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Functional Equations Characterizing Derivations: A Synthesis

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Abstract. Many articles have been written about functional equations characterizing derivations on integral domains (or sometimes commutative rings) of characteristic 0. Here we synthesize several recent results by various authors and extend them by showing that they hold more generally on commutative rings of sufficiently large finite characteristic.

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1. Introduction

Solutions of functional equations linking additive functions have a long and rich history, a sampling of which can be found in [1–3,6,8,10] and their references. As in other recent papers [2,3,6] we consider functional equations of the form

$$\sum_{k=1}^{n} x^{p_k} f_k(x^{q_k}) = 0, \qquad x \in R, \tag{1}$$

for additive functions $f_k : R \to S$, where S is a commutative ring with subring R. Here p_k and q_k are nonnegative integers. Under appropriate conditions on the ring S, it has been shown in [2,3,6] that the expected solutions are linear combinations of derivations of various orders and linear functions. The present author [1–3] took a direct approach, while Gselmann et al. [6] worked with a multivariate version. Here we unify those results while at the same time weakening the assumptions on the ring S.

We assume without further mention that our rings have a multiplicative identity denoted by 1. That a function $f : R \to S$ is *additive* means

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in R$. A *derivation* from R into S is an additive function $d : R \to S$ which satisfies also the *(Leibniz) product rule*

$$d(xy) = xd(y) + d(x)y,$$

for all $x, y \in R$. A mapping $B : R \times R \to S$ is called a *bi-derivation* if B is a derivation in each variable when the other variable is fixed. We define derivations of all nonnegative (integer) orders inductively as follows. Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 1. Let S be a commutative ring and let R be a subring of S. The zero function on R is the only *derivation of order 0*. For each $n \in \mathbb{N}$, suppose we have defined derivations of order at most $n-1$. If $f : R \to S$ is additive, then f is said to be a *derivation of order at most* n if there exists a function $B : R \times R \to S$ which is a derivation of order at most $n-1$ in each variable such that

$$f(xy) - xf(y) - f(x)y = B(x, y), \qquad x, y \in R. \tag{2}$$

We refer to such a function B as a *bi-derivation of order at most* $n-1$.

For each $n \in \mathbb{N}_0$ let $\mathcal{D}_n(R, S)$ denote the set of all derivations of order at most n from R into S. If the meaning of R and S are clear we may abbreviate $\mathcal{D}_n(R, S)$ by $\mathcal{D}_n$.

It is evident that the definition of derivation of order at most 1 agrees with the previous definition of derivation.

The terminology "derivation of order at most n" differs slightly from that of other papers on this topic. Many authors (including the present one) have referred to an element of $\mathcal{D}_n$ as a derivation of order n, and an element of $\mathcal{D}_n \backslash \mathcal{D}_{n-1}$ as a *nontrivial* derivation of order n. In the present paper we say that an element of $\mathcal{D}_n$ is a derivation of order at most n, and an element of $\mathcal{D}_n \backslash \mathcal{D}_{n-1}$ is a derivation of order n (exactly). The use of the qualifier "at most" is justified because of the nesting property

$$\mathcal{D}_0(R, S) \subseteq \mathcal{D}_1(R, S) \subseteq \cdots \subseteq \mathcal{D}_n(R, S) \subseteq \cdots$$

which holds for any commutative rings $R \subseteq S$ (see for example [4,7]).

One of the primary tools used in [1–3,6] is a basic homogeneity theorem which permits the separation of Eq. (1) into constituent equations that are homogeneous of different degrees. As an example, the equation

$$x^3 f_1(x) + x^2 f_2(x) + x^2 f_3(x^2) + x f_4(x^2) = 0$$

can be separated into the two equations

$$x^3 f_1(x) + x^2 f_3(x^2) = 0 \quad \text{and} \quad x^2 f_2(x) + x f_4(x^2) = 0$$

which are homogeneous of degree 4 and degree 3, respectively. Then various methods can be used to solve these homogeneous equations.

The fundamental homogeneity theorem (see Lemma 2.2 in [1] or Lemma 3 in [3]) enabling this separation of (1) into homogeneous equations required that R must be an integral domain of characteristic 0 with field of fractions S. These restrictions are lifted in the present paper, where we shall prove in Theorem 4 that R does not need to be an integral domain, nor must it have characteristic 0. This significantly broadens the applicability of the method.

