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Results in Mathematics

An Inverse Spectral Problem for Second Order Differential Operators with Retarded Argument

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Abstract. Non-self-adjoint second-order differential operators with a constant delay are studied. Properties of spectral characteristics are established and the inverse problem of recovering operators from their spectra is investigated. For this nonlinear inverse problem an algorithm for constructing the global solution is developed.

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1. Introduction

In various real-world processes, the future behavior of the system depends not only on its present state and rate of change of the state (corresponding to the values of the function and its derivatives at the current point), but also on the states in the past. Such processes described by functional differential equations with delay, arise in physics, biology and especially in engineering and control theory (see the monographs [5,8]).

In this paper we study an inverse spectral problem for non-self-adjoint Sturm-Liouville differential operators on a finite interval with a constant delay and with complex-valued potentials. Inverse spectral problems consist in recovering operators from their spectral characteristics. The greatest success in the inverse spectral theory has been achieved for the classical Sturm-Liouville operator (see [3,6,7,11] and the references therein) and afterwards for higher-order differential operators and other classes of differential operators and systems [11]. The classical methods of the inverse spectral theory (the transformation operator method [3,6,7] and the method of spectral mappings [3,11]), which allow one to obtain global solutions of inverse problems for differential operators, are not applicable for differential operators with deviating argument. Therefore, the general inverse spectral theory for differential operators with delay has not yet been constructed, and there are only isolated results in this direction [1,2,4,9,10].

We consider the boundary value problems $L_j = L_j(q,h)$, $j = 0,1$, of the form

$$-y''(x) + q(x)y(x-a) = \lambda y(x), \quad 0 < x < \pi, \tag{1}$$
$$y'(0) - hy(0) = y^{(j)}(\pi) = 0.$$

Here λ is the spectral parameter, $a \in [\pi/3, \pi/2)$, h is a complex number, $q(x)$ is a complex-valued function, $q(x) \in L(a,\pi)$ and $q(x) = 0$ a.e. on $(0,a)$.

Let $C(x,\lambda), S(x,\lambda)$ and $\varphi(x,\lambda)$ be solutions of Eq. (1) satisfying the initial conditions

$$C(0,\lambda) = S'(0,\lambda) = \varphi(0,\lambda) = 1,\ S(0,\lambda) = C'(0,\lambda) = 0,\ \varphi'(0,\lambda) = h.$$

Then $\varphi(x,\lambda) = C(x,\lambda) + hS(x,\lambda)$. For each fixed x, the functions $C^{(j)}(x,\lambda)$, $S^{(j)}(x,\lambda)$ and $\varphi^{(j)}(x,\lambda)$, $j = 0,1$, are entire in λ of order 1/2. Denote

$$\Delta_j(\lambda) := \varphi^{(j)}(\pi,\lambda), \quad j = 0,1.$$

The eigenvalues $\{\lambda_{nj}\}_{n\geq 0}$ of the boundary value problem L_j coincide with the zeros of the entire function $\Delta_j(\lambda)$. The function $\Delta_j(\lambda)$ is called the characteristic function for L_j. In this paper we study the inverse problem of recovering the potential $q(x)$ and the coefficient h provided that the spectra $\{\lambda_{nj}\}_{n\geq 0}$, $j = 0,1$, are given. If $a \geq \pi/2$, then the corresponding inverse problem becomes linear. In this paper we pay attention to the essentially nonlinear case when $a \in [\pi/3, \pi/2)$ (the case $a < \pi/3$ requires separate investigations). In this paper we obtain a global constructive procedure for the solution of the inverse problem and establish its uniqueness.

