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Results in Mathematics

Limits of Sequences of Bochner Integrable Functions Over Sequences of Probability Measures Spaces

Dumitru Popa

Abstract. We prove limits of sequences of Bochner integrable functions over sequences of probability measures spaces. A sample result: Let X be a bounded closed convex set in a Banach space F, $a \in X$ and E a non-null Banach space. Let $(\Omega_n, \Sigma_n, \mu_n)_{n\in\mathbb{N}}$ be a sequence of probability measure spaces, $\varphi_n : \Omega_n \to X$ a sequence of μ_n-Bochner integrable functions. Then the following assertions are equivalent:

(i) $\lim_{n\to\infty} \int_{\Omega_n} \|\varphi_n(\omega_n) - a\|_F \, d\mu_n(\omega_n) = 0$.

(ii) For each uniformly continuous and bounded function $f : X \to E$, the following equality holds

$$\lim_{n\to\infty} \int_{\Omega_n} f(\varphi_n(\omega_n)) \, d\mu_n(w_n) = f(a) \text{ in norm of } E.$$

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1. Introduction and Notation

The main purpose of this paper is to prove the result as stated in Abstract. For this we prove first a result analogous result to the Niculescu's characterization of weakly compact operators on $C(K)$-spaces, see Lemma 1, from which we deduce a very general result concerning the convergence of sequences of integrals, see Theorem 1. Then, we prove various results for limits/the limit of sequences of Bochner integrable functions in which sequences of probability measures may vary, see Propositions 1 and 2. Further, in the case of l_p spaces we give some results, see Propositions 3 and 4 which show that the result in the case of Hilbert spaces proved in Proposition 2 is not necessary true in general. We apply these results in the case of the Rademacher functions, see Corollaries 1 and 3. Let us recall some basic concepts and notation in vector integration; for more details we refer the reader to the "Vector measures" of Diestel and Uhl see [4]. Let (Ω, Σ, μ) be a measure space, E a Banach space over the scalar field $\mathbb{K} = \mathbb{R}\,(\text{or } \mathbb{C})$. A function $f : \Omega \to E$ is μ -Bochner integrable if f is μ-measurable and $\int_\Omega \|f(\omega)\|_E \, d\mu(\omega) < \infty$. In this case $\int_\Omega f(\omega)\, d\mu(\omega) \in E$ denote the Bochner integral. Let us recall the inequality $\left\| \int_\Omega f(\omega)\, d\mu(\omega) \right\|_E \le \int_\Omega \|f(\omega)\|_E \, d\mu(\omega)$. Recall that if (X, ρ) is a metric space, $x \in X$, $M > 0$ then $\overline{B}(x, M)$ is the closed ball with center at x and radius M and non-empty subset A of X is bounded if $diam\,(A) := \sup_{(a,b) \in A \times A} \rho(a, b) < \infty$. If (X, ρ) and (E, σ) are metric spaces, a function $f : X \to E$ is bounded if the set $f(X) \subset E$ is bounded. All notation and notion used and not defined in this paper are standard, e.g. see [4,5].

2. The Basic Results

The following lemma is the main result of this paper. It is analogous to the Niculescu's characterisation of weakly compact operators on $C(K)$-spaces, see [5, Theorem 15.2, page 309], [8, Theorem 1]. For other type results in various direction we recommend the reader to consult the papers of Niculescu, [8–10] and the papers [1,7].

Lemma 1. *Let (X, ρ) be a metric space, (E, σ) be a metric space, $f : X \to E$ a uniformly continuous and bounded function and $x \in X$. Let also $0 < p < \infty$. Then: $\forall \varepsilon > 0$, $\exists \eta_\varepsilon > 0$ such that for each finite measure space (Ω, Σ, μ), each μ -measurable function $\varphi : \Omega \to X$ the following relation holds*

$$\int_\Omega [\sigma(f(\varphi(\omega)), f(x))]^p \, d\mu(\omega) \le [\eta_\varepsilon diam\,(f(X))]^p \int_\Omega [\rho(\varphi(\omega), x)]^p \, d\mu(\omega) + \varepsilon^p \mu(\Omega).$$

Proof. Let us note that since φ is μ-measurable, f continuous, the function $\omega \to \sigma(f(\varphi(\omega)), f(x))$ is μ-measurable and bounded by $diam\,(f(X)) < \infty$ (f is bounded), thus μ-integrable ((Ω, Σ, μ) is a finite measure space). Let $\varepsilon > 0$. Since f is uniformly continuous, there exists $\delta_\varepsilon > 0$ such that for each $(u, v) \in X \times X$ with $\rho(u, v) < \delta_\varepsilon$ it follows that $\sigma(f(u), f(v)) < \varepsilon$. Let us define $M(\delta_\varepsilon) = \{\omega \in \Omega \mid \rho(\varphi(\omega), x) \ge \delta_\varepsilon\}$ and note that by Markov inequality

$$\mu(M(\delta_\varepsilon)) \le \frac{1}{[\delta_\varepsilon]^p} \int_\Omega [\rho(\varphi(\omega), x)]^p \, d\mu(\omega). \tag{1}$$

