Results Math (2019) 74:23

1422-6383/19/010001-27
published online January 5, 2019
https://doi.org/10.1007/s00025-018-0948-9

Results in Mathematics

A Left Linear Weighted Composition Operator on Quaternionic Fock Space

Yu-Xia Liang

Abstract. A left linear weighted composition operator $W_{f,\varphi}$ is defined on slice regular quaternionic Fock space $\mathcal{F}^2(\mathbb{H})$. We carry out a comprehensive analysis on its classical properties. Firstly, the boundedness and compactness of weighted composition operator on $\mathcal{F}^2(\mathbb{H})$ are investigated systematically, which can be seen new and brief characterizations. And then all normal bounded weighted composition operators are found, particularly, equivalent conditions for self-adjoint weighted operators on $\mathcal{F}^2(\mathbb{H})$ are developed. Finally, we describe all types of isometric weighted composition operators on $\mathcal{F}^2(\mathbb{H})$.

Mathematics Subject Classification. Primary 30G35, 47B38.

Keywords. Weighted composition operator, quaternionic Fock space, boundedness, compactness, self-adjoint, isometry.

1. Introduction

Recently, the theory of slice regular functions has been developed rapidly and found wide range of applications, for example in Schur analysis and to define some functional calculi. It's well-known that there are several different type definitions of regularity for functions in quaternions. Fueter, who expanded on the work of Moisil, defined regular functions those which satisfy a first order system of linear differential equations generalizing one of Cauchy–Riemann, see e.g. [4,8] etc. Indeed, a new functional calculus for slice functions was established which could be considered as the mathematically framework of the quaternionic quantum mechanics. All the time mathematicians have been interested in creating a theory of quaternionic valued functions of a quaternionic variable, which would somehow resemble the classical theory of holomorphic functions of one complex variable. For details on slice holomorphic functions one can refer to the excellent books [5,6]. Here we only recall some preliminaries about slice regular functions used later.

1.1. Quaternion

The symbol $\mathbb{H}$ denotes the noncommutative, associative, real algebra of quaternions with standard basis $\{1, i, j, k\}$, subject to the multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1.$$

That is to say $\mathbb{H}$ is the set of the quaternions

$$q = x_0 + x_1 i + x_2 j + x_3 k = Re(q) + Im(q),$$

with $Re(q) = x_0$ and $Im(q) = x_1 i + x_2 j + x_3 k$, where $x_j \in \mathbb{R}$ for $j = 0, 1, 2, 3$. And then $\overline{q} = x_0 - (x_1 i + x_2 j + x_3 k) = Re(q) - Im(q)$ represents the conjugate of q. The Euclidean norm of a quaternion q is given by

$$|q| = \sqrt{q\overline{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

By the symbol $\mathbb{S}$ we denote the two-dimensional unit sphere of purely imaginary quaternions, i.e.

$$\mathbb{S} = \{q = x_1 i + x_2 j + x_3 k : \ x_1^2 + x_2^2 + x_3^2 = 1\}.$$

That is, $I^2 = -1$ for $I \in \mathbb{S}$. For any fixed $I \in \mathbb{S}$ we define

$$\mathbb{C}_I := \{x + Iy : \ x, y \in \mathbb{R}\},$$

which can be identified with a complex plane. In the sequel, an element in the complex plane $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$ is denoted by $x + Iy$. Moreover, it holds that

$$\mathbb{H} = \bigcup_{I \in \mathbb{S}} \mathbb{C}_I.$$

Interestingly, the real axis belongs to $\mathbb{C}_I$ for every $I \in \mathbb{S}$ and thus a real number can be associated with any imaginary unit I. However any non real quaternion q is uniquely associated to the element $I_q \in \mathbb{S}$ given by

$$I_q := (ix_1 + jx_2 + kx_3)/|ix_1 + jx_2 + kx_3|,$$

and then q belongs to the complex pane $\mathbb{C}_{I_q}$. It's easy to check

$$\overline{pq} = \overline{q}\,\overline{p} \tag{1.1}$$

for $p,\ q \in \mathbb{H}$. Now, we are ready to give the key concept of this paper.

Definition 1.1. [6, Definition 2.1.1] Let U be an open set in $\mathbb{H}$ and a function $f : \ U \to \mathbb{H}$ be real differentiable. The function f is called *slice regular* or *slice hyperholomorphic* if, for every $I \in \mathbb{S}$, its restriction $f_I(x + Iy) = f(x + Iy)$ is holomorphic, *i.e.* it has continuous partial derivatives and satisfies

$$\overline{\partial_I} f(x + yI) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + yI) = 0$$

for all $x+yI \in U \cap \mathbb{C}_I$. The class of all slice regular functions on U is denoted by $\mathcal{R}(U)$. Particularly, $\mathcal{R}(\mathbb{H})$ is the collection of all entire regular functions on $\mathbb{H}$. And then $\mathcal{R}(U)$ is a right linear space on $\mathbb{H}$.

Let $I, J \in \mathbb{S}$ be such that I and J are orthogonal, so that I, J, IJ is an orthogonal basis of $\mathbb{H}$ and write the restriction f_I as the function

$$f = f_0 + If_1 + Jf_2 + IJf_3$$

on the complex plane $\mathbb{C}_I$. It can also be written as $f_I = F + GJ$ where $f_0 + If_1 = F$, and $f_2 + If_3 = G$. Hence we have the following splitting lemma, which relates slice regularity with classical holomorphy.

Lemma 1.2. [6, Lemma 2.1.4] (Splitting Lemma) *If f is a slice regular function on the domain U, then for every $I, J \in \mathbb{S}$, with $I \perp J$, there are two holomorphic functions F, $G: U_I = U \cap \mathbb{C}_I \to \mathbb{C}_I$ such that*

$$f_I(z) = F(z) + G(z)J \ \text{ for any } z = x + yI \in U_I.$$

Slice regular functions possess good properties on specific open sets that are called *axially symmetric slice domains*.

Definition 1.3. [6, Definition 2.2.1] Let $U \subset \mathbb{H}$ be a domain.
(1) U is called a *slice domain* (or *s-domain* for short) if it intersects the real axis and if, for any $I \in \mathbb{S}$, $U_I := \mathbb{C}_I \cap U$ is a domain in $\mathbb{C}_I$.
(2) U is *axially symmetric* if for every $x + yI \in U$ with $x, y \in \mathbb{R}$ and $I \in \mathbb{S}$, all the elements $x + y\mathbb{S} = \{x + yJ : J \in \mathbb{S}\}$ is contained in U.

The following representation formula of a slice regular function on an axially symmetric domain allows to recover all its values from its values on a single slice $\mathbb{C}_I$.

Proposition 1.4. [6, Theorem 2.2.4](Representation Formula) *Let f be a slice regular function on an axially symmetric s-domain $U \subset \mathbb{H}$. Let $J \in \mathbb{S}$ and let $x \pm yJ \in U \cap \mathbb{C}_J$, then the following equality holds for all $q = x + yI \in U$,*

$$f(x+yI) = \frac{1}{2}\left[(1+IJ)f(x-yJ) + (1-IJ)f(x+yJ)\right].$$

For more on the entire slice regular functions, we refer to the excellent books [5,6] and the references therein.

It's easy to verify the pointwise product of functions does not preserve slice regularity, a new multiplication operation, the $\star$-product, was introduced. In the special case of *power series*, the regular product (or $\star$-product) is given below.

Let U be a ball with center at a real point, $f(q) = \sum_{n=0}^{\infty} q^n a_n$, $a_n \in \mathbb{H}$ and $g(q) = \sum_{n=0}^{\infty} q^n b_n$ with $b_n \in \mathbb{H}$, the regular product of f and g is defined as

$$(f \star g)(q) := \sum_{n=0}^{\infty} q^n \left(\sum_{r=0}^{n} a_r b_{n-r} \right). \tag{1.2}$$

In this case, the notion $\star$-product coincides with the classical notion of product of series with coefficients in a ring. It's easy to see the function $f \star g$ is slice hyperholomorphic. The regular product was further generalized to the case of regular functions defined on *axially symmetric s-domains*.

Let $U \subset \mathbb{H}$ be an axially symmetric s-domain and let $f, g: \ U \to \mathbb{H}$ be slice regular functions. For any $I, J \in \mathbb{S}$ with $I \perp J$, the Splitting Lemma guarantees the existence of four holomorphic functions $F, G, H, K: U \cap \mathbb{C}_I \to \mathbb{C}_I$ such that for all $z = x + yI \in U \cap \mathbb{C}_I$,

$$f_I(z) = F(z) + G(z)J, \ \ g_I(z) = H(z) + K(z)J.$$

Then $f_I \star g_I: U \cap \mathbb{C}_I \to \mathbb{C}_I$ is defined as

$$\begin{aligned} & f_I \star g_I(z) \\ & \quad = [F(z)H(z) - G(z)\overline{K(\overline{z})}] + [F(z)K(z) + G(z)\overline{H(\overline{z})}]J. \end{aligned} \tag{1.3}$$

The new function $f_I \star g_I$ is a holomorphic map and it admits an unique slice regular extension to U defined by $ext(f_I \star g_I)(q)$.

Definition 1.5. Let $U \subset \mathbb{H}$ be an axially symmetric s-domain and let $f, g: U \to \mathbb{H}$ be slice regular. The function

$$(f \star g)(q) = ext(f_I \star g_I)(q)$$

defined as the extension of (1.3) (using Proposition 1.4) is called the slice regular product of f and g.

1.2. Quaternionic Fock Space

Due to the theory of regular functions is by now very well developed, especially, it can be extremely successful in replicating many important properties of holomorphic functions. Parallel to holomorphic function spaces, there appear many slice regular spaces of hyperholomorphic functions, such as, Dirichlet and Besov spaces [16], Hardy and Bergman spaces [3,14], Bloch space [11] and so on. Since Fock space plays an important role in quantum mechanics, and also in its quaternionic formulation, see the book of Adler [1] and the paper [9]. In the sequel, we collect some information for slice regular quaternionic Fock space from [2], which is a generalization of the excellent book [18].

Definition 1.6. [2, Definition 3.6] Let I be any elements in $\mathbb{S}$ and $p|_{\mathbb{C}_I} = z$, consider the set

$$\mathcal{F}^2(\mathbb{H}) = \{f \in \mathcal{R}(\mathbb{H}): \ \|f\|^2 := \int_{\mathbb{C}_I} |f_I(z)|^2 e^{-|z|^2} d\sigma(x,y) < \infty\}, \tag{1.4}$$

where $d\sigma(x,y) := \frac{1}{\pi} dxdy$. We call $\mathcal{F}^2(\mathbb{H})$ (slice hyperholomorphic or slice regular) quaternionic Fock space.

Moreover, $\mathcal{F}^2(\mathbb{H})$ is a Hilbert space under the inner product

$$\langle f, g \rangle = \int_{\mathbb{C}_I} \overline{g_I(z)} f_I(z) e^{-|z|^2} d\sigma(x,y), \tag{1.5}$$

which can induce a norm $\|\cdot\|$ given in (1.4). [2, Proposition 3.8] has shown the definition of Fock space does not depend on the imaginary unit $I \in \mathbb{S}$. Also the monomials p^n ($n \in \mathbb{N}$) form an orthogonal basis and satisfy $\langle p^n, p^n \rangle = n!$.

