{H}$-modules defined by (2.1) for which M_1 and M_2 satisfy the conditions in Theorem 2.5. Then $\big(\bigotimes_{i=1}^m \Omega(\lambda_i,\alpha_i,\beta_i)\big) \otimes \mathrm{Ind}(M_1)$ and $\big(\bigotimes_{j=1}^n \Omega(\mu_j,c_j,d_j)\big) \otimes \mathrm{Ind}(M_2)$ are isomorphic as $\mathcal{H}$-modules if and only if $m=n, \mathrm{card}(S')=\mathrm{card}(T'), \mathrm{Ind}(M_1)\cong \mathrm{Ind}(M_2)$ as $\mathcal{H}$-modules and $(\lambda_i,\alpha_i,\beta_i)=(\mu_{i'},c_{i'},d_{i'})$ and $(\lambda_j,\alpha_j)=(\mu_{j'},c_{j'}), \beta_j=d_{j'}=0$ for $i\in S', i'\in T', j\in S\backslash S', j'\in T\backslash T'$ ($\varphi_1 : S'\to T'$ and $\varphi_2 : S\backslash S'\to T\backslash T'$ are both bijections).

Proof. The sufficiency is trivial. We denote $\Omega(\lambda_i,\alpha_i,\beta_i)=\mathbb{C}[\partial_i], \Omega(\mu_j,c_j,d_j)=\mathbb{C}[\widetilde{\partial}_j], V_1=\big(\bigotimes_{i=1}^m \Omega(\lambda_i,\alpha_i,\beta_i)\big)\otimes \mathrm{Ind}(M_1)$ and $V_2=\big(\bigotimes_{j=1}^n \Omega(\mu_j,c_j,d_j)\big)\otimes \mathrm{Ind}(M_2)$, respectively.

Let ϕ be an isomorphism from V_1 to V_2. Take a nonzero element $\underbrace{1\otimes\cdots\otimes 1}_{m}\otimes w\in V_1$. Assume

$$\phi\left(\underbrace{1\otimes\cdots\otimes 1}_{m}\otimes w\right)=\sum_{\mathbf{p}\in I}\widetilde{\partial_1}^{p_1}\cdots\widetilde{\partial_n}^{p_n}\otimes v_{\mathbf{p}}, \tag{5.1}$$

where I is a finite subset of $\mathbb{Z}_+^n$ and $v_{\mathbf{p}}$ are nonzero vectors of V_2. There exists a positive integer $K=\max\{K(w),K(v_{\mathbf{p}})\mid \mathbf{p}\in\mathbb{Z}_+^n\}$ such that $I_m\cdot w=I_m\cdot v_{\mathbf{p}}=L_m\cdot w=L_m\cdot v_{\mathbf{p}}=0$ for all integers $m\geq K$ and $\mathbf{p}\in\mathbb{Z}_+^n$.

First consider $d_{j'}\in\mathbb{C}^*, d_j=0$ for $j'\in T', j\in T\backslash T'$. We note that $m=\mathrm{card}(S')+\mathrm{card}(S\backslash S'), n=\mathrm{card}(T')+\mathrm{card}(T\backslash T')$. For enough large $k\in\mathbb{Z}$, we know that

$$\begin{aligned}\sum_{j=1}^m \lambda_j^k\beta_j\phi\big(\underbrace{1\otimes\cdots\otimes 1}_{m}\otimes w\big)&=\phi\big(I_k\big(\underbrace{1\otimes\cdots\otimes 1}_{m}\otimes w\big)\big)=I_k\phi\big(\underbrace{1\otimes\cdots\otimes 1}_{m}\otimes w\big)\\ &=\sum_{j'=1}^n\sum_{\mathbf{p}\in\mathbb{Z}_+^n}\widetilde{\partial_1}^{p_1}\cdots\mu_{j'}^k d_{j'}(\widetilde{\partial_{j'}}-k)^{p_{j'}}\cdots\widetilde{\partial_n}^{p_n}\otimes v_{\mathbf{p}}.\end{aligned} \tag{5.2}$$

