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Results in Mathematics

Inverse Spectral Problems for Sturm–Liouville Operators with a Constant Delay Less than Half the Length of the Interval and Robin Boundary Conditions

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Abstract. The topic of this paper are non-self-adjoint second-order differential operators with a constant delay, which is less than half of the length of the interval. We consider the case when a delay is from $\tau \in [\frac{2\pi}{5}, \frac{\pi}{2})$, and the potential is a real-valued function which satisfy $q \in L^2[0,\pi]$. The inverse spectral problem of recovering the potential from the spectra of two boundary value problems with Robin boundary conditions has been studied. We have proved that the delay and the potential are uniquely determined by two spectra of boundary spectral problem, one with boundary conditions $y'(0) - hy(0) = 0$, $y'(\pi) + H_1 y(\pi) = 0$ and the other with boundary conditions $y'(0) - hy(0) = 0$, $y'(\pi) + H_2 y(\pi) = 0$.

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1. Introduction

The inverse problems in the spectral theory of operators, especially differential operators, have been studied since the 1930s [1]. More details on this topic can be found in the monograph [2] and its references. A separate chapter of this study deals with the inverse tasks for the boundary problems of the generated equations with a delay. The main results in the inverse spectral problems for classical Sturm–Liouville operators can be found in the monograph [2]. Inverse spectral problems for classical Sturm–Liouville operators with boundary conditions depending on the spectral parameter study in [3]. Some of the main methods in the inverse problem theory for classical Sturm–Liouville operators, such as transformation operator method and method of spectral mappings, turned out to be unsuitable for operators with delays. Therefore other effective methods have been created, like contour integral method and method of characteristic function. The delay can take different forms, for example in [5] and [6] authors study inverse spectral problem for differential operator with an integral delay. In this paper we study the problem with constant delay. The papers [4,7–10] present the latest results in this field. For example, in [8] we have generalisation of the result from [1] for the equation with constant delay. In [9], for $\tau = \frac{\pi}{2}$ from two subspectrum authors find necessary and sufficient conditions for the solvability of the inverse problem in terms of asymptotic. In [10] it was proven that potential is uniquely determined by two spectrum when $\tau \in [\frac{\pi}{2}, \pi)$ with Cauchy boundary condition.

In this paper we study two boundary spectral problems generated by differential equations with constant delay and Robin boundary conditions $D_k, k = 1, 2$:

$$-y''(x) + q(x)y(x-\tau) = \lambda y(x), x \in (0,\pi) \tag{1}$$

$$y'(0) - hy(0) = 0 \tag{2}$$

$$y'(\pi) + H_k y(\pi) = 0, k = 1, 2 \tag{3}$$

where λ is the spectral parameter, potential $q(x)$ is a real-valued function, which satisfy conditions $q \in L^2(0,\pi), q(x) = 0, x \in [0,\tau)$ and $h, H_1, H_2 \in R\backslash\{0\}$.

We assume that $\frac{2\pi}{5} < \tau < \frac{\pi}{2}$ and integral $I_2 = \int_{2\tau}^{\pi} q(t) \int_{\tau}^{t-\tau} q(s)dsdt$ is known.

It is known that the spectrum of D_1, D_2 is countable. We will prove that the delay τ and the potential q are uniquely determined from the spectrum of D_1 and D_2. More precisely, let $(\lambda_{n,1})_{n=1}^{\infty}$ be the eigenvalues of D_1 and $(\lambda_{n,2})_{n=1}^{\infty}$ be the eigenvalues of D_2. The inverse problem is to determine $q(x)$,τ and parameters h, H_1, H_2 from $(\lambda_{n,1})_{n=1}^{\infty}$ and $(\lambda_{n,2})_{n=1}^{\infty}$.

The inverse problem: Given $(\lambda_{n,k})_{n=0}^{\infty}, k = 1, 2$ determine potential q, delay τ, and parameters h, H_1, H_2 if $\frac{2\pi}{5} < \tau < \frac{\pi}{2}$ and integral $I_2 = \int_{2\tau}^{\pi} q(t) \int_{\tau}^{t-\tau}$ $q(s)dsdt$ is known.

In Sect. 2, we study the spectral properties of the boundary value problems $D_k, k = 1, 2$ and we introduce transformation of characteristic functions which is needed for constructing the integral equation with the potential. In Sect. 3, we prove that a delay and parameters h, H_1, H_2 are uniquely determined from the spectra. Then we prove that a potential is uniquely determined from Volterra linear integral equation whose kernel is equal to one.