The reproducing kernel of $\mathcal{F}^2(\mathbb{H})$ is given by

$$e_*^{pq} = \sum_{n=0}^{+\infty} \frac{(pq)^{\star n}}{n!} = \sum_{n=0}^{+\infty} \frac{p^n q^n}{n!}.$$

Without loss of generality, we also denote $K_q(p) := e_*^{p\overline{q}}$, and [2, Theorem 3.10] implies $\langle f, K_q \rangle = f(q)$ for any $f \in \mathcal{F}^2(\mathbb{H})$. Particularly, $K_w(z) = e^{z\overline{w}}$ for $z, w \in \mathbb{C}_I$. Besides, it is true that

$$\|K_q\|^2 = \langle K_q, K_q \rangle = K_q(q) = e^{|q|^2}. \tag{1.6}$$

Furthermore the function $k_q(p) = K_q(p)/\|K_q\|$ is a unit-vector in $\mathcal{F}^2(\mathbb{H})$. Based on (1.2), we conclude

$$K_z(q) \star K_w(q) = e_*^{q\overline{z}} \star e_*^{q\overline{w}} = e_*^{q(\overline{z+w})} = K_{z+w}(q), \tag{1.7}$$

for $z, w \in \mathbb{C}_I$ and $q \in \mathbb{H}$. Indeed, it is due to

$$\begin{aligned} e_*^{q\overline{z}} \star e_*^{q\overline{w}} &= \left[\sum_{n=0}^{\infty} \frac{q^n \overline{z}^n}{n!}\right] \star \left[\sum_{n=0}^{\infty} \frac{q^n \overline{w}^n}{n!}\right] \\ &= \sum_{n=0}^{\infty} q^n \left(\sum_{r=0}^{n} \frac{\overline{z}^r}{r!} \frac{\overline{w}^{n-r}}{(n-r)!}\right) \\ &= \sum_{n=0}^{\infty} \frac{q^n}{n!} \left(\sum_{r=0}^{n} \frac{n!}{r!(n-r)!} \overline{z}^r \overline{w}^{n-r}\right) \\ &= \sum_{n=0}^{\infty} \frac{q^n (\overline{z+w})^n}{n!} \\ &= e_*^{q(\overline{z+w})} = K_{z+w}(q). \end{aligned}$$

1.3. Weighted Composition Operator

Recently huge interest has arisen to characterize composition operators on various holomorphic function spaces of different domains in $\mathbb{C}$ or $\mathbb{C}_n$. For more details on composition operators and their applications one can refer to the two books [7] by Cowen and MacCluer, and [15] by Shapiro. Motivated by rich achievements in complex analytic function theory, many mathematicians have contributed to the characterizations of composition operator on several slice regular spaces of hyperholomorphic functions, we refer to [10,16] and their references therein. To the best of our knowledge, there has been no such descriptions for weighted composition operator on slice regular spaces of hyperholomorphic functions. Considering the significant applications of Fock space, these basic characterizations are in desired need of response.

In this paper, we will first define a left-linear weighted composition operator on $\mathcal{F}^2(\mathbb{H})$ in details. Let $\varphi: \mathbb{H} \to \mathbb{H}$ be a slice hyperholomorphic map such that $\varphi(\mathbb{C}_I) \subset \mathbb{C}_I$ for some $I \in \mathbb{S}$. The composition operator C_φ on $\mathcal{F}^2(\mathbb{H})$ induced by φ is defined by

$$(C_\varphi f)_I(z) = (f_I \circ \varphi_I)(z) = F \circ \varphi_I(z) + G \circ \varphi_I(z)J$$

for all $f \in \mathcal{F}^2(\mathbb{H})$ with $f_I(z) = F(z) + G(z)J$. By the representation formula (Proposition 1.4), we can obtain the extension $C_\varphi h$ to the whole $\mathbb{H}$.

And then, the weighted composition operator $W_{f,\varphi}: \mathcal{F}^2(\mathbb{H}) \to \mathcal{F}^2(\mathbb{H})$ with $f \in \mathcal{R}(\mathbb{H})$ and $\varphi: \mathbb{C}_I \to \mathbb{C}_I$ for some $I \in \mathbb{S}$, is defined as

$$(W_{f,\varphi} h)_I(z) = (h_I \circ \varphi_I)(z) \star f_I(z)$$

for all $h \in \mathcal{F}^2(\mathbb{H})$. The extension $W_{f,\varphi}h$ on $\mathbb{H}$ is also deduced by the representation formula. Here we also note that

$$[W_{f,\varphi}(ag + bh)]_I(z) = a(W_{f,\varphi}g)_I(z) + b(W_{f,\varphi}h)_I(z)$$

for any $g, h \in \mathcal{F}^2(\mathbb{H})$ and $a, b \in \mathbb{H}$, which means $W_{f,\varphi}$ is left linear on $\mathcal{F}^2(\mathbb{H})$. Indeed, we have also defined a right linear weighted composition operator in [12] through $\star$-product by $f_I(z)$ on the left, that is

$$(\tilde{W}_{f,\varphi} h)_I(z) = f_I(z) \star (h_I \circ \varphi_I)(z).$$

Especially, we have systematically investigated some interesting properties of $\tilde{W}_{f,\varphi}$ on $\mathcal{F}^2(\mathbb{H})$. Parallelling to these important results in [12], we include analogous characterizations about $W_{f,\varphi}$ on $\mathcal{F}^2(\mathbb{H})$ for completeness. Comparing these corresponding results, we may find many differences, which can reveal the influence of weight function's location on weighted composition operator.

In what follows, we will find the slice regular product can make the theory of weighted composition operator more complicated than that behaves on functions of one complex variable. Specifically, the paper consists of 3 sections and its outline is as follows. In Sect. 2 we concentrate on the descriptions for boundedness and compactness of weighted composition operator acting on the slice regular quaternionic Fock space. Section 3 is devoted to determine all normal weighted composition operators on $\mathcal{F}^2(\mathbb{H})$ and then the equivalent conditions for self-adjoint weighted composition operators are presented. Finally, the isometric weighted composition operators on $\mathcal{F}^2(\mathbb{H})$ are investigated in Sect. 4. These results can be seen the extensions of the corresponding parts in [13] concerning weighted composition operators on Fock space $\mathcal{F}^2(\mathbb{C})$.

2. Boundedness and Compactness

In this section we will systematically investigate the boundedness and compactness of weighted composition operators acting on $\mathcal{F}^2(\mathbb{H})$. Firstly, we exhibit an interesting lemma for our further use.

Lemma 2.1. *For every* $q = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$, *it holds that*

$$x_0 = \frac{q + \overline{q}}{2} \ \text{ and } \ x_1 = \frac{i\overline{q} - qi}{2}. \tag{2.1}$$

Hence

$$x_0 + x_1 i = \frac{q + \overline{q}}{2} + i\frac{i\overline{q} - qi}{2} = \frac{q - iqi}{2}. \tag{2.2}$$

Proof. It is true that $\overline{q} = x_0 - (x_1 i + x_2 j + x_3 k)$ and then it's obvious that $x_0 = (q + \overline{q})/2$. By the properties $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$, it follows that

$$i\overline{q} = x_0 i + x_1 - x_2 ij - x_3 ik = x_0 i + x_1 - x_2 k - x_3 j$$

and

$$qi = x_0 i - x_1 + x_2 ji + x_3 ki = x_0 i - x_1 - x_2 k - x_3 j.$$

Hence $x_1 = (i\overline{q} - qi)/2$. This completes the proof. □

Paralleling to [13, Proposition 2.1], we provide a vital result for the characterizations of bounded and compact weighted composition operators on $\mathcal{F}^2(\mathbb{H})$.

Proposition 2.2. *Let* f *and* φ *be two slice regular functions on* $\mathbb{H}$, *such that* f *is not identically zero and* $\varphi(\mathbb{C}_I) \subset \mathbb{C}_I$ *for some* $I \in \mathbb{S}$. *Suppose there is a positive constant* M *such that*

$$M(f, \varphi) := \sup_{p \in \mathbb{H}} e^{|\varphi(p)|^2 - |p|^2} |f(p)|^2 \leq M. \tag{2.3}$$

Then $\varphi(p) = \varphi(0) + p\lambda$ *for some* $|\lambda| \leq 1$, $\lambda \in \mathbb{C}_I$. *If* $|\lambda| = 1$, *then*

$$f(p) = e^{-p\overline{\beta}} f(0), \tag{2.4}$$

where $\beta = \frac{1}{2}(\varphi(0)\overline{\lambda} - I\varphi(0)\overline{\lambda}I) = \varphi(0)\overline{\lambda}$. *Furthermore, if*

$$\lim_{|p| \to \infty} e^{|\varphi(p)|^2 - |p|^2} |f(p)|^2 = 0, \tag{2.5}$$

then $\varphi(p) = \varphi(0) + p\lambda$ *with* $|\lambda| < 1$, $\lambda \in \mathbb{C}_I$.

Proof. For $f \in \mathcal{R}(\mathbb{H})$ not identically zero, there is a nonnegative integer k and a slice regular function g with $g(0) \neq 0$ such that $f(p) = p^k g(p)$ for $p \in \mathbb{H}$. Hence (2.3) becomes into

$$e^{|\varphi(p)|^2 - |p|^2} |p^k g(p)|^2 \leq M.$$

That is to say

$$e^{|\varphi(p)|^2 - |p|^2} |p|^{2k} |g(p)|^2 \leq M.$$

Taking logarithms on both sides of the above display, it follows that

$$2 \log |p|^k + 2 \log |g(p)| + |\varphi(p)|^2 - |p|^2 \leq \log M.$$

Hence

$$|\varphi(p)|^2 - |p|^2 + 2k\log|p| + 2\log|g(p)| \le \log M.$$

On $\mathbb{C}_I$, using polar coordinates we write $p = \rho e^{I\theta}$ for $\rho = |p| > 0$, and we have that

$$|\varphi(\rho e^{I\theta})|^2 - |\rho e^{I\theta}|^2 + 2k\log|\rho e^{I\theta}| + 2\log|g(\rho e^{I\theta})| \le \log M.$$

We integrate the above inequality with respect to θ on $[-\pi, \pi]$, and obtain

$$\int_{-\pi}^{\pi} |\varphi(\rho e^{I\theta})|^2 \frac{d\theta}{2\pi} - \rho^2 + 2k\log\rho + 2\int_{-\pi}^{\pi} \log|g(\rho e^{I\theta})| \frac{d\theta}{2\pi} \le \log M.$$

Employing

$$\int_{-\pi}^{\pi} \log|g(\rho e^{I\theta})| \frac{d\theta}{2\pi} \ge \log|g(0)|,$$

it yields that

$$\int_{-\pi}^{\pi} |\varphi(\rho e^{I\theta})|^2 \frac{d\theta}{2\pi} - \rho^2 + 2k\log\rho + 2\log|g(0)| \le \log M. \tag{2.6}$$

We consider the power expansion

$$\varphi(p) = \varphi(0) + pb_1 + p^2 b_2 + \cdots,$$

for $p \in \mathbb{H}$, hence

$$\varphi(\rho e^{I\theta}) = \varphi(0) + \rho e^{I\theta} b_1 + \rho^2 e^{2I\theta} b_2 + \cdots.$$

Furthermore, it follows that

$$\int_{-\pi}^{\pi} |\varphi(\rho e^{I\theta})|^2 \frac{d\theta}{2\pi} = |\varphi(0)|^2 + \rho^2|b_1|^2 + \rho^4|b_2|^2 + \cdots,$$

and then we deduce that

$$|\varphi(0)|^2 + \rho^2(|b_1|^2 - 1) + \sum_{j=2}^{\infty} \rho^{2j}|b_j|^2 + 2k\log\rho + 2\log|g(0)| \le \log M.$$

Due to the above inequality holds for all $\rho > 0$, letting $\rho \to \infty$, we can conclude that $|b_1| \le 1$, $b_j = 0$ for all $j \ge 2$ and $k = 0$. This entails that $\varphi(p) = \varphi(0) + pb_1$ with $|b_1| \le 1$. Under this case, we denote $\varphi(p) = \varphi(0) + p\lambda$ for $|\lambda| \le 1$, $\lambda \in \mathbb{C}_I$ due to $\varphi(\mathbb{C}_I) \subset \mathbb{C}_I$.

