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Results in Mathematics

An Inverse Problem for an Integro-Differential Equation with a Convolution Kernel Dependent on the Spectral Parameter

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Abstract. We consider a pencil of the first-order integro-differential operators with the convolution kernel dependent on the spectral parameter. The inverse problem is studied, which consists in recovering the kernel from the spectrum. We develop a constructive procedure for solution and obtain necessary and sufficient conditions for the solvability of the inverse problem.

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1. Introduction and Main Results

The paper concerns the inverse spectral theory for integro-differential operators. Inverse spectral problems consist in reconstruction of operators, by using their spectral characteristics. The most complete theory of such problems was developed for *differential* operators (see the monographs [1–4]). However, *integro-differential* operators are often more adequate for description of various processes in physics, biology, economics and engineering [5]. Inverse spectral theory for intergo-differential operators has not been sufficiently developed yet. It consists of fragmentary results, not forming a general picture (see [6–13] and references therein).

In this paper, we solve an inverse problem for a pencil of integro-differential operators with a kernel, depending on the spectral parameter. We note that investigation of inverse problems for differential pencils causes principal difficulties, comparing with usual differential operators (see e.g. [14]). Inverse problems for integro-differential equations with coefficients, depending on the spectral parameter, as far as we know, have not been studied before.

We investigate the boundary value problem L for the first-order integro-differential equation

$$iy'(x) + \int_0^x M(x-t,\lambda)y(t)\,dt = \lambda y(x), \quad 0 < x < \pi, \tag{1}$$

with the boundary condition $y(0) = y(\pi)$, where the convolution kernel depends linearly on the spectral parameter: $M(x,\lambda) = M_0(x) + \lambda M_1(x)$.

Define the class of functions

$$L_{2,\pi} := \{f \colon (\pi - x)f(x) \in L_2(0,\pi)\}.$$

We assume that the functions M_0 and M_1 are *real-valued*, the function M_1 is absolutely continuous on $[0,\pi)$, M_0 and M_1' belong to $L_{2,\pi}$, and $M_1(0) = 0$.

The main results of the paper are Theorems 1 and 2, providing necessary and sufficient conditions on the spectrum of the problem L.

Theorem 1. *The boundary value problem L has a countable set of complex eigenvalues, which can be numbered as $\{\lambda_n\}_{n\in\mathbb{Z}}$, counting with their multiplicities, so that the following asymptotic relation holds*

$$\lambda_n = 2n + \varkappa_n, \quad \{\varkappa_n\} \in l_2. \tag{2}$$

Theorem 2. *For arbitrary complex numbers $\{\lambda_n\}_{n\in\mathbb{Z}}$, satisfying the asymptotic relation (2), there exists a unique boundary value problem L in the form described above, such that $\{\lambda_n\}_{n\in\mathbb{Z}}$ is the spectrum of L.*

Moreover, we develop a constructive procedure for solving the following inverse problem.

Inverse Problem 1. Given the spectrum $\{\lambda_n\}_{n\in\mathbb{Z}}$ of L, construct the functions M_0 and M_1.

Note that the case, when $M_1 = 0$ and M_0 is complex-valued, has been studied in [15]. In that case, the spectrum $\{\lambda_n\}_{n\in\mathbb{Z}}$ is also sufficient for recovering L, and the theorem similar to Theorem 2 is valid.

In order to solve Inverse Problem 1 and to prove Theorem 2, we develop an approach of [7,12]. Our method is based on the reduction of the inverse problem to the system of nonlinear integral Eq. (15), called *the main equations*. In Sect. 2, we derive the system (15) and prove its unique solvability (Lemma 2). In Sect. 3, we prove the main results and obtain Algorithm 1 for solution of the inverse problem.

