Results Math (2019) 74:166

1422-6383/19/040001-13
published online August 27, 2019
https://doi.org/10.1007/s00025-019-1089-5

Results in Mathematics

Large Characteristic Subgroups with Restricted Conjugacy Classes

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Abstract. It is well-known that every virtually abelian group contains an abelian characteristic subgroup of finite index. We shall say that a group class $\mathfrak{X}$ is F-characteristic if any group containing an $\mathfrak{X}$-subgroup of finite index has also a characteristic subgroup of finite index that belongs to $\mathfrak{X}$. Thus the class $\mathfrak{A}$ of abelian groups is F-characteristic. The aim of this paper is to prove that many interesting classes of infinite groups are F-characteristic. Moreover, it is shown that the class of free groups and that of free abelian groups are not F-characteristic.

Mathematics Subject Classification. 20E07, 20F24.

Keywords. Characteristic subgroup, FC-group, metahamiltonian group.

1. Introduction

Let $\mathfrak{X}$ be a class of groups, and let G be a group containing an $\mathfrak{X}$-subgroup of finite index. If $\mathfrak{X}$ is closed with respect to forming normal subgroups, it is clear that G has also a normal $\mathfrak{X}$-subgroup of finite index, and it is useful to understand for which group classes $\mathfrak{X}$ the group G must even contain a characteristic subgroup of finite index which is in $\mathfrak{X}$. This is certainly true if we take as $\mathfrak{X}$ the class $\mathfrak{A}$ of abelian groups, since it is known that if a group G has an abelian subgroup of finite index n, then it also contains an abelian characteristic subgroup A of finite index (see [11], Lemma 21.1.4, and [19]), and A can be chosen of index smaller or equal to n^2 (see [20]). The existence of large characteristic $\mathfrak{X}$-subgroups in *virtually* $\mathfrak{X}$-*groups* (i.e. groups admitting an $\mathfrak{X}$-subgroup of finite index) is quite obvious when $\mathfrak{X}$ is either the class $\mathfrak{N}$

The authors are supported by GNSAGA (INdAM), and work within the ADV-AGTA project.

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of nilpotent groups (as in such case the Fitting subgroup is nilpotent and has finite index) or the class $\mathfrak{S}$ of soluble groups (by considering the soluble radical).

We shall say that a group class $\mathfrak{X}$ is *F-characteristic* if any virtually $\mathfrak{X}$-group contains a characteristic subgroup of finite index which is in the class $\mathfrak{X}$. Thus the above quoted result just says that the class $\mathfrak{A}$ is F-characteristic, and it has recently been proved that many other natural group classes have the same property (see for instance [1,6–9,12,13]).

This paper aims to show that further relevant group classes are F-characteristic, and it is organized as follows. Sections 2 and 3 provide some necessary preliminaries and discuss the easy case of group classes defined by restrictions on conjugacy classes. The main results are contained in Sect. 4, which is focused on the class of metahamiltonian groups, and Sect. 5, where it is proved that the class of free groups and that of free abelian groups are not F-characteristic.

2. Notation and Preliminaries

Most of our notation is standard and can be found in [22]. By *group class* we mean a class of groups $\mathfrak{X}$ such that all groups isomorphic to a group in $\mathfrak{X}$ likewise belong to $\mathfrak{X}$, but we do not require in general that $\mathfrak{X}$ contains a trivial group. We shall use the first chapter of [22] as a general reference on the main closure properties of group classes, but for the reader's convenience we recall that a group class $\mathfrak{X}$ is:

- $\boldsymbol{S}_n$*-closed* if all normal subgroups of any $\mathfrak{X}$-group belong to $\mathfrak{X}$;
- $\boldsymbol{H}$*-closed* if all homomorphic images of any $\mathfrak{X}$-group belong to $\mathfrak{X}$;
- $\boldsymbol{N}$*-closed* if $\mathfrak{X}$ contains any group which is the product of an arbitrary collection of normal $\mathfrak{X}$-subgroups;
- $\boldsymbol{N}_0$*-closed* if $\mathfrak{X}$ contains any group which is the product of two normal $\mathfrak{X}$-subgroups;
- $\boldsymbol{R}_0$*-closed* if, whenever N_1 and N_2 are normal subgroups of a group G such that G/N_1 and G/N_2 are in $\mathfrak{X}$, also the factor group $G/N_1 \cap N_2$ belongs to $\mathfrak{X}$.