Besides, if $|\lambda| = 1$, we have that $\varphi(p) = \varphi(0) + p\lambda$. Let $I, J \in \mathbb{S}$ be such that I and J are orthogonal. Here we denote $p|_I = z = x_0 + x_1 I$ and $\varphi(0) = \alpha_0 + \alpha_1 J$ and $\lambda = \beta_0 + \beta_1 J$ with $|\beta_0|^2 + |\beta_1|^2 = 1$, for $\alpha_l,\ \beta_l \in \mathbb{C}_I$ for $l = 0, 1$. Assume $f_I(z) = u(z) + v(z)J$ and we get

$$\begin{aligned}
\varphi_I(z) &= \varphi(0) + (x_0 + x_1 I)\lambda \\
&= \alpha_0 + \alpha_1 J + (x_0 + x_1 I)(\beta_0 + \beta_1 J) \\
&= \alpha_0 + \beta_0(x_0 + x_1 I) + (\alpha_1 + \beta_1(x_0 + x_1 I))J \\
&= F(x_0, x_1) + G(x_0, x_1)J,
\end{aligned} \tag{2.7}$$

where

$$F(x_0,x_1):=\alpha_0+\beta_0(x_0+x_1I)=\alpha_0+\beta_0 z,$$
$$G(x_0,x_1):=\alpha_1+\beta_1(x_0+x_1I)=\alpha_1+\beta_1 z.$$

Hence

$$\begin{aligned}|\varphi_I(z)|^2&=|F(x_0,x_1)|^2+|G(x_0,x_1)|^2\\&=|\alpha_0|^2+|\beta_0|^2|z|^2+2Re(\overline{\alpha_0}\beta_0 z)+|\alpha_1|^2+|\beta_1|^2|z|^2+2Re(\overline{\alpha_1}\beta_1 z)\\&=|\alpha_0|^2+|z|^2+|\alpha_1|^2+2Re(\overline{\alpha_0}\beta_0 z+\overline{\alpha_1}\beta_1 z).\end{aligned}\tag{2.8}$$

Furthermore, (2.8) entails that

$$\begin{aligned}|\varphi_I(z)|^2-|z|^2&=|\alpha_0|^2+|\alpha_1|^2+2Re(\overline{\alpha_0}\beta_0 z+\overline{\alpha_1}\beta_1 z)\\&=|\varphi_I(0)|^2+2Re(\overline{\alpha_0}\beta_0 z+\overline{\alpha_1}\beta_1 z),\end{aligned}$$

which together with (2.3) imply that

$$\begin{aligned}&e^{|\varphi_I(z)|^2-|z|^2}|f_I(z)|^2\\&\quad=e^{|\alpha_0|^2+|\alpha_1|^2+2Re(\overline{\alpha_0}\beta_0 z+\overline{\alpha_1}\beta_1 z)}|f_I(z)|^2\\&\quad\leq M\ \text{ for all } z\in\mathbb{C}_I.\end{aligned}$$

The above formulas entail that

$$\left[|u(z)|^2+|v(z)|^2\right]\left|e^{\overline{\alpha_0}\beta_0 z+\overline{\alpha_1}\beta_1 z}\right|^2\leq Me^{-(|\alpha_0|^2+|\alpha_1|^2)}=Me^{-|\varphi_I(0)|^2},\tag{2.9}$$

for all $z\in\mathbb{C}_I$. Denote $\delta:=\alpha_0\overline{\beta_0}+\alpha_1\overline{\beta_1}$. The display (2.9) can be interpreted into

$$\begin{aligned}\left|u(z)e^{\overline{\delta}z}\right|^2&\leq Me^{-|\varphi_I(0)|^2},\\\left|v(z)e^{\overline{\delta}z}\right|^2&\leq Me^{-|\varphi_I(0)|^2},\end{aligned}\tag{2.10}$$

for all $z\in\mathbb{C}_I$. Liouville's theorem implies that both $u(z)e^{\overline{\delta}z}$ and $v(z)e^{\overline{\delta}z}$ are constant functions. It's easy to check that

$$u(z)=u(0)e^{-\overline{\delta}z}\ \text{ and }\ v(z)=v(0)e^{-p\overline{\delta}z}.$$

Therefore, we obtain that $f_I(z)=u(z)+v(z)J=u(0)e^{-z\overline{\delta}}+v(0)e^{-z\overline{\delta}}J$, that is to say

$$f_I(z)=e^{-z\overline{\delta}}(u(0)+v(0)J)=e^{-z\overline{\delta}}f_I(0).$$

In this case, the left side of (2.3) is the constant

$$\begin{aligned}e^{|\varphi_I(z)|^2-|z|^2}|f_I(z)|^2&=e^{|\varphi_I(0)|^2+2Re(\overline{\delta}z)}|e^{-z\overline{\delta}}f_I(0)|^2\\&=e^{|\varphi_I(0)|^2+2Re(\overline{\delta}z)}e^{-2Re(z\overline{\delta})}|f_I(0)|^2\\&=e^{|\varphi_I(0)|^2}|f_I(0)|^2.\end{aligned}$$

Therefore, if (2.5) is true, then $|\lambda|<1$.

In the sequel, we prefer to use $\varphi(0)$ and λ to express δ. In general case, we suppose that $\varphi(0) = \alpha_0 + \alpha_1 J$ and $\lambda = \beta_0 + \beta_1 J$, and then

$$\begin{aligned}\varphi(0)\overline{\lambda} &= (\alpha_0 + \alpha_1 J)\overline{(\beta_0 + \beta_1 J)}\\ &= (\alpha_0 + \alpha_1 J)(\overline{\beta_0} + \overline{J}\,\overline{\beta_1})\\ &= \alpha_0\overline{\beta_0} + \alpha_1\overline{\beta_1} + (\alpha_1\beta_0 - \alpha_0\beta_1)J. \end{aligned} \tag{2.11}$$

Therefore using Lemma 2.1, we can deduce that

$$\delta := \alpha_0\overline{\beta_0} + \alpha_1\overline{\beta_1} = \frac{1}{2}(\varphi(0)\overline{\lambda} - I\varphi(0)\overline{\lambda}I).$$

Indeed, we note $\varphi(\mathbb{C}_I) \subset \mathbb{C}_I$, then it yields $\varphi_I(0) = \varphi(0) \in \mathbb{C}_I$. Due to $\lambda \in \mathbb{C}_I$, it turns out

$$\delta = \frac{1}{2}(\varphi(0)\overline{\lambda} - I^2\varphi(0)\overline{\lambda}) = \varphi(0)\overline{\lambda},$$

which completes the proof. □

Let f and φ be two slice regular functions on $\mathbb{H}$, such that f is not identically zero and $\varphi(\mathbb{C}_I) \subset \mathbb{C}_I$ for some $I \in \mathbb{S}$. Then we can obtain the adjoint of weighted composition operator denoted by $W^*_{f,\varphi}$. For the kernel $K_p \in \mathcal{F}^2(\mathbb{H})$, it yields that

$$\langle W^*_{f,\varphi}K_p, K_q\rangle = \langle K_p, W_{f,\varphi}K_q\rangle = \overline{\langle W_{f,\varphi}K_q, K_p\rangle}.$$

Furthermore, it turns out

$$W^*_{f,\varphi}K_p(q) = \overline{W_{f,\varphi}K_q(p)}, \tag{2.12}$$

for $p, q \in \mathbb{H}$. In the sequel, we always choose $J \in \mathbb{S}$ with $I \perp J$ and denote

$$f_I(z) = F(z) + G(z)J \tag{2.13}$$

with two holomorphic functions $F, G : \mathbb{C}_I \to \mathbb{C}_I$.

Based on Proposition 2.2 and the display (2.12), we present a characterization for bounded weighted composition operator on $\mathcal{F}^2(\mathbb{H})$.

Theorem 2.3. *Let f and φ be two slice regular functions on $\mathbb{H}$, such that f is not identically zero and $\varphi(\mathbb{C}_I) \subset \mathbb{C}_I$ for some $I \in \mathbb{S}$. If the operator $W_{f,\varphi}$: $\mathcal{F}^2(\mathbb{H}) \to \mathcal{F}^2(\mathbb{H})$ is bounded, then $f \in \mathcal{F}^2(\mathbb{H})$, $\varphi(p) = \varphi(0) + p\lambda$ with $|\lambda| \le 1$, $\lambda \in \mathbb{C}_I$ and*

$$M(f,\varphi) := \sup_{p\in\mathbb{H}} e^{|\varphi(p)|^2 - |p|^2}|f(p)|^2 < +\infty. \tag{2.14}$$

Conversely, under the additional assumption $\lambda^{-1}\varphi(0) \in \mathbb{R}$ for $\lambda \neq 0$, the above necessary conditions can ensure the boundedness of $W_{f,\varphi}$ on $\mathcal{F}^2(\mathbb{H})$.

Proof. Firstly, suppose $W_{f,\varphi}$ is bounded on $\mathcal{F}^2(\mathbb{H})$, then we can deduce that $f = 1 \star f = W_{f,\varphi}1 \in \mathcal{F}^2(\mathbb{H})$. On the other hand, it follows that

$$\|W_{f,\varphi}\|^2 = \|W^*_{f,\varphi}\|^2 \ge \frac{\|W^*_{f,\varphi}K_p\|^2}{\|K_p\|^2}. \tag{2.15}$$

We use the Eq. (2.12) to compute the term $\|W^*_{f,\varphi}K_p\|^2$ as below,

$$\begin{aligned}
&\|W^*_{f,\varphi}K_p\|^2 \\
&\quad= \|W^*_{f,\varphi}K_p(q)\|^2 = \|\overline{W_{f,\varphi}K_q(p)}\|^2 \\
&\quad= \|W_{f,\varphi}K_q(p)\|^2 = \|[K_q\circ\varphi(p)]\star f(p)\|^2 \\
&\quad= \int_{\mathbb{C}_I}\left|\left[\sum_{n=0}^{\infty}\frac{(\varphi(p))^{\star n}\overline{z}^n}{n!}\right]\star f(p)\right|^2 e^{-|z|^2}d\sigma(x,y) \\
&\quad\geq \int_{\mathbb{C}_I}\left|\left[\sum_{n=0}^{\infty}\frac{(\varphi_I(w))^{\star n}\overline{z}^n}{n!}\right]\star f_I(w)\right|^2 e^{-|z|^2}d\sigma(x,y), \qquad (2.16)
\end{aligned}$$

where $q|_{\mathbb{C}_I} = z = x+yI \in \mathbb{C}_I$ and $p|_{\mathbb{C}_I} = w = u+vI \in \mathbb{C}_I$. Since $\varphi:\ \mathbb{C}_I \to \mathbb{C}_I$, it follows that

$$\varphi(p)^{\star n} = ext(\varphi_I(w)^{\star n}) = ext(\varphi_I(w)^n).$$

Hence, on $\mathbb{C}_I$, by the operation rule (1.3) it turns out

$$\begin{aligned}
&\left[\sum_{n=0}^{\infty}\frac{(\varphi_I(w))^{\star n}\overline{z}^n}{n!}\right]\star f_I(w) \\
&\quad= \left[\sum_{n=0}^{\infty}\frac{(\varphi_I(w))^{n}\overline{z}^n}{n!}\right]\star (F(w)+G(w)J) \\
&\quad= \left[\sum_{n=0}^{\infty}\frac{(\varphi_I(w))^{n}\overline{z}^n}{n!}\right]\cdot F(w) + \left[\sum_{n=0}^{\infty}\frac{(\varphi_I(w))^{n}\overline{z}^n}{n!}\right]\cdot G(w)J,
\end{aligned}$$

which ensure

$$\begin{aligned}
&\left|\left[\sum_{n=0}^{\infty}\frac{(\varphi_I(w))^{\star n}\overline{z}^n}{n!}\right]\star f_I(w)\right|^2 \\
&\quad= \left|\sum_{n=0}^{\infty}\frac{(\varphi_I(w))^{n}\overline{z}^n}{n!}\right|^2\cdot|F(w)|^2 + \left|\sum_{n=0}^{\infty}\frac{(\varphi_I(w))^{n}\overline{z}^n}{n!}\right|^2\cdot|G(w)|^2 \\
&\quad= \left|\sum_{n=0}^{\infty}\frac{(\varphi_I(w))^{n}\overline{z}^n}{n!}\right|^2\left(|F(w)|^2+|G(w)|^2\right) \\
&\quad= \left|\sum_{n=0}^{\infty}\frac{(\varphi_I(w))^{n}\overline{z}^n}{n!}\right|^2 |f_I(w)|^2.
\end{aligned}$$

