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Results in Mathematics

Characterizations of Generalized Proximinal Subspaces in Real Banach Spaces

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Abstract. Let X be a real Banach space, C a closed bounded convex subset of X with the origin as an interior point, and p_C the Minkowski functional generated by the set C. This paper is concerned with the problem of generalized best approximation with respect to p_C. A property (ε_*) concerning a subspace of X^* is introduced to characterize generalized proximinal subspaces in X. A set C with feature as above in the space l_1 of absolutely summable sequences of real numbers and a continuous linear functional f on l_1 are constructed to show that each point in an open half space determined by the kernel of f admits a generalized best approximation from the kernel but each point in the other open half space does not.

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1. Introduction

Throughout this paper, $(X, \|\cdot\|)$ is a real Banach space with the topological dual X^*, and C denotes a bounded closed convex subset of X with the origin as an interior point. Recall that the Minkowski functional $p_C : X \to \mathbb{R}$ generated by the set C is defined by

$$p_C(x) := \inf\{t > 0 : \ x \in tC\}, \quad \forall\, x \in X. \tag{1}$$

Let M be a nonempty subset of X, $x \in X$, and $y_0 \in M$. Then y_0 is called a generalized best approximation following [1] or a best d_C-approximation following [8] to x from M if

$$p_C(y_0 - x) = d_C(x, M),$$

where

$$d_C(x, M) := \inf\{p_C(y - x) : y \in M\} \tag{2}$$

is the distance with respect to p_C from the point x to the set M. The set of all best d_C-approximations to x from M is denoted by $P_M^C(x)$, i.e.,

$$P_M^C(x) := \{y_0 \in M : p_C(y_0 - x) = d_C(x, M)\}, \tag{3}$$

which is called the generalized projection onto M. The set M is called d_C-proximinal if $P_M^C(x)$ is nonempty for each $x \in X$.

In 1998, De Blasi and Myjak [1] first investigated this kind of generalized best approximation problem and established the corresponding well posedness under the assumption that the modulus of convexity of C is positive in real Banach spaces, the main results of which then were extended by Li [6] to more general setting. Further, Li and Ni [7] explored the relationships between the existence of generalized best approximations and directional derivatives of the function $d_C(\cdot, M)$. Recently, Ferreira and Németh [5] considered characterization issue of generalized best approximations from closed convex sets in $\mathbb{R}^n$, and Luo and Wang [9] studied the representation and continuity issue of a generalized projection onto a closed hyperplane in real Banach spaces.

Note that when C is the closed unit ball B_X of X (hence, p_C is the norm of X), (2) and the generalized best approximation are reduced, respectively, to the distance function of M and to the classical best approximation, which has been studied deeply and extensively since the late 1950s; see [2,12,13] and references therein. In particular, via the Ascoli theorem (i.e., the distance formula from a point to a hyperplane in X) and the so-called property (ε_*) defined for a subspace in the dual space X^*, various characterizations of proximinal linear subspaces of normed spaces have been obtained; see [12, Theorem 2.1, p. 94–p. 95].

In this paper, we intend to study the possibility of extending [12, Theorem 2.1] to the setting of generalized approximation. The organization of this paper is as follows. In Sect. 2, we first list some basic properties of Minkowski functionals, and then give the characterizations for the kernel of a continuous linear functional to be d_C-proximinal. In Sect. 3, based on results obtained in Sect. 2 and some extension property of continuous linear functionals, we obtain further characterization results of d_C-proximinal linear subspaces in real normed spaces (see Theorem 1), which correspond partially to results given in [12, Theorem 2.1]. In Sect. 4, we construct a bounded closed convex set C with the origin as an interior point in the space l_1 of absolutely summable sequences of real numbers and a continuous linear functional f on l_1 such that f attains its supremum on C but $-f$ does not, which is different from the case of norm (in fact, in normed spaces X, $f \in X^*$ attains its supremum on the closed unit ball B_X if and only if $-f$ attains its supremum on B_X), and which also show that assertion (ii) of Theorem 1 of the present paper cannot be strengthened to the version of assertion (ii) of [12, Theorem 2.1] (see Theorem 1 and Corollary 1 below).

2. Preliminaries

For a nonempty subset A of X, we denote by int$A$, bd$A$, $\overline{A}$, and $\overline{A}^w$ the interior, the boundary, the closure, and the weak closure of A, respectively. Recall that C is a closed bounded convex subset of X with $0 \in \text{int}C$, and that p_C is the Minkowski functional given by (1). Define the polar C° of the set C by

$$C^\circ := \{f \in X^* : \langle f, x\rangle = f(x) \leq 1, \ \forall\, x \in C\}.$$

Then C° is a nonempty weakly* compact convex subset of X^* with $0 \in \text{int}C^\circ$. So we can define similarly the polar $C^{\circ\circ}$ of the set C°, i.e.,

$$C^{\circ\circ} := \{F \in X^{**} : \langle F, f\rangle = F(f) \leq 1, \ \forall\, f \in C^\circ\}.$$

We first list some useful properties of the Minkowski functional p_C which can be proved easily by definition.

