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Results in Mathematics

Convexity Properties of Some Entropies (II)

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Abstract. This is a continuation of the author's paper "Convexity properties of some entropies", published in Raşa (Results Math 73:105, 2018). We consider the sum $F_n(x)$ of the squared fundamental Bernstein polynomials of degree n, in relation with Rényi entropy and Tsallis entropy for the binomial distribution with parameters n and x. Several functional equations and inequalities for these functions are presented. In particular, we give a new and simpler proof of a conjecture asserting that F_n is logarithmically convex. New combinatorial identities are obtained as a byproduct. Rényi entropies and Tsallis entropies for more general families of probability distributions are considered. The paper ends with three new conjectures.

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1. Introduction

Consider the sum of the squared fundamental Bernstein polynomials, i.e.,

$$F_n(x) := \sum_{k=0}^{n} \left(\binom{n}{k} x^k (1-x)^{n-k} \right)^2, \; x \in [0,1].$$

For obvious reasons it is called also the index of coincidence for the binomial distribution with parameters n and x. In this context $F_n(x)$ is related to the Rényi entropy $R_n(x)$ and the Tsallis entropy $T_n(x)$ corresponding to the same distribution:

$$R_n(x) = -\log F_n(x), \; T_n(x) = 1 - F_n(x).$$

For more details, see [4, 5, 13, 14, 16] and the references therein.

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A conjecture of the author, asserting that F_n is a convex function, was validated in [8,11,12]; details can be found in [14] and [16]. A second conjecture, asserting the logarithmic convexity of F_n, was formulated by the author in [13]. It was proved in [16]; for related results see [2].

Several functional/differential equations satisfied by F_n and some associated functions were involved in the above mentioned proofs.

One aim of this paper is to give a new and simpler proof of the second conjecture. This will be achieved after establishing new functional equations satisfied by F_n.

Another aim is to present new inequalities for $F_n(x), R_n(x), T_n(x)$. Several representations of the polynomial $F_n(x)$ are known, and several combinatorial identities were deduced from them. We provide a new representation and new combinatorial identities.

The indices of coincidence, and the Rényi and Tsallis entropies for more general families of probability distributions are briefly discussed; details and proofs will appear elsewhere.

The last section is devoted to the presentation of three new conjectures.

Throughout the paper we use the notation $X := x(1-x),\ x \in [0,1]$. Then $X' = 1-2x,\ (X')^2 = 1-4X$.

2. A Functional Equation Satisfied by F_n

Consider the Legendre polynomials (see [3, 22.3.1])

$$P_n(t) := 2^{-n} \sum_{k=0}^{n} \binom{n}{k}^2 (t+1)^k (t-1)^{n-k},\ n \geq 0.$$

As in [11,12], set

$$t = \frac{2x^2 - 2x + 1}{1-2x} \in [1, \infty),\ x \in [0, \tfrac{1}{2}).$$

Then (see [12, (2.3)], [14, (24)]),

$$F_n(x) = \left(t - \sqrt{t^2-1}\right)^n P_n(t), \tag{2.1}$$

or, equivalently,

$$P_n(t) = (1-2x)^{-n} F_n(x). \tag{2.2}$$

The polynomials $P_n(t)$ satisfy several functional/differential equations. Using them and (2.1), (2.2), similar equations for $F_n(x)$ were obtained in [14, Th. 10]. Here we present another equation of this type.

Theorem 2.1. *The polynomials $F_n(x)$ satisfy*

$$x(1-x)F_n'(x) = n(1-2x)\left(F_n(x) - F_{n-1}(x)\right),\ n \geq 1. \tag{2.3}$$

Proof. According to [3, 22.8.1], we have

$$(1-t^2)P_n'(t) + ntP_n(t) = nP_{n-1}(t). \tag{2.4}$$

From $t = (1-2X)/X'$ it follows easily that

$$\frac{dx}{dt} = \frac{1-4X}{4X}. \tag{2.5}$$

Now (2.2) and (2.5) imply

$$P_n'(t) = \frac{(X')^{1-n}}{4X}\left(X'F_n'(x) + 2nF_n(x)\right). \tag{2.6}$$

Combining (2.4) with (2.2) and (2.6), we get

$$XX'F_n'(x) = n(1-4X)\left(F_n(x) - F_{n-1}(x)\right). \tag{2.7}$$

Since $1-4X = (X')^2$, (2.7) implies (2.3). □

Remark 2.1. The above proof of (2.3) is based on the property (2.4) of the Legendre polynomials. (2.3) can be also derived from (3.1) below; we shall use this fact, even in a more general context, in a forthcoming paper.

