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Results in Mathematics

Landau-Type Theorems for Certain Bounded Biharmonic Mappings

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Abstract. In this paper, we establish three sharp versions of the Landau-type theorems for bounded biharmonic mappings $F(z) = |z|^2G(z)+H(z)$, where $G(z)$ and $H(z)$ are harmonic in the unit disk U with $G(0) = H(0) = 0$ and $\lambda_F(0) = ||F_z(0)| - |F_{\overline{z}}(0)|| = 1$. Our results generalize (or improve) the corresponding results given in Liu et al. (Math Methods Appl Sci 40:2582–2595, 2017). Three conjectures for the sharp version of the Landau-type theorem for certain bounded biharmonic mappings are given in third section.

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1. Introduction

A function $f(z) = u(z) + iv(z)$, $z = x + iy$ is a *harmonic mapping* on the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ if and only if f is twice continuously differentiable and satisfies the Laplacian equation

$$\Delta f = 4f_{z\overline{z}} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

for $z \in U$, where we use the common notations for its formal derivatives:

$$f_z = \frac{1}{2}(f_x - if_y), \quad f_{\overline{z}} = \frac{1}{2}(f_x + if_y).$$

A function $f(z) = u(z) + iv(z)$ is a *biharmonic mapping* on U if and only if f is four times continuously differentiable and satisfies the biharmonic equation $\Delta(\Delta f) = 0$ for $z \in U$.

For such function f, let

$$\Lambda_f(z) = \max_{0\le\theta\le 2\pi} |e^{i\theta} f_z(z) + e^{-i\theta} f_{\overline{z}}(z)| = |f_z(z)| + |f_{\overline{z}}(z)|$$

and

$$\lambda_f(z) = \min_{0\le\theta\le 2\pi} |e^{i\theta} f_z(z) + e^{-i\theta} f_{\overline{z}}(z)| = ||f_z(z)| - |f_{\overline{z}}(z)||.$$

Biharmonic mappings arise in many physical situations, particularly in fluid dynamics and elasticity problems, and have many important applications in engineering (see [1] and the references therein for more details).

It is known that a harmonic mapping is locally univalent if and only if its Jacobian $J_f(z) = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2 \neq 0$ for $z \in U$ (cf. [19]). Since U is simply connected, $f(z)$ can be written as $f = h + \overline{g}$ with $f(0) = h(0)$, where g and h are analytic on U (for details see [11]). Thus,

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2.$$

It is well-known (cf. [1]) that a mapping $F(z)$ is biharmonic in a simply connected domain D if and only if $F(z)$ has the following representation:

$$F(z) = |z|^2 G(z) + H(z), \tag{1.1}$$

where $G(z)$ and $H(z)$ are complex-valued harmonic functions in D.

The classical Landau's theorem states that if f is an analytic function on the unit disk U with $f(0) = f'(0) - 1 = 0$ and $|f(z)| < M$ for $z \in U$, then f is univalent in the disk $U_{r_0} = \{z \in \mathbb{C} : |z| < r_0\}$ with

$$r_0 = \frac{1}{M + \sqrt{M^2 - 1}}, \tag{1.2}$$

and $f(U_{r_0})$ contains a disk $|w| < R_0$ with $R_0 = M r_0^2$. This result is sharp, with the extremal function $f_0(z) = Mz\frac{1-Mz}{M-z}$. The Bloch theorem asserts the existence of a positive constant number b such that if f is an analytic function on the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ with $f'(0) = 1$, then $f(U)$ contains a Schlicht disk of radius b, that is, a disk of radius b which is the univalent image of some region in U. The supremum of all such constants b is called the Bloch constant (see [3,13]).

In 2000, under a suitable restriction, Chen et al. [3] first established the Bloch and Landau theorems for harmonic mappings. Their results were not sharp. Better estimates were given in [12] and later in [4,7–10,14–16,21,22,24,27,28]. In 2008, Abdulhadi and Muhanna established two versions of Landau-type theorems of certain bounded biharmonic mappings in [2]. From that on, many authors also considered the Landau-type theorems for certain bounded biharmonic mappings (see [6,7,20,23,25,29]) and logharmonic mappings (see [26]). However, few sharp results were found. In 2017, Liu et al. established the following Landau-type theorems for certain biharmonic mappings.

Theorem A ([25]). *Let $F(z) = |z|^2 g(z) + h(z)$ be a biharmonic mapping of the unit disk U with $g(z)$, $h(z)$ are harmonic mappings in U, and $g(0) = h(0) = 0, \lambda_F(0) = \lambda_g(0) = \Lambda_g(0) = 1$, $\Lambda_g(z) \le \Lambda_1$ and $\Lambda_h(z) \le \Lambda_2$ for $z \in U$. Then $\Lambda_1 \ge 1$, $\Lambda_2 \ge 1$, and F is univalent in the disk U_{r_1}, where r_1 is the minimum positive root in $(0,1)$ of the equation:*

$$1 - 3r^2 - \frac{\Lambda_2^2 - 1}{\Lambda_2} \cdot \frac{r}{1-r} + \frac{\Lambda_1^2 - 1}{\Lambda_1} \cdot \left[2r^2 + 2r\ln(1-r) - \frac{r^3}{1-r}\right] = 0, \tag{1.3}$$

and $F(U_{r_1})$ contains a Schlicht disk U_{R_1}, with

$$R_1 = r_1 - r_1^3 + \left(r_1^2 \cdot \frac{\Lambda_1^2 - 1}{\Lambda_1} + \frac{\Lambda_2^2 - 1}{\Lambda_2}\right)\left[r_1 + \ln(1-r_1)\right]. \tag{1.4}$$

When $\Lambda_1 = \Lambda_2 = 1$, $r_1 = \frac{\sqrt{3}}{3}$ and $R_1 = \frac{2\sqrt{3}}{9}$ are sharp.

Theorem B ([25]). *Let $F(z) = |z|^2 g(z) + h(z)$ be a biharmonic mapping of the unit disk U with $g(z)$, $h(z)$ are harmonic mappings in U, and $g(0) = h(0) = 0, \lambda_F(0) = \lambda_g(0) = \Lambda_g(0) = 1$, $\Lambda_g(z) \le \Lambda$ and $|h(z)| \le M$ for $z \in U$. Then $\Lambda \ge 1$, $M \ge 1$, and F is univalent in the disk U_{r_2}, where $K(M) = \min\{\frac{4M}{\pi}, \sqrt{2M^2 - 2}\}$, and r_2 is the minimum positive root in $(0,1)$ of the equation:*

$$1 - 3r^2 - K(M) \cdot \frac{2r - r^2}{(1-r)^2} + \frac{\Lambda^2 - 1}{\Lambda} \cdot \left[2r^2 + 2r\ln(1-r) - \frac{r^3}{1-r}\right] = 0, \tag{1.5}$$

and $F(U_{r_2})$ contains a Schlicht disk U_{R_2} with

$$R_2 = r_2 - r_2^3 - K(M)\frac{r_2^2}{1-r_2} + \frac{\Lambda^2 - 1}{\Lambda}[r_2^3 + r_2^2 \ln(1-r_2)]. \tag{1.6}$$

When $\Lambda = M = 1$, $r_2 = \frac{\sqrt{3}}{3}$ and $R_2 = \frac{2\sqrt{3}}{9}$ are sharp.

However, Theorems A and B are not sharp if $\Lambda > 1\,(\Lambda_1 > 1)$ or $M > 1\,(\Lambda_2 > 1)$, and they both have the strong hypothesis $\lambda_g(0) = \Lambda_g(0) = 1$. In this paper, by extending the method and technique in [24], we will establish several sharp versions of the Landau-type theorems for bounded biharmonic mappings.

This paper is organized as follows. In Sect. 2, we should recall several notions and lemmas, and establish four new lemmas, which play a key role in the proofs of our main results. In Sect. 3, by establishing Theorems 3.1 and 3.3, we first establish the sharp versions of Landau-type theorems for Theorem A without the hypothesis $\lambda_g(0) = \Lambda_g(0) = 1$. Next, by establishing Theorem 3.4, we establish the sharp version of Landau-type theorem for the case $\Lambda \ge 1, M = 1$ of Theorem B without the hypothesis $\lambda_g(0) = \Lambda_g(0) = 1$. Then, by establishing Theorem 3.7 and Corollary 3.8, we provide two sharp versions of Landau-type theorems of biharmonic mappings for the case $\Lambda \ge 1, M > 1$. Finally, we also provide three conjectures for the sharp versions of Landau-type theorems of bounded harmonic mappings or biharmonic mappings.