Proposition 1. *Let $x, y \in X$, and $x^* \in X^*$. Then*

(i) $p_C(x) \geq 0$ *and* $p_C(x) = 0 \Leftrightarrow x = 0$.
(ii) $x \in C \Leftrightarrow p_C(x) \leq 1$ *and* $x \in \text{bd}C \Leftrightarrow p_C(x) = 1$.
(iii) $p(x + y) \leq p(x) + p(y)$ *and* $p_C(tx) = tp_C(x)$ *for each* $t \geq 0$.
(iv) $p_C(x) = \sup_{y^* \in C^\circ} y^*(x)$ *and* $p_{C^\circ}(x^*) = \sup_{x \in C} x^*(x)$.
(v) $x^*(x) \leq p_C(x)p_{C^\circ}(x^*)$.
(vi) *There exist positive numbers* m_1 *and* m_2 *such that*

$$m_1\|x\| \leq p_C(x) \leq m_2\|x\|.$$

In the remainder of this paper, we always assume that M is a closed subspace of X. For $f \in X^*$, $f|_M$ denotes the restriction to M of f and $\ker(f)$ stands for the kernel of f defined by $\ker(f) := \{x \in X : f(x) = 0\}$. For $x_1, x_2 \in X$, we say that x_1 and x_2 lie on different sides of $\ker(f)$ if $f(x_1)f(x_2) < 0$. Now define

$$M^\perp := \{f \in X^* : f|_M = 0\}, \text{ and } (M^\perp)_\perp := \{x \in X : f(x) = 0, \forall f \in M^\perp\},$$

Then $M^\perp$ is a weakly* closed subspaces of X^*, and

$$(M^\perp)_\perp = \overline{M}^w = \overline{M} = M, \tag{4}$$

where the first equality holds by the bipolar theorem (see, for e.g., [11, Theorem 1.5, p.126]), and the second one is due to the Mazur theorem (see, e.g., [10, Theorem 2.5.16, p.216]. For a linear subspace Γ of X^* and $x \in X$, define

$$p_{C,\Gamma}(x) := \sup\{f(x) : f \in \Gamma \cap C^\circ\}.$$

Then,

$$p_{C,M^\perp}(x) \le p_C(x) \tag{5}$$

by Proposition 1 (iv). Thus, by (5), the separation theorem, and the definition of d_C-approximation, it is not difficult to verify that the following characterization results of the best d_C-approximation from a closed subspace of X hold.

Lemma 1. *Let $x \in X\backslash M$ and $y_0 \in M$. Then the following statements are equivalent.*

(i) $y_0 \in P_M^C(x)$.

(ii) *There exists $f \in X^*$ such that*

$$f|_M = 0,\ p_{C^\circ}(f) = 1, \text{ and } f(y_0 - x) = p_C(y_0 - x). \tag{6}$$

(iii) $p_{C,M^\perp}(y_0 - x) = p_C(y_0 - x)$.

By [9, Proposition 5] (see also [3, Proposition 4.3] or [4, Proposition 2.5.6, p.177] for a slightly more general form), we have the following result.

Lemma 2. *Let $f \in X^*\backslash\{0\}$ and $x \in X$. Then*

$$d_C(x, \ker(f)) = \begin{cases} -\frac{f(x)}{p_{C^\circ}(f)}, & \text{if } f(x) < 0, \\ \frac{f(x)}{p_{C^\circ}(-f)}, & \text{if } f(x) \ge 0. \end{cases}$$

Lemma 3. *Let f be as in Lemma 2. Then the following statements are equivalent.*

(i) $\ker(f)$ *is d_C-proximinal.*

(ii) *There exist two distinct points $z_1, z_2 \in X\backslash\{0\}$ on the different sides of* $\ker(f)$ *such that* $0 \in P^C_{\ker(f)}(z_1) \cap P^C_{\ker(f)}(z_2)$.

(iii) $p_{C^\circ}(f)$ *and* $p_{C^\circ}(-f)$ *are attainable on C.*

Proof. (i)⇒(ii). Suppose that $\ker(f)$ is d_C-proximinal and take $x \in X\backslash\ker(f)$ with $f(x) > 0$. Then there exists $y_1 \in \ker(f)$ such that $p_C(y_1 - x) = d_C(x, \ker(f))$. Let $z_1 := x - y_1$. Then $f(z_1) > 0$ (clearly, $z_1 \neq 0$), and

$$p_C(0 - z_1) = d_C(x, \ker(f)) = \frac{f(x)}{p_{C^\circ}(-f)} = \frac{f(z_1)}{p_{C^\circ}(-f)} = d_C(z_1, \ker(f))$$

thanks to Lemma 2; hence, $0 \in P^C_{\ker(f)}(z_1)$. Similarly, we can show that there is $z_2 \in X\backslash\{0\}$ with $f(z_2) < 0$ such that $0 \in P^C_{\ker(f)}(z_2)$. Thus (i)⇒(ii) is true.