3. Inequalities for F_n, R_n, T_n

The following integral representation of F_n can be found in [14, (29)]; for a more general result see the proof of Theorem 1 in [7].

$$F_n(x) = \frac{1}{\pi}\int_0^1 \left[t + (1-t)(1-2x)^2\right]^n \frac{dt}{\sqrt{t(1-t)}},\ n \geq 0. \tag{3.1}$$

Using it, we can prove

Theorem 3.1. *The following inequalities hold for* $x \in [0,1],\ n \geq 1:$

$$F_n^2(x) \leq F_{n-1}(x)F_{n+1}(x), \tag{3.2}$$

$$F_n(x) \leq \left(F_{n-1}(x) + F_{n+1}(x)\right)/2, \tag{3.3}$$

$$(1-2X)F_{n-1}(x) \leq F_n(x) \leq F_{n-1}(x), \tag{3.4}$$

$$F_{n+1}(x) \leq \frac{1+(4n-2)X}{1+(4n+2)X}F_{n-1}(x). \tag{3.5}$$

Proof. It is easy to verify that

$$\left[t+(1-t)(1-2x)^2\right]^{n+1}s^2 + 2\left[t+(1-t)(1-2x)^2\right]^n s + \left[t+(1-t)(1-2x)^2\right]^{n-1} \geq 0, \tag{3.6}$$

for all $t \in [0,1],\ x \in [0,1],\ n \geq 1,\ s \in \mathbb{R}$.

From (3.1) and (3.6) we derive

$$F_{n+1}(x)s^2 + 2F_n(x)s + F_{n-1}(x) \geq 0,\ s \in \mathbb{R},$$

and this implies (3.2). For $s = -1$ we get (3.3). On the other hand, (3.2) entails

$$\frac{F_n(x)}{F_{n-1}(x)} \geq \frac{F_{n-1}(x)}{F_{n-2}(x)} \geq \cdots \geq \frac{F_1(x)}{F_0(x)} = \frac{1-2X}{1},$$

which proves the first inequality in (3.4). The second one is a direct consequence of (3.1).

It was proved in [14, (30)] that F_n satisfies the functional equation

$$(n+1)F_{n+1}(x) + n(1-4X)F_{n-1}(x) - (2n+1)(1-2X)F_n(x) = 0. \quad (3.7)$$

Using (3.3) and (3.7) we get

$$\frac{F_{n-1}(x) + F_{n+1}(x)}{2} \geq F_n(x) = \frac{(n+1)F_{n+1}(x) + n(1-4X)F_{n-1}(x)}{(2n+1)(1-2X)},$$

and this leads immediately to (3.5). □

Remark 3.1. The inequality (3.3) and the second inequality in (3.4) can be deduced also from [7, (15)].

Concerning the Rényi entropy $R_n(x) = -\log F_n(x)$ and the Tsallis entropy $T_n(x) = 1 - F_n(x)$, from (3.2) and (3.3) we obtain the following

Corollary 3.1. *The inequalities*

$$R_n(x) \geq \left(R_{n-1}(x) + R_{n+1}(x)\right)/2,$$
$$T_n(x) \geq \left(T_{n-1}(x) + T_{n+1}(x)\right)/2,$$

hold for all $x \in [0,1]$ and $n \geq 1$. In other words, the sequences $(R_n(x))_{n\geq 0}$ and $(T_n(x))_{n\geq 0}$ are concave.

4. $F_n(x)$ is a *log*-Convex Function: New Proof

As mentioned in the Introduction, the logarithmic convexity of the function $F_n(x)$ was conjectured in [13] and proved in [16]. Here we present a new and simpler proof.

First, from the above results we derive other functional equations satisfied by the polynomials $F_n(x)$.

Theorem 4.1. *The following equations hold for $n \geq 2$:*

$$X^2 F_n'' = n(n-1)\left(X'\right)^2 \left(F_n - 2F_{n-1} + F_{n-2}\right) + 2nX(F_{n-1} - F_n), \quad (4.1)$$

$$\begin{aligned}&\left(X^2/n\right)\left(F_n''F_n - \left(F_n'\right)^2\right)\\ &\quad = \left(X'\right)^2\left[(n-1)\left(F_nF_{n-2} - F_{n-1}^2\right) - \left(F_{n-1} - F_n\right)^2\right]\\ &\qquad + 2XF_n(F_{n-1} - F_n).\end{aligned} \quad (4.2)$$

Proof. Taking the derivative in (2.3) we get

$$XF_n'' - (n-1)X'F_n' + 2nF_n = 2nF_{n-1} - nX'F_{n-1}'. \tag{4.3}$$

On the other hand, from (2.3) we obtain

$$F_n' = (nX'/X)\,(F_n - F_{n-1})$$

and a corresponding expression of F_{n-1}'; substituting them into (4.3) yields (4.1).