2. Main Equations of the Inverse Problem

Denote by $e(x, \lambda)$ the solution of Eq. (1), satisfying the initial condition $e(0, \lambda) = 1$. Obviously, the eigenvalues of L coincide with the zeros of the entire characteristic function $\Delta(\lambda) := e(\pi, \lambda) - e(0, \lambda)$. Introduce the following notations for convolutions

$$(f * g)(x) = \int_0^x f(x-t) g(t)\, dt,$$
$$f^{*1} = f, \quad f^{*n} = f^{*(n-1)} * f,\ n \geq 1, \quad f^{*0} * g = g * f^{*0} = g.$$

The solution $e(x, \lambda)$ admits the representation

$$e(x, \lambda) = \exp(-i\lambda x) + \int_0^x P(x, t, \lambda) \exp(-i\lambda(x-t))\, dt, \tag{3}$$

where

$$P(x, t, \lambda) = \sum_{\nu=1}^{\infty} i^{\nu} \frac{(x-t)^{\nu}}{\nu!} M^{*\nu}(t, \lambda). \tag{4}$$

The relations (3) and (4) can be obtained similarly to [7], where Eq. (1) with $M(x, \lambda) \equiv M(x)$ has been considered.

Formal calculations show that

$$\Delta(\lambda) = \exp(-i\lambda\pi) - 1 + \int_0^{\pi} v(t, \lambda) \exp(-i\lambda(\pi - t))\, dt, \tag{5}$$

where

$$\begin{aligned} v(t, \lambda) &= \sum_{\nu=1}^{\infty} i^{\nu} \frac{(\pi-t)^{\nu}}{\nu!} (M_0 + \lambda M_1)^{*\nu}(t) \\ &= \sum_{\nu=1}^{\infty} i^{\nu} \frac{(\pi-t)^{\nu}}{\nu!} \sum_{n=0}^{\nu} \frac{\nu!}{n!(\nu-n)!} \lambda^n (M_0^{*(\nu-n)} * M_1^{*n})(t) = \sum_{n=0}^{\infty} \lambda^n F_n(t), \end{aligned} \tag{6}$$

$$F_n(t) = \sum_{s=\max\{1-n, 0\}}^{\infty} \frac{i^{n+s} (\pi-t)^{n+s}}{n! s!} (M_0^{*s} * M_1^{*n})(t), \quad n \geq 0. \tag{7}$$

Below we use the symbol C for various constants, independent of t, λ, etc. Denote the functions

$$g_n(x) := \frac{x^n}{n!},\ n \geq 0, \quad f * g_{-1} = g_{-1} * f = f, \quad \tilde{M}_1 := M_1'.$$

Lemma 1. *For $n \geq 0$, the function F_n belongs to $W_2^n[0, \pi]$, and the following estimate holds*

$$\|F_n^{(n)}\|_{L_2(0,\pi)} \leq \frac{C^n}{[n/2]!}.$$

Moreover, $F_n^{(k)}(0) = F_n^{(k)}(\pi) = 0$, $0 \leq k < n$.

Proof. Rewrite the relation (7) in the form

$$F_n(t) = \sum_{s=\max\{1-n,0\}}^{\infty} F_{ns}(t),$$
$$F_{ns}(t) := \frac{i^{n+s}(\pi - t)^{n+s}}{n!s!}(M_0^{*s} * M_1^{*n})(t), \quad n \geq 0. \tag{8}$$

Note that

$$M_1 = \tilde{M}_1 * g_0, \quad M_1^{*n} = \tilde{M}_1^{*n} * g_{n-1}, \quad (f * g_n)' = f * g_{n-1},\ n \geq 0.$$

Using the latter formulas, we obtain

$$F_{ns}^{(k)}(t) = \frac{i^{n+s}(n+s)!}{n!s!} \sum_{j=0}^{k} (-1)^j g_{n+s-j}(\pi - t)(M_0^{*s} * \tilde{M}_1^{*n} * g_{n-k+j-1})(t),$$
$$0 \leq k \leq n, \quad s \geq 0, \quad n + s \geq 1. \tag{9}$$