We observe that an additive function f belongs to $\mathcal{D}_n$ if and only if

$$\begin{aligned} f(x_1 x_2 \cdots x_{n+1}) &- \sum_{1\leq i\leq n+1} x_i f(x_1 \cdots \widehat{x_i} \cdots x_{n+1}) \\ &+ \sum_{1\leq i<j\leq n+1} x_i x_j f(x_1 \cdots \widehat{x_i} \cdots \widehat{x_j} \cdots x_{n+1}) - \cdots \\ &+ (-1)^n \sum_{1\leq i\leq n+1} x_1 \cdots \widehat{x_i} \cdots x_{n+1} f(x_i) = 0, \end{aligned}$$

for all $x_1, \ldots, x_{n+1} \in R$. Here the hat $\widehat{\ }$ over an argument indicates the omission of that argument. This fact about derivations is proved by a simple induction argument as can be found for example in Proposition 4.5 of [1] but goes back further (cf. [8,10]). We will use this fact freely.

Note that the equation in the previous paragraph can be written more concisely as

$$\sum_{j=0}^{n} (-1)^j \sum_{\mathrm{card}(I)=j} \left(\prod_{i\in I} x_i \right) f \left(\prod_{r\in\{1,\ldots,n+1\}\setminus I} x_r \right) = 0$$

for all $x_1, \ldots, x_{n+1} \in R$, where the second summation is taken over all subsets of cardinality j of the index set $\{1, \ldots, n+1\}$.

Gselmann et al. [6] proved several multivariate characterizations of derivations and derived from them some of the univariate results proved in [1,2] by the present author. The authors of [6] also stated that the multivariate characterizations are more effective than their univariate versions for the purpose of determining precise forms of the unknown functions. While we tend to agree generally with this statement, it is not universally true as we will demonstrate with examples.

In the present work we combine several of the results of [1–3] and [6] into a common framework. The most significant aspect of our paper is the extension of previous results by the weakening of assumptions on the rings R and S. In the next section we present a stronger fundamental homogeneity result (Theorem 4) for equations of the form (1). This new homogeneity theorem is the foundation for the paper's main results which appear in the third section. The fourth and final section contains some remarks and examples.

2. Fundamental Homogeneity Theorem

Let G and H be commutative semigroups. A map $\phi : G \to H$ is *homogeneous of degree* $n \in \mathbb{N}_0$ provided that

$$\phi(kx) = k^n \phi(x), \qquad x \in G,\ k \in \mathbb{N}.$$

In case R and S are rings and $\phi : R \to S$, we apply the previous definition to the additive groups of the rings. It was shown in [1] that if S is an integral domain of characteristic 0, the functions $h_j : R \to S$ are homogeneous of degree j for $0 \leq j \leq n$, and

$$\sum_{j=0}^{n} h_j(x) = 0, \qquad x \in R,$$

then $h_j = 0$ for each $j = 0, 1, \ldots, n$. That result also applies to functions of more than one variable.

Here we prove an extension of that result, borrowing from the ideas used by Gselmann et al. [6]. We will show that for equations of form (1) there is no need to assume that S is an integral domain nor that S is a vector space over the field $\mathbb{Q}$ of rationals. What is essential is an assumption about divisibility by sufficiently large positive integers in S.

We use basic facts about multi-additive functions as found for example in the work [9] of Székelyhidi. Let G and H be commutative semigroups and $n \in \mathbb{N}$. A function $A : G^n \to H$ is said to be n-*additive* if A is a homomorphism of G into H in each variable. We extend this definition to the case $n = 0$ by defining $G^0 := G$ and saying that a function $A : G^0 \to H$ is 0-*additive* if it is constant. The *diagonalization* of an n-additive function $A : G^n \to H$ is the function $A^* : G \to H$ defined by

$$A^*(x) := A(x, \ldots, x), \qquad x \in G.$$

Observe that an n-additive function $A : G^n \to H$ is homogeneous of degree 1 in each variable; that is,

$$A(x_1, \ldots, x_{j-1}, kx_j, x_{j+1}, \ldots, x_n) = kA(x_1, \ldots, x_n), \qquad x_1, \ldots, x_n \in G,\ k \in \mathbb{N},$$

for each $j \in \{1, \ldots, n\}$. Its diagonalization A^* is homogeneous of degree n:

$$A^*(kx) = k^n A(x), \qquad x \in G,\ k \in \mathbb{N}.$$

If G is a commutative semigroup, H is a commutative group, and $y \in G$, then the *difference operator* Δ_y acting on a function $f : G \to H$ is defined by

$$\Delta_y f(x) := f(x + y) - f(x), \qquad x \in G.$$

Two results of fundamental importance in the theory of multi-additive symmetric functions are the *polarization formula*, which is next, and the corollary which follows it.