Combining (4.1) and (2.3) we get, after some elementary calculation, Eq. (4.2). □

As a consequence of Theorems 3.1 and 4.1 , we obtain

Corollary 4.1. *F_n is logarithmically convex, i.e., $\log F_n$ is convex.*

Proof. According to (3.2), $F_nF_{n-2} - F_{n-1}^2 \geq 0$, and (4.2) shows that

$$\begin{aligned}\left(X^2/n\right)\left(F_n''F_n - (F_n')^2\right) &\geq 2XF_n\,(F_{n-1}-F_n) - (1-4X)\,(F_{n-1}-F_n)^2\\ &= (F_{n-1}-F_n)\,[(1-2X)F_n - (1-4X)\,F_{n-1}]\,.\end{aligned}$$

Using (3.4) we see that $(1-4X)\,F_{n-1} = (1-2X)F_{n-1} - 2XF_{n-1} \leq F_n - 2XF_{n-1} \leq (1-2X)F_n$.

This shows that $F_n''F_n - {F_n'}^2 \geq 0$, i.e., $\log F_n$ is a convex function. This concludes the proof. □

Remark 4.1. From the fact that $F_n(x)$ is log-convex it follows that the Rényi entropy $R_n(x)$ is concave. The Tsallis entropy $T_n(x)$ is log-concave as a consequence of the inequalities $F_n'' \geq 0$ and $F_n \leq 1$ (see also [16, Cor.1]). The corresponding properties of the Shannon entropy were studied in [15].

5. A New Representation of $F_n(x)$ and Some Combinatorial Identities

From (3.1) we get by elementary calculation

$$F_n(x) = \frac{1}{\pi}\sum_{k=0}^{n}\binom{n}{k}(1-4X)^k\,B\left(n-k+\frac{1}{2},k+\frac{1}{2}\right),$$

and consequently

$$F_n(x) = 4^{-n}\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}(1-4X)^k\,. \tag{5.1}$$

Under a slightly different form, this formula was obtained in [8, (6)] by using a different method.

We have also (see [4, (21)])

$$F_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2k}{k} X^k. \tag{5.2}$$

Comparing (5.1) and (5.2), the combinatorial identities (28) and (29) in [4] were obtained.

Here we will get a new representation of $F_n(x)$ and will compare it with (5.1) and (5.2).

Theorem 5.1. *For $n \geq 0$ and $x \in [0,1]$,*

$$F_n(x) = \sum_{j=0}^{n} \binom{n}{j} \binom{2j}{j} X^j \left(1 - 4X\right)^{n-j}. \tag{5.3}$$

Proof. For $0 < x \leq \frac{1}{2}$, let $s := \frac{1}{X} - 4$. Then s is invertible as a function of x, $s \geq 0$ and $s = 0$ iff $x = \frac{1}{2}$. Define

$$U_n(s) := \frac{F_n(x)}{X^n}, \ n \geq 0. \tag{5.4}$$

Then $U_n(0) = 4^n F_n(\frac{1}{2}) = \binom{2n}{n}$.

From (2.3) we derive immediately

$$\frac{d}{dx} \frac{F_n(x)}{X^n} = -n \frac{X'}{X^2} \frac{F_{n-1}(x)}{X^{n-1}}. \tag{5.5}$$

Now (5.5) combined with (5.4) and $\frac{ds}{dx} = -\frac{X'}{X^2}$ produces

$$\frac{d}{ds} U_n(s) = n U_{n-1}(s).$$

It follows that

$$U_n(s) = n \int_0^s U_{n-1}(\theta) d\theta + \binom{2n}{n}.$$

By induction we get

$$U_n(s) = \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} s^{n-k},$$

and (5.4) shows that

$$F_n(x) = X^n U_n \left(\frac{1 - 4X}{X} \right).$$

This implies (5.3) for $x \in (0, \frac{1}{2}]$. (5.3) is obviously valid for $x = 0$, and by symmetry it is valid on $[0,1]$. This concludes the proof. □