We have

$$\int_{\Omega}\left[\sigma\left(f\left(\varphi\left(\omega\right)\right),f\left(x\right)\right)\right]^{p}d\mu\left(\omega\right)=\int_{M\left(\delta_{\varepsilon}\right)}\left[\sigma\left(f\left(\varphi\left(\omega\right)\right),f\left(x\right)\right)\right]^{p}d\mu\left(\omega\right)$$
$$+\int_{\mathcal{C}M\left(\delta_{\varepsilon}\right)}\left[\sigma\left(f\left(\varphi\left(\omega\right)\right),f\left(x\right)\right)\right]^{p}d\mu\left(\omega\right). \tag{2}$$

For each $\omega\in\mathcal{C}M\left(\delta_{\varepsilon}\right)$ we have $\rho\left(\varphi\left(\omega\right),x\right)<\delta_{\varepsilon}$ from where

$$\sigma\left(f\left(\varphi\left(\omega\right)\right),f\left(x\right)\right)<\varepsilon$$

and, by integration,

$$\int_{\mathcal{C}M\left(\delta_{\varepsilon}\right)}\left[\sigma\left(f\left(\varphi\left(\omega\right)\right),f\left(x\right)\right)\right]^{p}d\mu\left(\omega\right)\leq\varepsilon^{p}\mu\left(\mathcal{C}M\left(\delta_{\varepsilon}\right)\right)\leq\varepsilon^{p}\mu\left(\Omega\right). \tag{3}$$

Also for each $\omega\in M\left(\delta_{\varepsilon}\right)$ we have

$$\sigma\left(f\left(\varphi\left(\omega\right)\right),f\left(x\right)\right)\leq\sup_{(a,b)\in X\times X}\sigma\left(f\left(a\right),f\left(b\right)\right)=diam\left(f\left(X\right)\right)$$

and then, by integration,

$$\int_{M\left(\delta_{\varepsilon}\right)}\left[\sigma\left(f\left(\varphi\left(\omega\right)\right),f\left(x\right)\right)\right]^{p}d\mu\left(\omega\right)\leq\left[diam\left(f\left(X\right)\right)\right]^{p}\mu\left(M\left(\delta_{\varepsilon}\right)\right)$$

which by (1) gives us

$$\int_{M\left(\delta_{\varepsilon}\right)}\left[\sigma\left(f\left(\varphi\left(\omega\right)\right),f\left(x\right)\right)\right]^{p}d\mu\left(\omega\right)\leq\frac{\left[diam\left(f\left(X\right)\right)\right]^{p}}{\left[\delta_{\varepsilon}\right]^{p}}\int_{\Omega}\left[\rho\left(\varphi\left(\omega\right),x\right)\right]^{p}d\mu\left(\omega\right). \tag{4}$$

Then, for $\eta_{\varepsilon}=\frac{1}{\delta_{\varepsilon}}$ from (2–4) we get the relation from the statement. □

Remark 1. (i) If (X,ρ) is a compact metric space, then each continuous function $f:X\to E$ is uniformly continuous and bounded, hence Lemma 1 holds. (ii) If (X,ρ) is a metric space and E a Banach space then, $f:X\to E$ is bounded if and only if $\left\|f\right\|_{\infty}:=\sup_{x\in X}\left\|f\left(x\right)\right\|<\infty$. In this, case $diam\left(f\left(X\right)\right)\leq 2\left\|f\right\|_{\infty}$.

The next result is a general result for limits of sequences of Bochner integrals over sequences of probability measure spaces.

Theorem 1. *Let* (X,ρ) *be a metric space,* $(x_n)_{n\in\mathbb{N}}\subset X$*,* (E,σ) *be a metric space. Let* $0<p<\infty$*,* $(\Omega_n,\Sigma_n,\mu_n)_{n\in\mathbb{N}}$ *be a sequence of measure spaces with* $\sup_{n\in\mathbb{N}}\mu_n\left(\Omega_n\right)<\infty$*,* $\varphi_n:\Omega_n\to X$ *a sequence of* μ_n*-measurable functions. If*

$$\lim_{n\to\infty}\int_{\Omega_n}\left[\rho\left(\varphi_n\left(\omega_n\right),x_n\right)\right]^{p}d\mu_n\left(\omega_n\right)=0$$

then, for each uniformly continuous and bounded function $f : X \to E$ the following equality holds

$$\lim_{n\to\infty} \int_{\Omega_n} \left[\sigma\left(f\left(\varphi_n\left(\omega_n\right)\right), f\left(x_n\right)\right)\right]^p d\mu_n\left(\omega_n\right) = 0.$$