According to Proposition 2.6 in (5.2), we get $p_{j'}=0, \lambda_j=\mu_{j'}, \mathrm{card}(T')=\mathrm{card}(S')$, where $j\in S', j'\in T'$ and $\varphi_1 : S'\to T'$ is bijection. Then $\phi(\underbrace{1\otimes\cdots\otimes 1}_{m}\otimes w)=\sum_{\widehat{\mathbf{p}}\in I}\widetilde{\partial_1}^{\widehat{p_1}}\cdots\widetilde{\partial_n}^{\widehat{p_n}}\otimes v_{\widehat{\mathbf{p}}}$, where $\widehat{p_{j'}}=0$ for $j'\in T'$. Now we consider $c_j\in\mathbb{C}^*$ for $j\in T\backslash T'$. For enough large $k\in\mathbb{Z}$, it follows from $\phi\big(L_k(\underbrace{1\otimes\cdots\otimes 1}_{m}\otimes w)\big)=L_k\phi(\underbrace{1\otimes\cdots\otimes 1}_{n}\otimes w)$ that we have

$$\sum_{i=1}^{m}\Big(\lambda_i^k\phi(1\otimes\cdots\otimes\partial_i\otimes\cdots\otimes 1\otimes w)+\lambda_i^k k\alpha_i\phi\Big(\underbrace{1\otimes\cdots\otimes 1}_{m}\otimes w\Big)\Big)$$
$$=\sum_{j=1}^{n}\sum_{\widehat{\mathbf{p}}\in\mathbb{Z}_+^n}\widetilde{\partial}_1^{\widehat{p_1}}\cdots\mu_j^k(\widetilde{\partial}_j+kc_j)(\widetilde{\partial}_j-k)^{\widehat{p_j}}\cdots\widetilde{\partial}_n^{\widehat{p_n}}\otimes v_{\widehat{\mathbf{p}}}. \tag{5.3}$$

Using Proposition 2.6 in (5.3), one can easily to check that $\widehat{p_j}=0,\lambda_i=\mu_j,\mathrm{card}(T\backslash T')=\mathrm{card}(S\backslash S')$, where $i\in S\backslash S', j\in T\backslash T'$ and $\varphi_2: S\backslash S'\to T\backslash T'$ is bijection. Then $m=\mathrm{card}(T\backslash T')+\mathrm{card}(T)=\mathrm{card}(S\backslash S')+\mathrm{card}(S)=n$ and

$$\phi\big(\underbrace{1\otimes\cdots\otimes 1}_{m}\otimes w\big)=\underbrace{1\otimes\cdots\otimes 1}_{m}\otimes v_{\mathbf{0}}$$

Thus, (5.2) can be rewritten as

$$\sum_{i=1}^{m}\lambda_i^k\beta_i\phi\big(\underbrace{1\otimes\cdots\otimes 1}_{m}\otimes w\big)=\sum_{i=1}^{m}\mu_i^k d_i\big(\underbrace{1\otimes\cdots\otimes 1}_{m}\otimes v_{\mathbf{0}}\big),$$

we obtain $\beta_i=d_{i'}$ for $i\in S', i'\in T'$, which can be obtained by $\phi\big(\underbrace{1\otimes\cdots\otimes 1}_{m}\otimes w\big)\neq 0,\lambda_i=\mu_{i'}$ and Proposition 2.6. We note that $\beta_j=d_{j'}=0$ for $j\in S\backslash S', j'\in T\backslash T'$. Then for enough large $k\in\mathbb{Z}$, by $\phi\big(L_k(\underbrace{1\otimes\cdots\otimes 1}_{m}\otimes w)\big)=L_k(\underbrace{1\otimes\cdots\otimes 1}_{m}\otimes v_{\mathbf{0}})$ and $\lambda_i=\mu_{i'},\lambda_j=\mu_{j'}$, we can easily check that $\alpha_i=c_{i'},\alpha_j=c_{j'}$ for $i\in S', i'\in T', j\in S\backslash S', j'\in T\backslash T'$.

There exists a linear bijection $\tau: V_1\to V_2$ such that

$$\phi\left(\underbrace{1\otimes\cdots\otimes 1}_{m}\otimes v\right)=\underbrace{1\otimes\cdots\otimes 1}_{m}\otimes\tau(v)$$

for all $v\in V_1$. Meanwhile we conclude that $\tau(L_kv)=L_k\tau(v)$ for all $k\in\mathbb{Z}, v\in V_1$. Then from

$$\phi\big(I_k\big(\underbrace{1\otimes\cdots\otimes 1}_{m}\otimes v\big)\big)=I_k\phi\big(\underbrace{1\otimes\cdots\otimes 1}_{m}\otimes v\big)$$

and

$$\phi\big(C_i\big(\underbrace{1\otimes\cdots\otimes 1}_{m}\otimes v\big)\big)=C_i\phi\big(\underbrace{1\otimes\cdots\otimes 1}_{m}\otimes v\big),$$

we see that $\tau(I_kv)=I_k\tau(v)$ and $\tau(C_iv)=C_i\tau(v)$ for $i=1,2,3,k\in\mathbb{Z}$, respectively. Thus, $V_1\cong V_2$ as $\mathcal{H}$-modules for $d_{j'},c_j\in\mathbb{C}^*,d_j=0$ for $j'\in T', j\in T\backslash T'$.