Therefore, the display (2.16) becomes into

$$\begin{aligned}\|W_{f,\varphi}^{*}K_p\|^2 &\geq \int_{\mathbb{C}_I}\left|\sum_{n=0}^{\infty}\frac{(\varphi_I(w))^n\overline{z}^n}{n!}\right|^2|f_I(w)|^2e^{-|z|^2}d\sigma(x,y)\\ &=\sum_{n=0}^{\infty}\frac{|\varphi_I(w)|^{2n}n!}{(n!)^2}|f_I(w)|^2\\ &=e^{|\varphi_I(w)|^2}|f_I(w)|^2. \end{aligned}\tag{2.17}$$

Employing the norm in (1.6), it yields that

$$\begin{aligned}\infty > \frac{\|W_{f,\varphi}^{*}K_p\|^2}{\|K_p\|^2} &= \frac{e^{|\varphi_I(w)|^2}|f_I(w)|^2}{e^{|w|^2}}\\ &= e^{|\varphi_I(w)|^2-|w|^2}|f_I(w)|^2.\end{aligned}$$

Therefore we can obtain

$$\sup_{w\in\mathbb{C}_I} e^{|\varphi_I(w)|^2-|w|^2}|f_I(w)|^2<\infty. \tag{2.18}$$

Due to the definition of Fock space does not depend on the imaginary unit $I\in\mathbb{S}$, then the inequality (2.18) entails

$$\sup_{p\in\mathbb{H}} e^{|\varphi(p)|^2-|p|^2}|f(p)|^2<+\infty. \tag{2.19}$$

(2.19) together with Proposition 2.2 can imply that $\varphi(p)=\varphi(0)+p\lambda$ for $|\lambda|\leq 1$ with $\lambda,\ \varphi(0)\ \in\mathbb{C}_I$.

Conversely, we will show $W_{f,\varphi}:\mathcal{F}^2(\mathbb{H})\to\mathcal{F}^2(\mathbb{H})$ is bounded under the assumption $\lambda^{-1}\varphi(0)\in\mathbb{R}$ with $\lambda\neq 0$. For any $h\in\mathbb{H}$, it follows that

$$\begin{aligned}\|W_{f,\varphi}h\|^2 &= \int_{\mathbb{C}_I}|[(h\circ\varphi)\star f]_I(p)|^2e^{-|p|^2}d\sigma(x,y)\\ &=\int_{\mathbb{C}_I}|(h\circ\varphi)_I(z)\star f_I(z)|^2e^{-|z|^2}d\sigma(x,y).\end{aligned}\tag{2.20}$$

Here we denote $(h\circ\varphi)_I(z)=F(z)+G(z)J$ and $f_I(z)=H(z)+K(z)J$, and then the display (1.3) implies that

$$\begin{aligned}&(h\circ\varphi)_I(z)\star f_I(z)\\ &\quad=(F(z)H(z)-G(z)\overline{K(\overline{z})})+(F(z)K(z)+G(z)\overline{H(\overline{z})})J.\end{aligned}$$

Thus

$$\begin{aligned}&|(h\circ\varphi)_I(z)\star f_I(z)|^2\\ &\quad=|F(z)H(z)-G(z)\overline{K(\overline{z})}|^2+|F(z)K(z)+G(z)\overline{H(\overline{z})}|^2\\ &\quad\leq 2(|F(z)H(z)|^2+|G(z)\overline{K(\overline{z})}|^2+|F(z)K(z)|^2+|G(z)\overline{H(\overline{z})}|^2)\\ &\quad=2(|F(z)H(z)|^2+|G(z)K(\overline{z})|^2+|F(z)K(z)|^2+|G(z)H(\overline{z})|^2).\end{aligned}\tag{2.21}$$

Putting (2.21) into (2.20), we obtain

$$\begin{aligned}
\|W_{f,\varphi}h\|^2 &\le 2\int_{\mathbb{C}_I}(|F(z)H(z)|^2+|F(z)K(z)|^2)e^{-|z|^2}d\sigma(x,y)\\
&\quad+2\int_{\mathbb{C}_I}(|G(z)K(\overline{z})|^2+|G(z)H(\overline{z})|^2)e^{-|z|^2}d\sigma(x,y)\\
&=2\int_{\mathbb{C}_I}|F(z)|^2(|H(z)|^2+|K(z)|^2)e^{-|z|^2}d\sigma(x,y)\\
&\quad+2\int_{\mathbb{C}_I}(|G(\overline{z})K(z)|^2+|G(\overline{z})H(z)|^2)e^{-|\overline{z}|^2}d\sigma(x,y)\\
&=2\int_{\mathbb{C}_I}|F(z)|^2|f_I(z)|^2e^{-|z|^2}d\sigma(x,y)\\
&\quad+2\int_{\mathbb{C}_I}|G(\overline{z})|^2|f_I(z)|^2e^{-|z|^2}d\sigma(x,y) \qquad (2.22)\\
&\le 2\left[\sup_{z\in\mathbb{C}_I}e^{|\varphi_I(z)|^2-|z|^2}|f_I(z)|^2\right]\int_{\mathbb{C}_I}|(h\circ\varphi)_I(z)|^2e^{-|\varphi_I(z)|^2}d\sigma(x,y)\\
&\quad+2\left[\sup_{z\in\mathbb{C}_I}e^{|\varphi_I(z)|^2-|z|^2}|f_I(z)|^2\right]\int_{\mathbb{C}_I}|(h\circ\varphi)_I(\overline{z})|^2e^{-|\varphi_I(z)|^2}d\sigma(x,y)\\
&\le 2M(f,\varphi)\int_{\mathbb{C}_I}|(h\circ\varphi)_I(z)|^2e^{-|\varphi_I(z)|^2}d\sigma(x,y)\\
&\quad+2M(f,\varphi)\int_{\mathbb{C}_I}|(h\circ\varphi)_I(z)|^2e^{-|\varphi_I(\overline{z})|^2}d\sigma(x,y)\\
&=4M(f,\varphi)\int_{\mathbb{C}_I}|h_I\circ\varphi_I(z)|^2e^{-|\varphi_I(z)|^2}d\sigma(x,y), \qquad (2.23)\\
&=4M(f,\varphi)|\lambda|^{-2}\|h\|^2<+\infty.
\end{aligned}$$

In the above chain of inequalities, the changes of variables $z\to\overline{z}$ and $w=\varphi_I(z)=\varphi(0)+\lambda z$ were used in (2.22) and the last line, respectively. Here the display (2.23) is due to

$$|\varphi_I(\overline{z})|=|\lambda\overline{z}+\varphi(0)|=|\lambda||\overline{z}+\lambda^{-1}\varphi(0)|=|\lambda||z+\lambda^{-1}\varphi(0)|=|\varphi_I(z)| \qquad (2.24)$$

for $\lambda^{-1}\varphi(0)\in\mathbb{R}$. Therefore, the operator $W_{f,\varphi}$ is bounded on $\mathcal{F}^2(\mathbb{H})$ for $\lambda\neq 0$.

For the case $\lambda=0$, it holds that $\varphi(z)=\varphi(0)$. And then

$$W_{f,\varphi}h=(h\circ\varphi(0))\star f\in\mathcal{F}^2(\mathbb{H})$$

due to $f\in\mathcal{F}^2(\mathbb{H})$. All in all, the operator $W_{f,\varphi}:\ \mathcal{F}^2(\mathbb{H})\to\mathcal{F}^2(\mathbb{H})$ is bounded under the additional assumption $\lambda^{-1}\varphi(0)\in\mathbb{R}$ with $\lambda\neq 0$. This ends the proof. $\square$

In the sequel, we describe the conditions for compactness of weighted composition operator $W_{f,\varphi}$ on $\mathcal{F}^2(\mathbb{H})$.

Theorem 2.4. *Let f and φ be two slice regular functions on $\mathbb{H}$, such that f is not identically zero and $\varphi(\mathbb{C}_I) \subset \mathbb{C}_I$ for some $I \in \mathbb{S}$. If the operator $W_{f,\varphi}$ is compact on $\mathcal{F}^2(\mathbb{H})$, then $\varphi(p) = \varphi(0) + p\lambda$ with $|\lambda| < 1$, $\lambda \in \mathbb{C}_I$ and*

$$\lim_{|p|\to\infty} e^{|\varphi(p)|^2 - |p|^2} |f(p)|^2 = 0. \tag{2.25}$$

Conversely, under the additional assumption $\lambda^{-1}\varphi(0) \in \mathbb{R}$ for $\lambda \neq 0$, the necessary conditions can ensure the compactness of $W_{f,\varphi}$ on $\mathcal{F}^2(\mathbb{H})$.

Proof. Firstly, suppose the operator $W_{f,\varphi}$ is compact on $\mathcal{F}^2(\mathbb{H})$, it must be bounded and then Theorem 2.3 implies that the display (2.14) holds and

$$\varphi(p) = \varphi(0) + p\lambda$$

with $|\lambda| \leq 1$, $\lambda \in \mathbb{C}_I$. For the case $|\lambda| = 1$, Proposition 2.2 ensures

$$f_I(w) = e^{-w\lambda\overline{\varphi(0)}} f_I(0).$$

Furthermore, we deduce

$$\begin{aligned} & e^{|\varphi_I(w)|^2 - |w|^2} |f_I(w)|^2 \\ & = e^{|\varphi(0) + w\lambda|^2 - |w|^2} |e^{-w\lambda\overline{\varphi(0)}}|^2 |f_I(0)|^2 \\ & = |f_I(0)|^2 e^{|\varphi(0)|^2}, \end{aligned}$$

which contradicts with (2.25).

Thus $|\lambda| < 1$ holds, and then we go on to show the display (2.25). If $W_{f,\varphi}$ is compact on $\mathcal{F}^2(\mathbb{H})$, then the adjoint operator $W^*_{f,\varphi}$ is also compact on $\mathcal{F}^2(\mathbb{H})$. Since

$$k_p = \|K_p\|^{-1} K_p \to 0$$

as $|p| \to \infty$, we have

$$\|K_p\|^{-2} \|W^*_{f,\varphi} K_p\|^2 = e^{-|p|^2} \|W^*_{f,\varphi} K_p\|^2 \to 0$$

as $|p| \to \infty$. Employing the computations in (2.16) and (2.17), it yields that

$$|f_I(w)|^2 e^{|\varphi_I(w)|^2 - |w|^2} \to 0,$$

as $|w| \to \infty$. Further, employing the definition of Fock space not depending on the imaginary unit $I \in \mathbb{S}$, (2.25) is true.