(ii)⇒(iii). Suppose that (ii) holds. Then $z_1, z_2 \notin \ker(f)$ because $f(z_1)f(z_2) < 0$. We may assume that $f(z_1) < 0$ and $f(z_2) > 0$. Thus Lemma 2 guarantees that

$$p_C(-z_1) = d_C(z_1, \ker(f)) = -\frac{f(z_1)}{p_{C^\circ}(f)}$$

and

$$p_C(-z_2) = d_C(z_2, \ker(f)) = \frac{f(z_2)}{p_{C^\circ}(-f)}.$$

It follows that $p_{C^\circ}(f) = f\left(\frac{-z_1}{p_C(-z_1)}\right)$ and $p_{C^\circ}(-f) = (-f)\left(\frac{-z_2}{p_C(-z_2)}\right)$. This means that both $p_{C^\circ}(f)$ and $p_{C^\circ}(-f)$ are attainable on C.

(iii)⇒(i). Suppose that (iii) holds. Since $p_{C^\circ}(f)$ is attainable on C, there is $\bar{x} \in C$ such that $f(\bar{x}) = p_{C^\circ}(f)$; hence $p_C(\bar{x}) = 1$ by Proposition 1 (ii) and (v). Let $x \in X$ with $f(x) < 0$, and let $y := x - \frac{f(x)}{f(\bar{x})}\bar{x}$. Then $y \in \ker(f)$, and

$$p_C(y - x) = -\frac{f(x)}{f(\bar{x})} p_C(\bar{x}) = -\frac{f(x)}{p_{C^\circ}(f)} = d_C(x, \ker(f)),$$

so $y \in P_M^C(x)$. In the same way, we can show that if $p_{C^\circ}(-f)$ is attainable on C, then $P_M^C(x) \neq \emptyset$ for each $x \in X$ with $f(x) > 0$. Therefore, $\ker(f)$ is d_C-proximinal.

□

Recall from [10, Definition 1.7.3, p.51] that X/M is the quotient space with respect to M, defined by

$$X/M := \{\widetilde{x} := x + M : x \in X\} \quad \text{with the norm} \quad \|\widetilde{x}\| := \inf_{y \in M} \|x - y\|,$$

and that $Q_M : X \to X/M$ is the corresponding quotient mapping defined by $Q_M(x) = \widetilde{x}$ for each $x \in X$. Then the dual mapping $Q_M^* : (X/M)^* \to M^\perp$ of Q_M is a linear isometry onto $M^\perp$ (see [10, Theorem 1.10.17, p.95]), where

$$\langle Q_M^* \phi, x \rangle = \langle \phi, Q_M x \rangle = \langle \phi, \widetilde{x} \rangle, \ \forall\, \phi \in (X/M)^*, \forall x \in X.$$

So, the dual mapping $Q_M^{**} : (M^\perp)^* \to (X/M)^{**}$ of Q_M^* is also a linear isometry onto $(X/M)^{**}$, where,

$$\langle Q_M^{**} F, \phi \rangle = \langle F, Q_M^* \phi \rangle, \ \forall F \in (X^\perp)^*, \ \forall\, \phi \in (X/M)^*.$$

Now, we define functionals $\widetilde{p_C}$ on X/M, $\widetilde{p_C}^*$ on $(X/M)^*$, and $\widetilde{p_C}^{**}$ on $(X/M)^{**}$ as follows:

$$\widetilde{p_C}(\widetilde{x}) := \inf_{y \in M} p_C(x - y), \ \forall\, \widetilde{x} \in X/M,$$
$$\widetilde{p_C}^*(\phi) := \sup\{\phi(\widetilde{x}) : \widetilde{x} \in X/M, \widetilde{p_C}(\widetilde{x}) \leq 1\}, \ \forall\, \phi \in (X/M)^*,$$

and

$$\widetilde{p_C}^{**}(\Phi) := \sup\{\Phi(\phi) : \phi \in (X/M)^*, \widetilde{p_C}^*(\phi) \leq 1\}, \ \forall\, \Phi \in (X/M)^{**}.$$

Then $\widetilde{p_C}$, $\widetilde{p_C}^*$, $\widetilde{p_C}^{**}$ are respectively sublinear on X/M, $(X/M)^*$, and $(X/M)^{**}$. Moreover, we have the following useful result.

Lemma 4. $p_{C^\circ}(Q_M^* \phi) = \widetilde{p_C}^*(\phi), \ \forall\, \phi \in (X/M)^*$.

Proof. Let $\phi \in (X/M)^*$. Then $Q_M^*\phi \in M^\perp \subseteq X^*$. Since $\{\widetilde{x} : x \in C\} \subseteq \{\widetilde{y} \in X/M : \widetilde{p_C}(\widetilde{y}) \le 1\}$, one has that

$$p_{C^\circ}(Q_M^*\phi) = \sup_{x\in C}\langle Q_M^*\phi, x\rangle = \sup_{x\in C}\langle \phi, \widetilde{x}\rangle \le \sup\{\langle \phi, \widetilde{y}\rangle : \widetilde{y} \in X/M, \widetilde{p_C}(\widetilde{y}) \le 1\} = \widetilde{p_C}^*(\phi).$$

On the other hand, let $x \in X$ and $y \in M$. Noting that $\langle Q_M^*\phi, y\rangle = 0$, we obtain from Proposition 1(v) that

$$\langle \phi, \widetilde{x}\rangle = \langle Q_M^*\phi, x\rangle = \langle Q_M^*\phi, x-y\rangle \le p_{C^\circ}(Q_M^*\phi)p_C(x-y).$$

It follows that $\langle \phi, \widetilde{x}\rangle \le p_{C^\circ}(Q_M^*\phi)\widetilde{p_C}(\widetilde{x})$ because $y \in M$ is arbitrary. This implies that $\widetilde{p_C}^*(\phi) \le p_{C^\circ}(Q_M^*\phi)$. Therefore, Lemma 4 follows. □

3. Main Results

We begin with the following notion, which is a generalization of the property (ε_*) (see [12, p. 195]) in the approximation based on norms to the context of generalized approximations.