Proof. Let $\varepsilon > 0$. Then by Lemma 1 there exists $\eta_\varepsilon > 0$ such that for each natural number n the following relation holds

$$\int_{\Omega_n} \left[\sigma\left(f\left(\varphi_n\left(\omega_n\right)\right), f\left(x_n\right)\right)\right]^p d\mu_n\left(\omega_n\right)$$
$$\leq \left[\eta_\varepsilon diam\left(f\left(X\right)\right)\right]^p \int_{\Omega_n} \left[\rho\left(\varphi_n\left(\omega_n\right), x_n\right)\right]^p d\mu_n\left(\omega_n\right) + \varepsilon^p \mu_n\left(\Omega_n\right).$$

Since $M = \sup_{n\in\mathbb{N}} \mu_n\left(\Omega_n\right) < \infty$ we deduce

$$\int_{\Omega_n} \left[\sigma\left(f\left(\varphi_n\left(\omega_n\right)\right), f\left(x_n\right)\right)\right]^p d\mu_n\left(\omega_n\right)$$
$$\leq \left[\eta_\varepsilon diam\left(f\left(X\right)\right)\right]^p \int_{\Omega_n} \left[\rho\left(\varphi_n\left(\omega_n\right), x_n\right)\right]^p d\mu_n\left(\omega_n\right) + \varepsilon^p M.$$

Then

$$\limsup \int_{\Omega_n} \left[\sigma\left(f\left(\varphi_n\left(\omega_n\right)\right), f\left(x_n\right)\right)\right]^p d\mu_n\left(\omega_n\right)$$
$$\leq \left[\eta_\varepsilon diam\left(f\left(X\right)\right)\right]^p \limsup \int_{\Omega_n} \left[\rho\left(\varphi_n\left(\omega_n\right), x_n\right)\right]^p d\mu_n\left(\omega_n\right) + \varepsilon^p M.$$

which, by the hypothesis, gives us that

$$\limsup \int_{\Omega_n} \left[\sigma\left(f\left(\varphi_n\left(\omega_n\right)\right), f\left(x_n\right)\right)\right]^p d\mu_n\left(\omega_n\right) \leq \varepsilon^p M.$$

Since $\varepsilon > 0$ is arbitrary we deduce

$$\limsup \int_{\Omega_n} \left[\sigma\left(f\left(\varphi_n\left(\omega_n\right)\right), f\left(x_n\right)\right)\right]^p d\mu_n\left(\omega_n\right) \leq 0$$

that is,

$$\lim_{n\to\infty} \int_{\Omega_n} \left[\sigma\left(f\left(\varphi_n\left(\omega_n\right)\right), f\left(x_n\right)\right)\right]^p d\mu_n\left(\omega_n\right) = 0.$$

□

3. Limits of Sequences of Bochner Integrable Functions Over Sequences of Probability Measure Spaces

In the sequel we use Theorem 1 to give the necessary and sufficient conditions that to obtain limits of sequences of Bochner integrable functions over sequences of probability measure spaces. We need the following well-known result, see [4, Corollary 8, page 48]: Let (Ω, Σ, μ) be a probability measure space, E a Banach space and $\varphi : \Omega \to E$ a μ-Bochner integrable function. Then $\int_\Omega \varphi d\mu \in \overline{co}\ \varphi(\Omega)$.

Proposition 1. *Let X be a bounded closed convex set in a Banach space F, $a \in X$ and E a non-null Banach space. Let $(\Omega_n, \Sigma_n, \mu_n)_{n\in\mathbb{N}}$ be a sequence of probability measure spaces, $\varphi_n : \Omega_n \to X$ a sequence of μ_n-Bochner integrable functions. Then the following assertions are equivalent:*

(i) $\lim_{n\to\infty} \int_{\Omega_n} \|\varphi_n(\omega_n) - a\|_F \, d\mu_n(\omega_n) = 0$.
(i) *For each uniformly continuous and bounded function $f : X \to E$, the following equality holds*

$$\lim_{n\to\infty} \int_{\Omega_n} f(\varphi_n(\omega_n)) = f(a) \ \textit{ in norm of } E.$$

Proof. As we already remarked since X is closed and convex, $\int_{\Omega_n} \varphi_n d\mu_n \in \overline{co}$ $\varphi(\Omega_n) \subset X$ for all natural numbers n.

(i)$\Rightarrow$(ii). Indeed, by (i) and Theorem 1 we have

$$\lim_{n\to\infty} \int_{\Omega_n} \|f(\varphi_n(\omega_n)) - f(a)\|_E \, d\mu_n(\omega_n) = 0$$

and from

$$\begin{aligned}\left\| \int_{\Omega_n} f(\varphi_n(\omega_n)) \, d\mu_n(\omega_n) - f(a) \right\| &= \left\| \int_{\Omega_n} [f(\varphi_n(\omega_n)) - f(a)] \, d\mu_n(\omega_n) \right\|_E \\ &\le \int_{\Omega_n} \|f(\varphi_n(\omega_n)) - f(a)\|_E \, d\mu_n(\omega_n)\end{aligned}$$

we get (ii).

(ii)$\Rightarrow$(i). Let $e \in E$ with $\|e\| = 1$ and take in (ii) $f : X \to E$, $f(x) = \|x - a\|_F \, e$. Note that since X is bounded f is bounded (obvious uniformly continuous). □

We need the following well-known result. For the sake of completeness we include its proof.