To sum up, we obtain $m=n, V_1\cong V_2,(\lambda_i,\alpha_i,\beta_i)=(\mu_{i'},c_{i'},d_{i'}),(\lambda_j,\alpha_j)=(\mu_{j'},c_{j'})$ and $\beta_j=d_{j'}=0$ for $i\in S', i'\in T', j\in S\backslash S', j'\in T\backslash T'$. This completes the proof. □

Remark 5.2. When $S' = T' = \emptyset$ in Theorem 5.1, it was investigated in [17].

The isomorphism problem of the $\mathcal{H}$-modules $\big(\bigotimes_{i=1}^m \Omega(\lambda_i, \alpha_i, \beta_i)\big) \otimes \mathrm{Ind}(M)$ was also given in Theorem 35 in [14].

6. New Irreducible Modules

In this section, we prove that $\big(\bigotimes_{i=1}^m \Omega(\lambda_i, \alpha_i, \beta_i)\big) \otimes \mathrm{Ind}(M)$ is not isomorphic to $\mathrm{Ind}(M)$ or $\mathcal{M}\big(V, \Omega(\lambda, \alpha, \beta)\big)$ or $\widetilde{\mathcal{M}}(W, \gamma(t))$ or $\mathcal{A}_{\alpha,\beta}$, i.e., $\big(\bigotimes_{i=1}^m \Omega(\lambda_i, \alpha_i, \beta_i)\big) \otimes \mathrm{Ind}(M)$ is a class of new irreducible $\mathcal{H}$-modules.

For any $l, m \in \mathbb{Z}$, $s \in \mathbb{Z}_+$, define a sequence of operators $T_{l,m}^{(s)}$ as follows

$$T_{l,m}^{(s)} = \sum_{i=0}^{s} (-1)^{s-i} \binom{s}{i} I_{l-m-i} I_{m+i}.$$

For $d \in \{0, 1\}, r \in \mathbb{Z}_+$, denote by $\mathcal{H}_{r,d}$ the Lie subalgebra of $\mathcal{H}_{+,d} = \mathrm{span}_{\mathbb{C}}\{L_i, I_j \mid i \geq 0, j \geq d\}$ generated by L_i, I_j for all $i > r, j > r+d$. Now we write $\bar{\mathcal{H}}_{r,d}$ the quotient algebra $\mathcal{H}_{+,d}/\mathcal{H}_{r,d}$, and $\bar{L}_i, \bar{I}_{i+d}$ the respective images of L_i, I_{i+d} in $\bar{\mathcal{H}}_{r,d}$.

Assume that $d \in \{0, 1\}, r \in \mathbb{Z}_+$ and V is an $\bar{\mathcal{H}}_{r,d}$-module. For any fixed $\gamma(t) = \sum_i c_i t^i \in \mathbb{C}[t, t^{-1}]$, the action of $\mathcal{H}$ on $V \otimes \mathbb{C}[t, t^{-1}]$ can be defined as follows

$$L_m \circ (v \otimes t^n) = (L_m + \sum_i c_i I_{m+i})(v \otimes t^n),$$

$$I_m \circ (v \otimes t^n) = I_m(v \otimes t^n),\ C_i \circ (v \otimes t^n) = 0,$$

where $m, n \in \mathbb{Z}, v \in V$ and $i = 1, 2, 3$. Then $V \otimes \mathbb{C}[t, t^{-1}]$ is made into an $\mathcal{H}$-module, which is denoted by $\widetilde{\mathcal{M}}(V, \gamma(t))$. It is clear that $\widetilde{\mathcal{M}}(V, \gamma(t))$ is a weight $\mathcal{H}$-module if and only if $\gamma(t) \in \mathbb{C}$. Moreover, the $\mathcal{H}$-module $\widetilde{\mathcal{M}}(V, \gamma(t))$ for $\gamma(t) \in \mathbb{C}[t, t^{-1}]$ is irreducible if and only if V is irreducible (see [4]).