2. Spectral Properties and Basics Transformation

One can easily show that if y is the solution of differential equation (1) under the initial conditions (2) and $q(x) = 0, x \in [0, \tau)$, then the following integral equation holds:

$$y(x,z) = \cos xz + \frac{h}{z}\sin xz + \frac{1}{z}\int_{\tau}^{x} q(t)y(t-\tau,z)\sin z(x-t)dt \tag{4}$$

where $\lambda = z^2$. We will solve Eq. (4) by the method of steps.

For $x \in [2\tau, \pi]$ the solution is:

$$\begin{aligned} y(x,z) = {} & \cos zx + \frac{h}{z}\sin zx + \frac{1}{z}b_{sc}(x,z) + \frac{h}{z^2}b_{s^2}(x,z) + \frac{1}{z^2}b_{s^2c}(x,z) \\ & + \frac{h}{z^3}b_{s^3}(x,z) \end{aligned} \tag{5}$$

where we use notation:

$$\begin{aligned} b_{sc}(x,z) &= \int_{\tau}^{x} q(t)\sin z(x-t)\cos z(t-\tau)dt, b_{sc}(\pi,z) = b_{sc}(z), \\ b_{s^2}(x,z) &= \int_{\tau}^{x} q(t)\sin z(x-t)\sin z(t-\tau)dt, b_{s^2}(\pi,z) = b_{s^2}(z), \\ b_{s^2c}(x,z) &= \int_{2\tau}^{x} q(t)\sin z(x-t)b_{sc}(t-\tau,z)dt, b_{s^2c}(\pi,z) = b_{s^2c}(z), \\ b_{s^3}(x,z) &= \int_{2\tau}^{x} q(t)\sin z(x-t)b_{s^2}(t-\tau,y)dt, b_{s^3}(\pi,z) = b_{s^3}(z). \end{aligned} \tag{6}$$

Let $\Delta_k(\lambda) = F_k(z) = y'(\pi) + H_k y(\pi), k = 1, 2$. Using (5) for $k \in \{1, 2\}$ we have:

$$\begin{aligned} \Delta_k(\lambda) = F_k(z) = {} & \left(-z + \frac{hH_k}{z}\right)\sin \pi z + (h + H_k)\cos \pi z + b_{c^2}(z) + \frac{h}{z}b_{cs}(z) \\ & + \frac{H_k}{z}b_{sc}(z) + \frac{hH_k}{z^2}b_{s^2}(z) + \frac{1}{z}b_{csc}(z) + \frac{h}{z^2}b_{cs^2}(z) + \frac{H_k}{z^2}b_{s^2c}(z) \\ & + \frac{hH_k}{z^3}b_{s^3}(z). \end{aligned} \tag{7}$$

where we use notation:

$$
\begin{aligned}
b_{c^2}(x,z) &= \int\limits_{\tau}^{x} q(t)\cos z(x-t)\cos z(t-\tau)dt, b_{c^2}(\pi,z)=b_{c^2}(z),\\
b_{cs}(x,z) &= \int\limits_{\tau}^{x} q(t)\cos z(x-t)\sin z(t-\tau)dt, b_{cs}(\pi,z)=b_{cs}(z),\\
b_{csc}(x,z) &= \int\limits_{2\tau}^{x} q(t)\cos z(x-t)b_{sc}(t-\tau,z)dt, b_{csc}(\pi,z)=b_{csc}(z),\\
b_{cs^2}(x,z) &= \int\limits_{2\tau}^{x} q(t)\cos z(x-t)b_{s^2}(t-\tau,y)dt, b_{cs^2}(\pi,z)=b_{cs^2}(z).
\end{aligned}
\tag{8}
$$

Obviously, by using (3) the set of zeros of functions $\Delta_k(\lambda)$ is equivalent to the spectrum of boundary spectral problems D_k, respectively ([8]). Therefore, the functions $\Delta_k(\lambda), k=1,2$ are the characteristic functions for $D_k, k=1,2$, respectively. Now we define new function,

$$
\widetilde{q}(t) = \left\{ \begin{array}{l} q\left(t+\frac{\tau}{2}\right);\ t\in\left[\frac{\tau}{2},\pi-\frac{\tau}{2}\right] \\ 0;\ t\in\left(0,\frac{\tau}{2}\right)\cup\left(\pi-\frac{\tau}{2},\pi\right) \end{array} \right\}
\tag{9}
$$

It is clear this function orders the potential, also we define functions:

$$
K(t)=\left\{ \begin{array}{c} q(t+\tau)\int\limits_{\tau}^{t} q(s)ds - q(t)\int\limits_{t+\tau}^{\pi} q(s)ds - \int\limits_{t+\tau}^{\pi} q(s-t)q(s)ds;\ t\in[\tau,\pi-\tau] \\ 0;\ t\in[0,\tau)\cup(\pi-\tau,\pi] \end{array} \right\}
$$

$$
U(t)=\left\{ \begin{array}{c} q(t+\tau)\int\limits_{\tau}^{t} q(s)ds - q(t)\int\limits_{t+\tau}^{\pi} q(s)ds + \int\limits_{t+\tau}^{\pi} q(s-t)q(s)ds;\ t\in[\tau,\pi-\tau] \\ 0;\ t\in[0,\tau)\cup(\pi-\tau,\pi] \end{array} \right\}
\tag{10}
$$