Since the functions M_0 and $\tilde{M}_1$ belong to $L_{2,\pi}$, one can easily show (see e.g. [12]), that for $n + s \geq 2$, $\nu = \overline{0, n+s}$, the function $g_\nu(\pi - t)(M_0^{*s} * \tilde{M}^{*n} * g_{n-1-\nu})(t)$ is absolutely continuous on $[0, \pi]$, and

$$\left| g_\nu(\pi - t)(M_0^{*s} * \tilde{M}_1^{*n} * g_{n+s-1-\nu})(t) \right| \leq \frac{C^{n+s}}{[(n+s)/2]!}, \quad t \in [0, \pi],$$

where $[x]$ is an integer part of x. Consequently, the relation (9) for $0 \leq k < n$ and $n + s \geq 2$ yields that the functions $F_{ns}^{(k)}$ are absolutely continuous on $[0, \pi]$, and $F_{ns}^{(k)}(0) = F_{ns}^{(k)}(\pi) = 0$. For $n + s \geq 2$ the functions $F_{ns}^{(n)}$ are also absolutely continuous on $[0, \pi]$ and satisfy the estimate

$$\left| F_{ns}^{(n)}(t) \right| \leq \frac{C^{n+s}}{[(n+s)/2]!}, \quad t \in [0, \pi].$$

It remains to consider the two cases, when $n + s = 1$:

$$F_{01}(t) = i(\pi - t)M_0(t) \in L_2(0, \pi),$$
$$F_{10}(t) = i(\pi - t)(\tilde{M}_1 * g_0)(t) \in W_2^1[0, \pi], \quad F_{10}(0) = F_{10}(\pi) = 0.$$

Thus, we have for all $n, s \geq 0$, $n + s \geq 1$, that $F_{ns}^{(n)} \in L_2(0, \pi)$ and

$$\left\| F_{ns}^{(n)} \right\|_{L_2(0,\pi)} \leq \frac{C^{n+s}}{[(n+s)/2]!}.$$

Note that

$$\sum_{s=0}^{\infty} \frac{C^{n+s}}{[(n+s)/2]!} \leq C_1 \cdot \frac{C^n}{[n/2]!},$$

where C_1 is a constant. Consequently, the series

$$\sum_{s=\max\{0,1-n\}}^{\infty} F_{ns}^{(n)}, \quad n \geq 0,$$

converge in $L_2(0,\pi)$. Taking (8) into account, we arrive at the assertion of the Lemma. □

In view of Lemma 1, integration by parts yields

$$\lambda^n \int_0^\pi F_n(t)\exp(-i\lambda(\pi-t))\,dt = i^n \int_0^\pi F_n^{(n)}(t)\exp(-i\lambda(\pi-t))\,dt. \quad (10)$$

The relations (5), (6) and (10) imply

$$\Delta(\lambda) = \exp(-i\lambda\pi) - 1 + \int_0^\pi w(t)\exp(-i\lambda(\pi-t))\,dt, \quad (11)$$

where

$$w(t) = \sum_{n=0}^{\infty} i^n F_n^{(n)}(t). \quad (12)$$

Lemma 1 implies $w \in L_2(0,\pi)$.

Define the following functions for $n \geq 1$, $j = \overline{0,n}$:

$$\left.\begin{aligned} &Q_{nj}[M] := M_0^{*(n-j)} * \tilde{M}_1^{*j}, \quad \varphi_{nj}(x) := \frac{i^{n+j}}{j!(n-j)!}(\pi-x)^{n-1}, \\ &\Phi_{nj}(x,t) := \frac{i^{n+j} n!}{j!(n-j)!}\sum_{s=1}^{j} g_{n-s}(\pi-x)g_{s-1}(x-t). \end{aligned}\right\} \quad (13)$$

Substituting (7) into (12), we derive the relation

$$w(x) = (\pi-x)\sum_{n=1}^{\infty}\sum_{j=0}^{n}\left(\varphi_{nj}(x)Q_{nj}[M](x) + \int_0^x \Phi_{nj}(x,t)Q_{nj}[M](t)\,dt\right). \quad (14)$$