Remark 2. Let (Ω, Σ, μ) be a probability measure space, H a Hilbert space, $f : \Omega \to H$ a μ -Bochner integrable function and $a \in H$. Then

$$\int_\Omega \|f(\omega) - a\|^2 \, d\mu(\omega) = \int_\Omega \|f(\omega)\|^2 \, d\mu(\omega) - 2\Re \left\langle \int_\Omega f(\omega) \, d\mu(\omega), a \right\rangle + \|a\|^2 .$$

Proof. If $z \in \mathbb{C}$ then $\Re z$ denote the real part of z. Let $x^* : H \to \mathbb{K}$, $x^*(x) = \Re\langle x, a\rangle$. We use the well-known result: $x^*\left(\int_\Omega f(\omega)\, d\mu(\omega)\right) = \int_\Omega x^*(f(\omega))\, d\mu(\omega)$, that is, $\int_\Omega \Re\langle f(\omega), a\rangle \, d\mu(\omega) = \Re\left\langle \int_\Omega f(\omega)\, d\mu(\omega), a\right\rangle$, see [4, The Hille Theorem, page 47]. Then

$$\int_{\Omega} \|f(\omega) - a\|^2 d\mu(\omega) = \int_{\Omega} \|f(\omega)\|^2 d\mu(\omega) - 2\int_{\Omega} \Re \langle f(\omega), a \rangle d\mu(\omega) + \|a\|^2$$
$$= \int_{\Omega} \|f(\omega)\|^2 d\mu(\omega) - 2\Re \left\langle \int_{\Omega} f(\omega) d\mu(\omega), a \right\rangle + \|a\|^2.$$

□

The next result extend problem 6.7 in [11] to the case of Hilbert spaces; for some concrete applications, see also [6, Problems III. 4.4–4.6] where, in the proofs, the authors use the law of large numbers.

Proposition 2. *Let X be a closed bounded convex non-empty set in a Hilbert space H, E a non-null Banach space. Let $(\Omega_n, \Sigma_n, \mu_n)_{n\in\mathbb{N}}$ be a sequence of probability measure spaces, $\varphi_n : \Omega_n \to X$ a sequence of μ_n-Bochner integrable functions with $\lim_{n\to\infty} \int_{\Omega_n} \varphi_n d\mu_n = a \in X$. Then the following assertions are equivalent:*

(i) $\lim_{n\to\infty} \int_{\Omega_n} \|\varphi_n\|^2 d\mu_n = \|a\|^2$.
(ii) *for each uniformly continuous and bounded function $f : X \to E$, the following equality holds*

$$\lim_{n\to\infty} \int_{\Omega_n} f(\varphi_n(\omega_n)) d\mu_n(\omega_n) = f\left(\lim_{n\to\infty} \int_{\Omega_n} \varphi_n d\mu_n\right) \text{ in norm of } E.$$

Proof. (i)⇒(ii). Let n be a natural number. Then since $\mu_n(\Omega_n) = 1$ we have

$$\int_{\Omega_n} \|\varphi_n(\omega_n) - a\|_H d\mu_n(\omega_n) \le \left(\int_{\Omega_n} \|\varphi_n(\omega_n) - a\|_H^2 d\mu_n(\omega_n)\right)^{\frac{1}{2}}$$

and since H is a Hilbert space, by Remark 2, we deduce

$$\int_{\Omega_n} \|\varphi_n(\omega_n) - a\|_H d\mu_n(\omega_n)$$
$$\le \sqrt{\int_{\Omega_n} \|\varphi_n\|_H^2 d\mu_n - 2\Re \left\langle \int_{\Omega} \varphi_n(\omega_n) d\mu(\omega_n), a \right\rangle + \|a\|^2}.$$

By the hypothesis (i)

$$\lim_{n\to\infty} \left(\int_{\Omega_n} \|\varphi_n\|_H^2 d\mu_n - 2\Re \left\langle \int_{\Omega} \varphi_n(\omega_n) d\mu(\omega_n), a \right\rangle + \|a\|^2\right) = 0$$

and then $\lim_{n\to\infty} \int_{\Omega_n} \|\varphi_n(\omega_n) - a\|_H d\mu_n(\omega_n) = 0$. From Proposition 1 we get (ii).

(ii)⇒(i). Let $e \in E$ with $\|e\| = 1$. Take in (ii) $f : X \to E$, $f(x) = \|x\|^2 e$ and note that since X is bounded, f is Lipschitz, thus uniformly continuous (obvious bounded). □

Corollary 1. *Let H be a Hilbert space, $(x_{nk})_{n\in\mathbb{N},\ 1\le k\le n} \subset H$ be a triangular matrix such $M = \sup_{n\in\mathbb{N}} (\|x_{n1}\| + \cdots + \|x_{nn}\|) < \infty$. Let also E be a non-null Banach space. Then the following assertions are equivalent:*

(i) $\lim_{n\to\infty} \left(\|x_{n1}\|^2 + \cdots + \|x_{nn}\|^2\right) = 0$.

