Results Math (2019) 74:182

1422-6383/19/040001-15
published online October 4, 2019
https://doi.org/10.1007/s00025-019-1105-9

Results in Mathematics

Edge Metric Dimension of Some Generalized Petersen Graphs

Vladimir Filipović, Aleksandar Kartelj, and Jozef Kratica

Abstract. The edge metric dimension problem was recently introduced, which initiated the study of its mathematical properties. The theoretical properties of the edge metric representations and the edge metric dimension of generalized Petersen graphs $GP(n,k)$ are studied in this paper. We prove the exact formulae for $GP(n,1)$ and $GP(n,2)$, while for other values of k a lower bound is stated.

Mathematics Subject Classification. 05C12, 68R10, 90C35.

Keywords. Edge metric dimension, generalized Petersen graphs, discrete mathematics.

1. Introduction

The concept of the metric dimension of graph G was introduced independently by Slater [1] and Harary and Melter [2]. This concept is based on the notion of a resolving set R of vertices, which has the property that each vertex is uniquely identified by its metric representation with respect to R. The minimal cardinality of resolving sets is called the metric dimension of the graph G.

1.1. Literature Review

Kelenc et al. [3] recently introduced a similar concept of edge metric dimension and initiated the study of its mathematical properties. They made a comparison between the edge metric dimension and the standard metric dimension of graphs while presenting realization results concerning the edge metric dimension and the standard metric dimension of graphs. They also proved that

This research was partially supported by Serbian Ministry of Education, Science and Technological Development under the Grants Nos. 174010 and 174033.

the edge metric dimension problem is NP-hard and provided approximation results. Additionally, for several classes of graphs, exact values for the edge metric dimension were presented, while several others were given upper and lower bounds. In [4] and [5], the authors presented results of the mixed metric dimension alongside the edge metric dimension for some classes of graphs. Additionally, Peterin and Yero [6], provided exact formulas for join, lexicographic and corona products of graphs.

Zubrilina [7], firstly proposed the classification of graphs of n vertices for which the edge metric dimension is equal to its upper bound $n-1$. The second result states that the ratio between the edge metric dimension and the metric dimension of an arbitrary graph is not bounded from above. The third result characterizes the change of the edge dimension of an arbitrary graph upon taking a Cartesian product with a path, and changes of the edge dimension upon adding a vertex adjacent to all the original vertices. The edge metric dimension of the Erdös-Rényi random graph $G(n, p)$ is given by Zubrilina [8] and it is equal to $(1 + o(1)) \cdot \frac{4log(n)}{log(1/q)}$, where $q = 1 - 2p(1 - p)^2(2 - p)$.

Independently, Epstein et al. [9], introduced another edge metric dimension definition related to the line graphs, where a line graph of a graph $G(V, E)$ is defined as: $L(G) = (E, F)$ where $F = \{e_ie_j | e_i, e_j \in E, e_i$ is incident with $e_j\}$. Their edge metric dimension of a graph G is defined as the metric dimension of $L(G)$, which is called edge variant of metric dimension by some authors, e.g. Liu et al. [10].

1.2. Generalized Petersen Graphs

Generalized Petersen graphs were first studied by Coxeter [11]. Each such graph, denoted as $GP(n, k)$, is defined for $n \geq 3$ and $1 \leq k < n/2$. It has $2n$ vertices and $3n$ edges, with vertex set $V(GP(n, k)) = \{u_i, v_i \mid 0 \leq i \leq n - 1\}$ and edge set $E(GP(n, k)) = \{u_iu_{i+1},\ u_iv_i,\ v_iv_{i+k} \mid 0 \leq i \leq n - 1\}$. It should be noted that vertex indices are taken modulo n.

Example. Consider the Petersen graph, numbered $GP(5, 2)$, shown in Fig. 1. It is easily calculated by using the total enumeration technique, that its edge metric dimension is equal to 4 (it is also presented in Table 6). Figure 2 shows $GP(6, 1)$ where the edge metric dimension equals 3. This can also be concluded by Theorem 2.4.

The metric dimension of generalized Petersen graphs $GP(n, k)$ is studied for different values of k:

- Case $k = 1$ is concluded in [12];
- Case $k = 2$ is proven in [13];
- Case $k = 3$ in [14].

Various other properties of generalized Petersen graphs have recently been theoretically investigated in the following areas: Hamiltonian property [15], the cop number [16], the total coloring [17], etc.