Considering the real part and the imaginary part of (14) separately, we arrive at the system of two nonlinear integral equations with respect to real functions M_0 and $\tilde{M}_1$:

$$f_\nu(x) = \sum_{n=1}^{\infty}\sum_{j=0}^{n}\left(\psi_{\nu nj}(x)Q_{nj}[M](x) + \int_0^x \Psi_{\nu nj}(x,t)Q_{nj}[M](t)\,dt\right), \quad \nu = 0,1, \quad (15)$$

where

$$\left.\begin{aligned} &\psi_{0nj}(x) = -i\operatorname{Im}\varphi_{nj}(x), \quad \psi_{1nj}(x) = -\operatorname{Re}\varphi_{nj}(x), \\ &\Psi_{0nj}(x,t) = -i\operatorname{Im}\Phi_{nj}(x,t), \quad \Psi_{1nj}(x,t) = -\operatorname{Re}\Phi_{nj}(x,t), \\ &f_0(x) = -i\operatorname{Im} w(x)/(\pi-x), \quad f_1(x) = -\operatorname{Re} w(x)/(\pi-x) \end{aligned}\right\} \quad (16)$$

The following Lemma claims the unique solvability of (15), and plays a crucial role in investigation of Inverse Problem 1.

Lemma 2. *The system of main Eq.* (15) *with the coefficients, defined by* (16), *has the unique solution* $(M_0, \tilde{M}_1)$, $M_0 \in L_{2,\pi}$, $\tilde{M}_1 \in L_{2,\pi}$, *for any* $w \in L_2(0,\pi)$.

The proof of Lemma 2 is based on Proposition 1, which is a special case of [16, Theorem 3].

Proposition 1. *Let $\psi_{\nu nj}(x)$ and $\Psi_{\nu nj}(x,t)$, $\nu = 0,1$, $n \in \mathbb{N}$, $j = \overline{0,n}$, be arbitrary functions, square integrable on $(0,b)$ and $\mathcal{S} := \{(x,t)\colon 0 < t < x < b\}$, respectively, and satisfying the estimates*

$$\|\psi_{\nu nj}\|_{L_2(0,b)} \le A^n, \quad \|\Psi_{\nu nj}\|_{L_2(\mathcal{S})} \le A^n, \quad \nu = 0,1,\ n \in \mathbb{N},\ j = \overline{0,n}, \tag{17}$$

for some fixed $A > 0$ independent of ν, n and j. Assume that $\psi_{\nu 1j}(x) = \delta_{\nu j}$, $\nu, j = 0,1$, where $\delta_{\nu j}$ is the Kronecker delta. Then for every functions $f_\nu \in L_2(0,b)$, $\nu = 0,1$, the system (15) has the unique solution $(M_0, \tilde{M}_1)$, $M_0 \in L_2(0,b)$, $\tilde{M}_1 \in L_2(0,b)$.

Proof of Lemma 2. Let w be an arbitrary function from $L_2(0,\pi)$. Obviously, the functions $\psi_{\nu nj}$, $\Psi_{\nu nj}$ and f_ν, $\nu = 0,1$, $n \in \mathbb{N}$, $j = \overline{0,n}$, defined by (16), satisfy the conditions of Proposition 1 for every $b \in (0,\pi)$. Hence (15) has the unique solution $(M_0, \tilde{M}_1)$, $M_0 \in L_2(0,b)$, $\tilde{M}_1 \in L_2(0,b)$ for every $b \in (0,\pi)$. It remains to prove that $M_0 \in L_{2,\pi}$ and $\tilde{M}_1 \in L_{2,\pi}$.

We represent the functions in the form $M_0(x) = M_{01}(x) + M_{02}(x)$, $\tilde{M}_1(x) = M_{11}(x) + M_{12}(x)$, so that $M_{\nu 1}(x) = 0$ for $x \in (\pi/2,\pi)$ and $M_{\nu 2}(x) = 0$ for $x \in (0,\pi/2)$, $\nu = 0,1$. Note that $M_{\nu_1 2} * M_{\nu_2 2} \equiv 0$ on $(0,\pi)$, $\nu_k = 0,1$, $k = 1,2$. Consequently, we have

$$\begin{aligned} Q_{nj}[M] = Q_{nj}[M_{(1)}] &+ (n-j) M_{02} * Q_{n-1,j}[M_{(1)}] \\ &+ j M_{12} * Q_{n-1,j-1}[M_{(1)}], \quad n \ge 2, \quad j = \overline{0,n}, \end{aligned} \tag{18}$$

where

$$Q_{nj}[M_{(1)}] := M_{01}^{*(n-j)} * M_{11}^{*j}, \quad n \ge 1,\ j = \overline{0,n}.$$