Conversely. Assume that $\varphi(p) = \varphi(0) + p\lambda$ with $|\lambda| < 1$, $\lambda \in \mathbb{C}_I$ and (2.25) holds. For the case $\lambda = 0$, this implication is obvious. In fact,

$$W_{f,\varphi} h = h(\varphi(0)) \star f,$$

which implies $W_{f,\varphi}$ has finite rank, thus it is compact.

Now suppose that $\lambda \neq 0$, and $\lambda^{-1}\varphi(0) \in \mathbb{R}$, we proceed to prove the weighted composition operator is compact on $\mathcal{F}^2(\mathbb{H})$. Let $\{h_m\}_{m=1}^{\infty}$ be a bounded sequence in $\mathcal{F}^2(\mathbb{H})$ with $C := \sup_{m\in\mathbb{N}} \|h_m\| < +\infty$ and converge weakly to 0 as $m \to \infty$. Then we know the sequence $\{h_m\}$ converges to zero uniformly on compact subsets of $\mathbb{H}$. In the sequel, we show that

$$\|W_{f,\varphi}h_m\|^2 \to 0$$

as $m \to \infty$. We denote $(h_m \circ \varphi)_I(z) = F_m(z) + G_m(z)J$ and $f_I(z) = H(z) + K(z)J$. By the similar calculations in (2.22), we have

$$\begin{aligned}
&\|W_{f,\varphi}h_m\|^2 \\
&= \int_{\mathbb{C}_I} |(h_m \circ \varphi)_I(z) \star f_I(z)|^2 e^{-|z|^2} d\sigma(x,y) \\
&\le 2\int_{\mathbb{C}_I} |F_m(z)|^2 |f_I(z)|^2 e^{-|z|^2} d\sigma(x,y) \\
&\quad + 2\int_{\mathbb{C}_I} |G_m(\overline{z})|^2 |f_I(z)|^2 e^{-|z|^2} d\sigma(x,y).
\end{aligned} \tag{2.26}$$

Since (2.25) holds, for any $\epsilon > 0$, there exists $R > 0$ such that

$$\sup_{|p|>R} e^{|\varphi(p)|^2 - |p|^2} |f(p)|^2 \le \epsilon. \tag{2.27}$$

The first part of (2.26) tends to zero as below,

$$\begin{aligned}
&\int_{\mathbb{C}_I} |F_m(z)|^2 |f_I(z)|^2 e^{-|z|^2} d\sigma(x,y) \\
&= \int_{|z|>R} |F_m(z)|^2 |f_I(z)|^2 e^{-|z|^2} d\sigma(x,y) + \int_{|z|\le R} |F_m(z)|^2 |f_I(z)|^2 e^{-|z|^2} d\sigma(x,y) \\
&\le \sup_{|z|>R} |f_I(z)|^2 e^{|\varphi_I(z)|^2 - |z|^2} \int_{|z|>R} |F_m(z)|^2 e^{-|\varphi_I(z)|^2} d\sigma(x,y) \\
&\quad + M(f,\varphi) \int_{|z|\le R} |F_m(z)|^2 e^{-|\varphi_I(z)|^2} d\sigma(x,y) \\
&\le \epsilon \int_{|z|>R} |(h_m)_I \circ \varphi_I(z)|^2 e^{-|\varphi_I(z)|^2} d\sigma(x,y) \\
&\quad + M(f,\varphi) \int_{|z|\le R} |(h_m)_I \circ \varphi_I(z)|^2 e^{-|\varphi_I(z)|^2} d\sigma(x,y) \\
&\le C^2 \epsilon, \text{ as } m \to \infty,
\end{aligned}$$

where we use (2.27) in the fifth line and employing $h_m \to 0$ on compact subsets of $\mathbb{H}$ in the last line. Furthermore, ϵ is arbitrary, then the first part tends to zero as $m \to \infty$.

Similar to the calculations for the first part in (2.26), the second part turns into

$$\begin{aligned}
&\int_{\mathbb{C}_I} |G_m(\overline{z})|^2 |f_I(z)|^2 e^{-|z|^2} d\sigma(x,y) \\
&= \int_{|z|>R} |G_m(\overline{z})|^2 |f_I(z)|^2 e^{-|z|^2} d\sigma(x,y)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{|z|\le R}|G_m(\overline{z})|^2|f_I(z)|^2e^{-|z|^2}d\sigma(x,y)\\
&\le \sup_{|z|>R}|f_I(z)|^2e^{|\varphi_I(z)|^2-|z|^2}\int_{|z|>R}|G_m(\overline{z})|^2e^{-|\varphi_I(z)|^2}d\sigma(x,y)\\
&\quad+M(f,\varphi)\int_{|z|\le R}|G_m(\overline{z})|^2e^{-|\varphi_I(z)|^2}d\sigma(x,y)\\
&= \sup_{|z|>R}|f_I(z)|^2e^{|\varphi_I(z)|^2-|z|^2}\int_{|z|>R}|G_m(z)|^2e^{-|\varphi_I(z)|^2}d\sigma(x,y)\\
&\quad+M(f,\varphi)\int_{|z|\le R}|G_m(z)|^2e^{-|\varphi_I(z)|^2}d\sigma(x,y)\\
&\le \epsilon\int_{|z|>R}|(h_m)_I\circ\varphi_I(z)|^2e^{-|\varphi_I(z)|^2}d\sigma(x,y)\\
&\quad+M(f,\varphi)\int_{|z|\le R}|(h_m)_I\circ\varphi_I(z)|^2e^{-|\varphi_I(z)|^2}d\sigma(x,y)\\
&\le C^2\epsilon,\ \text{as } m\to\infty.
\end{aligned}
$$

where we use the display (2.24) in seventh line. Putting the above two results into (2.26), we conclude that $\|W_{f,\varphi}h_m\|\to 0$ as $m\to\infty$. That is to say $W_{f,\varphi}$ is compact on $\mathcal{F}^2(\mathbb{H})$, which completes the proof. □

Question How can we delete *the assumption* $\lambda^{-1}\varphi(0)\in\mathbb{R}$ *for* $\lambda\neq 0$ in the proof for bounded or compact weighted composition operator on $\mathcal{F}^2(\mathbb{H})$?

3. Normal Weighted Composition Operators

Recall an operator $T:\mathcal{F}^2(\mathbb{H})\to\mathcal{F}^2(\mathbb{H})$ is normal if and only if $T^*T=TT^*$, where $T^*:\mathcal{F}^2(\mathbb{H})\to\mathcal{F}^2(\mathbb{H})$ is the adjoint operator of T. In this section, all normal weighted composition operators on $\mathcal{F}^2(\mathbb{H})$ will be described. First of all, we introduce a special weighted composition operator $W_{k_{\overline{\lambda}b},\varphi}$, which plays a vital role in following characterizations. Considering a special weight $k_{\overline{\lambda}b}$ with $\lambda,b\in\mathbb{C}_I$ and an entire regular composition symbol φ satisfying $\varphi(\mathbb{C}_I)\subset\mathbb{C}_I$, the weighted composition operator $W_{k_{\overline{\lambda}b},\varphi}$ on $\mathbb{C}_I$ is defined as

$$(W_{k_{\overline{\lambda}b},\varphi}h)_I(z)=h_I(\varphi_I(z))\star k_{\overline{\lambda}b}(z).$$

Similarly the extension $W_{k_{\overline{\lambda}b},\varphi}$ can also be deduced from the representation formula. By an easy modification for [12, Proposition 4.3], we obtain the left linear weighted composition operator $W_{k_{\overline{\lambda}b},\varphi}$ is unitary in $\mathcal{F}^2(\mathbb{H})$ under some special symbols.

Proposition 3.1. *Let* $\varphi(p)=p\lambda-b$ *with* $|\lambda|=1$ *and* $\lambda,\ b\in\mathbb{C}_I$ *such that* $\varphi(\mathbb{C}_I)\subset\mathbb{C}_I$ *for some* $I\in\mathbb{S}$. *Then* $W_{k_{\overline{\lambda}b},\varphi}$ *is a unitary operator in* $\mathcal{F}^2(\mathbb{H})$ *and it satisfies*

$$W^{-1}_{k_{\overline{\lambda}b},\varphi}=W^{*}_{k_{\overline{\lambda}b},\varphi}=W_{k_{-b},\varphi^{-1}}.$$

Remark 3.2. Putting $\lambda = 1$ in Proposition 3.1, it follows $\varphi_1(p) = p - b$ with $b \in \mathbb{C}_I$, and denote

$$W_b = W_{k_b,\varphi_1}. \tag{3.1}$$

The commutation relation is true for $u, v \in \mathbb{C}_I$,

$$W_u W_v = e^{iIm(u\overline{v})} W_{u+v},$$

which further implies $W_u^{-1} = W_{-u}$.

We are now in a position to describe all normal weighted composition operators on slice regular quaternionic Fock space $\mathcal{F}^2(\mathbb{H})$.

Theorem 3.3. *Let f and φ be two slice regular functions on $\mathbb{H}$, such that $f_I(z) = F(z) + G(z)J$ is not identically zero and $\varphi(\mathbb{C}_I) \subset \mathbb{C}_I$ for some $I \in \mathbb{S}$. Then $W_{f,\varphi}$ is a normal bounded operator on $\mathcal{F}^2(\mathbb{H})$ if and only if one of the following two cases occurs:* (i) *$\varphi(p) = p\lambda + b$ with $|\lambda| = 1$ and $f(p) = K_{-\overline{\lambda}b}(p)f(0)$. In this case $W_{f,\varphi}$ is the unitary operator $W_{k_{-\overline{\lambda}b},\varphi}$ multiply a constant $e^{|b|^2/2}f(0)$ on the right.*

(ii) *$\varphi(p) = p\lambda + b$ with $|\lambda| < 1$ and $f(p) = K_{a(1-\overline{\lambda})}(p)f(0)$ with $a = b(1-\lambda)^{-1} \in \mathbb{C}_I$. In this case, $W_{f,\varphi}$ is unitarily equivalent to a special weighted composition operator $W_{F(a),p\lambda}$.*

Proof. Necessity. Suppose $W_{f,\varphi}$ is a normal bounded operator on $\mathcal{F}^2(\mathbb{H})$. Theorem 2.3 ensures $\varphi(p) = p\lambda + b$ with $\lambda, b \in \mathbb{C}_I$, for $|\lambda| \leq 1$, and

$$e^{|\varphi_I(z)|^2 - |z|^2} |f_I(z)|^2 < +\infty$$

for all $z \in \mathbb{C}_I$. We divide the following discussion into two cases:

Case 1. For the case $|\lambda| = 1$. The display (2.4) of Proposition 2.2 asserts that

$$f(p) = e^{-p\lambda\overline{b}} f(0) = K_{-\overline{\lambda}b}(p) f(0) = k_{-\overline{\lambda}b}(p) e^{|b|^2/2} f(0) = k_{-\overline{\lambda}b}(p)c,$$

where $c = e^{|b|^2/2} f(0) \in \mathbb{H}$. That is to say

$$(W_{f,\varphi} h_I)(z) = h_I(\varphi_I(z)) \star f_I(z) = h_I(\varphi_I(z)) \star k_{-\overline{\lambda}b}(z)c = (W_{k_{-\overline{\lambda}b},\varphi} h_I)(z)c.$$

By the representation formula, it turns out $W_{f,\varphi} = W_{k_{-\overline{\lambda}b},\varphi} \cdot c$, which together with Proposition 3.1 imply $W_{f,\varphi}$ is the unitary operator $W_{k_{-\overline{\lambda}b},\varphi}$ multiply a constant on the right.