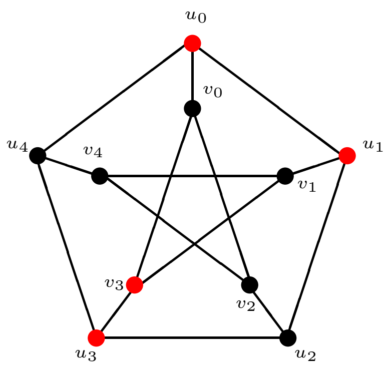

Figure 1. Petersen graph GP(5,2)—edge metric base is colored red (color figure online)

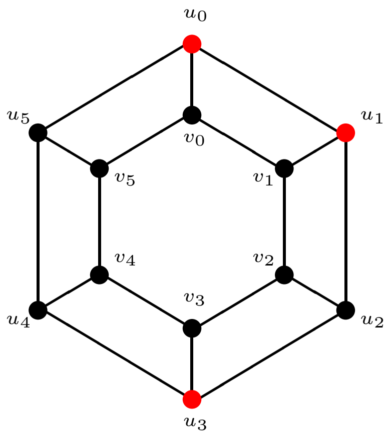

Figure 2. Graph GP(6,1)—edge metric base is colored red (color figure online)

1.3. Definitions and Previous Work

Given a simple connected undirected graph $G = (V, E)$, by $d(u, v)$ we denote the distance between two vertices $u, v \in V$, i.e. the length of a shortest $u - v$ path. A vertex x of the graph G is said to resolve two vertices u and v of G if $d(x, u) \neq d(x, v)$. An ordered vertex set $R = \{x_1, x_2, ..., x_k\}$ of G is a resolving set of G if every two distinct vertices of G are resolved by some vertex of R. A metric basis of G is a resolving set of minimum cardinality. The metric dimension of G, denoted by $\beta(G)$, is the cardinality of its metric basis.

Similarly, for a given connected graph G, a vertex $w \in V$, and an edge $uv \in E$, the distance between the vertex w and the edge uv is defined as

Table 1. Edge metric representations for $GP(6,1)$ with respect to S_1

e	$r(e)$	e	$r(e)$
u_0u_1	$(0,0,2)$	u_0u_5	$(0,1,2)$
u_0v_0	$(0,1,3)$	u_1u_2	$(1,0,1)$
u_1v_1	$(1,0,2)$	u_2u_3	$(2,1,0)$
u_2v_2	$(2,1,1)$	u_3u_4	$(2,2,0)$
u_3v_3	$(3,2,0)$	u_4u_5	$(1,2,1)$
u_4v_4	$(2,3,1)$	u_5v_5	$(1,2,2)$
v_0v_1	$(1,1,3)$	v_0v_5	$(1,2,3)$
v_1v_2	$(2,1,2)$	v_2v_3	$(3,2,1)$
v_3v_4	$(3,3,1)$	v_4v_5	$(2,3,2)$

$d(w,uv) = \min\{d(w,u), d(w,v)\}$. A vertex $w \in V$ resolves two edges e_1 and e_2 $(e_1, e_2 \in E)$, if $d(w,e1) \neq d(w,e2)$. A set S of vertices in a connected graph G is an edge metric generator for G if every two edges of G are resolved by some vertex of S. The smallest cardinality of an edge metric generator of G is called the edge metric dimension and is denoted by $\beta_E(G)$. An edge metric basis for G is the edge metric generator of G with cardinality $\beta_E(G)$. Given an edge $e \in E$ and an ordered vertex set $S = \{x_1, x_2, ..., x_k\}$, the k-touple $r(e,S) = (d(e,x_1), d(e,x_2), ..., d(e,x_k))$ is called the edge metric representation of e with respect to S.

Example. Consider the generalized Petersen graph $GP(6,1)$ given in Fig. 2. The set $S_1 = \{u_0, u_1, u_3\}$ is an edge metric generator for G since the vectors of metric coordinates for edges of G with respect to S_1 are mutually different. This can be seen in Table 1.

From Corollary 2.3, it holds that for $GP(6,1)$, as for any other generalized Petersen graph, the cardinality of an edge metric generator must be at least 3, so S_1 is an edge metric basis for $GP(6,1)$. This implies that its edge metric dimension is equal to 3, i.e. $\beta_E(GP(6,1)) = 3$.

Two edges are incident, if both contain one common endpoint. For a given vertex $v \in V$, its degree deg_v is equal to the number of its neighbors, i.e. the number of edges in which it is the endpoint. The maximum and minimum degrees over all vertices of graph G are denoted as $\Delta(G)$ and $\delta(G)$, respectively. Formally, $\Delta(G) = \max_{v\in V} \deg_v$ and $\delta(G) = \min_{v\in V} \deg_v$.

In the following text, we briefly show three propositions from [3] that are relevant for our work. The first proposition is related to paths and cycles and classification of graphs having the edge metric dimension equal to 1. The last two give bounds of the edge metric dimension based on the degree of vertices.