(ii) *For each uniformly continuous and bounded function $f : \overline{B}(0, M) \to E$, the following equality holds*

$$\lim_{n\to\infty} \int_{[0,1]} f(x_{n1} r_1(t) + \cdots + x_{nn} r_n(t))\, dt = f(0) \text{ in norm of } E.$$

Proof. Let $\varphi_n : [0,1] \to \overline{B}(0, M)$ be defined by $\varphi_n(t) = x_{n1} r_1(t) + \cdots + x_{nn} r_n(t)$. Then $\int_{[0,1]} \varphi_n(t)\, dt = 0$ and by orthonormality of Rademacher functions and since H is a Hilbert space, $\int_{[0,1]} \|\varphi_n(t)\|_H^2\, dt = \|x_{n1}\|^2 + \cdots + \|x_{nn}\|^2$. Thus $\lim_{n\to\infty} \int_{[0,1]} \|\varphi_n(t)\|_H^2\, dt = 0$ if and only if $\lim_{n\to\infty} \big(\|x_{n1}\|^2 + \cdots + \|x_{nn}\|^2\big) = 0$. The equivalence between (i) and (ii) follows from Proposition 2. □

A natural question is whether the result in Proposition 2 is true in arbitrary Banach spaces. As we will show in the sequel, in general, the answer is no. We recall Tannery's theorem, see [2, page 123]: If $|a_{nk}| \le b_k$ for all natural numbers n and k, the series $\sum_{k=1}^{\infty} b_k$ is convergent and for all $k \in \mathbb{N}$, $\lim_{k\to\infty} a_{nk} \in \mathbb{K}$, then $\lim_{n\to\infty} (\sum_{k=1}^{\infty} a_{nk}) = \sum_{k=1}^{\infty} \lim_{n\to\infty} a_{nk}$.

Proposition 3. *Let $1 \le p < \infty$, $a = (a_k)_{k\in\mathbb{N}} \in l_p$,*

$$C_a = \left\{(t_k)_{k\in\mathbb{N}} \in l_p \mid |t_k| \le |a_k|,\ \forall k \in \mathbb{N}\right\}$$

and $(x_{nk})_{n,k\in\mathbb{N}}$ a double sequence of scalars such that

$$|x_{nk}| \le |a_k|,\ \forall n, k \in \mathbb{N}.$$

Let $(\Omega_n, \Sigma_n, \mu_n)_{n\in\mathbb{N}}$ be a sequence of probability measure spaces, $\varphi_{nk} : \Omega_n \to \mathbb{K}$ a double sequence of μ_n-measurable functions such that

$$|\varphi_{nk}(\omega_n)| \le |a_k|,\ \forall n, k \in \mathbb{N}.$$

Let also E be a non-null Banach space. Then the following assertions are equivalent:

(i) $\lim_{n\to\infty} \int_{\Omega_n} |\varphi_{nk}(\omega_n) - x_{nk}|^p\, d\mu_n(\omega_n) = 0$ *for all $k \in \mathbb{N}$.*

(ii) *for each continuous function $f : C_a \to E$ the following equality holds*

$$\lim_{n\to\infty} \int_{\Omega_n} \left\|f\left((\varphi_{nk}(\omega_n))_{k\in\mathbb{N}}\right) - f\left((x_{nk})_{k\in\mathbb{N}}\right)\right\|_E^p d\mu_n(\omega_n) = 0.$$

In particular, if (i), holds then,

$$\lim_{n\to\infty} \left[\int_{\Omega_n} f\left((\varphi_{nk}(\omega_n))_{k\in\mathbb{N}}\right) d\mu_n(\omega_n) - f\left((x_{nk})_{k\in\mathbb{N}}\right)\right] = 0 \text{ in norm of } E.$$

Proof. First note that since $a \in l_p$, by the well-known characterization of compact sets in l_p, see [3, Exercise 6, page 6], C_a is a compact set in l_p and obvious a convex set. Also by the hypothesis, $x_n = (x_{nk})_{k\in\mathbb{N}} \in C_a$. For each natural number n, let $\varphi_n : \Omega_n \to C_a$ be defined by $\varphi_n(\omega_n) = (\varphi_{nk}(\omega_n))_{k\in\mathbb{N}}$ and note that φ_n are μ_n-measurable. For all natural numbers n we have

$$\int_{\Omega_n} \|\varphi_n(\omega_n) - x_n\|_{l_p}^p \, d\mu_n(\omega_n) = \sum_{k=1}^{\infty} \int_{\Omega_n} |\varphi_{nk}(\omega_n) - x_{nk}|^p \, d\mu_n(\omega_n). \tag{5}$$

We have also $|\varphi_{nk}(\omega_n) - x_{nk}| \le |\varphi_{nk}(\omega_n)| + |x_{nk}| \le 2|a_k|$, and by integration

$$\int_{\Omega_n} |\varphi_{nk}(\omega_n) - x_{nk}|^p \, d\mu_n(\omega_n) \le 2^p |a_k|^p, \forall n, k \in \mathbb{N}.$$

(i)⇒(ii). By (i) and Tannery's theorem from (5) it follows that

$$\lim_{n\to\infty} \int_{\Omega_n} \|\varphi_n(\omega_n) - x_n\|_{l_p}^p \, d\mu_n(\omega_n) = 0.$$

By Remark 1(i) and Theorem 1 we get (ii).