2. Preliminaries

In order to establish our main results, we need the following notions and lemmas.

We first introduce the notion of pseudo-disk [5,17,18]. For $z \in U$ and $0 < r < 1$, the pseudo-disk of pseudo-center z and pseudo-radius r is defined by

$$U_p(z,r) = \left\{ \zeta \in U : \left| \frac{\zeta - z}{1 - \bar{z}\zeta} \right| < r \right\},$$

and $\overline{U}_p(z,r) := \left\{ \zeta \in U : \left| \frac{\zeta - z}{1-\bar{z}\zeta} \right| \leq r \right\}$, $\partial U_p(z,r) := \left\{ \zeta \in U : \left| \frac{\zeta - z}{1-\bar{z}\zeta} \right| = r \right\}$.

It is obvious that $U_p(0,r) = U_r$, and that if $z \neq 0$, it is easy to verify that $U_p(z,r)$ is the Euclidian disk of diameter (a,b), where

$$a = e^{i\theta'} \cdot \frac{|z| - r}{1 - r|z|}, \quad b = e^{i\theta'} \cdot \frac{|z| + r}{1 + r|z|}, \quad \theta' = \arg z.$$

Next, we recall the classical Schwarz–Pick Lemma as follows.

Lemma 2.1 (Schwarz–Pick Lemma). *Suppose $f(z)$ is an analytic function in U and $f(U) \subset U$.*

(1) *For $z \in U$ and $0 < r < 1$, we have*

$$f\left(U_p(z,r)\right) \subseteq U_p\left(f(z), r\right), \quad f\left(\overline{U}_p(z,r)\right) \subseteq \overline{U}_p\left(f(z), r\right);$$

(2) *For $z' \in \partial U_p(z,r)$, $f(z') \in \partial U_p(f(z), r)$ if and only if f is a Möbius transformation of U onto itself.*

(3) $|f'(z)| / \left(1 - |f(z)|^2\right) \leq 1/(1 - |z|^2)$ *holds for $z \in U$, and the equality holds for some $z \in U$ if and only if f is a Möbius transformation.*

Now we establish two new lemmas, which play a key role in our proofs of the main results in this paper.

Lemma 2.2. *Suppose $\Lambda > 1$. Let $H(z)$ be a harmonic mapping of the unit disk U with $\lambda_H(0) = 1$ and $\Lambda_H(z) < \Lambda$ for all $z \in U$. Then for all $z_1, z_2 \in U_r$ $(0 < r < 1,\ z_1 \neq z_2)$, we have*

$$\left| \int_{\overline{z_1 z_2}} H_z(z)dz + H_{\bar{z}}(z)d\bar{z} \right| \geq \Lambda \frac{1 - \Lambda r}{\Lambda - r} |z_1 - z_2|, \tag{2.1}$$

where $\overline{z_1 z_2}$ is the line segment joining z_1 and z_2.

Proof. Let $\theta_0 = \arg(z_2 - z_1)$. Since $H(z)$ is a harmonic mapping in the unit disk U, we have $H(z)$ can be written as $H(z) = H_1(z) + \overline{H_2(z)}$ for $z \in U$, where H_1 and H_2 are analytic in U. Without lost of the generality, we may assume that

$|H_1'(0)| > |H_2'(0)|$, since $\lambda_H(0) = ||H_1'(0)| - |H_2'(0)|| = 1$ (if $|H_1'(0)| < |H_2'(0)|$, we can consider the harmonic mapping $\overline{H} = \overline{H_1} + H_2$ instead of H). Then

$$\Delta_{0\le\theta\le 2\pi} \arg\left\{H_1'(0)e^{i(\theta_0+\theta)} + H_2'(0)e^{i(\theta_0-\theta)}\right\}$$
$$= \Delta_{0\le\theta\le 2\pi} \arg\left\{H_1'(0)e^{i(\theta_0+\theta)}\right\} = 2\pi,$$

where $\Delta_{0\le\theta\le 2\pi}$ denotes the increment of the succeeding function as θ increasing from 0 to 2π. Thus there exists a $\theta_1 \in [0, 2\pi]$ such that

$$H_1'(0)e^{i(\theta_0+\theta_1)} + H_2'(0)e^{i(\theta_0-\theta_1)} > 0.$$

For $z \in U$, let

$$\omega(z) = \frac{H_1'(z)e^{i(\theta_0+\theta_1)} + H_2'(z)e^{i(\theta_0-\theta_1)}}{\Lambda}.$$

Then $\omega(z)$ is analytic with $|\omega(z)| \le \Lambda_H(z)/\Lambda < 1$ for $z \in U$ and

$$\alpha := \omega(0) = \frac{H_1'(0)e^{i(\theta_0+\theta_1)} + H_2'(0)e^{i(\theta_0-\theta_1)}}{\Lambda} \ge \frac{\lambda_H(0)}{\Lambda} = \frac{1}{\Lambda}.$$

Using Schwarz–Pick Lemma, we have

$$\operatorname{Re}\omega(z) \ge \frac{\alpha - r}{1-\alpha r} \ge \frac{\frac{1}{\Lambda} - r}{1 - \frac{r}{\Lambda}}, \quad z \in U_r.$$

That is

$$\operatorname{Re}\left\{H_1'(z)e^{i(\theta_0+\theta_1)} + H_2'(z)e^{i(\theta_0-\theta_1)}\right\} \ge \Lambda \frac{\frac{1}{\Lambda} - r}{1 - \frac{r}{\Lambda}}, \quad z \in U_r. \tag{2.2}$$

Then

$$\begin{aligned}\left|\int_{\overline{z_1 z_2}} H_z(z)dz + H_{\bar z}(z)d\bar z\right| &= \left|\int_{\overline{z_1 z_2}} \left(H_1'(z)e^{i(\theta_0+\theta_1)} + \overline{H_2'(z)}e^{-i(\theta_0-\theta_1)}\right)|dz|\right| \\ &\ge \int_{\overline{z_1 z_2}} \operatorname{Re}\left\{H_1'(z)e^{i(\theta_0+\theta_1)} + \overline{H_2'(z)}e^{-i(\theta_0-\theta_1)}\right\}|dz| \\ &= \int_{\overline{z_1 z_2}} \operatorname{Re}\left\{H_1'(z)e^{i(\theta_0+\theta_1)} + H_2'(z)e^{i(\theta_0-\theta_1)}\right\}|dz| \\ &\ge \int_{\overline{z_1 z_2}} \Lambda \frac{\frac{1}{\Lambda} - r}{1-\frac{r}{\Lambda}}|dz| = \Lambda\,\frac{1-\Lambda r}{\Lambda - r}\,|z_1 - z_2|.\end{aligned}$$

□

Lemma 2.3. *Suppose* $\Lambda > 1$. *Let* $H(z)$ *be a harmonic mapping of the unit disk* U *with* $\lambda_H(0) = 1$ *and* $\Lambda_H(z) < \Lambda$ *for all* $z \in U$. *Set* $\gamma = H^{-1}(\overline{ow'})$ *with* $w' \in H(\partial U_r)$ $(0 < r \le 1)$ *and* $\overline{ow'}$ *denotes the closed line segment joining the origin and* w', *then*

$$\left|\int_\gamma H_\zeta(\zeta)d\zeta + H_{\bar\zeta}(\zeta)d\bar\zeta\right| \ge \Lambda\int_0^r \frac{\frac{1}{\Lambda} - t}{1-\frac{t}{\Lambda}}dt. \tag{2.3}$$

Proof. Let $d\zeta = |d\zeta|e^{i\theta_\zeta}$, $\zeta \in \gamma$. Since $H(z)$ is a harmonic mapping in the unit disk U, we see that $H(z)$ can be written as $H(z) = H_1(z) + \overline{H_2(z)}$ for $z \in U$, where H_1 and H_2 are analytic in U. Similar to the proof of Lemma 2.2, we may assume that $|H_1'(0)| > |H_2'(0)|$ since $\lambda_H(0) = ||H_1'(0)| - |H_2'(0)|| = 1$. Then for a fixed $\zeta \in \gamma$, there is a $\theta_\zeta' \in [0, 2\pi]$ such that

$$H_1'(0)e^{i(\theta_\zeta+\theta_\zeta')} + H_2'(0)e^{i(\theta_\zeta-\theta_\zeta')} > 0.$$