Throughout the paper we use the notation:

$$
\begin{aligned}
I_1 &= \int\limits_{\tau}^{\pi} q(t)dt, I_2 = \int\limits_{2\tau}^{\pi} q(t)\int\limits_{\tau}^{t-\tau} q(s)dsdt,\\
\widetilde{a_c}(z) &= \int\limits_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \widetilde{q}(t)\cos z(\pi-2t)dt, \widetilde{a_s}(z) = \int\limits_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}} \widetilde{q}(t)\sin z(\pi-2t)dt,\\
k_c(z) &= \int\limits_{\tau}^{\pi-\tau} K(t)\cos z(\pi-2t)dt, k_s(z) = \int\limits_{\tau}^{\pi-\tau} K(t)\sin z(\pi-2t)dt,
\end{aligned}
$$

$$u_c(z) = \int_{\tau}^{\pi-\tau} U(t)\cos z(\pi-2t)dt, u_s(z) = \int_{\tau}^{\pi-\tau} U(t)\sin z(\pi-2t)dt. \quad (11)$$

Now we transform the products of trigonometric functions from (6) and (8) into sums/differences. It is easy to show that the characteristic functions have the form:

$$\begin{aligned}\Delta_k(\lambda) = F_k(z) &= \left(-z+\frac{hH_k}{z}\right)\sin\pi z + (h+H_k)\cos\pi z \\ &+\frac{1}{2}(\widetilde{a_c}(z) + I_1\cos z(\pi-\tau)) \\ &+\frac{h}{2z}(-\widetilde{a_s}(z) + I_1\sin z(\pi-\tau) \\ &+\frac{H_k}{2z}(\widetilde{a_s}(z)+I_1\sin z(\pi-\tau)) + \frac{hH_k}{2z^2}(\widetilde{a_c}(z) - I_1\cos z(\pi-\tau)) \\ &+\frac{1}{4z}(I_2\sin z(\pi-2\tau) - u_s(z)) - \frac{h}{4z^2}(I_2\cos z(\pi-2\tau)+k_c(z)) \\ &-\frac{H_k}{4z^2}(I_2\cos z(\pi-2\tau) - u_c(z)) - \frac{hH_k}{4z^3}(I_2\sin z(\pi-2\tau)+k_s(z)). \quad (12)\end{aligned}$$

The further consideration of the inverse problem requires the transformation of the characteristic functions (12). Then, integration by parts in integrals (11) gives:

$$\begin{aligned}\Delta_k(\lambda) = F_k(z) &= \left(-z+\tfrac{hH_k}{z}\right)\sin\pi z + (h+H_k)\cos\pi z + \tfrac{1}{2}\left(\widetilde{a_c}(z) + \tfrac{H_k}{z}\widetilde{a_s}(z)\right) \\ &-h\left(\widetilde{q_c}^{(1)}(z) + \tfrac{H_k}{z}\widetilde{q_s}^{(1)}(z)\right) - \tfrac{1}{2}\left(u_c^*(z) + \tfrac{H_k}{z}u_s^*(z)\right) + h\left(k_c^{**}(z) + \tfrac{H_k}{z}k_s^{**}(z)\right) \\ &+\tfrac{I_1}{2}\cos z(\pi-\tau) + \tfrac{2h+H_k}{2z}I_1\sin z(\pi-\tau) + \tfrac{1}{2z}\left(1 - \tfrac{hH_k}{2z^2}I_2\sin z(\pi-2\tau)\right) \\ &+\left(\tfrac{h}{2z}\sin z(\pi-2\tau) - \tfrac{hH_k}{2z^2}\cos z(\pi-2\tau)\right)\int_{\tau}^{\pi-\tau} K^*(t)dt,\end{aligned} \quad (13)$$

where:

$$\widetilde{q_c}^{(1)}(z) = \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\int_{\frac{\tau}{2}}^{t}\widetilde{q}(s)ds\cos z(\pi-2t)dt; \widetilde{q_s}^{(1)}(z) = \int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\int_{\frac{\tau}{2}}^{t}\widetilde{q}(s)ds\sin z(\pi-2t)dt,$$