(ii)⇒(i). Let $e \in E$ be such that $\|e\| = 1$. For all $k \in \mathbb{N}$ let $f : C_a \to E$ be defined by $f(t_1, ..., t_n, ...) = t_k \|e\|$ and note that

$$f\left((\varphi_{nk}(\omega_n))_{k\in\mathbb{N}}\right) - f\left((x_{nk})_{k\in\mathbb{N}}\right) = (\varphi_{nk}(\omega_n) - x_{nk})\, e.$$

From (ii) we have

$$\lim_{n\to\infty} \int_{\Omega_n} \left\| f\left((\varphi_{nk}(\omega_n))_{k\in\mathbb{N}}\right) - f\left((x_{nk})_{k\in\mathbb{N}}\right)\right\|_E^p d\mu_n(\omega_n) = 0$$

that is (i). Moreover, if (i) holds, then (ii) holds and from

$$\begin{aligned}
&\left\| \int_{\Omega_n} f\left((\varphi_{nk}(\omega_n))_{k\in\mathbb{N}}\right) d\mu_n(\omega_n) - f\left((x_{nk})_{k\in\mathbb{N}}\right)\right\|_E \\
&= \left\| \int_{\Omega_n} \left[f\left((\varphi_{nk}(\omega_n))_{k\in\mathbb{N}}\right) - f\left((x_{nk})_{k\in\mathbb{N}}\right)\right] d\mu_n(\omega_n)\right\|_E \\
&\le \int_{\Omega_n} \left\| f\left((\varphi_{nk}(\omega_n))_{k\in\mathbb{N}}\right) - f\left((x_{nk})_{k\in\mathbb{N}}\right)\right\|_E d\mu_n(\omega_n) \\
&\le \left(\int_{\Omega_n} \left\| f\left((\varphi_{nk}(\omega_n))_{k\in\mathbb{N}}\right) - f\left((x_{nk})_{k\in\mathbb{N}}\right)\right\|_E^p d\mu_n(\omega_n)\right)^{\frac{1}{p}}
\end{aligned}$$

we get the conclusion. □

Corollary 2. *Let* $1 \le p < \infty$, $a = (a_k)_{k\in\mathbb{N}} \in l_p$,

$$C_a = \left\{ (t_k)_{k\in\mathbb{N}} \in l_p \mid |t_k| \le |a_k|, \ \forall k \in \mathbb{N} \right\}.$$

Let $(\Omega_n, \Sigma_n, \mu_n)_{n\in\mathbb{N}}$ *be a sequence of probability measure spaces,* $\varphi_{nk} : \Omega_n \to \mathbb{K}$ *a double sequence of* μ_n*-measurable functions such that*

$$|\varphi_{nk}(\omega_n)| \le |a_k|, \forall n, k \in \mathbb{N}.$$

Let also E be a non-null Banach space. Then the following assertions are equivalent:

(i) $\lim_{n\to\infty}\int_{\Omega_n}|\varphi_{nk}(\omega_n)|^p\,d\mu_n(\omega_n)=0$ *for all* $k\in\mathbb{N}$.

(ii) *for each continuous function* $f:C_a\to E$ *the following equality holds*

$$\lim_{n\to\infty}\int_{\Omega_n}f\left((\varphi_{nk}(\omega_n))_{k\in\mathbb{N}}\right)d\mu_n(\omega_n)=f(0)\ \text{ in norm of } E.$$

Proof. (i)⇒(ii). From (i), by Proposition 3 we get (ii).

(ii)⇒(i). Let $e\in E$ be such that $\|e\|=1$. For all $k\in\mathbb{N}$ let $f:C_a\to E$ be defined by $f(t_1,...,t_n,...)=|t_k|^p\,e$. Then, $f(0)=0$ and

$$\int_{\Omega_n}f\left((\varphi_{nk}(\omega_n))_{k\in\mathbb{N}}\right)d\mu_n(\omega_n)=\left(\int_{\Omega_n}|\varphi_{nk}(\omega_n)|^p\,d\mu_n(\omega_n)\right)e.$$

From (ii) we get (i). □

Corollary 3. *Let* $1\le p<\infty$, $a=(a_k)_{k\in\mathbb{N}}\in l_p$ *and*

$$C_a=\left\{(t_k)_{k\in\mathbb{N}}\in l_p\mid |t_k|\le|a_k|,\ \forall k\in\mathbb{N}\right\}.$$

Let $(\alpha_{nk})_{n,k\in\mathbb{N}}\subset\mathbb{K}$ *be such that* $|\alpha_{n1}|+\cdots+|\alpha_{nk}|\le|a_k|$, $\forall n,k\in\mathbb{N}$. *Let also E be a non-null Banach space. The following assertions are equivalent:*

(i) $\lim_{n\to\infty}\sqrt{|\alpha_{n1}|^2+\cdots+|\alpha_{nk}|^2}=0$ *for all* $k\in\mathbb{N}$.