Let $d \in \{0, 1\}, r \in \mathbb{Z}_+$ and V be an $\bar{\mathcal{H}}_{r,d}$-module. For any $\lambda, \alpha, \beta \in \mathbb{C}$, define an $\mathcal{H}$-action on the vector space $\mathcal{M}\big(V, \Omega(\lambda, \alpha, \beta)\big) := V \otimes \mathbb{C}[t]$ as follows

$$L_m\big(v \otimes f(t)\big) = v \otimes \lambda^m (t - m\alpha) f(t-m) + \sum_{i=0}^{r} \Big(\frac{m^{i+1}}{(i+1)!} \bar{L}_i\Big) v \otimes \lambda^m f(t-m),$$

$$I_m\big(v \otimes f(t)\big) = \sum_{i=0}^{r} \Big(\frac{m^{i+d}}{(i+d)!} \bar{I}_{i+d}\Big) v \otimes \lambda^m \beta f(t-m),\ C_i\big(v \otimes f(t)\big) = 0,$$

where $i \in \{1, 2, 3\}, m \in \mathbb{Z}, v \in V, f(t) \in \mathbb{C}[t]$. We note that $\mathcal{M}\big(V, \Omega(\lambda, \alpha, \beta)\big)$ is reducible if and only if $V \cong V_{\alpha, \delta_{d,0}\tau}$ for some $\tau \in \mathbb{C}$ such that $\delta_{d,0}\beta\tau = 0$ (see [5]).

Lemma 6.1. *Assume that $\lambda_i \in \mathbb{C}^*, \alpha, \beta, \alpha_i, \beta_i \in \mathbb{C}, s \in \mathbb{Z}_+, d \in \{0, 1\}$ and M is an irreducible $\mathcal{H}_e$-module satisfying the conditions in Theorem 2.5. Let r' be the maximal nonnegative integer such that $\bar{I}_{r'+d} V \neq 0$. Then we obtain*

(i) *the action of* L_m *for* m *sufficiently large is not locally nilpotent on* $\big(\bigotimes_{i=1}^m \Omega(\lambda_i,\alpha_i,\beta_i)\big)\otimes \mathrm{Ind}(M)$;

(ii) *the action of* $T_{l,m}^{(s)}$ *on* $\mathcal{A}_{\alpha,\beta}$ *is trivial for* $l,m\in\mathbb{Z}$;

(iii) $T_{l,m}^{(1)}$ *acts nontrivially on* $\big(\bigotimes_{i=1}^m \Omega(\lambda_i,\alpha_i,\beta_i)\big)\otimes\mathrm{Ind}(M)$ *whenever* $m\ll 0$ *and* $l\ll m$;

(iv) *The action of* $T_{l,m}^{(s)}$ *on* $\mathcal{M}\big(V,\Omega(\lambda,\alpha,\beta)\big)$ *and* $\widetilde{\mathcal{M}}(V,\gamma(t))$ *are trivial for* $s>2(r'+d)$.

Proof. (i) follows from the local nilpotency of L_m on $\mathrm{Ind}(M)$ by Theorem 2.5 for m sufficiently large and its non-local nilpotency on $\Omega(\lambda,\alpha,\beta)$. (ii) follows from (3.1). (iii) can be obtained by the similar compute in Lemma 5.1 (v) of [5]. (iv) follows from [4, Lemma 3.3]. □

We are now ready to state the main result of this section.

Proposition 6.2. *Assume that* $d\in\{0,1\}$, $r,e\in\mathbb{Z}_+,\alpha,\beta\in\mathbb{C}$, M *is an irreducible* $\mathcal{H}_e$*-module satisfying the conditions in Theorem* 2.5. *Then we have* $\big(\bigotimes_{i=1}^m \Omega(\lambda_i,\alpha_i,\beta_i)\big)\otimes\mathrm{Ind}(M)$ *is not isomorphic to* $\mathrm{Ind}(M')$, *or* $\mathcal{M}\big(V,\Omega(\lambda,\alpha',\beta')\big)$, *or* $\widetilde{\mathcal{M}}(W,\gamma(t))$, *or* $\mathcal{A}_{\alpha,\beta}$.