Denote

$$b_{0nj} := n - j, \quad b_{1nj} := j, \quad n \ge 2,\ j = \overline{0,n}.$$

Substituting (18) into (15), we obtain the following relation for $x > \pi/2$:

$$\begin{aligned} f_\nu(x) = M_{\nu 2}(x) &+ \sum_{j=0}^{1} \int_0^x \Psi_{\nu 1j}(x,t) M_{j2}(t)\, dt + \sum_{n=2}^\infty \sum_{j=0}^n \psi_{\nu nj}(x) Q_{nj}[M_{(1)}](x) \\ &+ \sum_{n=1}^\infty \sum_{j=0}^n \int_0^x \Psi_{\nu nj}(x,t) Q_{nj}[M_{(1)}](t)\, dt + \sum_{n=2}^\infty \sum_{j=0}^n \sum_{\xi=0}^1 b_{\xi nj} \\ &\biggl(\psi_{\nu nj}(x) (M_{\xi 2} * Q_{n-1,j-\xi}[M_{(1)}])(x) \\ &+ \int_0^x \Psi_{\nu nj}(x,t) (M_{\xi 2} * Q_{n-1,j-\xi}[M_{(1)}])(t)\, dt \biggr), \quad \nu = 0,1. \end{aligned}$$

Multiplying this relation by $(\pi - x)$, we arrive at the system of linear Volterra integral equations

$$\mu_\nu(x) = z_\nu(x) + \sum_{\xi=0}^{1} \int_{\pi/2}^{x} K_{\nu\xi}(x,t) z_\xi(t)\, dt, \quad \pi/2 < x < \pi, \quad \nu = 0, 1, \tag{19}$$

where

$$z_\nu(x) = (\pi - x) M_{\nu 2}(x),$$

$$\mu_\nu(x) = (\pi - x)\Bigg(f_\nu(x) - \sum_{n=2}^{\infty}\sum_{j=0}^{n} \psi_{\nu nj}(x) Q_{nj}[M_{(1)}](x) - \sum_{n=1}^{\infty}\sum_{j=0}^{n} \int_0^x \Psi_{\nu nj}(x,t) Q_{nj}[M_{(1)}](t)\, dt \Bigg),$$

$$K_{\nu\xi}(x,t) = \frac{\pi - x}{\pi - t}\Bigg(\Psi_{\nu 1\xi}(x,t) + \sum_{n=2}^{\infty}\sum_{j=0}^{n} b_{\xi nj}\Bigg(\psi_{\nu nj}(x) Q_{n-1,j-\xi}[M_{(1)}](x-t) + \int_0^{x-t} \Psi_{\nu nj}(x, t+s) Q_{n-1,j-\xi}[M_{(1)}](s)\, ds \Bigg)\Bigg), \quad \nu, \xi = 0, 1.$$

Note that $f_\nu \in L_2(0,\pi)$, $\Psi_{\nu 1\xi} \in L_2(\mathcal{T})$, $\nu, \xi = 0, 1$, $\mathcal{T} := \{(x,t) \colon \pi/2 < t < x < \pi\}$. Using the estimates

$$\left| \frac{\pi - x}{\pi - t} \right| < 1, \quad \pi/2 < t < x < \pi, \qquad |Q_{nj}[M_{(1)}](x)| \le \frac{C^n}{[n/2]!}, \quad n \ge 2,$$

together with (17), we conclude that $\mu_\nu \in L_2(\pi/2, \pi)$, $K_{\nu\xi} \in L_2(\mathcal{T})$, $\nu, \xi = 0, 1$. Consequently, the Volterra integral Eq. (19) has the unique solution (z_0, z_1), $z_\nu \in L_2(0, \pi)$, $\nu = 0, 1$, so we arrive at the assertion of the Lemma. □

3. Proofs of the Main Results

In this section, we prove Theorems 1 and 2, and also provide an algorithm for solving Inverse Problem 1.