Let us return to (5.3) and expand $X^j(1-4X)^{n-j}$ in powers of $1-4X$; then write $F_n(x)$ as a polynomial of degree n in the variable $1-4X$ and compare with (5.1). The result is the following combinatorial identity:

$$\sum_{j=0}^{k}(-1)^{k-j}4^j\binom{n}{j}\binom{n-j}{n-k}\binom{2n-2j}{n-j}=\binom{2k}{k}\binom{2n-2k}{n-k}. \tag{5.6}$$

Similarly, if in (5.3) we write the right-hand side as a polynomial in X and then compare with (5.2), we obtain

$$\sum_{j=0}^{k}\left(-\frac{1}{4}\right)^j\binom{n}{j}\binom{2j}{j}\binom{n-j}{k-j}=4^{-k}\binom{n}{k}\binom{2k}{k}, \tag{5.7}$$

where $k=0,1,\dots,n$.

For $k=n$, both (5.6) and (5.7) reduce to (3.85) in [10].

Remark 5.1. Other combinatorial identities, obtained by comparing different expressions of the same function, can be found in [6]. Purely combinatorial proofs of such identities are presented in [1].

Remark 5.2. More general families of probability distributions are associated with other classes of positive linear operators, as the classical Baskakov, Szász-Mirakjan, Meyer-König and Zeller, Bleimann-Butzer-Hahn operators, or the operators introduced by Baskakov in 1957. The indices of coincidence and the corresponding Rényi and Tsallis entropies for these families were investigated in [13–16]. In a forthcoming paper, the above results involving $F_n(x)$, $R_n(x)$, $T_n(x)$ will be translated into this more general context, in specific forms. In particular, it will be shown that the sequences of Rényi and Tsallis entropies are concave, exactly as in Corollary 3.1.

6. New Conjectures

In relation with the above results, we present here three new conjectures.

6.1. For $k=0,1,\dots,n$ and $x\in[0,1]$, let

$$b_{n,k}(x):=\binom{n}{k}x^k(1-x)^{n-k} \tag{6.1}$$

be the fundamental Bernstein polynomials. Let B_n be the classical Bernstein operators on $C[0,1]$, i.e.,

$$B_nf(x)=\sum_{k=0}^{n}b_{n,k}(x)f\left(\frac{k}{n}\right),\ n\geq 1,\ f\in C[0,1],\ x\in[0,1].$$

Let $(a_k)_{k=0,1,\dots,n}$ be a convex sequence of non-negative numbers, i.e., $2a_k\leq a_{k-1}+a_{k+1}$, $k=1,\dots,n-1$. Consider the piecewise linear function $w_n\in C[0,1]$ with $w_n\left(\frac{2k-1}{2n}\right)=0$, for $k=1,\dots,n$, and $w_n\left(\frac{k}{n}\right)=a_k$ for $k=0,1,\dots,n$.

Conjecture 6.1. $B_{2n}w_n$ is a convex function.

Remark 6.1. It is a pleasant calculation to verify that the sequence $a_k := \binom{n}{k}^2 / \binom{2n}{2k}$, $k = 0, 1, \ldots, n$, is convex, and for the corresponding function w_n we have $F_n(x) = B_{2n}w_n(x)$.

6.2. Let $r \in [0, 1)$ be given. Consider the function

$$F_{n,r}(x) := \sum_{k=0}^{n} b_{n,k}(x) b_{n,k}(x - r), \quad x \in [r, 1].$$

Notice that $F_{n,0}(x) = F_n(x),\ x \in [0, 1]$.

Conjecture 6.2. $F_{n,r}$ is a log-convex function.

6.3. Following a suggestion of C. A. Micchelli, T.N.T. Goodman proved in [9] that $B_n f$ is log-concave whenever $f \in C[0, 1]$ is log-concave. In November 2017, at the University of Bielsko-Biała, the author of this paper formulated

Conjecture 6.3. If $f \in C[0, 1]$ is log-concave, then

$$\sum_{\substack{i+j=h \\ 0\le i\le n-1 \\ 0\le j\le n}}^{n} \binom{n-1}{i}\binom{n}{j}\left[(n-1-i) f\left(\frac{j}{n}\right) \Delta^2_{1/n} f\left(\frac{i}{n}\right) - (n-j)\Delta^1_{1/n} f\left(\frac{i}{n}\right) \Delta^1_{1/n} f\left(\frac{j}{n}\right)\right] \le 0,$$

for all $n \ge 1,\ h \in \{0, 1, \ldots, 2n-2\}$.(Here $\Delta^1_t f(a) = f(a+t) - f(a),\ \Delta^2_t f(a) = f(a + 2t) - 2f(a + t) + f(a)$).

If this conjecture is true, it implies Goodman's result.

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