2. Properties of Spectral Characteristics

Let $\lambda = \rho^2$. The functions $C(x,\lambda)$ and $S(x,\lambda)$ are the unique solutions of the following integral equations

$$C(x,\lambda) = \cos\rho x + \int_a^x G(x,t,\lambda)C(t-a,\lambda)\,dt,$$
$$S(x,\lambda) = \frac{\sin\rho x}{\rho} + \int_a^x G(x,t,\lambda)S(t-a,\lambda)\,dt,$$

where $G(x,t,\lambda) = \dfrac{q(t)\sin\rho(x-t)}{\rho}$. Solving these equations we get

$$\begin{aligned} C(x,\lambda) &= \cos\rho x + C_1(x,\lambda) + C_2(x,\lambda), \\ S(x,\lambda) &= \frac{\sin\rho x}{\rho} + S_1(x,\lambda) + S_2(x,\lambda), \end{aligned} \tag{2}$$

where

$$\begin{aligned} C_1(x,\lambda) &= \frac{1}{\rho}\int_a^x q(t)\sin\rho(x-t)\cos\rho(t-a)\,dt, \\ S_1(x,\lambda) &= \frac{1}{\rho^2}\int_a^x q(t)\sin\rho(x-t)\sin\rho(t-a)\,dt, \end{aligned}$$

for $x \ge a$, and $C_1(x,\lambda) = S_1(x,\lambda) = 0$ for $x \in [0,a]$. Similarly,

$$\begin{aligned} C_2(x,\lambda) &= \int_{2a}^x G(x,t,\lambda)C_1(t-a,\lambda)\,dt, \\ S_2(x,\lambda) &= \int_{2a}^x G(x,t,\lambda)S_1(t-a,\lambda)\,dt, \end{aligned} \tag{3}$$

for $x \ge 2a$, and $C_2(x,\lambda) = S_2(x,\lambda) = 0$ for $x \in [0,2a]$. In particular, this yields for $x \ge a$:

$$\left.\begin{aligned} C_1(x,\lambda) &= \frac{\sin\rho(x-a)}{2\rho}\int_a^x q(t)\,dt - \frac{1}{2\rho}\int_a^x q(t)\sin\rho(2t-x-a)\,dt, \\ S_1(x,\lambda) &= -\frac{\cos\rho(x-a)}{2\rho^2}\int_a^x q(t)\,dt + \frac{1}{2\rho^2}\int_a^x q(t)\cos\rho(2t-x-a)\,dt, \end{aligned}\right\} \tag{4}$$

and consequently,

$$\left.\begin{aligned} C_1(\pi,\lambda) &= \frac{A\sin\rho(\pi-a)}{\rho} - \frac{1}{2\rho}\int_a^\pi q(t)\sin\rho(2t-\pi-a)\,dt, \\ S_1(\pi,\lambda) &= -\frac{A\cos\rho(\pi-a)}{\rho^2} + \frac{1}{2\rho^2}\int_a^\pi q(t)\cos\rho(2t-\pi-a)\,dt, \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} C_1'(\pi,\lambda) &= A\cos\rho(\pi-a) + \frac{1}{2}\int_a^\pi q(t)\cos\rho(2t-\pi-a)\,dt, \\ S_1'(\pi,\lambda) &= \frac{A\sin\rho(\pi-a)}{\rho} + \frac{1}{2\rho}\int_a^\pi q(t)\sin\rho(2t-\pi-a)\,dt, \end{aligned}\right\} \tag{6}$$

where $A := \dfrac{1}{2}\displaystyle\int_a^\pi q(t)\,dt$. Substituting (4) into (3) we obtain for $x \ge 2a$:

$$\left.\begin{aligned} C_2(x,\lambda) &= \int_{2a}^x G(x,t,\lambda)\left(\frac{\sin\rho(t-2a)}{2\rho}\int_a^{t-a} q(s)\,ds - \frac{1}{2\rho}\int_a^{t-a} q(s)\sin\rho(2s-t)\,ds\right)dt, \\ S_2(x,\lambda) &= \int_{2a}^x G(x,t,\lambda)\left(-\frac{\cos\rho(t-2a)}{2\rho^2}\int_a^{t-a} q(s)\,ds + \frac{1}{2\rho^2}\int_a^{t-a} q(s)\cos\rho(2s-t)\,ds\right)dt. \end{aligned}\right\} \tag{7}$$