Proof. From Lemma 6.1 (i), we have $\big(\bigotimes_{i=1}^m \Omega(\lambda_i,\alpha_i,\beta_i)\big)\otimes\mathrm{Ind}(M)\ncong\mathrm{Ind}(M')$. Let $m\ll 0, l\ll m$ that $I_{l-m},I_m\notin\mathcal{H}_e$ and $s>2(r'+d)$. For any $1\otimes v\in\big(\bigotimes_{i=1}^m \Omega(\lambda_i,\alpha_i,\beta_i)\big)\otimes\mathrm{Ind}(M)$, noting that the action of I_m on 1 is scalar for any $m\in\mathbb{Z}$, we deduce that

$$\begin{aligned}
&T_{l,m}^{(s)}(1\otimes v)\\
&\quad=\sum_{i=0}^{s}(-1)^{s-i}\binom{s}{i}\big(I_{l-m-i}I_{m+i}(1)\otimes v+I_{m+i}(1)\otimes I_{l-m-i}v\\
&\qquad+I_{l-m-i}(1)\otimes I_{m+i}v+1\otimes I_{l-m-i}I_{m+i}v\big)\neq 0.
\end{aligned}$$

Then by Lemma 6.1 (iv), we obtain that $\big(\bigotimes_{i=1}^m \Omega(\lambda_i,\alpha_i,\beta_i)\big)\otimes\mathrm{Ind}(M)$ is not isomorphic to $\mathcal{M}\big(V,\Omega(\lambda,\alpha',\beta')\big)$, or $\widetilde{\mathcal{M}}(W,\gamma(t))$.

At last, by Lemma 6.1 (ii) and (iii), we get that $\big(\bigotimes_{i=1}^m \Omega(\lambda_i,\alpha_i,\beta_i)\big)\otimes\mathrm{Ind}(M)$ is not isomorphic to $\mathcal{A}_{\alpha,\beta}$. □

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References

[1] Arbarello, E., DeConcini, C., Kac, V.G., Procesi, C.: Moduli spaces of curves and representation theory. Commun. Math. Phys. **117**, 1–36 (1988)

[2] Chen, H., Guo, X.: New simple modules for the Heisenberg–Virasoro algebra. J. Algebra **390**, 77–86 (2013)

[3] Chen, H., Guo, X.: Non-weight modules over the Heisenberg–Virasoro algebra and the W algebra $W(2,2)$. J. Algebra Appl. **16**, 1750097 (2017)

[4] Chen, H., Han, J., Su, Y.: A class of simple weight modules over the twisted Heisenberg–Virasoro algebra. J. Math. Phys. **57**, 101705 (2016). 7 pp

[5] Chen, H., Han, J., Su, Y., Yue, X.: Two classes of non-weight modules over the twisted Heisenberg-Virasoro algebra. Manuscr. Math. **160**, 265–284 (2019)

[6] Han, J., Chen, Q., Su, Y.: Modules over the algebra $\mathcal{V}ir(a,b)$. Linear Algebra Appl. **515**, 11–23 (2017)

[7] Kaplansky, I., Santharoubane, L.J.: Harish–Chandra modules over the Virasoro algebra. Math. Sci. Res. Inst. Publ. **4**, pp. 217–231. Springer, New York (1985)

[8] Lü, R., Guo, X., Zhao, K.: Irreducible modules over the Virasoro algebra. Doc. Math. **16**, 709–721 (2011)

[9] Liu, D., Jiang, C.: Harish–Chandra modules over the twisted Heisenberg–Virasoro algebra. J. Math. Phys. **49**, 012901 (2008). 13 pp

[10] Liu, D., Wu, Y., Zhu, L.: Whittaker modules for the twisted Heisenberg–Virasoro algebra. J. Math. Phys. **51**, 023524 (2010). 12 pp

[11] Lü, R., Zhao, K.: Irreducible Virasoro modules from irreducible Weyl modules. J. Algebra **414**, 271–287 (2014)

[12] Lü, R., Zhao, K.: Classification of irreducible weight modules over the twisted Heisenberg–Virasoro algebra. Commun. Contemp. Math. **12**, 183–205 (2010)

[13] Lü, R., Zhao, K.: A family of simple weight Virasoro modules. J. Algebra **479**, 437–460 (2017)

[14] Lü, R., Zhao, K.: Generalized oscillator representations of the twisted Heisenberg–Virasoro algebra. Algebr. Represent. Theory (2019). https://doi.org/10.1007/s10468-019-09897-1

[15] Radobolja, G.: Subsingular vectors in Verma modules, and tensor product of weight modules over the twisted Heisenberg–Virasoro algebra and $W(2,2)$ algebra. J. Math. Phys. **54**, 071701 (2013). 24 pp

[16] Shen, R., Su, Y.: Classification of irreducible weight modules with a finite-dimensional weight space over twisted Heisenberg–Virasoro algebra. Acta Math. Sin. (E. S.) **23**, 189–192 (2007)

[17] Tan, H., Zhao, K.: Irreducible Virasoro modules from tensor products (II). J. Algebra **394**, 357–373 (2013)

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