Definition 1. Let Γ be a linear subspace of X^*. Then Γ is said to have the property (ε_*) with respect to C if for each $x \in X$ there exists $y \in X$ such that

$$f(y) = f(x) \text{ for each } f \in \Gamma \text{ and } p_C(y) = p_{C,\Gamma}(x). \tag{7}$$

The main results of this paper are contained in the following theorem, giving various characterizations for a closed subspace of X to be d_C-proximinal and extending [12, Theorem 2.1, p. 94–p. 95] to the setting of generalized approximation.

Theorem 1. *Let M be a closed subspace of X. Consider the following statements.*

- (i) *M is d_C-proximinal.*
- (ii) *In each linear subspace $M_x := M\dot{+}\text{span}\{x\}$ with $x \in X\backslash M$, there exist two distinct nonzero points z_1 and z_2 on the different sides of M such that $0 \in P_M^C(z_1) \cap P_M^C(z_2)$, where we regard M as a hyperplane in M_x.*
- (iii) *For each $x \in X\backslash M$ and each $f \in (M_x)^*$ with the property $f|_M = 0$, there is $z \in M_x\backslash\{0\}$ such that $f(z) = p_C(z)p_{(C\cap M_x)^\circ}(f)$.*
- (iv) *$M^\perp$ has the property (ε_*) with respect to C.*
- (v) *For each $F \in (M^\perp)^*$ there exists an element $y \in X$ such that*

$$f(y) = F(f) \text{ for all } f \in M^\perp \quad \text{and} \quad p_C(y) = p_{(C^\circ \bigcap M^\perp)^\circ}(F).$$

Then (i)⇔(ii)⇔(iii)⇔(iv)⇐(v). If, in addition, X/M is reflexive, then all the above statements are equivalent.

Proof. (i)⇒(ii). Suppose that (i) hold. Then for each $x \in X\backslash M$, M is d_C-proximinal in $M_x = M\dot{+}\text{span}\{x\}$. Hence, (ii) holds by Lemma 3 because M is a hyperplane in M_x.

(ii)⇒(iii). Suppose that (ii) holds. Let $x \in X\backslash M$ and let $f \in M_x^*$ satisfy $f|_M = 0$. If $f(x) = 0$, then the assertion in (iii) is trivial. If $f(x) \neq 0$, then $M = \ker(f)$. By (ii), we may take $z \in M_x\backslash\{0\}$ with $f(z) > 0$ such that $0 \in P_M^C(-z)$. Thus Lemma 1 guarantees that there exits $f_1 \in M_x^*$ such that

$$f_1|_M = 0,\ p_{(C\bigcap M_x)^\circ}(f_1) = 1,\ \text{and}\ f_1(z) = p_C(z). \tag{8}$$

Obviously, $f_1(x) \neq 0$ (otherwise, $f_1(M_x) = \{0\}$). Putting $\alpha := f(x)$, $\beta := f_1(x)$, and $\lambda := \alpha\beta^{-1}$, one has that

$$f(y + tx) = t\alpha = \alpha\beta^{-1}(t\beta) = \lambda f_1(y + tx), \quad \forall\, y \in M,\ \forall\, t \in \mathbb{R}.$$

Hence, $f = \lambda f_1$ and $\lambda > 0$ [noting that $f(z) > 0$ by the choice of z and $f_1(z) > 0$ by (8)]. Multiplying the equality $f_1(z) = p_C(z)p_{(C\bigcap M_x)^\circ}(f_1)$ by λ, one obtains that $f(z) = p_C(z)p_{(C\cap M_x)^\circ}(f)$, i.e., the assertion in (iii) holds for x with $f(x) \neq 0$.

(iii)⇒ (iv) Suppose that (iii) holds. Below we show that $M^\perp$ has the property (ε_*) with respect to C. To do this, let $x \in X$. If $x \in M$, it is easy to see that (7) holds for $y = 0$ and $\Gamma := M^\perp$. If $x \in X\backslash M$, by the separation theorem, there exists $f \in X^*$ such that $f(x) > 0$ and $f|_M = 0$. For convenience put $f_x := f|_{M_x}$. By (iii), there exists $z_0 \in M_x\backslash\{0\}$ with $p_C(z_0) = 1$ such that $f(z_0) = p_{(C\cap M_x)^\circ}(f_x) > 0$. Let $y_0 := -x + \frac{f(x)}{f(z_0)}z_0$. Then $y_0 \in M$. Moreover, since $M = \ker(f_x)$, Lemma 2 yields

$$p_C(y_0 + x) = \frac{f(x)}{f(z_0)} = \frac{f(x)}{p_{(C\cap M_x)^\circ}(f_x)} = d_C(-x, M),$$

that is, $y_0 \in P_M^C(-x)$. Thus by Lemma 1,

$$p_C(y_0 + x) = p_{C,M^\perp}(y_0 + x) = p_{C,M^\perp}(x).$$

Since $g(y_0 + x) = g(x)$ for all $g \in M^\perp$, it follows that $y := y_0 + x$ and $\Gamma := M^\perp$ satisfy the conditions from (7).