For $z \in U$, define

$$\omega_\zeta(z) = \frac{H_1'(z)e^{i(\theta_\zeta+\theta_\zeta')} + H_2'(z)e^{i(\theta_\zeta-\theta_\zeta')}}{\Lambda}.$$

Then $\omega_\zeta(z)$ is analytic in U, with $|\omega_\zeta(z)| \le \Lambda_H(z)/\Lambda < 1$ and

$$\alpha_\zeta := \omega_\zeta(0) = \frac{H_1'(0)e^{i(\theta_\zeta+\theta_\zeta')} + H_2'(0)e^{i(\theta_\zeta-\theta_\zeta')}}{\Lambda} \ge \frac{1}{\Lambda}.$$

According to Schwarz–Pick Lemma, we get that

$$\operatorname{Re}\{\omega_\zeta(\zeta)\} \ge \frac{\alpha_\zeta - |\zeta|}{1 - \alpha_\zeta|\zeta|} \ge \frac{\frac{1}{\Lambda} - |\zeta|}{1 - \frac{|\zeta|}{\Lambda}}, \quad \zeta \in \gamma. \tag{2.4}$$

Thus

$$\begin{aligned}\left|\int_\gamma H_\zeta(\zeta)d\zeta + H_{\bar\zeta}(\zeta)d\bar\zeta\right| &= \left|\int_\gamma \left(H_1'(\zeta)e^{i(\theta_\zeta+\theta_\zeta')} + \overline{H_2'(\zeta)}e^{-i(\theta_\zeta-\theta_\zeta')}\right)|d\zeta|\right| \\ &\ge \int_\gamma \operatorname{Re}\left\{H_1'(\zeta)e^{i(\theta_\zeta+\theta_\zeta')} + H_2'(\zeta)e^{-i(\theta_\zeta-\theta_\zeta')}\right\}|d\zeta| \\ &= \int_\gamma \operatorname{Re}\left\{H_1'(\zeta)e^{i(\theta_\zeta+\theta_\zeta')} + H_2'(\zeta)e^{i(\theta_\zeta-\theta_\zeta')}\right\}|d\zeta| \\ &= \Lambda\int_\gamma \operatorname{Re}\{\omega_\zeta(\zeta)\}|d\zeta| \ge \Lambda\int_0^r \frac{\frac{1}{\Lambda} - t}{1 - \frac{t}{\Lambda}}dt.\end{aligned}$$

□

Lemma 2.4 ([20]). *Suppose that $f(z) = g(z) + \overline{h(z)}$ is a harmonic mapping in U with $g(z) = \sum_{n=1}^\infty a_n z^n$ and $h(z) = \sum_{n=1}^\infty b_n z^n$ are analytic in U, and $\lambda_f(0) = 1$. If $\Lambda_f(z) \le \Lambda$ for $z \in U$, then $\Lambda \ge 1$ and*

$$|a_n| + |b_n| \le \frac{\Lambda^2 - 1}{n\Lambda}, \quad n = 2, 3, \ldots. \tag{2.5}$$

When $\Lambda > 1$, the above estimates are sharp for all $n = 2, 3, \ldots$, with the extremal functions $f_n(z)$ and $\overline{f_n(z)}$, where

$$f_n(z) = \Lambda^2 z - (\Lambda^3 - \Lambda)\int_0^z \frac{dz}{\Lambda + z^{n-1}}. \tag{2.6}$$

When $\Lambda = 1$, then $f(z) = a_1 z + \overline{b_1}\overline{z}$ with $||a_1| - |b_1|| = 1$.

Lemma 2.5. *Suppose* $\Lambda \geq 0$*. Let* $F(z) = a\Lambda|z|^2 z + b\overline{z}$ *be a biharmonic mapping of the unit disk* U *with* $|a| = |b| = 1$*. Then* F *is univalent in the disk* U_{ρ_1}*, and* $F(U_{\rho_1})$ *contains a Schlicht disk* U_{σ_1}*, where* $\rho_1 = 1$ *when* $0 \leq \Lambda \leq \frac{1}{3}$*,* $\rho_1 = \frac{1}{\sqrt{3\Lambda}}$ *when* $\Lambda > \frac{1}{3}$*, and*

$$\sigma_1 = \rho_1 - \Lambda\rho_1^3 = \begin{cases} 1 - \Lambda, & \text{if } 0 \leq \Lambda \leq \frac{1}{3}, \\ \frac{2}{3\sqrt{3\Lambda}}, & \text{if } \Lambda > \frac{1}{3}. \end{cases} \tag{2.7}$$

This result is sharp.

Proof. We first prove the case of $0 \leq \Lambda \leq \frac{1}{3}$.

To this end, for every $z_1, z_2 \in U$ with $z_1 \neq z_2$, because $|a| = |b| = 1$, we have

$$\begin{aligned} |F(z_1) - F(z_2)| &= \left|a\Lambda(|z_1|^2 z_1 - |z_2|^2 z_2) + b(\overline{z_1 - z_2})\right| \\ &\geq |b||z_1 - z_2| - |a|\Lambda|z_1^2\overline{z_1} - z_2^2\overline{z_2}| \\ &\geq |z_1 - z_2| - \Lambda|z_1^2 - z_2^2||\overline{z_1}| - \Lambda|z_2^2||\overline{z_1 - z_2}| \\ &\geq |z_1 - z_2|(1 - \Lambda|z_1|(|z_1| + |z_2|) - \Lambda|z_2|^2) > 0. \end{aligned}$$

This implies $F(z_1) \neq F(z_2)$, which proves the univalence of $F(z)$ in the disk U.

On the other hand, set $\theta_0 = \arg\frac{b}{a}$, for each $z' \in \partial U$ and $z_0' = e^{i\frac{\pi+\theta_0}{2}} \in \partial U$, because $F(0) = 0$, we have

$$|F(z') - F(0)| = |F(z')| \geq |b||z'| - |a|\Lambda|z'|^3 = 1 - \Lambda,$$

and

$$|F(z_0') - F(0)| = |ae^{i\frac{\pi+\theta_0}{2}}(\Lambda - 1)| = 1 - \Lambda.$$

Hence $F(U)$ contains a Schlicht disk $U_{1-\Lambda}$, and the radius $1 - \Lambda$ is sharp.

Next, we prove the case of $\Lambda > \frac{1}{3}$.

To this end, for every $z_1, z_2 \in U_{\frac{1}{\sqrt{3\Lambda}}}$ with $z_1 \neq z_2$, because $|a| = |b| = 1$, we have

$$\begin{aligned} |F(z_1) - F(z_2)| &= \left|a\Lambda(|z_1|^2 z_1 - |z_2|^2 z_2) + b(\overline{z_1 - z_2})\right| \\ &\geq |b||z_1 - z_2| - |a|\Lambda|z_1^2\overline{z_1} - z_2^2\overline{z_2}| \\ &\geq |z_1 - z_2| - \Lambda|z_1^2 - z_2^2||\overline{z_1}| - \Lambda|z_2^2||\overline{z_1 - z_2}| \\ &\geq |z_1 - z_2|(1 - \Lambda|z_1|(|z_1| + |z_2|) - \Lambda|z_2|^2) \\ &> |z_1 - z_2|\left(1 - 3\Lambda\left(\frac{1}{\sqrt{3\Lambda}}\right)^2\right) = 0. \end{aligned}$$

This implies $F(z_1) \neq F(z_2)$, which proves the univalence of F in the disk $U_{\frac{1}{\sqrt{3\Lambda}}}$.

Now we prove that F is not univalent in the disk U_r for each $r \in (\frac{1}{\sqrt{3\Lambda}}, 1]$.