Proposition 1.1 [3]. *For $n \geq 2$ it holds $\beta_E(P_n) = \beta(P_n) = 1$, $\beta_E(C_n) = \beta(C_n) = 2$, $\beta_E(K_n) = \beta(K_n) = n - 1$. Moreover, $\beta_E(G) = 1$ if and only if G is a path P_n.*

Proposition 1.2 [3]. *Let G be a connected graph and let $\Delta(G)$ be the maximum degree of G. Then $\beta_E(G) \geq log_2 \Delta(G)$*

Proposition 1.3 [3]. *Let G be a connected graph and let S be an edge metric basis with $|S| = k$. Then S does not contain a vertex with the degree greater than 2^{k-1}.*

From each of these propositions, given in [3], the next two corollaries follow:

Corollary 1.4. *The edge metric dimension of any 3-regular graph is at least 2.*

Corollary 1.5. $\beta_E(GP(n, k)) \geq 2$.

As previously mentioned, Epstein et al. [9] introduced another edge metric dimension definition based on line graphs. In order to avoid any misunderstanding, $\beta'_E(G)$ will denote this second definition of the edge metric dimension of graph G, also called the edge version of metric dimension [10], i.e. $\beta'_E(G) = \beta(L(G))$. Based on this definition, in [18], the authors obtained the results for n-sunlet graphs and prism graphs.

Difference between these definitions can be demonstrated with the following example. Let $G_1 = (V_1, E_1)$ be the graph with $V_1 = \{v_0, v_1, v_2, v_3\}$ and $E_1 = \{e_0, e_1, e_2, e_3, e_4\}$ such that $e_0 = v_0v_1,\ e_1 = v_1v_2,\ e_2 = v_0v_2,\ e_3 = v_1v_3,\ e_4 = v_2v_3$. Line graph of G_1 is $L(G_1) = (E_1, F_1)$ where $F_1 = \{e_0e_1, e_0e_2, e_0e_3, e_1e_2, e_1e_3, e_1e_4, e_2e_4, e_3e_4\}$. Graphs G_1 and $L(G_1)$ are presented in Fig. 3.

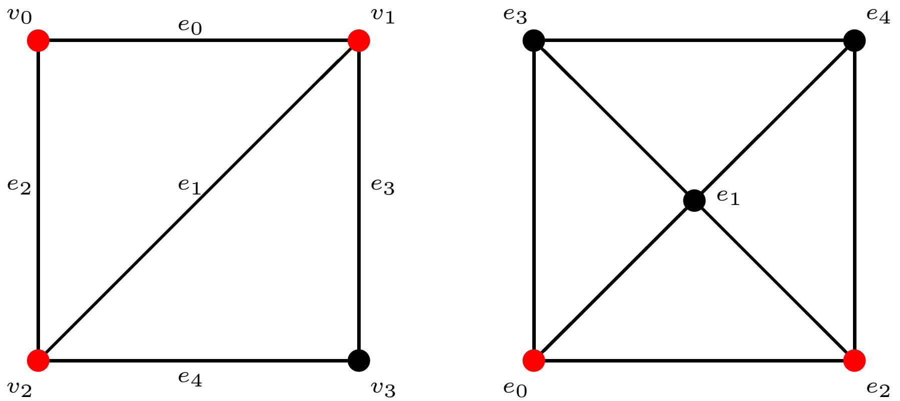

FIGURE 3. Graph from Example 1 and its corresponding line graph (edge metric base is colored red) (color figure online)

By using the total enumeration technique, it can be shown that $\beta_E(G_1) = 3$ with a edge metric base $\{v_0, v_1, v_2\}$. On the other hand, $\beta'_E(G_1) = \beta(L(G_1)) = 2$ with a metric base $\{e_0, e_2\} = \{v_0v_1, v_0v_2\}$.

2. Main Results

2.1. Lower Bound

Having in mind the fact that vertices from an edge metric base are also endpoints for some (incident) edges, the bound presented in Proposition 1.2, could be improved in some cases.

Theorem 2.1. *Let G be a connected graph and let $\delta(G)$ be the minimum degree of G. Then, $\beta_E(G) \geq 1 + \lceil log_2\delta(G)\rceil$.*

Proof. Let $S = \{w_1, w_2, ..., w_p\}$ be an edge metric generator of graph G with a minimal cardinality, i.e. $p = \beta_E(G)$. Vertex w_1 is incident to at least $\delta(G)$ edges. Name them e_1, ..., $e_{\delta(G)}$. Since w_1 is incident with e_1, ..., $e_{\delta(G)}$, it is obvious that $d(e_1, w_1) = ... = d(e_{\delta(G)}, w_1) = 0$. By the definition of the distance between vertex and edge, it is clear that for an arbitrary vertex $v \in V(G)$ there can be only two different distances to a set of incident edges. Then, for each i, $i = 2, ..., p$, distances $d(e_1, w_i)$, ..., $d(e_{\delta(G)}, w_i)$ have only two different values, so since $d(e_1, w_1) = ... = d(e_{\delta(G)}, w_1) = 0$, there exist at most 2^{p-1} different edge metric representations of edges e_1, ..., $e_{\delta(G)}$ with respect to S, so $\delta(G) \leq 2^{p-1}$. Next, because p is integer, it follows that $\lceil log_2\delta(G)\rceil \leq p-1$ $\Rightarrow \beta_E(G) = p \geq 1 + \lceil log_2\delta(G)\rceil$. □

In the case of regular graphs, the bound presented in Proposition 1.2 is improved by 1.