$$u_c^*(z) = \int_{\tau}^{\pi-\tau}\int_{\tau}^{t} U(s)ds\cos z(\pi-2t)dt; u_s^*(z) = \int_{\tau}^{\pi-\tau}\int_{\tau}^{t} U(s)ds\sin z(\pi-2t)dt,$$

$$k_c^{**}(z) = \int_{\tau}^{\pi-\tau}\int_{\tau}^{t} K^*(s)ds\cos z(\pi-2t)dt; k_s^{**}(z) = \int_{\tau}^{\pi-\tau}\int_{\tau}^{t} K^*(s)ds\sin z(\pi-2t)dt,$$

$$K^*(t) = \left\{\begin{array}{l}\int_{\tau}^{t} K(u)du; t\in[\tau,\pi-\tau] \\ 0; t\in[0,\tau)\cup(\pi-\tau,\pi].\end{array}\right\} \quad (14)$$

Let us calculate the integral $\int\limits_{\tau}^{\pi-\tau} K^*(t)dt$. Firstly, we calculate the integral

$$\int\limits_{\tau}^{\pi-\tau} K(t)dt = \int\limits_{\tau}^{\pi-\tau} q(t+\tau)\int\limits_{\tau}^{t} q(s)dsdt$$
$$-\int\limits_{\tau}^{\pi-\tau} q(t)\int\limits_{t+\tau}^{\pi} q(s)dsdt - \int\limits_{\tau}^{\pi-\tau}\int\limits_{t+\tau}^{\pi} q(s-t)q(s)dsdt$$

Changing of variables or/and interchanging the order of integration, we obtain

$$\int\limits_{\tau}^{\pi-\tau} K(t)dt = I_2 - I_2 - I_2 = -I_2. \tag{15}$$

Using:

$$\int\limits_{\tau}^{\pi-\tau} uK(u)du = \int\limits_{\tau}^{\pi-\tau} uq(u+\tau)\int\limits_{\tau}^{u} q(s)dsdu -$$
$$\int\limits_{\tau}^{\pi-\tau} uq(u)\int\limits_{u+\tau}^{\pi} q(s)dsdu - \int\limits_{\tau}^{\pi-\tau}$$
$$\int\limits_{u+\tau}^{\pi} uq(s-u)q(s)dsdu = -\tau I_2$$

we have

$$\int\limits_{\tau}^{\pi-\tau} K^*(t)dt = -(\pi-2\tau)I_2. \tag{16}$$

Putting (16) into (13) we obtain:

$$\begin{gathered}\Delta_k(\lambda) = F_k(z) = \left(-z + \frac{hH_k}{z}\right)\sin\pi z + (h+H_k)\cos\pi z + \frac{1}{2}\left(\widetilde{a_c}(z) + \frac{H_k}{z}\widetilde{a_s}(z)\right)\\ -h\left(\widetilde{q_c}^{(1)}(z) + \frac{H_k}{z}\widetilde{q_s}^{(1)}(z)\right) - \frac{1}{2}\left(u_c^*(z) + \frac{H_k}{z}u_s^*(z)\right) + h\left(k_c^{**}(z) + \frac{H_k}{z}k_s^{**}(z)\right)\\ +\frac{I_1}{2}\cos z(\pi-\tau) + \frac{2h+H_k}{2z}I_1\sin z(\pi-\tau) +\\ \frac{I_2}{2z}\left(1 - h(\pi-2\tau) - \frac{hH_k}{z^2}\right)\sin z(\pi-2\tau) + \frac{hH_k(\pi-2\tau)I_2}{2z^2}\cos z(\pi-2\tau).\end{gathered} \tag{17}$$

Characteristic functions $F_k(z)$ given with (17) are entire functions. Functions $F_k(z)$ obviously has only one singular point $z = 0$. It is easy to see that $\lim_{z\to 0} F_k(z)$ exists so $z = 0$ is an apparent singularity of the characteristic functions $F_k(z)$. We know that the spectrum of boundary spectral problems D_k is countable [2]. Now, if $(\lambda_{n,k})_{n=0}^{\infty}, k = 1, 2$ is the spectrum of boundary spectral problems D_k, using (17) by the well-known method ([2],Ch.1), we obtain

$$\lambda_{n,k} = n^2 + \frac{2}{\pi}(h + H_k) + \frac{I_1}{\pi}\cos n\tau + o(1), (n \to \infty) \tag{18}$$

Since the $\Delta_k, k = 1, 2$ are entire in λ in order $\frac{1}{2}$, by Hadamard's factorization theorem ([8], Lemma 1), the characteristic functions are uniquely determined up to a multiplicative constant by its zeros. The following lemma holds.