In fact, fixed $r \in (\frac{1}{\sqrt{3\Lambda}}, 1]$, set $\theta_1 = (\pi + \arg \frac{b}{a})/2, \varepsilon = \min\left\{(r - \frac{1}{\sqrt{3\Lambda}})/2, \frac{1}{2\sqrt{3\Lambda}}\right\} > 0$ and $r_1 = \frac{1}{\sqrt{3\Lambda}} + \varepsilon, r_2 = \frac{1}{\sqrt{3\Lambda}} - \delta$ with

$$\delta = \frac{\frac{3}{\sqrt{3\Lambda}} + \varepsilon - \sqrt{\left(\frac{3}{\sqrt{3\Lambda}} - 3\varepsilon\right)\left(\frac{3}{\sqrt{3\Lambda}} + \varepsilon\right)}}{2} \in (0, 2\varepsilon) \subseteq \left(0, \frac{1}{\sqrt{3\Lambda}}\right).$$

Direct computation yields $r_1^3 - \frac{1}{\Lambda}r_1 = r_2^3 - \frac{1}{\Lambda}r_2$. Thus there exist two points $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_1}$ in U_r with $z_1 \neq z_2$ such that

$$\begin{aligned} F(z_1) &= a\Lambda\left(r_1^3 e^{i\theta_1} + e^{i\arg\frac{b}{a}}\frac{1}{\Lambda}r_1 e^{-i\theta_1}\right) = a\Lambda e^{i\theta_1}\left(r_1^3 - \frac{r_1}{\Lambda}\right) \\ &= a\Lambda e^{i\theta_1}\left(r_2^3 - \frac{r_2}{\Lambda}\right) = F(z_2), \end{aligned}$$

which implies that $F(z)$ is not univalent in the disk U_r for each $r \in (\frac{1}{\sqrt{3\Lambda}}, 1]$. Hence, the univalent radius $\frac{1}{\sqrt{3\Lambda}}$ is sharp.

On the other hand, for each $z'' \in \partial U_{\frac{1}{\sqrt{3\Lambda}}}$ and $z_0'' = \frac{1}{\sqrt{3\Lambda}} e^{i\frac{\pi+\theta_0}{2}} \in \partial U_{\frac{1}{\sqrt{3\Lambda}}}$, where $\theta_0 = \arg\frac{b}{a}$, because $F(0) = 0$, we have

$$|F(z'') - F(0)| = |F(z'')| \geq |b||z''| - |a|\Lambda|z''|^3 = \frac{1}{\sqrt{3\Lambda}} - \Lambda\left(\frac{1}{\sqrt{3\Lambda}}\right)^3 = \frac{2}{3\sqrt{3\Lambda}},$$

and

$$|F(z_0'') - F(0)| = \left|ae^{i\frac{\pi+\theta_0}{2}}\left(\Lambda\left(\frac{1}{\sqrt{3\Lambda}}\right)^3 - \frac{1}{\sqrt{3\Lambda}}\right)\right| = \frac{2}{3\sqrt{3\Lambda}}.$$

Hence $F(U_{\frac{1}{\sqrt{3\Lambda}}})$ contains a Schlicht disk $U_{\frac{2}{3\sqrt{3\Lambda}}}$, and the radius $\frac{2}{3\sqrt{3\Lambda}}$ is sharp. This completes the proof. □

Lemma 2.6 ([20]). *Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of the unit disk U with $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$. If $\lambda_f(0) = 1$ and $|f(z)| < M$ for all $z \in U$, then $M \geq 1$, and*

$$|a_n| + |b_n| \leq \sqrt{2M^2 - 2}, \quad n = 2, 3, \ldots.$$

Lemma 2.7. *Let $F(z) = |z|^2 G(z) + H(z)$ be a biharmonic mapping of the unit disk U with $G(0) = H(0) = 0$ and $\Lambda_G(z) \leq \Lambda$ for all $z \in U$, where $G(z) = G_1(z) + \overline{G_2(z)} = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n} \bar{z}^n$, $H(z) = H_1(z) + \overline{H_2(z)} = \sum_{n=1}^{\infty} c_n z^n + \sum_{n=1}^{\infty} \overline{d_n} \bar{z}^n$ are harmonic mappings in U. Then for all $z_1, z_2 \in U_r (0 < r < 1)$ with $z_1 \neq z_2$, we have*

$$|F(z_1) - F(z_2)| \geq |z_1 - z_2|\left[||c_1| - |d_1|| - \sum_{n=2}^{\infty}(|c_n| + |d_n|)nr^{n-1} - 3\Lambda r^2\right].$$

Proof. According to Lemma 2.4, for any $z_1, z_2 \in U_r (0 < r < 1,\ z_1 \neq z_2)$, we have

$$|G(z)| = \left| \int_{[0,z]} G_z(z)dz + G_{\bar{z}}(z)d\bar{z} \right| \leq \int_{[0,z]} |\Lambda_G(z)||dz| \leq \Lambda|z|, \tag{2.8}$$

and

$$\begin{aligned} |F(z_1) - F(z_2)| &= \left| \int_{[z_1,z_2]} F_z(z)dz + F_{\bar{z}}(z)d\bar{z} \right| \\ &= \left| \int_{[z_1,z_2]} (\bar{z}G(z) + |z|^2 G_1'(z) + H_z(z))dz + (zG(z) + |z|^2 \overline{G_2'(z)} + H_{\bar{z}}(z))d\bar{z} \right| \\ &= \left| \int_{[z_1,z_2]} (H_z(0)dz + H_{\bar{z}}(0)d\bar{z}) + \int_{[z_1,z_2]} (H_z(z) - H_z(0))dz \right. \\ &\quad \left. + (H_{\bar{z}}(z) - H_{\bar{z}}(0))d\bar{z} + \int_{[z_1,z_2]} (\bar{z}G(z) + |z|^2 G_1'(z))dz + (zG(z) + |z|^2 \overline{G_2'(z)})d\bar{z} \right| \\ &\geq \left| \int_{[z_1,z_2]} H_z(0)dz + H_{\bar{z}}(0)d\bar{z} \right| - \left| \int_{[z_1,z_2]} (H_z(z) - H_z(0))dz \right. \\ &\quad \left. + (H_{\bar{z}}(z) - H_{\bar{z}}(0))d\bar{z} \right| - \int_{[z_1,z_2]} (2|z||G(z)| + |z|^2|G_1'(z)| + |z|^2|G_2'(z)|)|dz| \\ &\geq |z_1 - z_2| \left[||c_1| - |d_1|| - \sum_{n=2}^{\infty} (|c_n| + |d_n|) n r^{n-1} - 3\Lambda r^2 \right]. \end{aligned}$$

This completes the proof of Lemma 2.7. □

3. The Landau-Type Theorems of Biharmonic Mappings

We first prove the sharp version of the Landau-type theorem for biharmonic mappings $F(z)$ under the assumptions $G(0) = H(0) = 0$, $\lambda_F(0) = 1, \Lambda_G(z) \leq \Lambda_1$ and $\Lambda_H(z) < \Lambda_2$ for all $z \in U$, which is one of the main results in this paper.

Theorem 3.1. *Suppose that $\Lambda_1 \geq 0$ and $\Lambda_2 > 1$. Let $F(z) = |z|^2 G(z) + H(z)$ be a biharmonic mapping of the unit disk U, where $G(z)$ and $H(z)$ are harmonic in U, satisfying $G(0) = H(0) = 0$, $\lambda_F(0) = 1$, $\Lambda_G(z) \leq \Lambda_1$ and $\Lambda_H(z) < \Lambda_2$ for all $z \in U$. Then $F(z)$ is univalent in the disk U_{ρ_0} and $F(U_{\rho_0})$ contains a Schlicht disk U_{σ_0}, where ρ_0 is the unique root in $(0, 1)$ of the equation*

$$\Lambda_2 \frac{1 - \Lambda_2 r}{\Lambda_2 - r} - 3\Lambda_1 r^2 = 0, \tag{3.1}$$

and

$$\sigma_0 = \Lambda_2^2 \rho_0 + \left(\Lambda_2^3 - \Lambda_2\right) \ln\left(1 - \frac{\rho_0}{\Lambda_2}\right) - \Lambda_1 \rho_0^3. \tag{3.2}$$

This result is sharp, with an extremal function given by

$$F_0(z) = \Lambda_2 \int_{[0,z]} \frac{1-\Lambda_2 z}{\Lambda_2 - z}\, dz - \Lambda_1 |z|^2 z$$
$$= \Lambda_2^2 z - \Lambda_1 |z|^2 z + \left(\Lambda_2^3 - \Lambda_2\right) \ln\left(1 - \frac{z}{\Lambda_2}\right), \quad z \in U. \tag{3.3}$$