Corollary 2.2. *Let G be an r-regular graph. Then, $\beta_E(G) \geq 1 + \lceil log_2 r\rceil$.*

Since $GP(n, k)$ are 3-regular graphs, and $\lceil log_2 3\rceil = 2$ then the next corollary holds.

Corollary 2.3. $\beta_E(GP(n, k)) \geq 3$.

2.2. Exact Value for $GP(n, 1)$

In this section, we are giving the exact value of the edge metric dimension of the generalized Petersen graphs $GP(n, 1)$.

Theorem 2.4. $\beta_E(GP(n, 1)) = 3$.

Proof. Let $S = \{u_0, u_1, v_0\}$.

Case 1. $n = 2t$

Edge metric representations with respect to S are:

$$r(u_iu_{i+1},S)=\begin{cases}(0,0,1),\ i=0\\(i,i-1,i+1),\ 1\le i\le t-1\\(t-1,t-1,t),\ i=t\\(2t-1-i,2t-i,2t-i),\ t+1\le i\le 2t-1\end{cases}.$$

$$r(u_iv_i,S)=\begin{cases}(0,1,0),\ i=0\\(i,i-1,i),\ 1\le i\le t\\(2t-i,2t+1-i,2t-i),\ t+1\le i\le 2t-1\end{cases}.$$

$$r(v_iv_{i+1},S)=\begin{cases}(1,1,0),\ i=0\\(i+1,i,i),\ 1\le i\le t-1\\(t,t,t-1),\ i=t\\(2t-i,2t+1-i,2t-1-i),\ t+1\le i\le 2t-2\\(1,2,0),\ i=2t-1\end{cases}.$$

Since all edge metric representations with respect to S are pairwise different, we deduce that S is an edge metric generator. Since $|S|=3$, from Corollary 2.3, it follows $\beta_E(GP(2t,1))=3$.

Case 2. $n=2t+1$

Edge metric representations with respect to S are:

$$r(u_iu_{i+1},S)=\begin{cases}(0,0,1),\ i=0\\(i,i-1,i+1),\ 1\le i\le t\\(2t-i,2t+1-i,2t+1-i),\ t+1\le i\le 2t\end{cases}.$$

$$r(u_iv_i,S)=\begin{cases}(0,1,0),\ i=0\\(i,i-1,i),\ 1\le i\le t\\(t,t,t),\ i=t+1\\(2t+1-i,2t+2-i,2t+1-i),\ t+2\le i\le 2t\end{cases}.$$

$$r(v_iv_{i+1},S)=\begin{cases}(1,1,0),\ i=0\\(i+1,i,i),\ 1\le i\le t\\(2t+1-i,2t+2-i,2t-i),\ t+1\le i\le 2t\end{cases}.$$

Similarly to Case 1, all edge metric representations with respect to S are pairwise different, so S is an edge metric generator. Having in mind that $|S|=3$, from Corollary 2.3 it follows $\beta_E(GP(2t+1,1))=3$. □

In [3], the authors consider the relation between the metric dimension and the edge metric dimension of some graphs (called *realization question*). They concluded that it is possible to find all three cases, i.e. graphs G such that $\beta_E(G)=\beta(G)$, $\beta_E(G)>\beta(G)$ or $\beta_E(G)<\beta(G)$. For $GP(n,1)$ there