Lemma 1. *The specification of spectrum* $(\lambda_{n,k})_{n=0}^{\infty}, k = 1, 2$ *uniquely determines the characteristic functions* $\Delta_k, k = 1, 2$ *by the formula*

$$\Delta_k(\lambda) = \pi(\lambda_{0,k} - \lambda)\prod_{n=1}^{\infty}\frac{\lambda_{n,k} - \lambda}{n^2}; k = 1, 2 \tag{19}$$

3. Main Results

Lemma 2. *If* $(\lambda_{n,k})_{n=0}^{\infty}, k = 1, 2$ *are the spectra of boundary spectral problems* $D_k, k = 1, 2$ *respectively, then the delay* τ*, integral* I_1 *and sum* $h + H_k, k = 1, 2$ *are uniquely determined.*

Proof. From (18) we have

$$\lim_{n\to\infty}\frac{\lambda_{n-2,k} - (n-2)^2 - \lambda_{n+2,k} + (n+2)^2}{\lambda_{n-1,k} - (n-1)^2 - \lambda_{n+1,k} + (n+1)^2} =$$

$$\lim_{n\to\infty}\frac{\cos\tau(n-2) - \cos\tau(n+2)}{\cos\tau(n-1) - \cos\tau(n+1)} = \lim_{n\to\infty}\frac{\sin n\tau \sin 2\tau}{\sin n\tau \sin\tau} = 2\cos\tau.$$

Finally

$$\tau = arccos\left(\frac{1}{2}\lim_{n\to\infty}\frac{\lambda_{n-2,k} - (n-2)^2 - \lambda_{n+2,k} + (n+2)^2}{\lambda_{n-1.k} - (n-1)^2 - \lambda_{n+1,k} + (n+1)^2}\right).$$

Because $\frac{2\pi}{5} < \tau < \frac{\pi}{2}$, there are infinitely many $m \in N$ satisfying $\sin\frac{(2m+1)\tau}{2} \neq 0$, now we have:

$$\lambda_{m+1,k} - (m+1)^2 - \lambda_{m,k} + m^2 = \frac{I_1}{\pi}(\cos(m+1)\tau - \cos m\tau) + o(1), (m \to \infty)$$

finally

$$I_1 = \lim_{m\to\infty}\frac{\pi(\lambda_{m+1,k} - (m+1)^2 - \lambda_{m,k} + m^2)}{-2\sin\frac{\tau}{2}\sin\frac{(2m+1)\tau}{2}}$$

and, for $k = 1, 2$

$$h + H_k = \lim_{n\to\infty}\frac{\pi}{2}\left(\lambda_{n,k} - n^2 - \frac{I_1}{\pi}\cos n\tau\right)$$

□

Lemma 3. *If* $(\lambda_{n,k})_{n=0}^{\infty}, k=1,2$ *are the spectra of boundary spectral problems* $D_k, k=1,2$ *respectively, then parameters* $h, H_k, k=1,2$ *are uniquely determined by* $(\lambda_{n,k})_{n=0}^{\infty}, k=1,2$.

Proof. Using (12) we have:

$$F_2(z)-F_1(z)=\frac{h(H_2-H_1)}{z}\sin\pi z+(H_2-H_1)\cos\pi z$$
$$+\frac{H_2-H_1}{2z}(\widetilde{a_s}(z)+I_1\sin z(\pi-\tau))+\frac{h(H_2-H_1)}{2z^2}(\widetilde{a_c}(z)-I_1\cos z(\pi-\tau))$$
$$-\frac{H_2-H_1}{4z^2}(I_2\cos z(\pi-2\tau)-u_c(z))$$
$$-\frac{h(H_2-H_1)}{4z^3}(I_2\sin z(\pi-2\tau)+k_s(z)).$$

According to Lemma 1 this function is determined. Now we put $z=\frac{4m+1}{2}, m\in N$ in this function and we have

$$\lim_{m\to\infty}\frac{4m+1}{2}\left(F_2\left(\frac{4m+1}{2}\right)-F_1\right.$$
$$\left.\left(\frac{4m+1}{2}\right)-\frac{H_2-H_1}{4m+1}I_1\cos\frac{(4m+1)\tau}{2}\right)=h(H_2-H_1)$$

Using Lemma 2 parameters τ, H_2-H_1, I_1 are determined, so we have

$$h=\lim_{m\to\infty}\frac{4m+1}{2(H_2-H_1)}\left(F_2\left(\frac{4m+1}{2}\right)-F_1\left(\frac{4m+1}{2}\right)\right.$$
$$\left.-\frac{H_2-H_1}{4m+1}I_1\cos\frac{(4m+1)\tau}{2}\right)$$