Denote

$$A_1=\int_{2a}^{\pi} q(t)dt\int_a^{t-a} q(s)ds,\ Q_1(t)=q(t)\int_a^{t-a} q(s)ds,\ Q_2(t)=q(t)\int_{t+a}^{\pi} q(s)ds,$$

$$Q_3(t)=\int_{t+a}^{\pi} q(s)q(s-t)ds,\ Q_{\mp}(\xi)=Q_1(\xi/2+\pi/2+a)$$
$$-Q_2(\xi/2+\pi/2)\mp Q_3(\xi/2+\pi/2).$$

It follows from (7) that

$$\left.\begin{aligned} C_2(\pi,\lambda)&=-\frac{A_1\cos\rho(\pi-2a)}{4\rho^2}+\frac{1}{8\rho^2}\int_{-(\pi-2a)}^{\pi-2a} Q_+(\xi)\cos\rho\xi\,d\xi,\\ S_2(\pi,\lambda)&=-\frac{A_1\sin\rho(\pi-2a)}{4\rho^3}+\frac{1}{8\rho^3}\int_{-(\pi-2a)}^{\pi-2a} Q_-(\xi)\sin\rho\xi\,d\xi,\end{aligned}\right\}\tag{8}$$

$$\left.\begin{aligned} C_2'(\pi,\lambda)&=\frac{A_1\cos\rho(\pi-2a)}{4\rho}+\frac{1}{8\rho}\int_{-(\pi-2a)}^{\pi-2a} Q_+(\xi)\sin\rho\xi\,d\xi,\\ S_2'(\pi,\lambda)&=-\frac{A_1\sin\rho(\pi-2a)}{4\rho^2}-\frac{1}{8\rho^2}\int_{-(\pi-2a)}^{\pi-2a} Q_-(\xi)\cos\rho\xi\,d\xi.\end{aligned}\right\}\tag{9}$$

Since $\Delta_j(\lambda):=\varphi^{(j)}(\pi,\lambda),\ j=0,1,$ and $\varphi(x,\lambda)=C(x,\lambda)+hS(x,\lambda)$, it follows from (2), (5)–(6) and (8)–(9) that

$$\Delta_0(\lambda)=\cos\rho\pi+\frac{h\sin\rho\pi}{\rho}+\frac{A\sin\rho(\pi-a)}{\rho}-\frac{hA\cos\rho(\pi-a)}{\rho^2}+\frac{d_0(\rho)}{2\rho},\tag{10}$$

$$\Delta_1(\lambda)=-\rho\sin\rho\pi+h\cos\rho\pi+A\cos\rho(\pi-a)+\frac{hA\sin\rho(\pi-a)}{\rho}+\frac{d_1(\rho)}{2},\tag{11}$$

where

$$\begin{aligned} d_0(\rho)=&-\int_a^{\pi} q(t)\sin\rho(2t-\pi-a)\,dt+\frac{h}{\rho}\int_a^{\pi} q(t)\cos\rho(2t-\pi-a)\,dt\\ &-\frac{A_1\cos\rho(\pi-2a)}{2\rho}\\ &-\frac{hA_1\sin\rho(\pi-2a)}{2\rho^2}+\frac{1}{4\rho}\int_{-(\pi-2a)}^{(\pi-2a)} Q_+(\xi)\cos\rho\xi\,d\xi\\ &+\frac{h}{4\rho^2}\int_{-(\pi-2a)}^{(\pi-2a)} Q_-(\xi)\sin\rho\xi\,d\xi,\end{aligned}\tag{12}$$

$$\begin{aligned} d_1(\rho)=&\int_a^{\pi} q(t)\cos\rho(2t-\pi-a)\,dt+\frac{h}{\rho}\int_a^{\pi} q(t)\sin\rho(2t-\pi-a)\,dt\\ &+\frac{A_1\sin\rho(\pi-2a)}{2\rho}\end{aligned}$$

$$-\frac{hA_1\cos\rho(\pi-2a)}{2\rho^2}+\frac{1}{4\rho}\int_{-(\pi-2a)}^{(\pi-2a)}Q_+(\xi)\sin\rho\xi\,d\xi$$
$$-\frac{h}{4\rho^2}\int_{-(\pi-2a)}^{(\pi-2a)}Q_-(\xi)\cos\rho\xi\,d\xi. \tag{13}$$

Using (10)–(13) by the well-known arguments (see, for example [3]) we obtain the following facts.