TABLE 2. Edge metric representations for $GP(4t, 2)$

e	$r(e)$	Condition
$u_{2i}u_{2i+1}$	$(0, 2, t)$	$i = 0$
	$(2, 1, t + 1)$	$i = 1$
	$(i + 2, i, t + 2 - i)$	$2 \leq i \leq t$
	$(2t + 2 - i, 2t + 2 - i, i - t)$	$t + 1 \leq i \leq 2t - 3$
	$(3, 4, t - 2)$	$i = 2t - 2$
	$(1, 3, t - 1)$	$i = 2t - 1$
$u_{2i+1}u_{2i+2}$	$(1, 2, t)$	$i = 0$
	$(3, 1, t + 1)$	$i = 1$
	$(i + 3, i, t + 2 - i)$	$2 \leq i \leq t - 1$
	$(t + 1, t, 2)$	$i = t$
	$(2t + 1 - i, 2t + 2 - i, i - t)$	$t + 1 \leq i \leq 2t - 3$
	$(2, 4, t - 2)$	$i = 2t - 2$
	$(0, 3, t - 1)$	$i = 2t - 1$
$u_{2i}v_{2i}$	$(0, 3, t)$	$i = 0$
	$(2, 2, t + 1)$	$i = 1$
	$(i + 1, i, t + 3 - i)$	$2 \leq i \leq t$
	$(t, t + 1, 2)$	$i = t + 1$
	$(2t + 1 - i, 2t + 3 - i, i - t)$	$t + 2 \leq i \leq 2t - 1$
$u_{2i+1}v_{2i+1}$	$(1, 1, t - 1)$	$i = 0$
	$(i + 2, i - 1, t + 1 - i)$	$1 \leq i \leq t - 1$
	$(t + 1, t - 1, 1)$	$i = t$
	$(2t + 1 - i, 2t + 1 - i, i - t - 1)$	$t + 1 \leq i \leq 2t - 2$
	$(1, 2, t - 2)$	$i = 2t - 1$
$v_{2i}v_{2i+2}$	$(1, 3, t + 1)$	$i = 0$
	$(2, 3, t + 2)$	$i = 1$
	$(i + 1, i + 1, t + 3 - i)$	$2 \leq i \leq t - 1$
	$(t, t + 1, 3)$	$i = t$
	$(t - 1, t + 2, 3)$	$i = t + 1$
	$(2t - i, 2t + 3 - i, i + 1 - t)$	$t + 2 \leq i \leq 2t - 2$
	$(1, 4, t)$	$i = 2t - 1$
$v_{2i+1}v_{2i+3}$	$(2, 0, t - 1)$	$i = 0$
	$(i + 2, i - 1, t - i)$	$1 \leq i \leq t - 1$
	$(t, t - 1, 0)$	$i = t$
	$(2t - i, 2t - i, i - t - 1)$	$t + 1 \leq i \leq 2t - 2$
	$(2, 1, t - 2)$	$i = 2t - 1$

TABLE 3. Edge metric representations for $GP(4t+1,2)$

e	$r(e)$	Condition
$u_{2i}u_{2i+1}$	$(0,t-2,t-1)$	$i=0$
	$(2,t-3,t-2)$	$i=1$
	$(i+2,t-2-i,t-1-i)$	$2\le i\le t-3$
	$(i+2,i+4-t,i+3-t)$	$t-2\le i\le t$
	$(2t+2-i,i+4-t,i+3-t)$	$t+1\le i\le 2t-3$
	$(4t-2i,3t-1-i,3t-1-i)$	$2t-2\le i\le 2t$
$u_{2i+1}u_{2i+2}$	$(1,t-2,t-2)$	$i=0$
	$(3,t-3,t-3)$	$i=1$
	$(i+3,t-2-i,t-2-i)$	$2\le i\le t-3$
	$(t+1,2,2)$	$i=t-2$
	$(t+2,3,3)$	$i=t-1$
	$(2t+2-i,i+4-t,i+4-t)$	$t\le i\le 2t-3$
	$(3,t,t+1)$	$i=2t-2$
	$(1,t-1,t)$	$i=2t-1$
$u_{2i}v_{2i}$	$(0,t-1,t-2)$	$i=0$
	$(2,t-2,t-3)$	$i=1$
	$(i+1,t-1-i,t-2-i)$	$2\le i\le t-3$
	$(i+1,i+4-t,i+2-t)$	$t-2\le i\le t$
	$(2t+2-i,i+4-t,i+2-t)$	$t+1\le i\le 2t-3$
	$(4,t,t)$	$i=2t-2$
	$(3,t-1,t+1)$	$i=2t-1$
	$(1,t-2,t)$	$i=2t$
$u_{2i+1}v_{2i+1}$	$(1,t-3,t-1)$	$i=0$
	$(i+2,t-3-i,t-1-i)$	$1\le i\le t-3$
	$(t,1,2)$	$i=t-2$
	$(t+1,2,3)$	$i=t-1$
	$(2t+1-i,i+3-t,i+4-t)$	$t\le i\le 2t-3$
	$(3,t+1,t)$	$i=2t-2$
	$(2,t,t-1)$	$i=2t-1$
$v_{2i}v_{2i+2}$	$(i+1,t-1-i,t-3-i)$	$0\le i\le t-4$
	$(t-2,3,0)$	$i=t-3$
	$(t-1,3,0)$	$i=t-2$
	$(t,4,1)$	$i=t-1$
	$(2t+1-i,i+5-t,i+2-t)$	$t\le i\le 2t-4$
	$(2t+1-i,3t-3-i,i+2-t)$	$2t-3\le i\le 2t-1$
	$(2,t-3,t)$	$i=2t$

Table 3. continued

e	$r(e)$	Condition
$v_{2i+1}v_{2i+3}$	$(2, t-4, t-1)$	$i = 0$
	$(i+2, t-4-i, t-1-i)$	$1 \leq i \leq t-4$
	$(t-1, 0, 3)$	$i = t-3$
	$(t, 1, 3)$	$i = t-2$
	$(2t-i, i+3-t, i+5-t)$	$t-1 \leq i \leq 2t-4$
	$(3, t, t)$	$i = 2t-3$
	$(2, t+1, t-1)$	$i = 2t-2$
	$(1, t, t-2)$	$i = 2t-1$

are only two cases, since, from [12], it follows $\beta(GP(n,1)) = \begin{cases} 2, & n \textit{ is odd} \\ 3, & n \textit{ is even} \end{cases}$. When $n = 2t$, it holds $\beta_E(GP(n,1)) = \beta(GP(n,1)) = 3$, while for $n = 2t+1$, it holds $3 = \beta_E(GP(n,1)) > \beta(GP(n,1)) = 2$.