Using Lemma 2 parameters H_1, H_2 are ordered. □

In order to recover the potential from the spectra, at the beginning we introduce the functions

$$A(z)=\frac{2}{H_2-H_1}(H_2F_1(z)-H_1F_2(z))+2z\sin\pi z-2h\cos\pi z-I_1\cos z(\pi-\tau), \tag{20}$$

$$B(z)=\frac{2z}{H_2-H_1}(F_2(z)-F_1(z))-2h\sin\pi z-2z\cos\pi z-I_1\sin z(\pi-\tau). \tag{21}$$

According to (17) we obtain

$$A(z)=\widetilde{a_c}(z)-2h\widetilde{q_c}^{(1)}(z)-u_c^*(z)+2hk_c^{**}(z)+\alpha(z), \tag{22}$$
$$B(z)=\widetilde{a_s}(z)-2h\widetilde{q_s}^{(1)}(z)-u_s^*(z)+2hk_s^{**}(z)+\beta(z), \tag{23}$$

where

$$\alpha(z)=\frac{2hI_1}{z}\sin z(\pi-\tau)+\frac{I_2}{z}(1-h(\pi-2\tau))\sin z(\pi-2\tau), \tag{24}$$

$$\beta(z)=\frac{hI_2}{z}(\pi-2\tau)\cos z(\pi-2\tau)-\frac{hI_2}{z^2}\sin z(\pi-2\tau). \tag{25}$$

Obviously, functions $\alpha(z),\beta(z)$ are known if I_2 is known. If we put $z=m, m\in N$ into (22) and (23) and denote

$$A_{2m}=\frac{2}{\pi}(-1)^m(A(m)-\alpha(m)); B_{2m}=\frac{2}{\pi}(-1)^{m+1}(B(m)-\beta(m). \tag{26}$$

we have:

$$A_{2m}=\frac{2}{\pi}\widetilde{a}_{2m}-\frac{4}{\pi}h\widetilde{q}^{(1)}_{2m,c}-\frac{2}{\pi}u^*_{2m,c}+\frac{4}{\pi}hk^{**}_{2m,c}, \tag{27}$$

$$B_{2m}=\frac{2}{\pi}\widetilde{b}_{2m}-\frac{4}{\pi}h\widetilde{q}^{(1)}_{2m,s}-\frac{2}{\pi}u^*_{2m,s}+\frac{4}{\pi}hk^{**}_{2m,s}, \tag{28}$$

where:

$$\widetilde{a}_{2m}=\int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\widetilde{q}(t)\cos 2mtdt; \widetilde{b}_{2m}=\int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\widetilde{q}(t)\sin 2mtdt$$

$$\widetilde{q}^{(1)}_{2m,c}=\int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\left(\int_{\frac{\tau}{2}}^{t}\widetilde{q}(s)ds\right)\cos 2mtdt; \widetilde{q}^{(1)}_{2m,s}=\int_{\frac{\tau}{2}}^{\pi-\frac{\tau}{2}}\left(\int_{\frac{\tau}{2}}^{t}\widetilde{q}(s)ds\right)\sin 2mtdt$$

$$u^*_{2m,c}=\int_{\tau}^{\pi-\tau}\left(\int_{\tau}^{t}U(s)ds\right)\cos 2mtdt; u^*_{2m,s}=\int_{\tau}^{\pi-\tau}\left(\int_{\tau}^{t}U(s)ds\right)\sin 2mtdt$$

$$k^{**}_{2m,c}=\int_{\tau}^{\pi-\tau}\left(\int_{\tau}^{t}K^*(s)ds\right)\cos 2mtdt; k^{**}_{2m,s}=\int_{\tau}^{\pi-\tau}\left(\int_{\tau}^{t}K^*(s)ds\right)\sin 2mtdt.$$

Further, it can be easily verified that the following relations hold

$$\alpha_0=\lim_{z\to 0}\alpha(z)=2hI_1(\pi-\tau)+I_2(1-h(\pi-2\tau))(\pi-2\tau)$$

$$\lim_{z\to 0}A(z)=\widetilde{a_0}-2h\widetilde{q_{0,c}}^{(1)}-u^*_{0,c}+2hk^{**}_{0,c}+\alpha_0$$

$$\beta_0=\lim_{z\to 0}\beta(z)=0; B_0=\lim_{z\to 0}B(z)=0.$$

Denote

$$A_0=\frac{2}{\pi}\left(\lim_{z\to 0}A(z)-\widetilde{\alpha}_0\right), \tag{29}$$

then $A_0=\frac{2}{\pi}\widetilde{a_0}-\frac{4}{\pi}h\widetilde{q}^{(1)}_{0,c}-\frac{2}{\pi}u^*_{0,c}+\frac{4}{\pi}hk^{**}_{0,c}+\alpha_0$. One can easily prove that sequences A_{2m} and B_{2m} belong to the space l^2, hence by virtue of Riesz-Fischers theorem, there exists a function f from $L^2[0,\pi]$ such that