Lemma 1. *For* $n\to\infty$,

$$\sqrt{\lambda_{n0}}=(n+1/2)+(h+A\cos(n+1/2)a)/(\pi n)+o(1/n),$$
$$\sqrt{\lambda_{n1}}=n+(h+A\cos na)/(\pi n)+o(1/n). \tag{14}$$

Lemma 2. *The specification of the spectra* $\{\lambda_{nj}\}_{n\geq 0}$, $j=0,1$, *uniquely determines the characteristic functions via*

$$\Delta_0(\lambda)=\prod_{n=0}^{\infty}\frac{\lambda_{n0}-\lambda}{(n+1/2)^2},\quad \Delta_1(\lambda)=\pi(\lambda_{01}-\lambda)\prod_{n=1}^{\infty}\frac{\lambda_{n1}-\lambda}{n^2}. \tag{15}$$

3. Solution of the Inverse Problem

Let the spectra $\{\lambda_{nj}\}_{n\geq 0}$, $j=0,1$, be given. Our goal is to find the potential $q(x)$ and the coefficient h. First of all, by (15) we construct the characteristic functions $\Delta_j(\lambda)$, $j=0,1$. Then, using (10) or (11) we can find the coefficients h and A. Indeed, it follows from (10) that $A\sin an=(-1)^{n+1}(\Delta_0(n^2)-(-1)^n)n+o(1)$, as $n\to\infty$, and consequently,

$$A=\lim_{n_k\to\infty}(-1)^{n_k+1}(\sin an_k)^{-1}(\Delta_0(n_k^2)-(-1)^{n_k})n_k, \tag{16}$$

where n_k are such that $|\sin an_k|>\delta>0$. Using (10) again we infer

$$h=\lim_{n\to\infty}\Big((2n+1/2)\Delta_0((2n+1/2)^2)-A\sin(2n+1/2)(\pi-a)\Big). \tag{17}$$

Note that we can also calculate A and h using (14). Since A and h are known, we can construct the functions $d_j(\rho)$, $j=0,1$, with the help of (10) and (11).

In order to simplify calculations we assume that $q(x)$ and $q'(x)$ are absolutely continuous on $[a,\pi]$. The general case requires slightly different calculations. Integration by parts in (12)–(13) yields

$$2\rho d_0(\rho)=B_0\cos\rho(\pi-a)+\int_a^{\pi}g(t)\cos\rho(2t-\pi-a)dt-A_1\cos\rho(\pi-2a)$$
$$-\frac{hA_1\sin\rho(\pi-2a)}{\rho}+\frac{1}{2}\int_{-(\pi-2a)}^{(\pi-2a)}Q_+(\xi)\cos\rho\xi\,d\xi$$
$$+\frac{h}{2\rho}\int_{-(\pi-2a)}^{(\pi-2a)}Q_-(\xi)\sin\rho\xi\,d\xi, \tag{18}$$

$$2\rho d_1(\rho) = B_1 \sin\rho(\pi - a) + \int_a^{\pi} g(t)\sin\rho(2t - \pi - a)dt + A_1 \sin\rho(\pi - 2a)$$
$$-\frac{hA_1\cos\rho(\pi-2a)}{\rho} + \frac{1}{2}\int_{-(\pi-2a)}^{(\pi-2a)} Q_+(\xi)\sin\rho\xi\, d\xi$$
$$-\frac{h}{2\rho}\int_{-(\pi-2a)}^{(\pi-2a)} Q_-(\xi)\cos\rho\xi\, d\xi, \tag{19}$$