(ii) *for each continuous function* $f:C_a\to E$ *the following equality holds*

$$\lim_{n\to\infty}\int_0^1 f\left((\alpha_{n1}r_1(t)+\cdots+\alpha_{nk}r_n(t))_{k\in\mathbb{N}}\right)dt=f(0)\ \text{ in norm of } E.$$

Proof. Let $\varphi_{nk}:[0,1]\to\mathbb{K}$ be defined by $\varphi_{nk}(t)=\alpha_{n1}r_1(t)+\cdots+\alpha_{nk}r_k(t)$ and note that $|\varphi_{nk}(t)|\le|a_k|$, $\forall n,k\in\mathbb{N}$. Also $\int_0^1\varphi_{nk}(t)\,dt=0$, $\forall n,k\in\mathbb{N}$. By Khinchin's inequality we have

$$A_p\sqrt{|\alpha_{n1}|^2+\cdots+|\alpha_{nk}|^2}\le\left(\int_0^1|\varphi_{nk}(t)|^p\,dt\right)^{\frac{1}{p}}\le B_p\sqrt{|\alpha_{n1}|^2+\cdots+|\alpha_{nk}|^2}$$

where A_p, B_p are Khinchin's constants, see [5, page 10]. Then for all $k\in\mathbb{N}$, $\lim_{n\to\infty}\int_0^1|\varphi_{nk}(t)|^p\,dt=0$ if and only if $\lim_{n\to\infty}\sqrt{|\alpha_{n1}|^2+\cdots+|\alpha_{nk}|^2}=0$. The equivalence between (i) and (ii) follows from Corollary 2. □

The next result show that Proposition 2 does not hold, in general Banach spaces.

Proposition 4. *Let* $2\le p<\infty$, $m=[p/2]$ *be the integer part of* $p/2$, $a=(a_k)_{k\in\mathbb{N}}\in l_p$ *and*

$$C_a=\left\{(t_k)_{k\in\mathbb{N}}\in l_p\mid |t_k|\le|a_k|,\ \forall k\in\mathbb{N}\right\}.$$

Let $(\Omega_n, \Sigma_n, \mu_n)_{n\in\mathbb{N}}$ *be a sequence of probability measure spaces,* $\varphi_{nk} : \Omega_n \to \mathbb{R}$ *a double sequence of* μ_n*-measurable functions for all* $n, k \in \mathbb{N}$ *such that*

$$|\varphi_{nk}(\omega_n)| \leq |a_k|, \forall n, k \in \mathbb{N}$$

and

$$\lim_{n\to\infty} \int_{\Omega_n} \varphi_{nk}(\omega_n)\, d\mu_n(\omega_n) = \lambda_k \in \mathbb{K} \text{ for all } k \in \mathbb{N}.$$

Let also E *be a non-null Banach space. Then the following assertions are equivalent:*

(i) $\lim_{n\to\infty} \int_{\Omega_n} [\varphi_{nk}(\omega_n)]^i \, d\mu_n(\omega_n) = (\lambda_k)^i$ *for all* $2 \leq i \leq 2m$.

(ii) *For each continuous function* $f : C_a \to E$ *the following equality holds*

$$\begin{aligned}&\lim_{n\to\infty} \int_{\Omega_n} f\left((\varphi_{nk}(\omega_n))_{k\in\mathbb{N}}\right) d\mu_n(\omega_n) \\ &\quad = f\left(\left(\lim_{n\to\infty} \int_{\Omega_n} \varphi_{nk}(\omega_n)\, d\mu_n(\omega_n)\right)_{k\in\mathbb{N}}\right) \text{ in norm of } E.\end{aligned}$$

Proof. First note that $\left|\int_{\Omega_n} \varphi_{nk} d\mu_n\right| \leq \int_{\Omega_n} |\varphi_{nk}| \, d\mu_n \leq |a_k|, \ , \forall n, k \in \mathbb{N}$. Passing to the limit as $n \to \infty$ we get $|\lambda_k| \leq |a_k|$ for all $k \in \mathbb{N}$. Then $\lambda = (\lambda_k)_{k\in\mathbb{N}} \in C_a$, $x_n = \left(\int_{\Omega_n} \varphi_{nk}(\omega_n)\, d\mu_n(\omega_n)\right)_{k\in\mathbb{N}} \in C_a$ and as we will prove next $x_n \to \lambda$ in norm of l_p. Indeed, for each natural number n we have

$$\|x_n - \lambda\|_{l_p}^p = \sum_{k=1}^{\infty} \left| \int_{\Omega_n} \varphi_{nk}(\omega_n)\, d\mu_n(\omega_n) - \lambda_k \right|^p$$

and $\left|\int_{\Omega_n} \varphi_{nk}(\omega_n)\, d\mu_n(\omega_n) - \lambda_k\right| \leq 2|\lambda_k| \leq 2|a_k|$. By Tannery's theorem, $\lim_{n\to\infty} \|x_n - \lambda\|_{l_p} = 0$.