Another interesting discussion is comparison between $\beta_E(GP(n,1))$ and $\beta'_E(GP(n,1))$. From [18], it follows that $\beta'_E(GP(n,1)) = 3$, which matches $\beta_E(GP(n,1)) = 3$ from Theorem 2.4.

2.3. Exact Value for $GP(n,2)$

In this section, we are giving the exact value of the edge metric dimension of the generalized Petersen graphs $GP(n,2)$.

Theorem 2.5. $\beta_E(GP(n,2)) = \begin{cases} 3, & n = 8 \vee n \geq 10 \\ 4, & n \in \{5,6,7,9\} \end{cases}$

Proof. In the case of $n = 4t$, $t \geq 4$, let $S = \{u_0, v_3, v_{2t+3}\}$. All edge metric representations with respect to S are given in Table 2. The first column is related to edge $e \in E(GP(4t,2))$, the second column presents its edge metric representation $r(e)$, while the last column gives the condition in which the statement in the second column is true. As can be observed from Table 2, all edge metric representations with respect to S are pairwise different, so S is an edge metric generator for $GP(4t,2)$. Having in mind that $|S| = 3$, from Corollary 2.3 it follows that, for $t \geq 4$, $\beta_E(GP(4t,2)) = 3$ holds.

If $n = 4t+1$, $t \geq 4$, then let $S = \{u_0, v_{2t-5}, v_{2t-4}\}$. All edge metric representations with respect to S are given in Table 3. As can be seen in Table 3 all edge metric representations with respect to S are pairwise different, so S is an edge metric generator for $GP(4t+1,2)$. Again, having in mind that $|S| = 3$, from Corollary 2.3 it follows that, for $t \geq 4$, $\beta_E(GP(4t+1,2)) = 3$ holds.

For $t \geq 4$, in cases when $n = 4t+2$ or $n = 4t+3$, let us define $S = \{u_0, v_{2t-2}, v_{2t-1}\}$. All edge metric representations of $GP(4t+2,2)$ and $GP(4t+$

TABLE 4. Edge metric representations for $GP(4t+2,2)$

e	$r(e)$	Condition
$u_{2i}u_{2i+1}$	$(0,t,t+1)$	$i=0$
	$(2,t-1,t)$	$i=1$
	$(i+2,t-i,t+1-i)$	$2\leq i\leq t-1$
	$(t+2,2,1)$	$i=t$
	$(2t+3-i,i+2-t,i+1-t)$	$t+1\leq i\leq 2t-2$
	$(3,t+1,t)$	$i=2t-1$
	$(1,t+1,t+1)$	$i=2t$
$u_{2i+1}u_{2i+2}$	$(1,t,t)$	$i=0$
	$(3,t-1,t-1)$	$i=1$
	$(i+3,t-i,t-i)$	$2\leq i\leq t-1$
	$(2t+2-i,i+2-t,i+2-t)$	$t\leq i\leq 2t-2$
	$(2,t+1,t+1)$	$i=2t-1$
	$(0,t+1,t+1)$	$i=2t$
$u_{2i}v_{2i}$	$(0,t+1,t)$	$i=0$
	$(2,t,t-1)$	$i=1$
	$(i+1,t+1-i,t-i)$	$2\leq i\leq t-1$
	$(t+1,2,0)$	$i=t$
	$(2t+2-i,i+2-t,i-t)$	$t+1\leq i\leq 2t$
$u_{2i+1}v_{2i+1}$	$(1,t-1,t+1)$	$i=0$
	$(i+2,t-1-i,t+1-i)$	$1\leq i\leq t-1$
	$(2t+2-i,i+1-t,i+2-t)$	$t\leq i\leq 2t-1$
	$(1,t,t+2)$	$i=2t$
$v_{2i}v_{2i+2}$	$(i+1,t+1-i,t-1-i)$	$0\leq i\leq t-2$
	$(t,3,0)$	$i=t-1$
	$(2t+1-i,i+3-t,i-t)$	$t\leq i\leq 2t-1$
	$(1,t+2,t)$	$i=2t$
$v_{2i+1}v_{2i+3}$	$(i+2,t-2-i,t+1-i)$	$0\leq i\leq t-2$
	$(t+1,0,3)$	$i=t-1$
	$(2t+1-i,i+1-t,i+3-t)$	$t\leq i\leq 2t-1$
	$(2,t-1,t+2)$	$i=2t$

$3,2)$, with respect to S, are given in Tables 4 and 5, respectively. It can be seen in Table 4 that all edge metric representations of $GP(4t+2,2)$, with respect to S, are pairwise different, so S is an edge metric generator for $GP(4t+2,2)$. Again, having in mind that $|S|=3$, from Corollary 2.3 it follows that, for $t\geq 4$, $\beta_E(GP(4t+2,2))=3$ holds. The same conclusion can be drawn for $GP(4t+3,2)$, since all its edge metric representations presented in Table 5 are also pairwise different, so for $t\geq 4$, $\beta_E(GP(4t+3,2))=3$ holds.