where $g(x) = -q'(x) + 2hq(x)$, $B_0 = q(\pi) - q(a)$, $B_1 = q(\pi) + q(a)$. Using (18)–(19) we can find B_0, B_1 and A_1. Indeed, it follows from (18)–(19) that for real ρ, $|\rho| \to \infty$,

$$2\rho d_0(\rho) = B_0\cos\rho(\pi - a) - A_1\cos\rho(\pi - 2a) + o(1), \tag{20}$$
$$2\rho d_1(\rho) = B_1\sin\rho(\pi - a) + A_1\sin\rho(\pi - 2a) + o(1). \tag{21}$$

Taking in (21) $\rho_n = n\pi/(\pi - a)$, we get for $n \to \infty$:

$$2\rho_n d_1(\rho_n) = A_1\sin(\alpha n\pi) + o(1),\ \alpha = (\pi - 2a)/(\pi - a) < 1,$$

and consequently,

$$A_1 = 2\lim_{m_k\to\infty}\Big(\rho_{m_k} d_1(\rho_{m_k})(\sin\alpha m_k\pi)^{-1}\Big), \tag{22}$$

where m_k are such that $|\sin\alpha m_k\pi| > \delta > 0$. Using (20)–(21) we infer

$$\left.\begin{aligned} B_1 &= \lim_{n\to\infty}\Big(2\rho_{n1}d_1(\rho_{n1}) - A_1\sin\rho_{n1}(\pi - 2a)\Big),\ \rho_{n1} = (2n + 1/2)\pi/(\pi - a),\\ B_0 &= \lim_{n\to\infty}\Big(2\rho_{n0}d_0(\rho_{n0}) + A_1\cos\rho_{n0}(\pi - 2a)\Big),\ \rho_{n0} = 2n\pi/(\pi - a). \end{aligned}\right\} \tag{23}$$

Since B_0 and B_1 are known, we calculate $q(a)$ and $q(\pi)$ by the formulas $q(\pi) = (B_1 + B_0)/2$ and $q(a) = (B_1 - B_0)/2$. Let us now construct the functions

$$\left.\begin{aligned} d_0^*(\rho) &= 2\rho d_0(\rho) - B_0\cos\rho(\pi - a) + A_1\cos\rho(\pi - 2a) + \frac{hA_1\sin\rho(\pi - 2a)}{\rho},\\ d_1^*(\rho) &= 2\rho d_1(\rho) - B_1\sin\rho(\pi - a) - A_1\sin\rho(\pi - 2a) + \frac{hA_1\cos\rho(\pi - 2a)}{\rho}. \end{aligned}\right\} \tag{24}$$

It follows from (18)–(19) that

$$d_0^*(\rho) = \int_a^{\pi} g(t)\cos\rho(2t - \pi - a)dt + \frac{1}{2}\int_{-(\pi-2a)}^{(\pi-2a)} Q_+(\xi)\cos\rho\xi\, d\xi$$
$$+\frac{h}{2\rho}\int_{-(\pi-2a)}^{(\pi-2a)} Q_-(\xi)\sin\rho\xi\, d\xi,$$

$$d_1^*(\rho) = \int_a^\pi g(t)\sin\rho(2t-\pi-a)dt + \frac{1}{2}\int_{-(\pi-2a)}^{(\pi-2a)} Q_+(\xi)\sin\rho\xi\, d\xi + \frac{h}{2\rho}\int_{-(\pi-2a)}^{(\pi-2a)} Q_-(\xi)\cos\rho\xi\, d\xi.$$