Recall also that if $\mathfrak{X}$ and $\mathfrak{Y}$ are arbitrary group classes, the *product* $\mathfrak{X}\mathfrak{Y}$ is the group class consisting of all $\mathfrak{X}$-by-$\mathfrak{Y}$ groups, i.e. groups G containing a normal $\mathfrak{X}$-subgroup N such that G/N belongs to $\mathfrak{Y}$.

Let $\mathfrak{X}$ be a group class. If G is a group containing some normal $\mathfrak{X}$-subgroup, the $\mathfrak{X}$*-radical* of G is the subgroup generated by all normal subgroups of G which belong to $\mathfrak{X}$. Thus the $\mathfrak{N}$-radical and the $\mathfrak{L}\mathfrak{N}$-radical of a group G coincide with the Fitting subgroup and the Hirsch-Plotkin radical of G, respectively (here of course $\mathfrak{L}\mathfrak{N}$ denotes the class of locally nilpotent groups). Clearly, the $\mathfrak{X}$-radical of a group is always defined when $\{1\} \in \mathfrak{X}$.

Notice that the $\mathfrak{X}$-radical need not be an $\mathfrak{X}$-group, but this is true when the class $\mathfrak{X}$ is $\boldsymbol{N}$-closed.

A group class $\mathfrak{X}$ is said to be F-*radical* if the $\mathfrak{X}$-radical of a group G belongs to $\mathfrak{X}$, whenever G contains a normal $\mathfrak{X}$-subgroup of finite index. The concept of an F-radical group class has been introduced in [9], where among other results it has been remarked that $\boldsymbol{S}_n$-closed F-radical group classes are F-characteristic. Notice also that any group class $\mathfrak{X}$ such that $\mathfrak{X}\mathfrak{F} = \mathfrak{X}$ is F-radical, where $\mathfrak{F}$ denotes the class of all finite groups.

Since the $\mathfrak{X}$-radical of any $\mathfrak{X}$-by-finite group is the product of finitely many normal $\mathfrak{X}$-subgroups, it is clear that all $\boldsymbol{N}_0$-closed group classes are F-radical. Finally, it turns out that if $\mathfrak{X}$ is any group class such that $\{1\} \in \mathfrak{X}$ and $\mathfrak{X} \cap \mathfrak{F}$ is not $\boldsymbol{N}_0$-closed, then $\mathfrak{X}$ is not F-radical.

3. Restrictions on Conjugacy Classes

If G is any group, the elements of G admitting only finitely many conjugates form a characteristic subgroup $FC(G)$, called the FC-*centre* of G, and G is an FC-*group* if $FC(G) = G$, i.e. if all conjugacy classes of elements of G are finite. Thus a group G has the FC-property if and only if the index $|G : C_G(x)|$ is finite for each element x of G. Abelian groups and finite groups obviously have the FC-property, and the study of FC-groups was initially developed with the purpose of finding properties common to these two relevant group classes. We refer to the monographs [27] and [3] for a detailed account of this important chapter of the theory of infinite groups.

If X is any subgroup of finite index of a group G, then the FC-centre of X lies in the FC-centre of G. In particular, if G contains a subgroup X of finite index with the FC-property, then $X \leq FC(G)$, so that the index $|G : FC(G)|$ is finite and of course $FC(G)$ is a characteristic subgroup with the FC-property. It follows that the class of FC-groups is F-characteristic. The aim of this section is to show that certain relevant subclasses of this latter class likewise are F-characteristic, as well as certain group classes defined by restrictions of similar type.

First of all, it can be easily seen that any group with a finite commutator subgroup has the FC-property, and it has been shown in [14] that finite-by-abelian groups form an F-characteristic group class (see also [9]).

Lemma 3.1. *The class of groups with a finite commutator subgroup is F-characteristic.*

Since it was proved by Neumann that a group is finite-by-abelian if and only if it has boundedly finite conjugacy classes (see [22] Part 1, Theorem 4.35), it follows that the class of groups with boundedly finite conjugacy classes is F-characteristic.

A subgroup X of a group G is said to be *nearly normal* if it has finite index in its normal closure X^G. Obviously, all normal subgroups and all subgroups of finite index are nearly normal. Moreover, it is easy to show that a group has the FC-property if and only if all its cyclic subgroups are nearly normal. A famous theorem of Neumann [17] states that all subgroups of a group G are nearly normal if and only if the commutator subgroup G' of G is finite. It follows in particular that the class of groups in which all subgroups are nearly normal is F-characteristic.