Combining the above two cases, one sees that $M^\perp$ has the property (ε_*) with respect to C.

(iv)⇒(i). Suppose (iv) holds. Let $x \in X\backslash M$. Then by (iv), there exists $y_0 \in X$ such that $f(y_0) = f(-x)$ for each $f \in M^\perp$ and $p_C(y_0) = p_{C,M^\perp}(-x)$. Let $z_0 := y_0 + x$. Then $f(z_0) = 0$ for each $f \in M^\perp$. It follows from (4) that $z_0 \in (M^\perp)_\perp = M$. Moreover,

$$p_C(z_0 - x) = p_C(y_0) = p_{C,M^\perp}(-x) = p_{C,M^\perp}(z_0 - x).$$

This and Lemma 1 imply that $z_0 \in P_M^C(x)$, and so (i) holds.

(v)⇒(iv). Let $x \in X$. Then by the proof of (iii)⇒(iv), we may assume that $x \in X\backslash M$. Now define a continuous linear functional F on $M^\perp$ by $F(f) := f(x)$ for each $f \in M^\perp$. Then, by (v), there exists $y \in X$ such that

$$f(y) = F(f) \text{ for each } f \in M^\perp \quad \text{and} \quad p_C(y) = p_{(C^\circ \cap M^\perp)^\circ}(F).$$

This implies that $f(y) = f(x)$ for each $f \in M^\perp$, and

$$p_C(y) = \sup\{F(f) : f \in C^\circ \cap M^\perp\} = \sup\{f(x) : f \in C^\circ \cap M^\perp\} = p_{C,M^\perp}(x);$$

hence, $M^\perp$ has the property (ε_*) with respect to C. Thus the proofs of (i)⇔(ii)⇔(iii)⇔(iv)⇐(v) are complete.

If X/M is additionally reflexive, to complete the proof of Theorem 1, it suffices to show that (i)⇒(v). To this end, suppose that (i) holds and let $F \in (M^\perp)^*$. Then $Q_M^{**}F \in (X/M)^{**}$. Since X/M is reflexive, there is $x \in X$ such that

$$\langle Q_M^{**}F, \phi\rangle = \langle \phi, \widetilde{x}\rangle, \ \forall \phi \in (X/M)^*. \tag{9}$$

But M being d_C-proximinal, there exists $z \in M$ such that $\widetilde{p_C}(\widetilde{x}) = p_C(z+x)$. Let $y := z + x$. Then

$$\widetilde{y} = \widetilde{x} \quad \text{and} \quad \widetilde{p_C}(\widetilde{x}) = p_C(y). \tag{10}$$

Further, for each $f \in M^\perp$, take $\phi \in (X/M)^*$ such that $Q_M^*\phi = f$. Then

$$\langle F, f\rangle = \langle F, Q_M^*\phi\rangle = \langle Q_M^{**}F, \phi\rangle = \langle \phi, \widetilde{x}\rangle = \langle Q_M^*\phi, y\rangle = \langle f, y\rangle,$$

where the third equality is true by (9) and the forth one is due to (10). Hence, the first assertion in (v) holds.

To show the second assertion in (v), we first show the following result:

$$\widetilde{p_C}^{**}(Q_M^{**}F) = \widetilde{p_C}(\widetilde{x}). \tag{11}$$

In fact, if $x \in M$, then $\widetilde{x} = 0$, and hence $\langle Q_M^{**}F, \phi\rangle = 0$ for each $\phi \in (X/M)^*$ by (9). Thus, $\widetilde{p_C}^{**}(Q_M^{**}F) = \widetilde{p_C}(\widetilde{x}) = 0$. If $x \notin M$, then $\widetilde{p_C}(\widetilde{x}) \neq 0$ as M is closed, and we obtain from (9) and (10) that

$$\begin{aligned}
\widetilde{p_C}^{**}(Q_M^{**}F) &= \sup\{\langle Q_M^{**}F, \phi\rangle : \phi \in (X/M)^*, \widetilde{p_C}^*(\phi) \leq 1\} \\
&= \sup\{\langle \phi, \widetilde{x}\rangle : \phi \in (X/M)^*, \widetilde{p_C}^*(\phi) \leq 1\} \\
&= \widetilde{p_C}(\widetilde{x}) \sup\left\{\left\langle \phi, \frac{\widetilde{x}}{\widetilde{p_C}(\widetilde{x})}\right\rangle : \phi \in (X/M)^*, \widetilde{p_C}^*(\phi) \leq 1\right\} \\
&\leq \widetilde{p_C}(\widetilde{x}).
\end{aligned}$$