Case 2. For the case $|\lambda| < 1$. The map $\varphi(p) = p\lambda + b$ has a unique fixed point $a = b(1-\lambda)^{-1} \in \mathbb{C}_I$. By the formula (2.12), we can deduce

$$\begin{aligned}(W^*_{f,\varphi} K_a)_I(z) &= \overline{W_{f_I,\varphi_I} K_z(a)} \\ &= \overline{K_z(\varphi_I(a)) \star f_I(a)} \\ &= \overline{K_z(a)(F(a) + G(a)J)}\end{aligned}$$

$$= \overline{K_z(a)F(a)} + \overline{J}\ \overline{G(a)}\ \overline{K_z(a)}$$
$$= (\overline{F(a)} - G(a)J)K_a(z). \tag{3.2}$$

The normality entails $W_{f,\varphi}W^*_{f,\varphi} = W^*_{f,\varphi}W_{f,\varphi}$. Let the operator $W_{f,\varphi}$ act on both sides of (3.2), then it turns out that

$$\begin{aligned}(W^*_{f,\varphi}W_{f,\varphi}K_a)_I(z) &= (W_{f,\varphi}W^*_{f,\varphi}K_a)_I(z)\\ &= [W_{f,\varphi}(\overline{F(a)} - G(a)J)K_a]_I(z)\\ &= (\overline{F(a)} - G(a)J)K_a(\varphi(z)) \star f_I(z)\\ &= (\overline{F(a)} - G(a)J)(W_{f,\varphi}K_a)_I(z).\end{aligned}$$

Comparing with (3.2), we conclude that $(W_{f,\varphi}K_a)_I(z) = \beta K_a(z)$ with $\beta \in \mathbb{H}$. Taking $z = a$ in the above formula, we obtain that

$$K_a(\varphi(a)) \star f_I(a) = K_a(a)f_I(a) = \beta K_a(a),$$

which implies $\beta = f_I(a) = F(a) + G(a)J$. Hence

$$(W_{f,\varphi}K_a)_I(z) = f_I(a)K_a(z) = (F(a) + G(a)J)K_a(z).$$

On the other hand,

$$(W_{f,\varphi}K_a)_I(z) = K_a(\varphi(z))F(z) + K_a(\varphi(z))G(z)J.$$

Combining the above two formulas, it yields that

$$\begin{cases} K_a(\varphi(z))F(z) = F(a)K_a(z),\\ K_a(\varphi(z))G(z) = G(a)\overline{K_a(z)}.\end{cases}$$

Since the analytic functions $K_a(\varphi(z))$ and $K_a(z)$ are both not zero, the second equation implies $G(a) = 0$ and then $G(z) = 0$. Furthermore, the first equation ensures

$$F(z) = F(a)K_a(z - \varphi(z)) = F(a)K_a(z(1-\lambda) - b).$$

That is to say

$$f_I(z) = F(z) = F(a)\exp(-\overline{a}b)K_{a(1-\overline{\lambda})}(z),$$

which is a holomorphic map on $\mathbb{C}_I$. And then

$$f(p) = \exp(-\overline{a}b)K_{a(1-\overline{\lambda})}(p)F(a).$$

Particularly, letting $z = 0$ in the above equation, it yields that $f(0) = \exp(-\overline{a}b)F(a) \in \mathbb{C}_I$. That entails $f(p) = K_{a(1-\overline{\lambda})}(p)f(0)$.

In view of Proposition 3.1 and Remark 3.2, we obtain

$$W_a^*W_{f,\varphi}W_a = W_{-a}W_{f,\varphi}W_a,$$

which is denoted by $W_{g,\psi}$ with

$$\begin{aligned}g_I(z) &= k_a(\varphi(z+a)) \star [F(a)K_a((z+a) - \varphi(z+a))] \star k_{-a}(z)\\ &= k_a(\varphi(z+a))[F(a)K_a((z+a) - \varphi(z+a))]k_{-a}(z)\end{aligned}$$

$$= e^{((z+a)\lambda+b)\overline{a}-|a|^2/2} \cdot \left[F(a)e^{(z+a-(z+a)\lambda-b)\overline{a}}\right] \cdot e^{-z\overline{a}-|a|^2/2}$$
$$= F(a),$$

and

$$\psi_I(z) = \varphi_I(z+a) - a = (z+a)\lambda - a = \lambda z.$$

The above formulas ensure

$$(W_a^* W_{f,\varphi} W_a h)_I(z) = (W_{g,\psi} h)_I(z) = h_I(\lambda z) \star F(a) = h_I(\lambda z)F(a)$$

for any $h \in \mathcal{F}^2(\mathbb{H})$. That means $W_a^* W_{f,\varphi} W_a = W_{F(a),p\lambda}$. In other words, $W_{f,\varphi}$ is unitarily equivalent to the special weighted composition operator $W_{F(a),p\lambda}$ on $\mathbb{C}_I$.

Sufficiency. If the **Case 1** holds, $W_{f,\varphi} = W_{k_{-\overline{\lambda}b},\varphi} \cdot c$. Proposition 3.1 entails

$$W_{f,\varphi}^* W_{f,\varphi} = \overline{c} W_{k_{-\overline{\lambda}b},\varphi}^* W_{k_{-\overline{\lambda}b},\varphi} \cdot c = |c|^2 = W_{f,\varphi} W_{f,\varphi}^*,$$

which clearly means $W_{f,\varphi}$ is normal.

If the **Case 2** holds, then $W_{f,\varphi}$ is unitarily equivalent to $W_{F(a),p\lambda}$, which is diagonalizable with respect to the standard orthonormal basis $\{p^m/\sqrt{m!}\}_{m=0}^{\infty}$ in $\mathcal{F}^2(\mathbb{H})$. Hence $W_{f,\varphi}$ is a normal operator. This ends the proof. □

Subsequently, we will present the equivalent characterizations for a self-adjoint operator $W_{f,\varphi}$ on $\mathcal{F}^2(\mathbb{H})$, which is a special case of Theorem 3.3. Given a bounded linear operator T acting from a Hilbert space $\mathcal{H}$ to itself, we say that T is self-adjoint on $\mathcal{H}$ if $T^* = T$. Equivalently, a bounded operator T on $\mathcal{F}^2(\mathbb{H})$ is self-adjoint if $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x,\ y \in \mathcal{F}^2(\mathbb{H})$. It's evident that if $W_{f,\varphi}$ is self-adjoint on $\mathcal{F}^2(\mathbb{H})$, then it is normal on $\mathcal{F}^2(\mathbb{H})$. We generally investigate the necessary conditions for the adjoint of weighted composition operator is another weighted composition operator in Theorem 3.4. And then we can deduce a corollary for a self-adjoint weighted composition operator on $\mathcal{F}^2(\mathbb{H})$, which is an extension of [17, Corollary 2.9].

Theorem 3.4. *Let f, g and φ, ψ be four slice regular functions on $\mathbb{H}$. Denote $f_I(z) = F(z) + G(z)J$ and $g_I(z) = H(z) + K(z)J$ nonzero functions and suppose $\varphi(\mathbb{C}_I) \subset \mathbb{C}_I$, $\psi(\mathbb{C}_I) \subset \mathbb{C}_I$ for some $I \in \mathbb{S}$. If a bounded weighted composition operator $W_{f,\varphi}$ on $\mathcal{F}^2(\mathbb{H})$ satisfies $W_{f,\varphi}^* = W_{g,\psi}$, then one of the following two cases occurs:* (i) *$f(p) = K_{\psi_I(0)}(p)F(0)$ and $g(p) = K_{\varphi_I(0)}(p)\overline{F(0)}$, $\varphi(p) = \varphi(0) + p\overline{\lambda}$ and $\psi(p) = \psi(0) + p\lambda$, with $\lambda \in \mathbb{C}_I$ and $|\lambda| \le 1$.*

(ii) *$f(p) = f(0)$, $g(p) = g(0)$ with $f_I(0) = F(0) + G(0)J$ and $g_I(0) = \overline{F(0)} - G(0)J$; $\varphi(p) = \psi(p) = 0$.*

Proof. Since $W_{f,\varphi}^* = W_{g,\psi}$, it follows that

$$(W_{f,\varphi}^* K_w)_I(z) = (W_{g,\psi} K_w)_I(z)$$

for all $z, w \in \mathbb{C}_I$. On the one hand,

$$\begin{aligned}(W^*_{f,\varphi}K_w)_I(z) &= \overline{W^*_{f_I,\varphi_I}K_z(w)}\\ &= \overline{K_z(\varphi_I(w)) \star f_I(w)} = \overline{K_z(\varphi_I(w)) \star (F(w)+G(w)J)}\\ &= \overline{K_z(\varphi_I(w))F(w) + K_z(\varphi_I(w))G(w)J}\\ &= K_{\varphi_I(w)}(z)\overline{F(w)} - G(w)K_z(\varphi_I(w))J.\end{aligned}$$

On the other hand,

$$(W_{g,\psi}K_w)_I(z) = K_w(\psi_I(z))H(z) + K_w(\psi_I(z))K(z)J.$$

Combining the above formulas, we obtain that

$$\begin{cases} K_{\varphi_I(w)}(z)\overline{F(w)} = K_w(\psi_I(z))H(z),\\ -G(w)K_z(\varphi_I(w)) = K_w(\psi_I(z))K(z).\end{cases} \tag{3.3}$$

Firstly, let $z = w = 0$ in (3.3), it yields that

$$H(0) = \overline{F(0)} \text{ and } K(0) = -G(0). \tag{3.4}$$

Secondly, let $z = 0$ in (3.3), we deduce

$$\begin{cases} K_{\varphi_I(w)}(0)\overline{F(w)} = K_w(\psi_I(0))H(0),\\ -G(w)K_0(\varphi_I(w)) = K_w(\psi_I(0))K(0),\end{cases}$$

which together with (3.4) entail that

$$\begin{cases} F(w) = \overline{H(0)}K_{\psi_I(0)}(w) = F(0)K_{\psi_I(0)}(w),\\ G(w) = -K(0)K_w(\psi_I(0)) = G(0)K_w(\psi_I(0)).\end{cases} \tag{3.5}$$

Similarly, also letting $w = 0$ in (3.3), it follows that

$$\begin{cases} H(z) = \overline{F(0)}K_{\varphi_I(0)}(z) = H(0)K_{\varphi_I(0)}(z),\\ K(z) = -G(0)K_z(\varphi_I(0)) = K(0)K_z(\varphi_I(0)).\end{cases} \tag{3.6}$$

Observing the second formula in (3.5), it holds if and only if the following two cases occur, $G(0) = 0$ or $G(0) \neq 0$ and $\psi_I(0) = 0$. Specifically, we divide into Case 1 and Case 2.