Proof. By the hypothesis of Theorem 3.1, we have

$$|G(z)| = \left| \int_{[0,z]} G_z(z)dz + G_{\bar{z}}(z)d\bar{z} \right| \leq \int_{[0,z]} |\Lambda_G(z)||dz| \leq \Lambda_1 |z|, \quad z \in U.$$

We first prove that F is univalent in the disk U_{ρ_0}. Indeed, for all $z_1, z_2 \in U_r (0 < r < \rho_0,\ z_1 \neq z_2)$, note that $\lambda_F(0) = \lambda_H(0) = 1$ and $\Lambda_H(z) < \Lambda_2$ for all $z \in U$, we obtain from Lemma 2.2 that

$$|F(z_2) - F(z_1)| = \left| \int_{\overline{z_1 z_2}} F_z(z)dz + F_{\bar{z}}(z)d\bar{z} \right|$$
$$= \left| \int_{\overline{z_1 z_2}} \left(\bar{z}G(z) + |z|^2 G_z(z) + H_z(z)\right) dz + \left(zG(z) + |z|^2 G_{\bar{z}}(z) + H_{\bar{z}}(z)\right) d\bar{z} \right|$$
$$\geq \left| \int_{\overline{z_1 z_2}} H_z(z)dz + H_{\bar{z}}(z)d\bar{z} \right| - \int_{\overline{z_1 z_2}} 3\Lambda_1 r^2 |dz|$$
$$\geq |z_1 - z_2| \left(\Lambda_2 \frac{1-\Lambda_2 r}{\Lambda_2 - r} - 3\Lambda_1 r^2 \right). \tag{3.4}$$

It is easy to verify that the function

$$g_0(r) := \Lambda_2 \frac{1-\Lambda_2 r}{\Lambda_2 - r} - 3\Lambda_1 r^2$$

is continuous and strictly decreasing on $[0, 1]$, $g_0(0) = 1 > 0$, and

$$g_0(1) = -(\Lambda_2 + 3\Lambda_1) < 0.$$

Therefore, by the mean value theorem, there is a unique real $\rho_0 \in (0, 1)$ such that $g_0(\rho_0) = 0$. We obtain that

$$|F(z_2) - F(z_1)| > |z_1 - z_2| \left(\Lambda_2 \frac{1-\Lambda_2 \rho_0}{\Lambda_2 - \rho_0} - 3\Lambda_1 \rho_0^2 \right) = 0.$$

This implies $F(z_1) \neq F(z_2)$, which proves the univalence of F in the disk U_{ρ_0}.

Next, we prove that $F(U_{\rho_0}) \supseteq U_{\sigma_0}$.

Indeed, note that $F(0) = 0$, for $z' \in \partial U_{\rho_0}$ with $w' = F(z') \in F(\partial U_{\rho_0})$ and $|w'| = \min\{|w| : w \in F(\partial U_{\rho_0})\}$. Let $\gamma = F^{-1}\left(\overline{ow'}\right)$, by Lemma 2.3, we have

$$
\begin{aligned}
|w'| = \left||z'|^2G(z') + H(z')\right| &\geq |H(z')| - \Lambda_1\rho_0^3 \\
&= \left|\int_\gamma H_\zeta(\zeta)d\zeta + H_{\bar{\zeta}}(\zeta)d\bar{\zeta}\right| - \Lambda_1\rho_0^3 \\
&\geq \Lambda_2\int_0^{\rho_0}\frac{\frac{1}{\Lambda_2}-t}{1-\frac{t}{\Lambda_2}}dt - \Lambda_1\rho_0^3 \\
&= \Lambda_2^2\rho_0 + \left(\Lambda_2^3-\Lambda_2\right)\ln\left(1-\frac{\rho_0}{\Lambda_2}\right) - \Lambda_1\rho_0^3 = \sigma_0,
\end{aligned}
$$

which implies that $F(U_{\rho_0}) \supseteq U_{\sigma_0}$.

Now, we prove the sharpness of ρ_0 and σ_0. To this end, we consider a biharmonic mapping $F_0(z)$ which is given by (3.3). It is easy to verify that $F_0(z)$ satisfies the hypothesis of Theorem 3.1, and thus, we have that $F_0(z)$ is univalent in the disk U_{ρ_0}, and $F_0(U_{\rho_0}) \supseteq U_{\sigma_0}$.

To show that the univalent radius ρ_0 is sharp, we need to prove that $F_0(z)$ is not univalent in U_r for each $r \in (\rho_0, 1]$. In fact, considering the real differentiable function

$$
h_0(x) = \Lambda_2^2x - \Lambda_1x^3 + \left(\Lambda_2^3-\Lambda_2\right)\ln\left(1-\frac{x}{\Lambda_2}\right), \quad x \in [0,1]. \tag{3.5}
$$

Because the continuous function

$$
h_0'(x) = \Lambda_2^2 - 3\Lambda_1x^2 + \frac{\Lambda_2-\Lambda_2^3}{\Lambda_2-x} = g_0(x)
$$

is strictly decreasing on $[0,1]$ and $h_0'(\rho_0) = g_0(\rho_0) = 0$, we see that $h_0'(x) = 0$ for $x \in [0,1]$ if and only if $x = \rho_0$. So $h_0(x)$ is strictly increasing on $[0,\rho_0)$ and strictly decreasing on $[\rho_0,1]$. Since $h_0(0) = 0$, there is a unique real $r_3 \in (\rho_0,1]$ such that $h_0(r_3) = 0$ if $h_0(1) \leq 0$, and

$$
\sigma_0 = \Lambda_2^2\rho_0 + \left(\Lambda_2^3-\Lambda_2\right)\ln\left(1-\frac{\rho_0}{\Lambda_2}\right) - \Lambda_1\rho_0^3 = h_0(\rho_0) > h_0(0) = 0. \tag{3.6}
$$

For every fixed $r \in (\rho_0,1]$, set $x_1 = \rho_0 + \varepsilon$, where

$$
\varepsilon = \begin{cases} \min\left\{\frac{r-\rho_0}{2}, \frac{r_3-\rho_0}{2}\right\}, & \text{if } f_0(1) \leq 0, \\ \frac{r-\rho_0}{2}, & \text{if } f_0(1) > 0, \end{cases}
$$

by the mean value theorem, there is a unique $\delta \in (0,\rho_0)$ such that $x_2 := \rho_0 - \delta \in (0,\rho_0)$ and $h_0(x_1) = h_0(x_2)$.

Let $z_1 = x_1$ and $z_2 = x_2$. Then $z_1,\ z_2 \in U_r$ with $z_1 \neq z_2$. Directly computation leads to

$$
F_0(z_1) = F_0(x_1) = h_0(x_1) = h_0(x_2) = F_0(z_2).
$$

Hence F_0 is not univalent in the disk U_r for each $r \in (\rho_0,1]$, and the univalent radius ρ_0 is sharp.

Finally, note that $F_0(0) = 0$ and picking up $z' = \rho_0 \in \partial U_{\rho_0}$, by (3.3), (3.5) and (3.6), we have

$$|F_0(z') - F_0(0)| = |F_0(\rho_0)| = |h_0(\rho_0)| = h_0(\rho_0) = \sigma_0.$$

Hence, the covering radius σ_0 is also sharp. □

Remark 3.2. Corollary 1 in [24] is just a special case of Theorem 3.1 when $\Lambda_1 = 0$.

For the harmonic mapping $H(z)$ of the unit disk U with $\lambda_H(0) = 1$ and $\Lambda_H(z) \leq \Lambda_2$ for all $z \in U$, it follows from Lemma 2.4 that $\Lambda_2 \geq 1$. Theorem 3.1 provides the sharp version of Landau-type theorem of biharmonic mappings for the case $\Lambda_1 \geq 0$ and $\Lambda_2 > 1$. If $\Lambda_1 \geq 0$ and $\Lambda_2 = 1$, then we prove the following sharp version of Landau-type theorem for biharmonic mappings using Lemmas 2.4 and 2.5.