On the other hand, since $\widetilde{p_C}$ is a nonnegative sublinear functional on X/M and $\widetilde{p_C}(\widetilde{x}) \neq 0$, by [3, Proposition 3.2], there exists a linear functional ϕ_0 on X/M satisfying

$$\phi_0(\widetilde{w}) \leq L\widetilde{p_C}(\widetilde{w}), \quad \forall \widetilde{w} \in X/M \tag{12}$$

for some $L > 0$, such that

$$\sup\{\phi_0(\widetilde{w}) : \widetilde{w} \in X/M, \widetilde{p_C}(\widetilde{w}) \leq 1\} = 1 \quad \text{and} \quad \phi_0(\widetilde{x}) = \widetilde{p_C}(\widetilde{x}). \tag{13}$$

It follows from (12), the definition of $\widetilde{p_C}$, and Proposition 1 (v) that $\phi_0(\widetilde{w}) \leq Lm_2\|\widetilde{w}\|$, and hence $|\phi_0(\widetilde{w})| \leq Lm_2\|\widetilde{w}\|$ for each $\widetilde{w} \in X/M$. This means $\phi_0 \in (X/M)^*$. Furthermore, by (9), (13) and the definition of $\widetilde{p_C}^*(\phi_0)$, one has that

$$\widetilde{p_C}(\widetilde{x}) = \langle \phi_0, \widetilde{x}\rangle = \langle Q_M^{**}F, \phi_0\rangle \leq \widetilde{p_C}^{**}(Q_M^{**}F)\widetilde{p_C}^*(\phi_0) = \widetilde{p_C}^{**}(Q_M^{**}F);$$

hence (11) holds. Note from Lemma 4 and the isometry property of Q_M^* that

$$\begin{aligned}\widetilde{p_C}^{**}(Q_M^{**}F) &= \sup\{\langle F, Q_M^*\phi\rangle : \phi \in (X/M)^*, \widetilde{p_C}^*(\phi) \leq 1\}\\ &= \sup\{\langle F, f\rangle : f \in M^\perp, p_{C^\circ}(f) \leq 1\}\\ &= p_{(C^\circ \cap M^\perp)^\circ}(F).\end{aligned}$$

One obtains from (10) and (11) that

$$p_C(y) = p_{(C^\circ \cap M^\perp)^\circ}(F).$$

Hence, the second assertion in (v) is true, and the proof of (i)⇒(v) is complete. □

When C is the closed unit ball of X, we have the following corollary, which is exactly [12, Theorem 2.1, p. 94–p. 95] in the real case.

Corollary 1. *Let M be a closed subspace of X. Consider the following statements.*

(i) *M is proximinal.*
(ii) *In each linear subspace $M_x := M \dot{+} \mathrm{span}\{x\}$ with $x \in X\backslash M$, there exists a nonzero point $z \in M_x$ such that $0 \in P_M(z)$, where and in the sequel, P_M denotes the usual metric projection onto M.*
(iii) *For each $x \in X\backslash M$ and each $f \in (M_x)^*$ with $f|_M = 0$, there exists $y \in M_x\backslash\{0\}$ such that $f(y) = \|y\|\|f\|$.*
(iv) *$M^\perp$ has the property (ε_*), i.e., for each $x \in X$ there exists $y \in X$ such that $f(y) = f(x)$ for all $f \in M^\perp$ and $\|y\| = \sup\{|f(x)| : f \in X^*, \|f\| \leq 1\}$.*
(v) *For each $F \in (M^\perp)^*$ there exists an element $y \in X$ such that*

$$f(y) = F(f) \text{ for all } f \in M^\perp \quad \text{and} \quad \|y\| = \sup\{|F(f)| : f \in M^\perp, \|f\| \leq 1\}.$$

Then (i)⇔(ii)⇔(iii)⇔(iv)⇐(v). If, in addition, X/M is reflexive, then all the above statements are equivalent.

4. An Example

The following example illustrates that there exist a Banach space X, a bounded closed convex set C in X with $0 \in \mathrm{int}C$, and a continuous linear functional f on X such that $p_{C^\circ}(f)$ is attainable on C, but $p_{C^\circ}(-f)$ is not.

Example 1. Let l_1 be the space of all absolutely summable sequences in $\mathbb{R}$ endowed the norm defined by

$$\|x\| := \sum_{i=1}^{\infty} |\xi_i|, \quad \forall\, x := \sum_{i=1}^{\infty} \xi_i e_i \in l_1,$$

where $\{e_i\}_{i=1}^{\infty}$ is the natural base of l_1. Then the dual of l_1 is l_∞, i.e., the space of all bounded sequences in $\mathbb{R}$ with the norm defined by

$$\|f\| := \sup_{i\geq 1} |\eta_i|, \quad \forall\, f := (\eta_i) \in l_\infty.$$

Furthermore, let $f := (1, \frac{1}{2}, \ldots, \frac{n-1}{n}, \ldots) \in l_1^* = l_\infty$. Then $\|f\| = 1$. Let

$$W_1 := \mathrm{co}(\{e_1\} \cup (\ker(f) \cap B_{l_1})),$$
$$W_2 := \mathrm{co}\left(\left\{-\frac{1}{2}e_1\right\} \cup \{-e_i\}_{i=2}^{\infty} \cup (\ker(f) \cap B_{l_1})\right),$$

and $C = \overline{W_1 \cup W_2}$, where and in the sequel, B_{l_1} denotes the closed unit ball of l_1, and $\mathrm{co}(A)$ stands for the convex hull of the set $A \subseteq X$. Then C is bounded and closed and $C \subseteq B_{l_1}$. Below we first show that C is a convex set containing the origin as an interior.