Case 1. $G(0) = 0$. Since f is nonzero, $F(0) \neq 0$ and then (3.4) ensures $K(0) = 0$ and $H(0) \neq 0$. Combining the equations (3.5) and (3.6), we obtain that

$$\begin{cases} F(w) = K_{\psi_I(0)}(w)F(0),\\ G(w) = 0,\\ H(z) = K_{\varphi_I(0)}(z)H(0),\\ K(z) = 0.\end{cases} \tag{3.7}$$

That is to say

$$f_I(w) = K_{\psi_I(0)}(w)F(0) \text{ and } g_I(z) = K_{\varphi_I(0)}(z)\overline{F(0)}.$$

Furthermore, representation formula gives the expressions of f and g on the whole $\mathbb{H}$ by

$$f(p) = K_{\psi_I(0)}(p)F(0) \;\text{ and }\; g(p) = K_{\varphi_I(0)}(p)\overline{F(0)}. \tag{3.8}$$

Subsequently, we put the expressions in (3.7) into (3.3), it entails

$$\begin{cases} K_{\varphi_I(w)}(z)\overline{K_{\psi_I(0)}(w)F(0)} = K_w(\psi_I(z))K_{\varphi_I(0)}(z)\overline{F(0)}, \\ 0\cdot K_z(\varphi_I(w)) = K_w(\psi_I(z))\cdot 0. \end{cases}$$

We can further infer that

$$K_{\varphi_I(w)}(z)K_w(\psi_I(0)) = K_w(\psi_I(z))K_{\varphi_I(0)}(z),$$

holds for all $z, w \in \mathbb{C}_I$, due to $\overline{F(0)} \neq 0$. Considering $K_w(z) = e^{z\overline{w}}$ for $z, w \in \mathbb{C}_I$, it entails that

$$z\overline{\varphi_I(w)} + \psi_I(0)\overline{w} = \psi_I(z)\overline{w} + z\overline{\varphi_I(0)} + n(z,w)2\pi i, \tag{3.9}$$

where $n(z,w)$ is a continuous integer-valued function. Employing $z = w = 0$ in the above display, it turns out $n(0,0) = 0$. And then the continuous integer-valued function $n(z,w) = 0$ for all $z, w \in \mathbb{C}_I$. The formula (3.9) can be rewritten into

$$\frac{\overline{\varphi_I(w) - \varphi_I(0)}}{\overline{w}} = \frac{\psi_I(z) - \psi_I(0)}{z}. \tag{3.10}$$

We note the left part is a function of $\overline{w} \in \mathbb{C}_I$ and the right part is a holomorphic function of $z \in \mathbb{C}_I$. Hence there exists $\lambda \in \mathbb{C}_I$, such that

$$\varphi_I(z) = z\overline{\lambda} + \varphi_I(0) \;\text{ and }\; \psi_I(z) = z\lambda + \psi_I(0). \tag{3.11}$$

Suppose the weighted composition operator $W_{f,\varphi}$ is bounded on $\mathcal{F}^2(\mathbb{H})$, then the parameter $|\lambda| \leq 1$. Furthermore, employing the representation formula (Proposition 1.4), we extend φ_I and ψ_I into the whole $\mathbb{H}$ by

$$\varphi(p) = p\overline{\lambda} + \varphi(0), \;\text{ and }\; \psi(p) = p\lambda + \psi(0).$$

with $\lambda \in \mathbb{C}_I$ and $|\lambda| \leq 1$. Here we only proceed with $\varphi(p)$ as below. Indeed, for $p = x + yI$ and $q = x + yJ$, we have

$$\begin{aligned} \varphi(p) &= \frac{1}{2}(1 - IJ)\varphi(q) + \frac{1}{2}(1 + IJ)\varphi(\overline{q}) \\ &= \frac{1}{2}(1 - IJ)(\varphi(0) + q\overline{\lambda}) + \frac{1}{2}(1 + IJ)(\varphi(0) + \overline{q}\overline{\lambda}) \\ &= \varphi(0) + \frac{1}{2}[(q + \overline{q}) + IJ(\overline{q} - q)\overline{\lambda}] \\ &= \varphi(0) + \frac{1}{2}[2x + IJ(-2Jy)\overline{\lambda}] \\ &= \varphi(0) + (x + Iy)\overline{\lambda} \\ &= \varphi(0) + p\overline{\lambda}. \end{aligned}$$

Summarizing the above calculations, the statement (i) holds.

Case 2. $G(0) \neq 0$. The second equation in (3.5) implies $\psi_I(0) = 0$. Since $K(0) = -G(0) \neq 0$, which together with the second formula in (3.6) imply $\varphi_I(0) = 0$. Observing the first equations in (3.5) and (3.6), it turns out

$$F(w) = F(0),\ G(w) = G(0),\ H(z) = H(0),\ K(z) = K(0)$$

for $z, w \in \mathbb{C}_I$. That means

$$f_I(z) = f(0) = F(0) + G(0)J \text{ and } g_I(z) = g(0) = G(0) + H(0)J. \quad (3.12)$$

Putting the above formulas into (3.3), it turns out

$$\begin{cases} K_{\varphi_I(w)}(z)\overline{F(0)} = K_w(\psi_I(z))H(0), \\ -G(0)K_z(\varphi_I(w)) = K_w(\psi_I(z))K(0). \end{cases}$$

The second one implies $\varphi_I(w)\overline{z} = \psi_I(z)\overline{w} + n(z, w)2\pi i$. Similarly, it follows the integer function $n(z, w) = 0$ due to $n(0, 0) = 0$. That entails $\varphi_I(w)\overline{z} = \psi_I(z)\overline{w}$, which implies $\varphi_I(z) = \psi_I(z) = 0$ for all $z \in \mathbb{C}_I$. Again employing representation formula, it implies $\varphi(p) = \psi(p) = 0$. Summarizing the above calculations, the statement (ii) holds. This completes the proof. □

Corollary 3.5. *Let f and φ be two slice regular functions on $\mathbb{H}$. Denote $f_I(z) = F(z)+G(z)J$ nonzero functions on $\mathbb{H}$ and suppose $\varphi(\mathbb{C}_I) \subset \mathbb{C}_I$ for some $I \in \mathbb{S}$. Then weighted composition operator $W_{f,\varphi}$ is a bounded self-adjoint operator on $\mathcal{F}^2(\mathbb{H})$ if and only if $f(p) = K_{\varphi_I(0)}(p)F(0)$, $\varphi(p) = \varphi(0)+p\lambda$, with $F(0),\ \lambda \in \mathbb{R}$ and $|\lambda| \leq 1$. That means $W_{f,\varphi}$ is the unitary operator $W_{k_{\varphi_I(0)},\varphi}$ multiply a constant $e^{|\varphi_I(0)|^2/2}F(0)$, with $F(0) \in \mathbb{R}$.*

Proof. Necessity. Let $f(p) = g(p)$ and $\varphi(p) = \psi(p)$ in Theorem 3.4, then the case (i) is stated as $f(p) = K_{\varphi_I(0)}(p)F(0)$, $\varphi(p) = \varphi(0)+p\lambda$, with $F(0),\ \lambda \in \mathbb{R}$ and $|\lambda| \leq 1$. Besides, the case (ii) becomes into $f(p) = F(0)$ with $F(0) \in \mathbb{R}$ and $\varphi(p) = 0$, which can be contained in case (i) with $\varphi(p) = 0$.

Sufficiency. Our aim is to show $W^*_{f,\varphi}h(q) = W_{f,\varphi}h(q)$ for any $h \in \mathcal{F}^2(\mathbb{H})$. Considering the facts

$$W^*_{f,\varphi}h(q) = \langle W^*_{f,\varphi}h, K_q\rangle = \langle h, W_{f,\varphi}K_q\rangle,$$

and

$$W_{f,\varphi}h(q) = \langle W_{f,\varphi}h, K_q\rangle = \langle h, W^*_{f,\varphi}K_q\rangle,$$

for any $h \in \mathcal{F}^2(\mathbb{H})$. Hence we only need to show

$$(W_{f,\varphi}K_q)_I(z) = (W^*_{f,\varphi}K_q)_I(z). \quad (3.13)$$

On the one hand,

$$\begin{aligned} (W_{f,\varphi}K_q)_I(z) &= K_q(\varphi_I(z)) \star f_I(z) \\ &= K_q(\varphi_I(0) + z\lambda) \star (K_{\varphi_I(0)}(z)F(0)) \\ &= K_q(\varphi_I(0) + z\lambda)K_{\varphi_I(0)}(z)F(0). \end{aligned}$$

On the other hand,

$$\begin{aligned}(W_{f,\varphi}^{*}K_q)_I(z) &= \overline{W_{f,\varphi}K_z(q)}\\ &= \overline{K_z(\varphi(0)+q\lambda)\star f(q)}\\ &= \overline{K_z(\varphi(0)+q\lambda)\star K_{\varphi_I(0)}(q)F(0)}\\ &= \overline{K_z(\varphi(0))K_z(q\lambda)\star K_{\varphi_I(0)}(q)F(0)}\\ &= \overline{K_z(\varphi(0))e_*^{q\lambda\bar{z}}\star e_*^{q\overline{\varphi_I(0)}}F(0)}\\ &= \overline{K_z(\varphi(0))e_*^{q(\lambda\bar{z}+\overline{\varphi_I(0)})}F(0)}\\ &= \overline{K_z(\varphi(0))K_{(\lambda z+\varphi_I(0))}(q)F(0)}\\ &= F(0)K_q(\lambda z+\varphi_I(0))K_{\varphi(0)}(z),\end{aligned}$$

where the display (1.7) is used in the third line from the bottom and (1.1) is used in the last line. Combining the above two formulas with $\varphi_I(0)=\varphi(0)$ and $F(0)\in\mathbb{R}$, the formula (3.13) is true. This ends the proof. □

Remark 3.6. Observing Theorem 3.3 and Corollary 3.5, it's obvious Corollary 3.5 is a special case of Theorem 3.3, which coheres with the fact a self-adjoint operator is normal.

4. Isometric Weighted Composition Operators

Recall that an operator T on $\mathcal{F}^2(\mathbb{H})$ is called isometric if $||Th||=||h||$ for all $h\in\mathcal{F}^2(\mathbb{H})$, or equivalently, $\langle Th,Tg\rangle=\langle h,g\rangle$ for all $h,g\in\mathcal{F}^2(\mathbb{H})$. In this section, we describe all isometric weighted composition operators on slice regular quaternionic Fock space $\mathcal{F}^2(\mathbb{H})$. First of all, we deduce an equivalent characterization for a special isometric weighted composition operator on $\mathcal{F}^2(\mathbb{H})$.

Proposition 4.1. *Let $\eta(p)=p\lambda$ with $|\lambda|\le 1$ and $\lambda\in\mathbb{C}_I$ for some $I\in\mathbb{S}$. Denote $\xi\in\mathcal{F}^2(\mathbb{H})$ with restriction $\xi_I(z)=F(z)+G(z)J$ on $\mathbb{C}_I$, then $W_{\xi,\eta}$ is an isometry on $\mathcal{F}^2(\mathbb{H})$ if and only if $|\lambda|=1$, both $F(z)$ and $G(z)$ are constants and satisfy $|F(z)|^2+|G(z)|^2=1$.*