Integration by parts yields

$$2\rho d_0^*(\rho) = b_0\sin\rho(\pi-a) + \omega_0\sin\rho(\pi-2a) - \int_{-(\pi-a)}^{(\pi-a)} g_0(\xi)\sin\rho\xi\, d\xi - \int_{-(\pi-2a)}^{(\pi-2a)} G(\xi)\sin\rho\xi\, d\xi, \tag{25}$$

$$2\rho d_1^*(\rho) = b_1\cos\rho(\pi-a) + \omega_1\cos\rho(\pi-2a) + \int_{-(\pi-a)}^{(\pi-a)} g_0(\xi)\cos\rho\xi\, d\xi + \int_{-(\pi-2a)}^{(\pi-2a)} G(\xi)\cos\rho\xi\, d\xi, \tag{26}$$

where $G(\xi) = Q_+'(\xi) - hQ_-(\xi)$, $g_0(\xi) = g_1((\xi+\pi+a)/2)/2$, $g_1(x) = g'(x)$, $b_0 = g(a) + g(\pi)$, $b_1 = g(a) - g(\pi)$, $\omega_0 = Q_+(\pi-2a) + Q_+(-(\pi-2a))$, $\omega_1 = Q_+(\pi-2a) - Q_+(-(\pi-2a))$. Using (25)–(26) by similar arguments as above we can find b_0, b_1, ω_0 and ω_1:

$$\left.\begin{aligned} \omega_0 &= 2\lim_{m_k\to\infty}\left(\rho_{m_k} d_0^*(\rho_{m_k})(\sin\alpha m_k\pi)^{-1}\right),\\ \omega_1 &= 2\lim_{r_k\to\infty}\left(\rho_{r_k} d_1^*(\rho_{r_k})(\cos\alpha(2r_k+1/2)\pi)^{-1}\right),\end{aligned}\right\} \tag{27}$$

$$\left.\begin{aligned} b_0 &= \lim_{n\to\infty}\left(2\rho_n^0 d_0^*(\rho_n^0) - \omega_0\sin\rho_n^0(\pi-2a)\right),\ \rho_n^0 = (2n+1/2)\pi/(\pi-a),\\ b_1 &= \lim_{n\to\infty}\left(2\rho_n^1 d_1^*(\rho_n^1) - \omega_1\cos\rho_n^1(\pi-2a)\right),\ \rho_n^1 = 2n\pi/(\pi-a),\end{aligned}\right\} \tag{28}$$

where r_k are such that $|\cos\alpha(2r_k+1/2)\pi| > \delta > 0$.

Since b_0 and b_1 are known, we calculate $g(a)$ and $g(\pi)$ by the formulas $g(\pi) = (b_0 - b_1)/2$ and $g(a) = (b_0 + b_1)/2$, and consequently, we can find $q'(a)$ and $q'(\pi)$ via $q'(a) = -g(a) + 2hq(a)$, $q'(\pi) = -g(\pi) + 2hq(\pi)$. Let us now construct the functions

$$\left.\begin{aligned} D_0(\rho) &= 2\rho d_0^*(\rho) - b_0\sin\rho(\pi-a) - \omega_0\sin\rho(\pi-2a),\\ D_1(\rho) &= 2\rho d_1^*(\rho) - b_1\cos\rho(\pi-a) - \omega_1\cos\rho(\pi-2a).\end{aligned}\right\} \tag{29}$$

It follows from (25)–(26) that

$$D_0(\rho) = -\int_{-(\pi-a)}^{(\pi-a)} R(\xi)\sin\rho\xi\, d\xi, \quad D_1(\rho) = \int_{-(\pi-a)}^{(\pi-a)} R(\xi)\cos\rho\xi\, d\xi, \tag{30}$$

where

$$R(\xi) = g_0(\xi) + G(\xi), \tag{31}$$

and $G(\xi) \equiv 0$ for $\xi \notin (-(\pi - 2a), \pi - 2a)$. Using (30) we construct the function $R(\xi)$. Since $G(\xi) \equiv 0$ for $\xi \notin (-(\pi - 2a), \pi - 2a)$, we find the function $g_0(\xi)$ for $\xi \notin (-(\pi - 2a), \pi - 2a)$ via $g_0(\xi) = R(\xi)$. This yields

$$q''(x) - 2hq'(x) = -2R_1(x), \quad x \in [a, 3a/2] \cup [\pi - a/2, \pi], \tag{32}$$

where $R_1(x) := R(2x - \pi - a)$. Since $q(a), q'(a), q(\pi)$ and $q'(\pi)$ are known, we can construct the potential $q(x)$ for $x \in [a, 3a/2] \cup [\pi - a/2, \pi]$ by solving the linear equation (32).