(i) C is convex.

It suffices to show that $W_1 \cup W_2$ is convex. To do this, let $x, y \in W_1 \cup W_2$, and $\lambda \in (0, 1)$. We have to show that $z := (1-\lambda)x + \lambda y \in W_1 \cup W_2$. Clearly, $z \in B_{l_1}$ as $x, y \in B_{l_1}$. Since W_1 and W_2 are convex, we may assume that $x \in W_1$ and $y \in W_2$, so $f(x) \geq 0$ and $f(y) \leq 0$ by the representation of $f \in l_1^*$. When $f(x) = 0$, one has $x \in \ker(f) \cap B_{l_1} \subseteq W_2$; hence, $z \in W_2$. Similarly, when $f(y) = 0$, one obtains that $z \in W_1$. Below we assume that $f(x) > 0$ and $f(y) < 0$. Define the function ϕ by

$$\phi(t) := f(x + t(y - x)),\ \forall\ t \in \mathbb{R}.$$

Then ϕ is strictly decreasing on $\mathbb{R}$, $\phi(0) > 0$, and $\phi(1) < 0$. Take $t_0 := \frac{f(x)}{f(x)-f(y)}$ and set $z' := x + t_0(y - x)$. Then, $f(z') = \phi(t_0) = 0$, and $\|z'\| \leq 1$ (noting that $t_0 \in (0, 1)$); hence $z' \in \ker(f) \cap B_{l_1} \subseteq W_1 \cap W_2$. Consider the following two cases. When $\phi(\lambda) = f(z) > 0$, then $t_0 \in (\lambda, 1)$, and z can be rewritten as $z = (1-\lambda')x + \lambda' z'$ with $\lambda' = \frac{\lambda}{t_0} \in (0, 1)$. Hence, $z \in W_1$ (noting that $x, z' \in W_1$). When $\phi(\lambda) = f(z) \leq 0$, then $t_0 \in (0, \lambda]$, and

$$z = (1-\widetilde{\lambda})z' + \widetilde{\lambda} y \text{ with } \widetilde{\lambda} = \frac{\lambda - t_0}{1 - t_0} \in [0, 1);$$

hence $z \in W_2$. Therefore, $W_1 \cup W_2$ is convex.

(ii) C contains the origin as an interior.

In fact, let $W_3 := \mathrm{co}(\{-\frac{1}{2}e_1\} \cup (\ker(f) \cap B_{l_1}))$. It suffices to show that $B_{l_1}(\frac{1}{4}) \subseteq W_1 \cup W_3$ because $W_1 \cup W_3 \subset C$, where and in the sequel, $B_{l_1}(\frac{1}{4})$

stands for the closed ball centered at zero with radius $\frac{1}{4}$. To do this, let $x := \sum_{i=1}^{\infty} \xi_i e_i \in B_{l_1}(\frac{1}{4})$. Then

$$|f(x)| \leq \|x\| = \sum_{i=1}^{\infty} |\xi_i| \leq \frac{1}{4}. \tag{14}$$

Consider two cases as follows.

Case 1: $f(x) \geq 0$. Let

$$\widetilde{x} := e_1 + \widetilde{t}(x - e_1) \text{ with } \widetilde{t} = \frac{1}{1 - f(x)} \geq 1.$$

Then $f(\widetilde{x}) = 0$ (noting that $f(e_1) = 1$). Moreover, one checks that

$$\|\widetilde{x}\| = |1 + \widetilde{t}(\xi_1 - 1)| + \widetilde{t} \sum_{i=2}^{\infty} |\xi_i| \leq \frac{|\xi_1 - f(x)|}{1 - f(x)} + \frac{\frac{1}{4} - |\xi_1|}{1 - f(x)} < 1$$

by (14). Hence, $\widetilde{x} \in \ker(f) \cap B_{l_1} \subseteq W_1$, and we can rewrite x as $x = (1 - \frac{1}{\widetilde{t}})e_1 + \frac{1}{\widetilde{t}}\widetilde{x} \in W_1$.