Recall also that a subgroup X of a group G is *almost normal* if it has only finitely many conjugates or equivalently if its normalizer $N_G(X)$ has finite index in G. Again, the property of being almost normal generalizes both normality and the property of having a finite index, and it turns out that a group has the FC-property if and only if all its cyclic subgroups are almost normal. In the same paper, Neumann also proved that all subgroups of a group G are almost normal if and only if the centre $Z(G)$ has finite index in G. Therefore a group contains a subgroup of finite index in which all subgroups are almost normal if and only if it has an abelian subgroup of finite index, and so the result on the existence of abelian characteristic subgroups applies also to prove that the class of groups in which all subgroups are almost normal is F-characteristic.

A third type of generalized normal subgroup defined by the obstruction of a finite section was introduced in [2]. A subgroup X of a group G is said to be *normal-by-finite* if the core X_G of X in G has finite index in X. Thus all normal subgroups and all finite subgroups of an arbitrary group are normal-by-finite. A group is called a *CF-group* if all its subgroups are normal-by-finite, and it was proved in [26] that any CF-group contains an abelian subgroup of finite index, provided that all its periodic homomorphic images are locally finite. Then it follows from this theorem that if a locally (soluble-by-finite) group contains a subgroup of finite index with the CF-property, then it also has a characteristic subgroup of finite index which is a CF-group. Therefore the class of locally (soluble-by-finite) CF-groups is F-characteristic.

If G is a group, the *upper FC-central series* of G is the ascending characteristic series $\{FC_\alpha(G)\}_\alpha$ defined by setting $FC_0(G) = \{1\}$,

$$FC_{\alpha+1}(G)/FC_\alpha(G) = FC\big(G/FC_\alpha(G)\big)$$

for each ordinal α and

$$FC_\lambda(G) = \bigcup_{\alpha<\lambda} FC_\alpha(G)$$

if λ is a limit ordinal. The last term of the upper FC-central series of G is called the *FC-hypercentre* of G, and G is said to be *FC-hypercentral* if it coincides with the FC-hypercentre. Moreover, G is called *FC-nilpotent* if $FC_k(G) = G$ for some non-negative integer k, and in this case the smallest such k is the *FC-nilpotency class* of G; then a group has the FC-property if and only if it is FC-nilpotent of class ≤ 1. Of course, every nilpotent-by-finite group is

FC-nilpotent, and any finite extension of an FC-nilpotent group is likewise FC-nilpotent. Thus the class of FC-nilpotent groups is trivially F-radical, and so also F-characteristic.

Theorem 3.2. *Let k be a positive integer. Then the class of FC-nilpotent groups of class at most k is F-characteristic.*

Proof. Let G be a group containing a subgroup X of finite index which is FC-nilpotent of class at most k. As $|G : X|$ is finite, we have that $FC_i(X) \leq FC_i(G)$, for each non-negative integer i. In particular, $X = FC_k(X)$ is contained in $FC_k(G)$, and so the characteristic subgroup $FC_k(G)$ has finite index in G. Therefore the class of FC-nilpotent groups of class at most k is F-characteristic. □

We point out that FC-hypercentral groups, like hypercentral ones, form a group class which is $\boldsymbol{N}_0$-closed (see for instance [22] Part 1, p. 130), and so also F-radical. Observe that, in contrast to the previous remark, the class of FC-groups is not F-radical, since it is easy to produce examples of abelian-by-finite groups that do not have the FC-property but can be decomposed into the product of two abelian normal subgroups.

If $\mathfrak{X}$ is a group class, the $\mathfrak{X}C$-*centre* of a group G is the set $\mathfrak{X}C(G)$ of all elements x of G for which the factor group $G/C_G(\langle x\rangle^G)$ belongs to $\mathfrak{X}$, and G is said to have $\mathfrak{X}$-*conjugacy classes* (or to be an $\mathfrak{X}C$-*group*) if $\mathfrak{X}C(G) = G$. In particular, $\mathfrak{F}C(G)$ is the FC-centre of G, and G has $\mathfrak{F}$-conjugacy classes if and only if it is an FC-group, so that $\mathfrak{X}C$-groups arise as a natural generalization of groups with finite conjugacy classes. It is clear that if the group class $\mathfrak{X}$ is $\boldsymbol{H}$ and $\boldsymbol{R}_0$-closed, then the $\mathfrak{X}C$-centre of any group is a characteristic subgroup.