Theorem 3.3. *Suppose that $\Lambda \geq 0$. Let $F(z) = |z|^2G(z) + H(z)$ be a biharmonic mapping of U, where $G(z)$, $H(z)$ are harmonic in U, satisfying $G(0) = H(0) = 0$, $\lambda_F(0) = 1, \Lambda_G(z) \leq \Lambda$ and $\Lambda_H(z) \leq 1$ for all $z \in U$. Then F is univalent in the disk U_{ρ_1} and $F(U_{\rho_1})$ contains a Schlicht disk U_{σ_1}, where $\rho_1 = 1$ when $0 \leq \Lambda \leq \frac{1}{3}$, $\rho_1 = \frac{1}{\sqrt{3\Lambda}}$ when $\Lambda > \frac{1}{3}$, and $\sigma_1 = \rho_1 - \Lambda\rho_1^3$ is defined by (2.7). This result is sharp.*

Proof. Because $F(z) = |z|^2G(z) + H(z)$ satisfies the hypothesis of Theorem 3.3, where $G(z) = G_1(z) + \overline{G_2(z)}$ and $H(z) = H_1(z) + \overline{H_2(z)}$ with $G_1(z) = \sum_{n=1}^{\infty} a_n z^n$, $G_2(z) = \sum_{n=1}^{\infty} b_n z^n$ and $H_1(z) = \sum_{n=1}^{\infty} c_n z^n$, $H_2(z) = \sum_{n=1}^{\infty} d_n z^n$ are analytic on U. Then

$$\lambda_F(0) = \lambda_H(0) = ||c_1| - |d_1|| = 1.$$

Since $\Lambda_H(z) \leq 1$, using Lemma 2.4, we have

$$c_n = d_n = 0, \quad n = 2, 3, \ldots.$$

Hence $H(z) = c_1 z + \overline{d_1 z}$ with $||c_1| - |d_1|| = 1$.

Now we prove F is univalent in the disk U_{ρ_1}. To this end, for any $z_1, z_2 \in U_r (0 < r < \rho_1)$ with $z_1 \neq z_2$, by (3.4), we have

$$\begin{aligned} |F(z_1) - F(z_2)| &\geq \left| \int_{\overline{z_1 z_2}} H_z(z)dz + H_{\bar{z}}(z)d\bar{z} \right| - \int_{\overline{z_1 z_2}} 3\Lambda_1 r^2 |dz| \\ &\geq |z_1 - z_2|(||c_1| - |d_1|| - 3\Lambda r^2) \\ &= |z_1 - z_2|(1 - 3\Lambda r^2) > 0. \end{aligned}$$

Then, we have $F(z_1) \neq F(z_2)$, which proves the univalence of F in the disk U_{ρ_1}.

Noticing that $F(0) = 0$, for any $z = \rho_1 e^{i\theta} \in \partial U_{\rho_1}$, we have

$$\begin{aligned} |F(z)| &= ||z|^2G(z) + H(z)| \geq |H(z)| - \rho_1^2|G(z)| \\ &\geq \rho_1||c_1| - |d_1|| - \Lambda\rho_1^3 = \rho_1 - \Lambda\rho_1^3 = \sigma_1. \end{aligned}$$

Hence, $F(U_{\rho_1})$ contains a Schlicht disk U_{σ_1}.

Finally, for $F(z) = a_1\Lambda|z|^2 z + \overline{d_1}\overline{z}$ with $|a_1| = |d_1| = 1$, we have $G(z) = a_1\Lambda z, H(z) = \overline{d_1}\overline{z}$. Direct computation yields

$$G(0) = H(0) = 0,\ \lambda_F(0) = |-\overline{d_1}| = 1,\ \Lambda_G(z) = |a_1\Lambda| \le \Lambda$$

and $|\Lambda_H(z)| = |\overline{d_1}| \le 1$ for all $z \in U$. Applying Lemma 2.5, we obtain that ρ_1, σ_1 are sharp. This completes the proof. □

Next, we prove the sharp version of Landau-type theorems for certain biharmonic mappings $F(z) = |z|^2 G(z) + H(z)$ with $G(0) = H(0) = 0$, $\lambda_F(0) = 1, \Lambda_G(z) \le \Lambda$ and $|H(z)| < 1$ for all $z \in U$, which is also one of the main results in this paper.

Theorem 3.4. *Suppose that $\Lambda \ge 0$. Let $F(z) = |z|^2 G(z) + H(z)$ be a biharmonic mapping of U, where $G(z)$, $H(z)$ are harmonic in U, satisfying $G(0) = H(0) = 0$, $\lambda_F(0) = 1, \Lambda_G(z) \le \Lambda$ and $|H(z)| < 1$ for all $z \in U$. Then F is univalent in the disk U_{ρ_1} and $F(U_{\rho_1})$ contains a Schlicht disk U_{σ_1}, where $\rho_1 = 1$ when $0 \le \Lambda \le \frac{1}{3}$, $\rho_1 = \frac{1}{\sqrt{3\Lambda}}$ when $\Lambda > \frac{1}{3}$, and $\sigma_1 = \rho_1 - \Lambda\rho_1^3$ is defined by (2.7). This result is sharp.*

Proof. Because $F(z) = |z|^2 G(z) + H(z)$ satisfies the hypothesis of Theorem 3.4, where $G(z) = g_1(z) + \overline{g_2(z)}$ and $H(z) = h_1(z) + \overline{h_2(z)}$ with $g_1(z) = \sum_{n=1}^{\infty} a_n z^n$, $g_2(z) = \sum_{n=1}^{\infty} b_n z^n$ and $h_1(z) = \sum_{n=1}^{\infty} c_n z^n$, $h_2(z) = \sum_{n=1}^{\infty} d_n z^n$ are analytic on U. Then

Since $|H(z)| < 1$, using Lemma 2.6, we have

$$c_n = d_n = 0, \quad n = 2, 3, \ldots.$$

Hence $H(z) = c_1 z + \overline{d_1 z}$ with $||c_1| - |d_1|| = 1$.

Now we prove F is univalent in the disk U_{ρ_1}, where

$$\rho_1 = \begin{cases} 1 & \text{if } 0 \le \Lambda \le \frac{1}{3}, \\ \frac{1}{\sqrt{3\Lambda}} & \text{if } \Lambda > \frac{1}{3}. \end{cases}$$

To this end, for any $z_1, z_2 \in U_r (0 < r < \rho_1)$ with $z_1 \ne z_2$, by the means of Lemma 2.7, we have

$$|F(z_1) - F(z_2)| \ge |z_1 - z_2|(1 - 3\Lambda r^2) > |z_1 - z_2|(1 - 3\Lambda\rho_0^2) \ge 0.$$

Then, we have $F(z_1) \ne F(z_2)$, which proves the univalence of F in the disk U_{ρ_1}.

Noticing that $F(0) = 0$, for any $z = \rho_0 e^{i\theta} \in \partial U_{\rho_1}$, it follows from (2.8) that

$$\begin{aligned} |F(z) - F(0)| &= ||z|^2 G(z) + H(z)| \ge |H(z)| - \rho_1^2 |G(z)| \\ &\ge \rho_1 ||c_1| - |d_1|| - \Lambda\rho_1^3 = \rho_1 - \Lambda\rho_1^3 = \sigma_1. \end{aligned}$$

Hence, $F(U_{\rho_1})$ contains a Schlicht disk U_{σ_1}.

Finally, for $F_1(z) = a_1\Lambda|z|^2\overline{z} + \overline{d_1}z$ with $|a_1| = |d_1| = 1$, we have $G_1(z) = a_1\Lambda\overline{z}, H_1(z) = \overline{d_1}z$. It is easy to verify that $G_1(0) = H_1(0) = 0$, $\lambda_F(0) =$

$1, \Lambda_{G_1}(z) = |a_1\Lambda| \leq \Lambda$ and $|H_1(z)| = |\overline{d_1}||z| < 1$ for all $z \in U$. Applying Lemma 2.5, we obtain that both of ρ_1 and σ_1 are sharp. This completes the proof. □

Remark 3.5. For the harmonic mapping $H(z)$ in the unit disk U with $\lambda_H(0) = 1$ and $|H(z)| < M$ for all $z \in U$, it follows from Lemma 2.6 that $M \geq 1$. Theorem 3.4 provides the sharp version of Landau-type theorem of biharmonic mappings for the case $\Lambda \geq 0$ and $M = 1$. For the case of $\Lambda \geq 0$ and $M > 1$, we first consider an example as follows.