Case 2: $f(x) < 0$. In the same manner as above, let

$$\hat{x} := -\frac{e_1}{2} + \hat{t}\left(x + \frac{e_1}{2}\right) \text{ with } \hat{t} = \frac{1}{1 + 2f(x)} > 1.$$

Then $f(\hat{x}) = 0$, and

$$\|\hat{x}\| = \left| -\frac{1}{2} + \hat{t}\left(\xi_1 + \frac{1}{2}\right) \right| + \hat{t} \sum_{i=2}^{\infty} |\xi_i| \leq \frac{|\xi_1 - f(x)|}{1 + 2f(x)} + \frac{\frac{1}{4} - |\xi_1|}{1 + 2f(x)} \leq 1$$

again by (14). This implies that $\hat{x} \in \ker(f) \cap B_{l_1}$, so x can be expressed as

$$x = \left(1 - \frac{1}{\hat{t}}\right)\left(-\frac{e_1}{2}\right) + \frac{1}{\hat{t}}\hat{x} \in W_3.$$

Combining the two cases above, we have that $B_{l_1}(\frac{1}{4}) \subseteq W_1 \cup W_3$.

Then we show that $p_{C^\circ}(f) = 1$ and $p_{C^\circ}(f)$ is attainable on C. Indeed, let $x := \sum_{i=1}^{\infty} \xi_i e_i \in W_1$. Then

$$f(x) = \xi_1 + \sum_{i=2}^{\infty} \frac{i-1}{i} \xi_i \leq \sum_{i=1}^{\infty} |\xi_i| = \|x\| \leq 1. \tag{15}$$

Note that $C = \overline{W_1 \cup W_2}$ and that $f(y) \leq 0$ for all $y \in W_2$. One has from Proposition 1 (iv) that

$$p_{C^\circ}(f) = \sup\{f(x) : x \in W_1 \cup W_2\} = \sup\{f(x) : x \in W_1\}.$$

This and (15) imply that $p_{C^\circ}(f) \leq 1$. Thus, $p_{C^\circ}(f) = 1$ because $f(e_1) = 1$ (noting that $e_1 \in W_1$), and $p_{C^\circ}(f)$ is attainable on C.

Finally we show that $p_{C^\circ}(-f) = 1$ and $p_{C^\circ}(-f)$ cannot attain on C. We only verify the latter assertion because the proof of the former is similar to that for $p_{C^\circ}(f) = 1$. Suppose, on the contrary, that there exists $\overline{x} := \sum_{i=1}^{\infty} \bar{\xi}_i e_i \in C$ such that $-f(\overline{x}) = 1$. Then,

$$-1 = f(\overline{x}) = \bar{\xi}_1 + \sum_{i=2}^{\infty} \frac{i-1}{i} \bar{\xi}_i.$$

This implies that $\bar{\xi}_i = 0 (i \geq 2)$ and $\bar{\xi}_1 = -1$; hence, $\overline{x} = -e_1$. Since $C = \overline{W_1} \cup \overline{W_2}$ and $-e_1 \notin \overline{W_1}$ (noting that $f(x) \geq 0$ for all $x \in \overline{W_1}$), one has that $-e_1 \in \overline{W_2}$, which will derive a contradiction. In fact, take $g := (1, 0, 0, \ldots) \in l_\infty$. Then

$$g(w) \geq -\frac{1}{2}, \quad \forall\, w \in W_2; \tag{16}$$

hence, $-e_1 \notin \overline{W_2}$ because $g(-e_1) = -1$, as desired. To show assertion (16), we first show that (16) holds for all $w \in \ker(f) \cap B_{l_1}$. To do this, let $w := (\eta_i) \in \ker(f) \cap B_{l_1}$. Then

$$\sum_{i=1}^{\infty} |\eta_i| \leq 1 \quad \text{and} \quad \eta_1 + \sum_{i=2}^{\infty} \frac{i-1}{i} \eta_i = 0. \tag{17}$$

It follows from the equality in (17) that $|\eta_1| \leq \sum_{i=2}^{\infty} |\eta_i|$, and hence, $|\eta_1| \leq \frac{1}{2}$ by the inequality in (17). Therefore, $g(w) = \eta_1 \geq -\frac{1}{2}$, and (16) holds for all $w \in \ker(f) \cap B_{l_1}$. To proceed, let $w \in W_2$ and set

$$w := \lambda_1 \left(-\frac{e_1}{2}\right) + \sum_{i=2}^{k} \lambda_i(-e_i) + \mu w', \tag{18}$$

where $k \in \mathbb{N}$, each λ_i and μ belong to $[0, 1]$,

$$\sum_{i=1}^{k} \lambda_i + \mu = 1, \tag{19}$$

and $w' \in \ker(f) \cap B_{l_1}$. Then, by the definition of g and the conclusion obtained just, one has from (18) and (19) that

$$g(w) = -\frac{1}{2}\lambda_1 + \mu g(w') \geq -\frac{1}{2}\lambda_1 - \frac{1}{2}\mu \geq -\frac{1}{2},$$

which completes the proof of (16).

Remark 1. With the notations in Example 1 and the proofs of (iii)⇒(i) and (i)⇒(ii) in Lemma 3, we see that for each $x \in X$ with $f(x) < 0$, $P^C_{\ker(f)}(x) \neq \emptyset$, and that for each $x \in X$ with $f(x) > 0$, $P^C_{\ker(f)}(x) = \emptyset$. This differs completely from the case of norm: For a subspace M of a normed space X and $x \in X$, $P_M(x) \neq \emptyset$ if and only if $P_M(-x) \neq \emptyset$.

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