Lemma 3.3. *Let $\mathfrak{X}$ be a group class which is $\boldsymbol{S}_n$, $\boldsymbol{H}$ and $\boldsymbol{R}_0$-closed. If $\mathfrak{X}\mathfrak{F} = \mathfrak{X}$, then the class of $\mathfrak{X}C$-groups is F-characteristic.*

Proof. Let G be a group containing a subgroup X of finite index which has the $\mathfrak{X}C$-property. It follows from the $\boldsymbol{S}_n$-closure of $\mathfrak{X}$ that the subgroup X can be chosen to be normal in G. If x is any element of X, the factor group $X/C_X(\langle x\rangle^X)$ belongs to $\mathfrak{X}$. Moreover, the subgroup $C_X(\langle x\rangle^X)$ has finitely many conjugates in G and

$$\bigcap_{g\in G} C_X(\langle x\rangle^X)^g = C_X(\langle x\rangle^G),$$

so that also $X/C_X(\langle x\rangle^G)$ is an $\mathfrak{X}$-group. Since X has finite index in G and $\mathfrak{X}\mathfrak{F} = \mathfrak{X}$, it follows that $G/C_G(\langle x\rangle^G)$ belongs to $\mathfrak{X}$, and so X is contained in the $\mathfrak{X}C$-centre of G. Therefore $\mathfrak{X}C(G)$ has finite index in G, and the statement is proved because $\mathfrak{X}C(G)$ is a characteristic $\mathfrak{X}C$-subgroup of G. □

If the group class $\mathfrak{X}$ is chosen to be either the class $\mathfrak{C}$ of Černikov groups or the class $\mathfrak{P}\mathfrak{F}$ of polycyclic-by-finite groups, we obtain the so-called CC-*centre* and PC-*centre* of a group, respectively, and correspondingly the concept of groups with Černikov conjugacy classes (CC-*groups*) and of groups with polycyclic-by-finite conjugacy classes (PC-*groups*). Groups with the CC-property were introduced by Y.D. Polovickiĭ (see [22] Part 1, p. 127), and later investigated by several authors, while the study of PC-groups was started in [10]. In particular, it is known that a periodic group has the CC-property if and only if it is covered by Černikov normal subgroups, and that the PC-property is equivalent to the existence of a covering consisting of polycyclic-by-finite normal subgroups.

Our next result is a direct application of Lemma 3.3.

Corollary 3.4. *The class of CC-groups and the class of PC-groups are F-characteristic.*

Let G be a group, and let k be a positive integer or ∞. The k-*layer* G_k of G is the subgroup generated by all elements of G of order k, and G is said to be an FL-*group* if all its layers are finite. Thus any FL-group is a periodic FC-group, and the structure of FL-groups was described by Polovickiĭ (see [22] Part 1, Theorem 4.43).

Lemma 3.5. *Let G be an FC-group containing a subgroup X of finite index with the FL-property. Then G is an FL-group.*

Proof. Clearly, G is periodic, and so it has a finite normal subgroup N such that $G = XN$. Thus the factor group G/N has the FL-property since N is finite. Let k be any positive integer, and let $\pi(k)$ be the set of positive divisors of k. Since G/N has finite layers, its normal subgroup

$$L/N = \langle (G/N)_h \mid h \in \pi(k)\rangle$$

is finite, and L/N obviously contains G_kN/N. Therefore G_k is finite, and G is an FL-group. □

Corollary 3.6. *The class of FL-groups is F-characteristic.*

Proof. Let G be any group containing a subgroup X of finite index which has the FL-property. Clearly, X is contained in the FC-centre F of G, and so the characteristic subgroup F is an FL-group by Lemma 3.5. Therefore the class of FL-groups is F-characteristic. □

Recall that a group G is a CL-*group* if all its layers are Černikov groups. In particular, CL-groups are periodic and have the CC-property. Moreover, it follows from a result of Polovickiĭ that all homomorphic images of a CL-group likewise have the CL-property (see [22] Part 1, Theorem 4.42). Thus the same arguments used in the proofs of Lemma 3.5 and Corollary 3.6 give the following results.