$$f(t)=\frac{A_0}{2}+\sum_{n=1}^{+\infty}A_{2m}\cos 2mt+B_{2m}\sin 2mt.$$

Now multiplying (29) with 1/2, (27) with $\cos 2mt$ and (28) with $\sin 2mt$, and then summing, we get the integral equation:

$$\widetilde{q}(t) - 2h\int\limits_{\frac{\tau}{2}}^{t}\widetilde{q}(s)ds - \int\limits_{\tau}^{t}U(s)ds + 2h\int\limits_{\tau}^{t}K^{*}(s)ds = f(t). \tag{30}$$

Substituting $\widetilde{q}$ from (9) into (30), and then putting $t+\frac{\tau}{2}=x$ and $s+\frac{\tau}{2}=u$,we obtain

$$q(x) - 2h\int\limits_{\tau}^{x}q(u)du - \int\limits_{\frac{3\tau}{2}}^{x}U(u-\frac{\tau}{2})du + 2h\int\limits_{\frac{3\tau}{2}}^{x}K^{*}(u-\frac{\tau}{2})du = f(x). \tag{31}$$

Finally, we come to our main result.

Theorem 1. *Let* $(\lambda_{n,k})_{n=0}^{\infty}, k=1,2$ *be the spectra of boundary spectral problems* $D_k, k=1,2$ *respectively, then potential* q *are uniquely determined by* $(\lambda_{n,k})_{n=0}^{\infty}, k=1,2$ *if* $\frac{2\pi}{5}<\tau<\frac{\pi}{2}$ *and integral* $I_2=\int_{2\tau}^{\pi}q(t)\int_{\tau}^{t-\tau}q(s)dsdt$ *is known.*

Proof. The potential $q(x)$ satisfies integral equation (31), we will show uniqueness of solution of this equation.

- For $x\in\left[\tau,\frac{3\tau}{2}\right]$ it is obvious that (31) is the Volterra linear integral equation:

  $$q(x) = f(x) + 2h\int\limits_{\tau}^{x}q(u)du.$$

  This integral equation has a unique solution on $\left[\tau,\frac{3\tau}{2}\right]$ ([14]).
- For $x\in\left(\pi-\frac{\tau}{2},\pi\right]$, according to (10) and (14) $U\left(u-\frac{\tau}{2}\right)=K^{*}\left(u-\frac{\tau}{2}\right)=0, u\in\left(\pi-\frac{\tau}{2},\pi+\frac{\tau}{2}\right]$ and therefore (31) leads to the integral equation

  $$q(x) - 2h\int\limits_{\tau}^{x}q(u)du - \int\limits_{\frac{3\tau}{2}}^{\pi-\frac{\tau}{2}}U(u-\frac{\tau}{2})du$$
  $$+2h\int\limits_{\frac{3\tau}{2}}^{\pi-\frac{\tau}{2}}K^{*}(u-\frac{\tau}{2})du = f(x). \tag{32}$$

  Further, from (16) we have $\int_{\tau}^{\pi-\tau}K^{*}(t)dt=\int_{\frac{3\tau}{2}}^{\pi-\frac{\tau}{2}}K^{*}\left(u-\frac{\tau}{2}\right)du=-(\pi-2\tau)I_2$. Also, in the same way as in (15), we calculate $\int_{\tau}^{\pi-\tau}U(t)dt=I_2$ and $\int_{\tau}^{\pi-\tau}U(t)dt=\int_{\frac{3\tau}{2}}^{\pi-\frac{\tau}{2}}U\left(u-\frac{\tau}{2}\right)du=I_2$.

Since $\int_\tau^x q(u)du = I_1 - \int_x^\pi q(u)du$ integral equation (32) is Volterra linear integral equation

$$q(x) = f_1(x) - 2h\int_x^\pi q(u)du,$$

where $f_1(x) = f(x) + 2hI_1 + I_2 + 2h(\pi - 2\tau)I_2$ is a known function. This integral equation has a unique solution on $\left(\pi - \frac{\tau}{2}, \pi\right]$ (see [14])

- For $x \in \left(\frac{3\tau}{2}, \pi - \frac{\tau}{2}\right]$ we get the integral equation

$$q(x) - 2h\int_\tau^{\frac{3\tau}{2}} q(u)du - 2h\int_{\frac{3\tau}{2}}^x q(u)du - \int_{\frac{3\tau}{2}}^x U(u-\frac{\tau}{2})du$$
$$+2h\int_{\frac{3\tau}{2}}^x K^*(u-\frac{\tau}{2})du = f(x). \tag{33}$$