Table 5. Edge metric representations for $GP(4t+3,2)$

e	$r(e)$	Condition
$u_{2i}u_{2i+1}$	$(0,t,t+1)$	$i=0$
	$(2,t-1,t)$	$i=1$
	$(i+2,t-i,t+1-i)$	$2\le i\le t-1$
	$(t+2,2,1)$	$i=t$
	$(2t+3-i,i+2-t,i+1-t)$	$t+1\le i\le 2t-1$
	$(2,t+2,t+1)$	$i=2t$
	$(0,t+1,t+1)$	$i=2t+1$
$u_{2i+1}u_{2i+2}$	$(1,t,t)$	$i=0$
	$(3,t-1,t-1)$	$i=1$
	$(i+3,t-i,t-i)$	$2\le i\le t-1$
	$(2t+3-i,i+2-t,i+2-t)$	$t\le i\le 2t-2$
	$(3,t+1,t+1)$	$i=2t-1$
	$(1,t+1,t+2)$	$i=2t$
$u_{2i}v_{2i}$	$(0,t+1,t)$	$i=0$
	$(2,t,t-1)$	$i=1$
	$(i+1,t+1-i,t-i)$	$2\le i\le t-1$
	$(t+1,2,0)$	$i=t$
	$(2t+3-i,i+2-t,i-t)$	$t+1\le i\le 2t-1$
	$(3,t+1,t)$	$i=2t$
	$(1,t,t+1)$	$i=2t+1$
$u_{2i+1}v_{2i+1}$	$(1,t-1,t+1)$	$i=0$
	$(i+2,t-1-i,t+1-i)$	$1\le i\le t-1$
	$(2t+2-i,i+1-t,i+2-t)$	$t\le i\le 2t-1$
	$(2,t+1,t+1)$	$i=2t$
$v_{2i}v_{2i+2}$	$(i+1,t+1-i,t-1-i)$	$0\le i\le t-2$
	$(t,3,0)$	$i=t-1$
	$(t+1,3,0)$	$i=t$
	$(2t+2-i,i+3-t,i-t)$	$t+1\le i\le 2t-2$
	$(3,t+1,t-1)$	$i=2t-1$
	$(2,t,t)$	$i=2t$
	$(2,t-1,t+1)$	$i=2t+1$
$v_{2i+1}v_{2i+3}$	$(i+2,t-2-i,t+1-i)$	$0\le i\le t-2$
	$(t+1,0,3)$	$i=t-1$
	$(2t+1-i,i+1-t,i+3-t)$	$t\le i\le 2t-2$
	$(2,t,t+1)$	$i=2t-1$
	$(1,t+1,t)$	$i=2t$

Table 6. Edge resolving bases of $GP(n, 2)$

n	Basis	$\beta_E(GP(n, 2))$
5	$\{u_0, u_1, u_3, v_3\}$	4
6	$\{u_0, u_1, u_2, u_3\}$	4
7	$\{u_0, u_1, u_4, v_2\}$	4
8	$\{u_0, u_2, v_4\}$	3
9	$\{u_0, u_1, u_2, v_5\}$	4
10	$\{u_0, u_3, v_6\}$	3
11	$\{u_0, u_3, v_4\}$	3
12	$\{u_0, u_3, v_4\}$	3
13	$\{u_0, v_3, v_4\}$	3
14	$\{u_0, u_4, v_1\}$	3
15	$\{u_0, u_5, v_1\}$	3
$n = 4t \wedge t \geq 4$	$\{u_0, v_3, v_{2t+3}\}$	3
$n = 4t + 1 \wedge t \geq 4$	$\{u_0, v_{2t-5}, v_{2t-4}\}$	3
$(n = 4t + 2 \vee n = 4t + 3) \wedge t \geq 4$	$\{u_0, v_{2t-2}, v_{2t-1}\}$	3

For the remaining cases when $n \leq 15$, the edge metric dimension of $GP(n, 2)$ is found by the total enumeration technique, and it is presented in Table 6, along with the corresponding edge metric bases. It should be stated that the edge metric dimension is equal to 3, except in cases for $n \in \{5, 6, 7, 9\}$, when it is equal to 4. $\square$

For $GP(n, 2)$ there are only two cases for the realization question given in [4]. From [13] it follows that $\beta(GP(n, 2)) = 3$, so for $n \notin \{5, 6, 7, 9\}$ the edge metric dimension of $GP(n, 2)$ is equal to its metric dimension. Only in cases when $n \in \{5, 6, 7, 9\}$, it holds $4 = \beta_E(GP(n, 2)) > \beta(GP(n, 2)) = 3$.