Example 3.6. Suppose that $M > 1$ and $\Lambda \geq 0$. Let $F_2(z) = -\Lambda|z|^2 z + Mz\frac{1-Mz}{M-z}$ be a biharmonic mapping of U. Then $F_2(z)$ is univalent in the disk U_{ρ_2}, where ρ_2 is the unique positive root in $(0, 1)$ of the equation

$$M^2 - \frac{M^2(M^2-1)}{(M-r)^2} - 3\Lambda r^2 = 0, \tag{3.7}$$

and $F_2(U_{\rho_2})$ contains a Schlicht disk U_{σ_2}, with

$$\sigma_2 = M\rho_2\frac{1-M\rho_2}{M-\rho_2} - \Lambda\rho_2^3. \tag{3.8}$$

Both of ρ_2 and σ_2 are sharp.

Proof. We first prove $F_2(z)$ is univalent in the disk U_{ρ_2}. To this end, for any $z_1, z_2 \in U_r (0 < r < \rho_2)$ with $z_1 \neq z_2$, simple computation yields that

$$\begin{aligned}
|F_2(z_1) - F_2(z_2)| &\geq |H(z_1) - H(z_2)| - \left|\Lambda|z_1|^2 z_1 - \Lambda|z_2|^2 z_2\right| \\
&= M^2|z_1 - z_2|\left|\frac{1 - M(z_1+z_2) + z_1z_2}{(M-z_1)(M-z_2)}\right| \\
&\quad - \Lambda\Big||z_1|^2(z_1 - z_2) + z_2(|z_1| + |z_2|)(|z_1| - |z_2|)\Big| \\
&\geq M^2|z_1 - z_2|\left|1 - \frac{M^2-1}{(M-z_1)(M-z_2)}\right| - 3\Lambda r^2|z_1 - z_2| \\
&\geq |z_1 - z_2|\left[M^2\left(1 - \frac{M^2-1}{(M-r)^2}\right) - 3\Lambda r^2\right].
\end{aligned}$$

It is easy to verify that the function

$$g_1(r) := M^2 - \frac{M^2(M^2-1)}{(M-r)^2} - 3\Lambda r^2$$

is continuous and strictly decreasing on $[0, 1]$, $g_0(0) = 1 > 0$, and

$$g_1(1) = -\left(\frac{2M^2}{M-1} + 3\Lambda\right) < 0.$$

There, by the mean value theorem, there is a unique real $\rho_2 \in (0, 1)$ such that $g_1(\rho_2) = 0$. We obtain that

$$|F_2(z_1) - F_2(z_2)| > |z_1 - z_2|\left(M^2 - \frac{M^2(M^2-1)}{(M-\rho_2)^2} - 3\Lambda\rho_2^2\right) = 0.$$

Then, we have $F_2(z_1) \neq F_2(z_2)$, which proves the univalence of $F_2(z)$ in the disk U_{ρ_2}.

Next, we prove the sharpness of ρ_2. To this end, we need to prove that $F_2(z)$ is not univalent in the disk U_r for each $r \in (\rho_2, 1]$.

In fact, consider the real differentiable function

$$h_1(x) = Mx\frac{1 - Mx}{M - x} - \Lambda x^3, \quad x \in [0, 1]. \tag{3.9}$$

Because the continuous function

$$h_1'(x) = M\frac{1 - Mx}{M - x} + Mx\frac{1 - M^2}{(M - x)^2} - 3\Lambda x^2 = g_1(x)$$

is strictly decreasing on $[0, 1]$ and $h_1'(\rho_2) = g_1(\rho_2) = 0$, we obtain that $h_1'(x) = 0$ for $x \in [0, 1]$ if and only if $x = \rho_2$. So $h_1(x)$ is strictly increasing on $[0, \rho_2]$ and strictly decreasing on $[\rho_2, 1]$. Since $h_1(0) = 0$, there is a unique real $r_4 \in (\rho_2, 1]$ such that $h_1(r_4) = 0$ if $h_1(1) \leq 0$, and

$$\sigma_2 = M\rho_2\frac{1 - M\rho_2}{M - \rho_2} - \Lambda\rho_2^3 = h_1(\rho_2) > h_1(0) = 0. \tag{3.10}$$

For every fixed $r \in (\rho_1, 1]$, set $x_1 = \rho_1 + \varepsilon$, where

$$\varepsilon = \begin{cases} \min\left\{\frac{r-\rho_2}{2}, \frac{r_4-\rho_2}{2}\right\}, & \text{if } h_1(1) \leq 0, \\ \frac{r-\rho_2}{2}, & \text{if } h_1(1) > 0, \end{cases}$$

by the mean value theorem, there is a unique $\delta \in (0, \rho_2)$ such that $x_2 := \rho_2 - \delta \in (0, \rho_2)$ and $h_1(x_1) = h_1(x_2)$.

Let $z_1 = x_1$ and $z_2 = x_2$, then $z_1,\ z_2 \in U_r\,(r \in (\rho_1, 1])$ with $z_1 \neq z_2$. Directly computation leads to

$$F_2(z_1) = F_2(x_1) = h_1(x_1) = h_1(x_2) = F_2(z_2),$$

which implies that $F_2(z)$ is not univalent in the disk U_r for each $r \in (\rho_2, 1]$. Hence, the univalent radius ρ_2 is sharp.

Finally, noticing that $F_2(0) = 0$, for any $z = \rho_2 e^{i\theta} \in \partial U_{\rho_2}$ and $z' = \rho_2 \in \partial U_{\rho_2}$, we have

$$\begin{aligned} |F_2(z) - F_2(0)| &= \left| -\Lambda|z|^2 z + Mz\frac{1 - Mz}{M - z}\right| \\ &\geq M\rho_2\left|\frac{1 - Mz}{M - z}\right| - \Lambda\rho_2^3 = M\rho_2\left|\frac{\frac{1}{M} - z}{1 - \frac{1}{M}z}\right| - \Lambda\rho_2^3 \\ &\geq M\rho_2\frac{\frac{1}{M} - |z|}{1 - \frac{1}{M}|z|} - \Lambda\rho_2^3 = M\rho_2\frac{1 - M\rho_2}{M - \rho_2} - \Lambda\rho_2^3 = \sigma_2, \end{aligned}$$

and

$$|F_2(z') - F_2(0)| = |F_2(\rho_2)| = |h_1(\rho_2)| = h_1(\rho_2) = \sigma_2.$$

Hence, $F_2(U_{\rho_2})$ contains a Schlicht disk U_{σ_2}, and the radius σ_2 is sharp. This completes the proof. □

Next, applying Lemma 2.7 and Example 3.6, we may verify the following sharp form of the Landau-type theorem for certain biharmonic mapping in the unit disk U.

Theorem 3.7. *Suppose that $M > 1$ and $\Lambda \geq 0$. Let $F(z) = |z|^2G(z) + H(z)$ be a biharmonic mapping of U, where $G(z)$ is harmonic in U, satisfying $G(0) = \lambda_F(0) - 1 = 0, \Lambda_G(z) \leq \Lambda$ for all $z \in U$, and $H(z) = \sum_{n=1}^{\infty} c_n z^n + \overline{\sum_{n=1}^{\infty} d_n z^n}$ is harmonic in U, satisfying the following inequality*

$$\sum_{n=2}^{\infty} n(|c_n| + |d_n|)r^{n-2} \leq \frac{(M^2-1)(2M-r)}{(M-r)^2}, \quad 0 \leq r \leq r_0$$
$$= \frac{1}{M+\sqrt{M^2-1}}. \tag{3.11}$$

Then F is univalent in the disk U_{ρ_2} and $F(U_{\rho_2})$ contains a Schlicht disk U_{σ_2}, where ρ_2 is the unique positive root in $(0,1)$ of Eq. (3.7) and σ_2 is given by (3.8). This result is sharp with $F_2(z) = -\Lambda|z|^2 z + Mz\frac{1-Mz}{M-z}$ being an extremal mapping.