Let us consider the integral

$$\int_{\frac{3\tau}{2}}^x U\left(u-\frac{\tau}{2}\right)du = \int_{\frac{3\tau}{2}}^x q\left(u+\frac{\tau}{2}\right)\int_\tau^{u-\frac{\tau}{2}} q(p)dpdu$$
$$-\int_{\frac{3\tau}{2}}^x q\left(u-\frac{\tau}{2}\right)\int_{u+\frac{\tau}{2}}^\pi q(p)dpdu$$
$$+\int_{\frac{3\tau}{2}}^x\int_{u+\frac{\tau}{2}}^\pi q\left(p-u+\frac{\tau}{2}\right)q(p)dpdu.$$

One can easily show that arguments of the potential q appearing in this function, belong to the intervals $[2\tau, \pi] \subset \left[\pi - \frac{\tau}{2}\right]$ and $[\tau, \pi - \tau] \subset \left[\tau, \frac{3\tau}{2}\right]$. Therefore, $\int_{\frac{3\tau}{2}}^x U\left(u - \frac{\tau}{2}\right)du$ is a known function. Further, according to (5), (6) and (14), arguments of the potential q in the function $\int_{\frac{3\tau}{2}}^x K^*\left(u - \frac{\tau}{2}\right)du$ are the same as arguments of the function $\int_{\frac{3\tau}{2}}^x U\left(u - \frac{\tau}{2}\right)du$. Consequently, $\int_{\frac{3\tau}{2}}^x K^*\left(u - \frac{\tau}{2}\right)du$ is a known function, too. Then (33) leads to the Volterra linear integral equation:

$$q(x) = f_2(x) + 2h\int_{\frac{3\tau}{2}}^x q(u)du,$$

where: $f_2(x) = f(x) + 2h\int_\tau^{\frac{3\tau}{2}} q(u)du + \int_{\frac{3\tau}{2}}^x U\left(u-\frac{\tau}{2}\right)du - 2h\int_{\frac{3\tau}{2}}^x K^*\left(u-\frac{\tau}{2}\right)du$ is a known function.

This integral equation has a unique solution on $\left(\frac{3\tau}{2}, \pi - \frac{\tau}{2}\right]$ (see [14]). The theorem is proved. $\square$

References

[1] Ambarzumjan, V.: Uber eine Frage der Eigenwerttheorie. Zeitshr. fr Physik -Bd. **53**, 690–695 (1929)

[2] Freiling, G., Yurko, V.: Inverse Sturm–Liouville Problems and Their Applications. Nova Science Publishers, Inc., Huntigton (2008)

[3] Sat, M.: Inverse problems for Sturm–Liouville operators with boundary conditions depending on a spectral parameter. Electr. J. Differ. Equ. **2017**(26), 1–7 (2017)

[4] Yurko, V.A., Yang, C.F.: Recovering differential operators with nonlocal boundary conditions. Anal. Math. Phys. **6**(4), 315–326 (2016)

[5] Bondarenko, N., Buterin, S.: On recovering the Dirac operator with an integral delay from the spectrum. Results Math. **71**(3), 1521–1529 (2017)

[6] Buterin, S.A., Sat, M.: On the half inverse spectral problem for an integrodifferential operator. Inverse Probl. Sci. Eng. **25**(10), 1508–1518 (2017)

[7] Yang, C.-F.: Trace and inverse problem of a discontinuous Sturm–Liouville operator with retarded argument. J. Math. Anal. Appl. **395**, 30–41 (2012)

[8] Freiling, G., Yurko, V.L.: Inverse problems for Sturm–Liouville diferential operators with a constant delay. Appl. Math. Lett. **25**(11), 1999–2004 (2012)

[9] Buterin, S., Yurko, V.: An inverse spectral problem for Sturm–Liouville operators with a large constant delay. Anal. Math. Phys. (2017). https://doi.org/10.1007/s13324-017-0176-6

[10] Vladicic, V., Pikula, M.: An inverse problems for Sturm–Liouville-type differential equation with a constant delay. Sarajevo J. Math. **12**(1), 83–88 (2016)

[11] Pikula, M.: Determination of a Stur–Liouville-type differential operator with delay argument from two spectra. Mat. Vesnik **43**(34), 159–171 (1991)

[12] Pikula, M., Vladicic, V., Markovic, O.: A solution to the inverse problem for Sturm–Liouville-type equation with a delay. Filomat **27**(7), 1237–1245 (2013)

[13] Buterin, S.A., Choque, A.E.: Rivero on inverse problem for a convolution integro-differential operator with Robin boundary conditions. Appl. Math. Lett **48**, 150–155 (2015)

[14] Hochstadt, H.: Integral Equations. Wiley, New York (1989)

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