Lemma 3.7. *Let G be a CC-group containing a subgroup of finite index with the CL-property. Then G is a CL-group.*

Corollary 3.8. *The class of CL-groups is F-characteristic.*

We shall say that a group G is a PL*-group* if all its layers are polycyclic-by-finite. It was proved by Robinson [21] that an arbitrary group with the PL-property is either polycyclic-by-finite or an FL-group. Thus the class of PL-groups can be realized as the join of the class of polycyclic-by-finite groups and the class of FL-groups. On the other hand, it is known that the union of any collection of F-characteristic group classes is likewise F-characteristic (see [9], Lemma 2.1), and so we have the following statement.

Corollary 3.9. *The class of PL-groups is F-characteristic.*

4. Metahamiltonian Groups

A specially relevant class of groups with finite conjugacy classes is that of metahamiltonian locally graded groups. Recall that a group is said to be *metahamiltonian* if all its non-abelian subgroups are normal. Metahamiltonian groups were introduced and investigated by Romalis and Sesekin ([23–25]), who proved in particular that if G is a soluble metahamiltonian group, then the commutator subgroup G' of G is finite of prime-power order, and hence G has boundedly finite conjugacy classes. Although Tarski groups (i.e. infinite simple groups whose proper non-trivial subgroups have prime order) are obviously metahamiltonian, the result of Romalis and Sesekin actually holds within the large universe of locally graded groups (see [4]). Here a group G is said to be *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index. Thus every locally (soluble-by-finite) group is locally graded, and so locally graded groups form a wide class of generalized soluble groups.

In a long series of papers, Kuzennyi and Semko obtained a detailed description of the structure of locally graded metahamiltonian groups, depending essentially on the behaviour of the commutator subgroup. Obviously, every group with a commutator subgroup of prime order is metahamiltonian, and they proved that a locally graded group G is metahamiltonian if and only if all its non-abelian subgroups contain G'. Moreover, it turns out that locally graded metahamiltonian groups are soluble and have derived length at most 3.

In order to prove that the class of metahamiltonian locally graded groups is F-characteristic, we need the following result dealing with nilpotent metahamiltonian groups (see [15,16]).

Lemma 4.1. *Let G be a nilpotent metahamiltonian group. Then either G contains an abelian subgroup of finite index or the commutator subgroup G' of G has prime order.*

Theorem 4.2. *The class of metahamiltonian locally graded groups is F-characteristic.*

Proof. Let G be a group containing a locally graded metahamiltonian subgroup of finite index. Then of course G itself is locally graded. If the group G contains an abelian subgroup of finite index, then it also has an abelian characteristic subgroup of finite index.

Suppose now that G has no abelian subgroups of finite index. As the commutator subgroup of any metahamiltonian locally graded group is finite, it follows from Lemma 3.1 that G contains a characteristic subgroup K of finite index whose commutator subgroup K' is finite. Then the centralizer $C = C_K(K')$ is a nilpotent characteristic subgroup of G and the index $|G : C|$ is finite. Let X be any metahamiltonian subgroup of finite index of C. Clearly X cannot contain abelian subgroups of finite index, and hence its commutator subgroup X' has prime order by Lemma 4.1. If Y is any other metahamiltonian subgroup of finite index of C, the intersection $X \cap Y$ has finite index in G, and so it cannot be abelian. Thus $X' \cap Y' \neq \{1\}$ and hence $X' = Y'$. It follows that the subgroup $N = X'$ is characteristic in C, and so also in G. As the factor group G/N is abelian-by-finite, it contains an abelian characteristic subgroup of finite index A/N. Then $A' = N$, and so A is a metahamiltonian characteristic subgroup of finite index of G. Therefore the class of locally graded metahamiltonian groups is F-characteristic. □

Notice here that the class of metahamiltonian locally graded groups is not F-radical, since the class of finite metahamiltonian groups is not $\boldsymbol{N}_0$-closed.

The situation seems to be more complicated in the case of metahamiltonian groups which are not locally graded. If G is an unsoluble metahamiltonian group, it is known that the second commutator subgroup G'' of G is a minimal non-abelian group and $G''/Z(G'')$ is an infinite simple group (see [5]).