Proof. Since $\lambda_F(0) = 1$, we have

$$||c_1| - |d_1|| = \lambda_H(0) = \lambda_F(0) = 1. \tag{3.12}$$

We first prove F is univalent in the disk U_{ρ_2}. To this end, note that $\rho_2 \leq r_0$, for any $z_1, z_2 \in U_r$ $(0 < r < \rho_2)$ with $z_1 \neq z_2$, by all hypotheses of Theorem 3.7 and Lemma 2.7, we obtain that

$$\begin{aligned}
|F(z_1) - F(z_2)| &\geq |z_1 - z_2|\left[||c_1| - |d_1|| - \sum_{n=2}^{\infty}(|c_n| + |d_n|)nr^{n-1} - 3\Lambda r^2\right] \\
&\geq |z_1 - z_2|\left[1 - \frac{(M^2-1)(2Mr - r^2)}{(M-r)^2} - 3\Lambda r^2\right] \\
&= |z_1 - z_2|\left[M^2 - \frac{M^2(M^2-1)}{(M-r)^2} - 3\Lambda r^2\right] \\
&> |z_1 - z_2|\left[M^2 - \frac{M^2(M^2-1)}{(M-\rho_2)^2} - 3\Lambda \rho_2^2\right] = 0.
\end{aligned}$$

Then, we have $F(z_1) \neq F(z_2)$, which proves the univalence of F in the disk U_{ρ_1}.

Next, noticing that $F(0) = 0$, for any $z = \rho_2 e^{i\theta} \in \partial U_{\rho_2}$, it follows from (2.8) that

$$\begin{aligned}|F(z)-F(0)| &= ||z|^2G(z)+H(z)| \geq \left|\sum_{n=1}^{\infty}(c_nz^n+\overline{d_nz^n})\right| - \rho_2^2|G(z)|\\ &\geq \rho_2||c_1|-|d_1|| - \sum_{n=2}^{\infty}(|c_n|+|d_n|)\rho_2^n - \Lambda\rho_2^3\\ &= \rho_2 - \int_0^{\rho_2}\sum_{n=2}^{\infty}n(|c_n|+|d_n|)r^{n-1}dr - \Lambda\rho_2^3\\ &\geq \rho_2 - \int_0^{\rho_2}\frac{(M^2-1)(2Mr-r^2)}{(M-r)^2}dr - \Lambda\rho_2^3\\ &= \rho_2 - \frac{(M^2-1)\rho_2^2}{M-\rho_2} - \Lambda\rho_2^3 = M\rho_2\frac{1-M\rho_2}{M-\rho_2} - \Lambda\rho_2^3 = \sigma_2.\end{aligned}$$

Hence, $F(U_{\rho_2})$ contains a Schlicht disk U_{σ_2}.

Finally, we prove the sharpness of ρ_2 and σ_2. We consider the biharmonic mapping $F_2(z) = -\Lambda|z|^2z + Mz\frac{1-Mz}{M-z}$. Let $G(z) = -\Lambda z$, $H(z) = Mz\frac{1-Mz}{M-z}$ for $z \in U$. Then $G(z), H(z)$ are harmonic mappings in U, and $G(0) = H(0) = 0$, $\lambda_F(0) = 1, \Lambda_G(z) = \Lambda \leq \Lambda$ for all $z \in U$. Note that

$$Mz\frac{1-Mz}{M-z} = z - \sum_{n=2}^{\infty}\frac{M^2-1}{M^{n-1}}z^n, \quad z \in U,$$

$$\sum_{n=2}^{\infty}\frac{M^2-1}{M^{n-1}}nr^{n-1} = \frac{(M^2-1)(2Mr-r^2)}{(M-r)^2}, \quad 0 \leq r < 1,$$

we obtain that $F_2(z)$ satisfies all hypotheses of Theorem 3.7. Applying Example 3.6, we obtain that both of ρ_2 and σ_2 are sharp. The proof of Theorem 3.7 is complete. □

Corollary 3.8. *Suppose that $M > 1$ and $\Lambda \geq 0$. Let $F(z) = |z|^2G(z) + H(z)$ be a biharmonic mapping of U, where $G(z)$ is harmonic in U, satisfying $G(0) = \lambda_F(0) - 1 = 0, \Lambda_G(z) \leq \Lambda$ for all $z \in U$, and $H(z) = \sum_{n=1}^{\infty}c_nz^n + \overline{\sum_{n=1}^{\infty}d_nz^n}$ is harmonic in U, satisfying the following inequality*

$$|c_n| + |d_n| \leq \frac{M^2-1}{M^{n-1}}, \quad n = 2, 3, \ldots. \tag{3.13}$$

Then F is univalent in the disk U_{ρ_2} and $F(U_{\rho_2})$ contains a Schlicht disk U_{σ_2}, where ρ_2 is the unique positive root in $(0, 1)$ of Eq. (3.7) and σ_2 is given by (3.8). This result is sharp with $F_2(z) = -\Lambda|z|^2z + Mz\frac{1-Mz}{M-z}$ being an extremal mapping.

Setting $\Lambda = 0$ in Theorem 3.7, we have the following corollary.

Corollary 3.9. *Suppose that $M > 1$. Let $H(z)$ be a harmonic mapping of U with $\lambda_H(0) = 1$, and $H(z) = \sum_{n=1}^{\infty}c_nz^n + \overline{\sum_{n=1}^{\infty}d_nz^n}$ satisfying the inequality (3.11). Then H is univalent in the disk U_{r_0} and $H(U_{r_0})$ contains a Schlicht* *disk* U_{R_0}*, where* $R_0 = Mr_0^2$ *and* r_0 *is given by* (1.2)*. This result is sharp with* $H_0(z) = Mz\frac{1-Mz}{M-z}$ *being an extremal mapping.*

Finally, note that

$$|H_0(z)| = M|z|\left|\frac{1-Mz}{M-z}\right| < M \quad \text{for all } z \in U,$$

and $H_0(z)$ satisfies (3.11) and (3.12). It is natural to pose three conjectures as follows:

Conjecture 3.10. *Suppose that* $M > 1$ *and* $\Lambda \geq 0$*. Let* $F(z) = |z|^2G(z) + H(z)$ *be a biharmonic mapping of the unit disk* U*, where* $G(z)$*,* $H(z)$ *are harmonic in* U*, satisfying* $G(0) = H(0) = 0$*,* $\lambda_F(0) = 1, \Lambda_G(z) \leq \Lambda$ *and* $|H(z)| < M$ *for all* $z \in U$*. Then* F *is univalent in the disk* U_{ρ_2} *and* $F(U_{\rho_2})$ *contains a Schlicht disk* U_{σ_2}*, where* ρ_2 *is the unique positive root in* $(0, 1)$ *of Eq.* (3.7) *and* σ_2 *is given by* (3.8)*. The two radiuses* ρ_2, σ_2 *are sharp, with the extremal mapping* $F_2(z) = -\Lambda|z|^2z + Mz\frac{1-Mz}{M-z}$*.*

Conjecture 3.11. *Suppose that* $M > 1$*. Let* $H(z)$ *be a harmonic mapping of the unit disk* U *with* $H(0) = 0, \lambda_H(0) = 1$*, and* $|H(z)| < M$ *for all* $z \in U$*. Then* H *is univalent in the disk* U_{r_0} *and* $H(U_{r_0})$ *contains a Schlicht disk* U_{R_0}*, where* $R_0 = Mr_0^2$ *and* r_0 *is given by* (1.2)*. This result is sharp with* $H_0(z) = Mz\frac{1-Mz}{M-z}$ *being an extremal mapping.*

Conjecture 3.12. *Suppose that* $M > 1$*. Let* $H(z) = \sum_{n=1}^{\infty} c_nz^n + \overline{\sum_{n=1}^{\infty} d_nz^n}$ *be a harmonic mapping of the unit disk* U *with* $\lambda_H(0) = 1$ *and* $|H(z)| < M$ *for all* $z \in U$*. Then*

$$\sum_{n=2}^{\infty} n(|c_n| + |d_n|)r^{n-2} \leq \frac{(M^2-1)(2M-r)}{(M-r)^2}, \quad 0 \leq r \leq r_0 = \frac{1}{M+\sqrt{M^2-1}}.$$

The inequality is sharp, with the extremal mapping $H_0(z) = Mz\frac{1-Mz}{M-z}$*.*

Remark 3.13. If Conjecture 3.12 holds true, it follows from Theorem 3.7 and Corollary 3.9 that Conjectures 3.10 and 3.11 also hold true.

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