Moreover, it follows from (31) that

$$\begin{aligned} q''(x) - 2hq'(x) &= -2R_1(x) + Q_1'(x + a/2) - Q_2'(x - a/2) + Q_3'(x - a/2) \\ &\quad -2hQ_1(x + a/2) + 2hQ_2(x - a/2) + 2hQ_3(x - a/2),\ x \in [3a/2, \pi - a/2]. \end{aligned} \tag{33}$$

Since $q(x)$ is known for $x \in [a, 3a/2] \cup [\pi - a/2, \pi]$, then equation (33) is linear with respect to $q(x)$, and the solution exists. In particular, if $a \in [2\pi/5, \pi/2)$, then the right-hand side in (33) is the known function. Solving linear equation (33), we can find $q(x)$ for $x \in [3a/2, \pi - a/2]$. Thus, the solution of the inverse problem can be found by the following algorithm.

Algorithm 1. *Let the spectra $\{\lambda_{nj}\}_{n\geq 0}$, $j = 0, 1$, be given.*
1) Construct the characteristic functions $\Delta_j(\lambda)$, $j = 0, 1$ by (15).
2) Calculate A and h using (16)–(17).
3) Find the functions $d_j(\rho)$, $j = 0, 1$, with the help of (10) and (11).
4) Calculate B_0, B_1 and A_1, using (22)–(23).
5) Find $q(\pi) = (B_1 + B_0)/2$ and $q(a) = (B_1 - B_0)/2$.
6) Construct the functions $d_j^*(\rho)$, $j = 0, 1$, by (24).
7) Calculate ω_0, ω_1, b_0 and b_1, using (27)–(28).
8) Find $g(a) = (b_0 + b_1)/2$ and $g(\pi) = (b_0 - b_1)/2$.
9) Calculate $q'(a) = -g(a) + 2hq(a)$ and $q'(\pi) = -g(\pi) + 2hq(\pi)$.
10) Construct the functions $D_j(\rho)$, $j = 0, 1$, by (29).
11) Find the function $R(\xi)$ using (30).
12) Calculate the potential $q(x)$ for $x \in [a, 3a/2] \cup [\pi - a/2, \pi]$ by solving equation (32).
13) Calculate the potential $q(x)$ for $x \in [3a/2, \pi - a/2]$ using (33) and knowledge $q(x)$ for $x \in [a, 3a/2] \cup [\pi - a/2, \pi]$.

We note that for $a \geq 2\pi/5$ the uniqueness of the solution of the inverse problem is obvious; for $a < 2\pi/5$ one needs a separate investigation.

Remark. Similar results are valid for the boundary value problems P_j, $j = 1, 2$, generated by Eq. (1) and the Robin boundary conditions

$$y'(0) - hy(0) = 0, \quad y'(\pi) + H_j y(\pi) = 0,$$

where H_j are complex numbers, and $H_1 \neq H_2$. The eigenvalues $\{\mu_{nj}\}_{n\geq 0}$ of P_j coincide with the zeros of the entire functions

$$\delta_j(\lambda) := \varphi'(\pi, \lambda) + H_j\varphi(\pi, \lambda). \tag{34}$$

The inverse problem here is formulated as follows: Given the spectra $\{\mu_{nj}\}_{n\geq 0}$, $j = 1, 2$, construct the potential $q(x)$ and the coefficients h, H_1, H_2. It follows from (34) that

$$\delta_j(\lambda) = \Delta_1(\lambda) + H_j\Delta_0(\lambda). \tag{35}$$

Using (35) one can reduce this inverse problem to the inverse problem for the boundary value problem L_j, which was solved above.

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