Proof. Sufficiency. If $|\lambda|=1$ and $\xi_I(z)=F+GJ$ with F and G are constants satisfying $|F|^2+|G|^2=1$. We consider a slice regular function $h(p)\in\mathcal{F}^2(\mathbb{H})$ with the power series expansion on $\mathbb{C}_I$,

$$h_I(z)=\sum_{k=0}^{\infty}z^k(a_k+b_kJ),$$

where $a_k,b_k\in\mathbb{C}_I$. Subsequently,

$$\|h\|^2=\sum_{k=0}^{\infty}|a_k|^2k!+\sum_{k=0}^{\infty}|b_k|^2k!.$$

Applying the above assumptions, we validate

$$
\begin{aligned}
&\|W_{\xi,\eta}h\|^2\\
&= \int_{\mathbb{C}_I} |[(h\circ\eta)]_I(z)\star\xi_I(z)|^2 e^{-|z|^2} d\sigma(x,y)\\
&= \int_{\mathbb{C}_I} \left| \left(\sum_{k=0}^{\infty} z^k\lambda^k a_k + \sum_{k=0}^{\infty} z^k\lambda^k b_k J \right)\star (F+GJ) \right|^2 e^{-|z|^2} d\sigma(x,y)\\
&= \int_{\mathbb{C}_I} \left| \left(\sum_{k=0}^{\infty} z^k\lambda^k a_k \right) F - \left(\sum_{k=0}^{\infty} z^k\lambda^k b_k \right) \overline{G} \right|^2 e^{-|z|^2} d\sigma(x,y)\\
&\quad + \int_{\mathbb{C}_I} \left| \left(\sum_{k=0}^{\infty} z^k\lambda^k a_k \right) G + \left(\sum_{k=0}^{\infty} z^k\lambda^k b_k \right) \overline{F} \right|^2 e^{-|z|^2} d\sigma(x,y)\\
&= \int_{\mathbb{C}_I} \left[\left| \sum_{k=0}^{\infty} z^k\lambda^k a_k \right|^2 (|F|^2+|G|^2) + \left| \sum_{k=0}^{\infty} z^k\lambda^k b_k \right|^2 (|G|^2+|F|^2) \right] e^{-|z|^2} d\sigma(x,y)\\
&\quad - \left(\sum_{k=0}^{\infty} k! a_k \overline{b_k} \right) FG - \left(\sum_{k=0}^{\infty} k! \overline{a_k} b_k \right) \overline{FG}\\
&\quad + \left(\sum_{k=0}^{\infty} k! a_k \overline{b_k} \right) FG + \left(\sum_{k=0}^{\infty} k! \overline{a_k} b_k \right) \overline{FG}\\
&= \left(\sum_{k=0}^{\infty} |a_k|^2 k! + \sum_{k=0}^{\infty} |b_k|^2 k! \right) (|F|^2+|G|^2)\\
&= \|h\|^2(|F|^2+|G|^2) = \|h\|^2,
\end{aligned}
$$

where we use the inner product (1.5) in $\mathcal{F}^2(\mathbb{H})$ on the second and third lines from the bottom. Hence $W_{\xi,\eta}$ is an isometry on $\mathcal{F}^2(\mathbb{H})$.

Necessity. Suppose the operator $W_{\xi,\eta}$ is an isometry on $\mathcal{F}^2(\mathbb{H})$, then $\lambda \neq 0$. Indeed, if $\lambda = 0$, $W_{\xi,\eta}p^N = 0$ for all monomials p^N with $N \geq 1$, which is a contradiction.

Taking $h \in \mathcal{F}^2(\mathbb{H})$ satisfying $h(\mathbb{C}_I) \subset \mathbb{C}_I$, we deduce that

$$
\begin{aligned}
\|W_{\xi,\eta}h\|^2 &= \int_{\mathbb{C}_I} |h(\eta_I(z))\star\xi_I(z)|^2 e^{-|z|^2} d\sigma(x,y)\\
&= \int_{\mathbb{C}_I} |h(z\lambda)|^2(|F(z)|^2+|G(z)|^2) e^{-|z|^2} d\sigma(x,y)\\
&= \|h\|^2.
\end{aligned}
$$

The above computations imply $(W_{\xi,\eta})_I$ acting on $\mathcal{F}^2(\mathbb{C}_I)$ is same as a weighted composition operator defined by a compositional symbol $\varphi_I(z) = \lambda z$ and a weight with norm $\sqrt{|F(z)|^2+|G(z)|^2}$. Employing the result [13, Propsition 4.2] in complex variables, it follows that $|\lambda| = 1$ and $|F(z)|^2 + |G(z)|^2 = 1$. Considering $F(z)$ and $G(z)$ are entire functions, Liouville Theorem can further imply that both $F(z)$ and $G(z)$ are constants. This completes the proof. $\square$

Immediately, we present the equivalent characterizations for isometric weighted composition operators on $\mathcal{F}^2(\mathbb{H})$.

Theorem 4.2. *Let f and φ be two slice regular functions on $\mathbb{H}$, such that $\varphi(\mathbb{C}_I) \subset \mathbb{C}_I$ for some $I \in \mathbb{S}$ and $f_I(z) = F(z) + G(z)J$ not identically zero. Then $W_{f,\varphi}$ is isometric on $\mathcal{F}^2(\mathbb{H})$ if and only if*

$$W_{f,\varphi} = W_{k^2_{-\overline{\lambda}b}\cdot\gamma,\varphi},$$

with $\gamma \in \mathbb{H}$ and $|\gamma| = 1$, for any $h \in \mathcal{F}^2(\mathbb{H})$.

Proof. Necessity. Assume the operator $W_{f,\varphi}$ is an isometry on $\mathcal{F}^2(\mathbb{H})$, then it's obviously bounded. Theorem 2.3 implies $\varphi(p) = b + p\lambda$ for some $|\lambda| \leq 1$ with $\lambda,\ b \in \mathbb{C}_I$.

Taking W_b defined in (3.1) with $b \in \mathbb{C}_I$ and $\varphi_1(p) = p - b$, we define $W_{\hat{f},\hat{\varphi}} = W_{f,\varphi}W_b$. For any $h \in \mathcal{F}^2(\mathbb{H})$, it yields that

$$\begin{aligned}
&(W_{\hat{f},\hat{\varphi}}h)_I(z)\\
&\quad = (W_{f,\varphi}W_b h)_I(z)\\
&\quad = h_I(z\lambda) \star [k_b(b + z\lambda)F(z) + k_b(b + z\lambda)G(z)J].
\end{aligned}$$

That is to say $\hat{\varphi}(p) = p\lambda$ and

$$\hat{f}_I(z) = k_b(b + z\lambda)F(z) + k_b(b + z\lambda)G(z)J.$$

Since the operator W_b is unitary, and then the operator $W_{\hat{f},\hat{\varphi}}$ is isometric, Proposition 4.1 entails that $|\lambda| = 1$, $k_b(b+z\lambda)F(z) = \alpha$ and $k_b(b+z\lambda)G(z) = \beta$, where $\alpha, \beta \in \mathbb{C}_I$ constants with $|\alpha|^2 + |\beta|^2 = 1$. In other words,

$$F(z) = \alpha k_{-\overline{\lambda}b}(z) \quad \text{and} \quad G(z) = \beta k_{-\overline{\lambda}b}(z).$$

This means

$$\hat{f}_I(z) = k_{-\overline{\lambda}b}(z)(\alpha + \beta J),$$

with $|\alpha|^2 + |\beta|^2 = 1$. Hence, by Proposition 3.1 and Remark 3.2, we obtain $W_{f,\varphi} = W_{\hat{f},\hat{\varphi}}W_{-b}$, which can be also rewritten as

$$\begin{aligned}
&(W_{f,\varphi}h)_I(z)\\
&\quad = (W_{\hat{f},\hat{\varphi}}W_{-b}h)_I(z)\\
&\quad = [h_I(z\lambda + b) \star k_{-b}(\lambda z)] \star \hat{f}_I(z)\\
&\quad = h_I(z\lambda + b) \star k_{-\overline{\lambda}b}(z) \star (k_{-\overline{\lambda}b}(z)(\alpha + \beta J))\\
&\quad = h_I(z\lambda + b) \star [k^2_{-\overline{\lambda}b}(z)(\alpha + \beta J)]\\
&\quad = (W_{k^2_{-\overline{\lambda}b}\cdot\gamma,\varphi}h)_I(z)
\end{aligned}$$

with $\gamma = \alpha + \beta J$ and $|\gamma| = 1$.

Sufficiency. Suppose $W_{f,\varphi} = W_{k^2_{-\overline{\lambda}b}\cdot\gamma,\varphi}$ with $\gamma \in \mathbb{C}_I$ and $|\gamma| = 1$. It holds that

$$W_{f,\varphi} = W_{\hat{f},\hat{\varphi}} W_{-b}.$$

For any element $h \in \mathcal{F}^2(\mathbb{H})$, there exists $g \in \mathcal{F}^2(\mathbb{H})$ such that $g = W_{-b}h$. Based on the fact W_{-b} is an unitary operator on $\mathcal{F}^2(\mathbb{H})$, it turns out that $\|h\| = \|g\|$. Since $W_{\hat{f},\hat{\varphi}}$ is isometric on $\mathcal{F}^2(\mathbb{H})$, it yields that

$$\|W_{f,\varphi}h\| = \|W_{\hat{f},\hat{\varphi}} W_{-b}h\| = \|W_{\hat{f},\hat{\varphi}} g\| = \|g\| = \|h\|,$$

which ensures $W_{f,\varphi}$ is isometric on $\mathcal{F}^2(\mathbb{H})$. This ends the proof. □

Acknowledgements

Y. X. Liang is supported by the National Natural Science Foundation of China (Grant No. 11701422).

References

[1] Adler, S.: Quaternionic Quantum Field Theory. Oxford University Press, Oxford (1995)

[2] Alpay, D., Colombo, F., Sabadini, I., Salomon, G.: The Fock Space in the Slice Hyperholomorphic Setting, pp. 43–59. Springer, Berlin (2014)

[3] Colombo, F., Gonzalez-Cervantes, J.O., et al.: On two approaches to the Bergman theory for slice regular functions. In: Advances in Hypercomplex Analysis, vol. 1, pp. 39–54. Springer INdAM Series, Berlin (2013)

[4] Colombo, F., Sabadini, I., Sommen, F., Struppa, D.C.: Analysis of Dirac Systems and Computational Algebra, Progress in Mathematical Physics, vol. 39. Birkhäuser, Boston (2004)

[5] Colombo, F., Sabadini, I., Struppa, D.C.: Slice Monogenic Functions, Noncommutative Functional Calculus. Springer, Basel (2011)

[6] Colombo, F., Sabadini, I., Struppa, D.C.: Entire Slice Regular Functions. Springer, Berlin (2016)

[7] Cowen, C.C., MacCluer, B.D.: Composition Operators on Spaces of Analytic Functions. CRC Press, Boca Raton (1995)

[8] Gürlebeck, K., Habetha, K., Sprößig, W.: Holomorphic Functions in the Plane and n-Dimensional Space. Birkhäuser, Basel (2008)

[9] Horwitz, L.P., Razon, A.: Tensor product of quaternion Hilbert modules. In: Classical and Quantum Systems (Goslar), pp. 266–268 (1991)

[10] Kumar, S., Manzoor, K.: Slice regular Besov spaces of hyperholomorphic functions and composition operators (2016). arXiv:1609.02394v1

[11] Kumar, S., Manzoor, K., Singh, P.: Composition operators in Bloch spaces of slice hyperholomorphic functions. Adv. Appl. Clifford Algebra **27**(2), 1459–1477 (2017)

[12] Lian, P., Liang, Y.X.: Weighted composition operator on quaternionic Fock space, arXiv: 1803.0678v1, pp. 1–39. (2018)

[13] Le, T.: Normal and isometric weighted composition operators on the Fock space. Bull. Lond. Math. Soc. **46**(4), 847–856 (2014)

[14] Sabadini, I., Saracco, A.: Carleson measures for Hardy and Bergman spaces in the quaternionic unit ball. J. Lond. Math. Soc. **95**(3), 853–874 (2017)

[15] Shapiro, J.H.: Composition Operators and Classical Function Theory. Springer, New York (1993)

[16] Villalba, C., Colombo, F., Gantner, J., et al.: Bloch, Besov and Dirichlet spaces of slice hyperholomorphic functions. Complex Anal. Oper. Theory **9**(2), 479–517 (2015)

[17] Zhao, L., Pang, C.: A class of weighted composition operators on the Fock space. J. Math. Res. Appl. **35**(3), 303–310 (2015)

[18] Zhu, K.H.: Analysis on Fock Spaces. Springer, New York (2012)

Yu-Xia Liang
School of Mathematical Sciences
Tianjin Normal University
Tianjin 300387
People's Republic of China
e-mail: liangyx1986@126.com

Received: June 1, 2018.
Accepted: December 26, 2018.