Another interesting discussion is the comparison between $\beta_E(GP(n, 2))$ and $\beta'_E(GP(n, 2))$. In contrast to $GP(n, 1)$, the values of $\beta_E(GP(n, 2))$ and $\beta'_E(GP(n, 2))$ sometimes differ. For example, $\beta_E(GP(9, 2)) = 4$, while $\beta'_E(GP(n, 2)) = 3$.

3. Conclusions

In this article, the recently introduced edge metric dimension problem is considered. The exact formulae for generalized Petersen graphs $GP(n, 1)$ and $GP(n, 2)$ are stated and proved. Moreover, a lower bound for 3-regular graphs, which holds for all generalized Petersen graphs, is given.

Possible future research could be the finding of the edge metric dimension of other challenging classes of graphs or the construction of the metaheuristic approach for solving the edge metric dimension problem.

References

[1] Slater, P.J.: Leaves of trees. Congr. Numer. **14**, 549–559 (1975)

[2] Harary, F., Melter, R.: On the metric dimension of a graph. Ars Comb. **2**(191–195), 1 (1976)

[3] Kelenc, A., Tratnik, N., Yero, I.G.: Uniquely identifying the edges of a graph: the edge metric dimension. Discrete Appl. Math. **251**, 204–220 (2018)

[4] Yero, I.G.: Vertices, edges, distances and metric dimension in graphs. Electron. Notes Discrete Math. **55**, 191–194 (2016)

[5] Kelenc, A., Kuziak, D., Taranenko, A., Yero, I.G.: Mixed metric dimension of graphs. Appl. Math. Comput. **314**, 429–438 (2017)

[6] Peterin, I., Yero, I.G.: Edge metric dimension of some graph operations. arXiv:1809.08900

[7] Zubrilina, N.: On the edge dimension of a graph. Discrete Math. **341**(7), 2083–2088 (2018)

[8] Zubrilina, N.: On the edge metric dimension for the random graph. arXiv:1612.06936

[9] Epstein, L., Levin, A., Woeginger, G.J.: The (weighted) metric dimension of graphs: hard and easy cases. Algorithmica **72**(4), 1130–1171 (2015)

[10] Liu, J.-B., Zahid, Z., Nasir, R., Nazeer, W.: Edge version of metric dimension and doubly resolving sets of the necklace graph. Mathematics **6**(11), 243 (2018)

[11] Coxeter, H.: Self-dual configurations and regular graphs. Bull. Am. Math. Soc. **56**, 413–455 (1950)

[12] Cáceres, J., Hernando, C., Mora, M., Pelayo, I.M., Puertas, M.L., Seara, C., Wood, D.R.: On the metric dimension of Cartesian products of graphs. SIAM J. Discrete Math. **21**(2), 423–441 (2007)

[13] Javaid, I., Rahim, M.T., Ali, K.: Families of regular graphs with constant metric dimension. Util. Math. **75**, 21–34 (2008)

[14] Imran, M., Baig, A.Q., Shafiq, M.K., Tomescu, I.: On metric dimension of generalized Petersen graphs P(n,3). Ars Comb. **117**, 113–130 (2014)

[15] Wang, X.: All double generalized Petersen graphs are Hamiltonian. Discrete Math. **340**(12), 3016–3019 (2017)

[16] Ball, T., Bell, R.W., Guzman, J., Hanson-Colvin, M., Schonsheck, N.: On the cop number of generalized Petersen graphs. Discrete Math. **340**(6), 1381–1388 (2017)

[17] Dantas, S., de Figueiredo, C.M., Mazzuoccolo, G., Preissmann, M., Dos Santos, V.F., Sasaki, D.: On the total coloring of generalized Petersen graphs. Discrete Math. **339**(5), 1471–1475 (2016)

[18] Nasir, R., Zafar, S., Zahid, Z.: Edge metric dimension of graphs. Ars Comb. (**in press**). https://www.researchgate.net/publication/322634658_Edge_metric_dimension_of_graphs. Accessed 8 Feb 2019

Vladimir Filipović and Aleksandar Kartelj
Faculty of Mathematics
University of Belgrade
Studentski trg 16/IV
Belgrade 11 000
Serbia
e-mail: vladofilipovic@hotmail.com

Aleksandar Kartelj
e-mail: aleksandar.kartelj@gmail.com

Jozef Kratica
Mathematical Institute
Serbian Academy of Sciences and Arts
Kneza Mihaila 36/III
Belgrade 11 000
Serbia
e-mail: jkratica@mi.sanu.ac.rs

Received: February 8, 2019.
Accepted: September 25, 2019.