Proof of Theorem 1. Using (11), we obtain the following relation for the characteristic function:

$$\Delta(\lambda) = \exp(-i\lambda\pi/2)\left(-2i \sin\tfrac{\lambda\pi}{2} + \int_{-\pi/2}^{\pi/2} w\big(s + \tfrac{\pi}{2}\big) \exp(i\lambda s)\, ds \right). \tag{20}$$

Applying to (20) the standard technique (see [4, Theorem 1.1.3]), based on Rouché‘s theorem, we derive the asymptotic relations (2) for the zeros of $\Delta(\lambda)$. □

Relying on (2) and (20), one can prove the following Proposition similarly to [11, Lemmas 1 and 2].

Proposition 2. *The characteristic function is uniquely determined by its zeros by the formula*

$$\Delta(\lambda) = -i\pi \exp(-i\lambda\pi/2)(\lambda - \lambda_0) \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\lambda_n - \lambda}{2n} \exp\left(\frac{\lambda}{2n}\right). \tag{21}$$

For arbitrary complex numbers $\{\lambda_n\}_{n\in\mathbb{Z}}$ of the form (2), the function $\Delta(\lambda)$, determined by (21), has the form (20) with a certain function $w \in L_2(0,\pi)$.

Proof of Theorem 2. Consider an arbitrary sequence of complex numbers $\{\lambda_n\}_{n\in\mathbb{Z}}$, satisfying the asymptotic relations (2). Let $\Delta(\lambda)$ be the functions, constructed by (21). By Proposition 2, $\Delta(\lambda)$ admits the representation (20) with some function $w \in L_2(0,\pi)$. Define the functions f_ν, $\psi_{\nu nj}$ and $\Psi_{\nu nj}$ for $\nu = 0, 1$, $n \in \mathbb{N}$, $j = \overline{0,n}$, by (16). Then, by Lemma 2, the main Eq. (15) have the unique solution $(M_0, \tilde{M}_1)$, $M_0 \in L_{2,\pi}$, $\tilde{M}_1 \in L_{2,\pi}$. Define $M_1(x) := \int_0^x \tilde{M}_1(t)\,dt$, and conisder the boundary value problem L with the kernel $M(x,\lambda) = M_0(x) + \lambda M_1(x)$, constructed by the found functions. By necessity, the characteristic function of L has the form (20) with the function w, satisfying the relation (14), equivalent to the system of the main Eq. (15). Thus, the characteristic function of L coincides with the function $\Delta(\lambda)$, constructed by the given numbers $\{\lambda_n\}_{n\in\mathbb{Z}}$. Hence the spectrum of L coincides with $\{\lambda_n\}_{n\in\mathbb{Z}}$. □

The proof of Theorem 2 leads to the following algorithm for solving Inverse Problem 1.

Algorithm 1. Let the complex numbers $\{\lambda_n\}_{n\in\mathbb{Z}}$ be given.

1. Construct the function $\Delta(\lambda)$ as an infinite product by (21).
2. Find the function $w(t)$, inverting the Fourier transform (20) by the formula

$$w(t) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \Delta(2n) \exp(-2int).$$

3. Construct the functions $\varphi_{nj}(x)$, $\Phi_{nj}(x,t)$, $n \in \mathbb{N}$, $j = \overline{0,n}$, using (13), and then $f_\nu(x)$, $\psi_{\nu nj}(x)$, $\Psi_{\nu nj}(x,t)$, $\nu = 0, 1$, $n \in \mathbb{N}$, $j = \overline{0,n}$, using (16).
4. Find the functions $M_0(x)$ and $\tilde{M}_1(x)$ as the solution of the main Eq. (15), put $M_1(x) := \int_0^x \tilde{M}_1(t)\,dt$.

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