(i)⇒(ii). For each $n, k \in \mathbb{N}$, $\left|\varphi_{nk}(\omega_n) - \int_{\Omega_n} \varphi_{nk}(\omega_n)\, d\mu_n(\omega_n)\right| \leq 2|a_k|$ and thus since $2m \leq p$,

$$\left|\varphi_{nk}(\omega_n) - \int_{\Omega_n} \varphi_{nk} d\mu_n\right|^p \leq (2|a_k|)^{p-2m} \left|\varphi_{nk}(\omega_n) - \int_{\Omega_n} \varphi_{nk} d\mu_n\right|^{2m}.$$

Here we use: if $0 \leq \alpha \leq \beta$, $x_1 \leq x_2$, then $\alpha^{x_2} \leq \alpha^{x_1}\beta^{x_2 - x_1}$. Since for $\alpha \in \mathbb{R}$, $|\alpha|^{2m} = \alpha^{2m}$ by integration

$$\begin{aligned}&\int_{\Omega_n} \left|\varphi_{nk}(\omega_n) - \int_{\Omega_n} \varphi_{nk} d\mu_n\right|^p d\mu_n(\omega_n) \\ &\quad \leq (2|a_k|)^{p-2m} \int_{\Omega_n} \left(\varphi_{nk}(\omega_n) - \int_{\Omega_n} \varphi_{nk} d\mu_n\right)^{2m} d\mu_n(\omega_n). \qquad (6)\end{aligned}$$

Now, by the Newton binomial formula, and the hypothesis (i) we have

$$\lim_{n\to\infty}\int_{\Omega_n}\left(\varphi_{nk}\left(\omega_n\right)-\int_{\Omega_n}\varphi_{nk}d\mu_n\right)^{2m}d\mu_n\left(\omega_n\right)$$
$$=\sum_{i=0}^{2m}\left(-1\right)^i C_{2m}^i\lim_{n\to\infty}\left(\int_{\Omega_n}\varphi_{nk}^{2m-i}\left(\omega_n\right)d\mu_n\left(\omega_n\right)\right)\left(\int_{\Omega_n}\varphi_{nk}\left(\omega_n\right)d\mu_n\left(\omega_n\right)\right)^i$$
$$=\lambda_k^{2m}\sum_{i=0}^{2m}\left(-1\right)^i C_{2m}^i=0. \tag{7}$$

From (6) and (7) we deduce that for all natural numbers k

$$\lim_{n\to\infty}\int_{\Omega_n}\left|\varphi_{nk}\left(\omega_n\right)-\int_{\Omega_n}\varphi_{nk}d\mu_n\right|^p d\mu_n\left(\omega_n\right)=0.$$

Then, from Proposition 3, $\lim_{n\to\infty}\left[\int_{\Omega_n}f\left(\left(\varphi_{nk}\left(\omega_n\right)\right)_{k\in\mathbb{N}}\right)d\mu_n\left(\omega_n\right)-f\left(x_n\right)\right]$ $=0$ in norm of E. Since $x_n\to\lambda$ in norm of l_p and f is continuous, $\lim_{n\to\infty}f\left(x_n\right)$ $=f\left(\lambda\right)$ in norm of E and (ii) follows.

(ii)⇒(i). Let $e\in E$ with $\|e\|=1$. For each $2\leq i\leq 2m$ let $f_i: C_a\to E$, $f_i\left(\left(t_k\right)_{k\in\mathbb{N}}\right)=\left(t_i\right)^i e$. Then $f\left(\left(\varphi_{nk}\left(\omega_n\right)\right)_{k\in\mathbb{N}}\right)=\left(\varphi_{ni}\left(\omega_n\right)\right)^i e$ and $f\left(\left(\lambda_k\right)_{k\in\mathbb{N}}\right)=\left(\lambda_i\right)^i e$. By (ii) we get (i). □

We show now that $\varphi_{nk}:[0,1]^n\to[0,1]$, $\varphi_{nk}\left(x_1,...,x_n\right)=\frac{x_1^{k-1}+\cdots+x_n^{k-1}}{n}$ verifies the condition (i) in Proposition 4. Indeed, we have

$$\int_{[0,1]^n}\varphi_{nk}\left(x_1,...,x_n\right)dx_1\cdots dx_n=\frac{1}{k}$$

and, as it follows from Proposition 2 for $H=\mathbb{R}$,

$$\lim_{n\to\infty}\int_{[0,1]^n}\left[\varphi_{nk}\left(x_1,...,x_n\right)\right]^i dx_1\cdots dx_n=\frac{1}{k^i}$$

for all natural numbers i, see also [6,11].

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Dumitru Popa
Department of Mathematics
Ovidius University of Constanta
Bd. Mamaia 124
900527 Constanţa
Romania
e-mail: dpopa@univ-ovidius.ro

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