Results Math (2019) 74:13

1422-6383/19/010001-4
published online December 6, 2018
https://doi.org/10.1007/s00025-018-0938-y

Results in Mathematics

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An Answer to an Open Problem on the Multivariate Bernstein Polynomials on a Simplex

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Dedicated to Professor Heiner Gonska on the occasion of his 70th birthday

Abstract. We answer and generalize an open problem on the two-dimensional Bernstein polynomials on the unit triangle.

Mathematics Subject Classification. 32E20, 41A36, 41A17, 42A16.

Keywords. Multivariate Bernstein polynomials, simplex, convexity.

1. The Open Problem

There is an increasing interest in studying the properties of the sums of squared Bernstein polynomials (see, e.g., [1,3–6] and many papers in http://arxiv.org/).

In this paper, we focus on a very recent open problem on the subject.

Let $T_2 := \{(x,y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0,\ x+y \leq 1\}$ and n be a positive integer. For $(x,y) \in T_2$,

$$A_{2,n}(x,y) := \sum_{i+j\leq n} \left(\frac{n!}{i!j!(n-i-j)!} x^i y^j (1-x-y)^{n-i-j} \right)^2,$$

is the sum of squares of the functions from the Bernstein basis on the unit triangle T_2.

The following open problem concerning $A_{2,n}$ was recently posed.

Problem 1 [7, Problem 1]. *Is $A_{2,n}$ convex on T_2?*

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2. Notation

Let m, n be positive integers and $f_k\colon \mathbb{R}^m \to \mathbb{R}$, $k = 1, \ldots, m$, be affine functions. Define $\mathcal{S}_{m,n}\colon \mathbb{R}^m \to \mathbb{R}$,

$$\mathcal{S}_{m,n}(x) := \sum_{k_1+\cdots+k_m=n} \left(\frac{n!}{k_1! \cdots k_m!} f_1(x)^{k_1} \ldots f_m(x)^{k_m} \right)^2, \qquad x \in \mathbb{R}^m.$$

(the sum is over all m-tuples of nonnegative integers $(k_1, \ldots, k_m)$ satisfying the constraint $k_1 + \cdots + k_m = n$).

In particular, $\mathcal{S}_{m,n}(x)$ may be the sum of squares of the functions from the multivariate Bernstein basis on a non-standard simplex.

3. The Main Result

The following is the main result of the paper.

Theorem 2. *$\mathcal{S}_{m,n}$ is convex on $\mathbb{R}^m$.*

Proof. We have

$$\begin{aligned} &\left(\sum_{k=1}^m f_k(x) e^{it_k}\right)^n \\ &\quad = \sum_{k_1+\cdots+k_m=n} \frac{n!}{k_1! \ldots k_m!} f_1(x)^{k_1} e^{it_1 k_1} \ldots f_m(x)^{k_m} e^{it_m k_m} \end{aligned} \tag{1}$$

From (1), using the Parseval identity, we obtain

$$\mathcal{S}_{m,n}(x) = \frac{1}{(2\pi)^m} \int_{[-\pi,\pi]^m} \left| \sum_{k=1}^m f_k(x)\, e^{it_k} \right|^{2n} \mathrm{d}t_1 \ldots \mathrm{d}t_m.$$

Using the triangle inequality and the fact that the function $t \mapsto t^{2n}$ is convex on $\mathbb{R}$, we get

$$\big|(1-\lambda)a + \lambda b\big|^{2n} \leq (1-\lambda)|a|^{2n} + \lambda |b|^{2n},$$

for any $a, b \in \mathbb{C}$, $\lambda \in [0,1]$ and $n \geq 1$. Let $x, y \in \mathbb{R}^m$ and $\lambda \in [0,1]$. We note that the affine functions f_k satisfy the equalities:

$$f_k\big((1-\lambda)x + \lambda y\big) = (1-\lambda) f_k(x) + \lambda f_k(y), \qquad \lambda \in \mathrm{R},\ x, y \in \mathrm{R}^m.$$

We have:

$$\begin{aligned} &\left| \sum_{k=1}^m f_k\big((1-\lambda)x + \lambda y\big) e^{it_k} \right|^{2n} \\ &\quad = \left| (1-\lambda) \sum_{k=1}^m f_k(x) e^{it_k} + \lambda \sum_{k=1}^m f_k(y) e^{it_k} \right|^{2n} \end{aligned}$$

$$\leq (1-\lambda)\left|\sum_{k=1}^{m} f_k(x)e^{it_k}\right|^{2n} + \lambda\left|\sum_{k=1}^{m} f_k(y)e^{it_k}\right|^{2n} \tag{2}$$

Integrating both sides of inequality (2) we obtain

$$\mathcal{S}_{m,n}\big((1-\lambda)x+\lambda y\big) \leq (1-\lambda)\mathcal{S}_{m,n}(x) + \lambda\mathcal{S}_{m,n}(y),$$

and the proof is complete. □

4. An Answer to the Open Problem

Let

$$A_{m,n}(x) := \sum_{k_1+\cdots+k_{m+1}=n}\left(\frac{n!}{k_1!\,\ldots\,k_{m+1}!}\,{x_1}^{k_1}\ldots{x_m}^{k_m}(1-x_1-\ldots-x_m)^{k_{m+1}}\right)^2$$

be the sum of squares of the functions from the m-dimensional Bernstein basis on the standard simplex

$$T_m := \big\{(x_1,\ldots,x_m)\in\mathbb{R}^m \mid x_1\geq 0,\ldots,x_m\geq 0,\ x_1+\cdots+x_m\leq 1\big\},$$

(see, e.g., [2, (5.2.89)]).

The following direct consequence of Theorem 2 answers and generalizes the Open Problem 1. In [6,7], using properties of the Legendre polynomials, the authors restrict themselves to the interval $[0,1]$. Using a Parseval identity and the definition of the convexity, we prove that $A_{m,n}$ is convex not only on T_m, but on $\mathbb{R}^m$.

Corollary 3. *$A_{m,n}$ is convex on $\mathbb{R}^m$.*

Proof. We note that $A_{m,n}(x)=\mathcal{S}_{m+1,n}(x)$ for $f_k(x)=x_k$, $k=1,\ldots,m$, and $f_{m+1}(x)=1-x_1-\cdots-x_m$.

The proof is complete. □

Remark 4. We note that:

- For $m=1$, Corollary 3 gives a simple affirmative answer to [5, Conjecture 3.2];
- For $m=2$, Corollary 3 answers the Open Problem 1.

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Received: July 29, 2018.
Accepted: November 30, 2018.