Lemma 4.3. *Let G be a group, and let X be a subgroup of finite index of G which is metahamiltonian and unsoluble. Then the subgroup X'' is characteristic in G.*

Proof. Since the subgroup X cannot be soluble-by-finite, also its core $Y = X_G$ is a metahamiltonian unsoluble subgroup of finite index, and $Y'' = X''$ because all proper subgroups of X'' are abelian. Thus it can be assumed without loss of generality that X is normal in G.

Let N be any normal subgroup of G such that the factor group G/N is abelian-by-finite. Then N cannot be soluble-by-finite, so that $N \cap X''$ is not abelian and hence $X'' \leq N$. On the other hand, all subgroups of X/X'' are obviously normal, and so the factor group G/X'' is abelian-by-finite. Therefore X'' is the residual of G with respect to the class $\mathfrak{AF}$ of all abelian-by-finite groups, and in particular it is characteristic in G. □

A group G is said to have *finite abelian section rank* if it has no infinite abelian sections of prime exponent. Of course, every finite extension of a group with finite abelian section rank likewise has finite abelian section rank.

Theorem 4.4. *The class of metahamiltonian groups with finite abelian section rank is F-characteristic.*

Proof. Let G be a group containing a subgroup X of finite index, which is metahamiltonian and has finite abelian section rank. Obviously, G itself has finite abelian section rank, and X can be chosen to be normal in G, so that $G^k \leq X$, where $|G : X| = k$. In order to prove that G contains a metahamiltonian characteristic subgroup of finite index, we may suppose by Theorem 4.2 that X is unsoluble, so that the subgroup X'' is characteristic in G by Lemma 4.3. It follows that $G^k X''$ is a metahamiltonian characteristic subgroup of G, and the index $|G : G^k X''|$ is finite, because G/X'' is an abelian-by-finite group with finite abelian section rank. □

5. Examples

It was proved in [8] that the class of soluble groups in which normality is a transitive relation is not F-characteristic, and the aim of this section is to provide further examples of relevant group classes which are not F-characteristic. In particular, we shall prove that the class of free abelian groups and that of free groups are not F-characteristic.

Lemma 5.1. *Let a group $G = A \times B$ be the direct product of a free abelian non-cyclic subgroup A and a subgroup B. If K is a characteristic subgroup of G, then $K = A^n \times C$, where n is a suitable non-negative integer and C is a characteristic subgroup of B.*

Proof. Since B is a direct factor of G, the intersection $C = B \cap K$ is a characteristic subgroup of B, and so it can be assumed without loss of generality that K is not contained in B. Put

$$A = \underset{i \in I}{\mathrm{Dr}} \langle a_i \rangle,$$

where each $\langle a_i \rangle$ is infinite cyclic and $|I| \geq 2$. Of course, each element of $K \backslash B$ can be uniquely written in the form $a_{i_1}^{\varepsilon_1} \dots a_{i_t}^{\varepsilon_t} b$, where $i_1, \dots, i_t$ are pairwise different indices in I, $\varepsilon_1, \dots, \varepsilon_t \neq 0$ and b belongs to B. Let m be the smallest positive integer which occurs in such expressions, and choose an element

$$x = a_{i_1}^{\varepsilon_1} \dots a_{i_t}^{\varepsilon_t} b$$

in $K \backslash B$ such that $\varepsilon_1 = m$. If $1 < h \leq t$, there exist integers q_h and r_h such that $\varepsilon_h = mq_h + r_h$ and $0 \leq r_h < m$. Let α_h be the automorphism of G defined by putting

$$a_{i_1}^{\alpha_h} = a_{i_1} a_{i_h}^{-q_h}, \quad a_i^{\alpha_h} = a_i \ \text{ if } i \neq i_1 \quad \text{and} \quad [B, \alpha_h] = \{1\}.$$

Then

$$x^{\alpha_h} = a_{i_1}^m \dots a_{i_{h-1}}^{\varepsilon_{i_{h-1}}} a_{i_h}^{r_h} a_{i_{h+1}}^{\varepsilon_{j_{h+1}}} \dots a_{i_t}^{\varepsilon_t} b$$

belongs to K, and hence $r_h = 0$, i.e. $\varepsilon_h = mq_h$ for all $h = i_2, \ldots, i_t$. It follows that

$$x^{\alpha_2 \ldots \alpha_t} = a_{i_1}^m b$$

is an element of K. Applying now the automorphism of G which only exchanges a_{i_1} with a_u for some $u \neq i_1$, we obtain that $a_u^m b \in K$, so that also $a_{i_1}^m a_u^{-m}$ belongs to K. The consideration of an automorphism of G mapping a_{i_1} into $a_{i_1} a_u$ and fixing a_u allows us to say that $a_{i_1}^m$ is in K. Therefore, K contains the subgroup A^m.

Let y be an arbitrary element of $K \backslash B$. Then

$$y = a_{j_1}^{\delta_1} \ldots a_{j_s}^{\delta_s} b_1,$$

where $j_1, \ldots, j_s$ are pairwise different elements of I, $\delta_1, \ldots, \delta_s$ are non-zero integers and $b_1 \in B$. Of course, there exist integers q and r such that $\delta_1 = mq + r$ and $0 \leq r < m$. Since $A^m \leq K$, also the product

$$a_{j_1}^{-mq} y = a_{j_1}^r a_{j_2}^{\delta_2} \ldots a_{j_s}^{\delta_s} b_1$$

is an element of K, so that $r = 0$. Then

$$a_{j_1}^{m(1-q)} y = a_{j_1}^m a_{j_2}^{\delta_2} \ldots a_{j_s}^{\delta_s} b_1$$

belongs to K, and it follows from the first part of the proof that m also divides each of the exponents $\delta_2, \ldots, \delta_s$. Therefore y lies in $A^m B$, so that $K \leq A^m B$ and hence

$$K = A^m B \cap K = A^m (B \cap K) = A^m \times C.$$

The statement is proved. □

In the above lemma the subgroup A cannot be chosen to be is infinite cyclic. In fact, if n is any integer ≥ 3 and G is the direct product of $\mathbb{Z}$ and the symmetric group S_n, the subset consisting of all pairs (h, ς) where the integer h and the permutation ς have the same parity, is a characteristic subgroup of G which does not admit the natural decomposition demanded by the statement.

Consider now the direct product $G = A \times P$, where A is a free abelian group of infinite rank and P is a group of prime order. Since A is not characteristic in G, it follows from Lemma 5.1 that G has no proper characteristic subgroups of finite index. Therefore neither the class of torsion-free groups nor that of free abelian groups are F-characteristic.

Notice also that Lemma 5.1 has the following interesting consequence, which is certainly well-known.

Corollary 5.2. *Let A be a free abelian group, and let K be a characteristic subgroup of A. Then $K = A^n$ for some non-negative integer n.*

The above corollary shows in particular that a free abelian group has only countably many characteristic subgroups. This is in contrast to the behaviour of free non-abelian groups, as it was proved by Ol'shanskiĭ [18] that any free non-abelian group contains uncountably many characteristic subgroups. On the other hand, it can be proved that neither arbitrary free groups form an F-characteristic group class.

Theorem 5.3. *The class of free groups is not F-characteristic.*

Proof. Let F be a free group of infinite rank with basis $\{x_i \mid i \in I\}$, and put

$$G = F \times \langle a \rangle,$$

where a is an element of prime order p. Consider now a torsion-free characteristic non-trivial subgroup K of G, and let y be an arbitrary non-trivial element of K. Then

$$y = cx_{i_1}^{\varepsilon_1} \dots x_{i_t}^{\varepsilon_t} a^{\varepsilon},$$

where c belongs to F', $i_1, \dots, i_t$ are pairwise different indices in I, $\varepsilon_1, \dots, \varepsilon_t \neq 0$ and $0 \leq \varepsilon < p$. For each $h = 1, \dots, t$, let α_h be the automorphism of G defined by putting $x_{i_h}^{\alpha_h} = x_{i_h} a$, $x_j^{\alpha_h} = x_j$ if $j \neq i_h$ and $a^{\alpha_h} = a$. Clearly, each α_h acts trivially on $F' = G'$ and so $y^{-1} y^{\alpha_h} = a^{\varepsilon_h}$ belongs to K. Thus $a^{\varepsilon_h} = 1$ and hence p divides ε_h for all h. It follows that K is contained in $F^p F' \langle a \rangle$, and so the index $|G : K|$ is infinite. Therefore G cannot contain torsion-free characteristic subgroups of finite index, and in particular it has no free characteristic subgroups of finite index. □

Acknowledgements

The authors are grateful to the referee, for his very useful comments.

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Received: September 25, 2018.
Accepted: August 19, 2019.