Proposition 2. *Let G be a commutative semigroup, H a commutative group, and $n \in \mathbb{N}_0$. If $A : G^n \to H$ is n-additive and symmetric, then for all $y_1, \ldots, y_m \in G$ we have*

$$\Delta_{y_1,\ldots,y_m} A^* = \begin{cases} n! A(y_1, \ldots, y_n) & \text{for } m = n \\ 0 & \text{for } m > n. \end{cases}$$

Corollary 3. *Let G be a commutative semigroup, H a commutative group, and $n \in \mathbb{N}_0$. Suppose multiplication by $n!$ is either surjective in G or injective in H. If $A : G^n \to H$ is n-additive and symmetric, then $A^* = 0$ implies $A = 0$.*

Now we use these tools to construct an improved homogeneity theorem for equations of form (1). For convenience of notation we adopt the convention that $x^0 = 1$ for all $x \in R$. We also abbreviate the double subscripted objects $p_{j,k}$ and $f_{j,k}$ respectively as p_{jk} and f_{jk}.

Theorem 4. *Let $n \in \mathbb{N}_0$, let S be a commutative ring with subring R, and let multiplication by $n!$ be bijective in S. For each $j \in \{0, \ldots, n\}$, let $m_j \in \mathbb{N}_0$ and suppose $h_j : R \to S$ has the form*

$$h_j(x) = \sum_{k=1}^{m_j} x^{p_{jk}} f_{jk}(x^{j-p_{jk}}), \qquad x \in R, \tag{3}$$

where $f_{jk} : R \to S$ is additive and $p_{j1} \ldots, p_{jm_j} \in \{0, \ldots, j\}$. If

$$\sum_{j=0}^{n} h_j = 0,$$

then $h_j = 0$ for every $j \in \{0, \ldots, n\}$.

Proof. The proof is by induction on n. For $n = 0$ there is nothing to prove. Now let $N \in \mathbb{N}$, suppose the statement is true for $0 \leq n \leq N - 1$, and let S be a commutative ring with subring R such that multiplication by $N!$ is bijective in S. Also suppose for each $0 \leq j \leq N$ that h_j has the form (3) and

$$\sum_{j=0}^{N} h_j = 0. \tag{4}$$

Define for each $j \in \{0, \ldots, N\}$ the function $\Phi_j : R^j \to S$ by

$$\Phi_j(x_1, \ldots, x_j) := \sum_{k=1}^{m_j} \frac{1}{\binom{j}{p_{jk}}} \sum_{\mathrm{card}(I)=p_{jk}} \left(\prod_{i \in I} x_i \right) f_{jk} \left(\prod_{r \in \{1,\ldots,j\} \setminus I} x_r \right)$$

for all $x_1, \ldots, x_j \in R$, where the second summation is taken over all subsets of cardinality p_{jk} of the index set $\{1, \ldots, j\}$. This definition is possible because of the bijectivity of multiplication by $N!$ in S. By the additivity of f_{jk}, for each j the function Φ_j is a symmetric j-additive function with diagonal

$$\Phi_j^*(x) = \Phi_j(x, \ldots, x) = \sum_{k=1}^{m_j} x^{p_{jk}} f_{jk}(x^{j-p_{jk}}) = h_j(x), \qquad x \in R.$$

Thus by hypothesis we have

$$\sum_{j=0}^{N} \Phi_j^* = 0.$$

Now for any $y_1, \ldots, y_N \in R$, applying $\Delta_{y_1,\ldots,y_N}$ to the previous equation we get by the polarization formula (Proposition 2)

$$0 = \Delta_{y_1,\ldots,y_N} \sum_{j=0}^{N} \Phi_j^* = \Delta_{y_1,\ldots,y_N} \Phi_N^* = N! \Phi_N(y_1, \ldots, y_n).$$

Since multiplication by $N!$ is injective in S is we conclude that $\Phi_N = 0$. Hence $h_N = \Phi_N^* = 0$ and Eq. (4) reduces to

$$\sum_{j=0}^{N-1} h_j = 0.$$

By the inductive hypothesis the proof is finished. □

This theorem, which we will call the Fundamental Homogeneity Theorem, allows us to restrict our attention to *homogeneous* functional equations of the form (1). That is, we need only consider such equations in which there is an $h \in \mathbb{N}_0$ with $p_k + q_k = h$ for all $1 \leq k \leq n$, since any non-homogeneous equation of form (1) can be decomposed into a set of such homogeneous equations.

We will see also in the next section that both the Fundamental Homogeneity Theorem itself and the idea of its proof enable us to strengthen many known results about functional equations characterizing derivations.

3. Main Results

In this section we combine and extend known results from [1–3,6] by weakening the assumptions on R and S.

The first result strengthens Proposition 5 of [3], where it was stated for the case that R is an integral domain of characteristic zero and S is the fraction field of R (see also Corollary 2 in [6]). Here we do not assume that R is an integral domain, and we allow rings of sufficiently large finite characteristic.

Theorem 5. *Let $n \in \mathbb{N}$, let S be a commutative ring with subring R, and suppose $f : R \to S$ is additive. Also suppose multiplication by $(n+1)!$ is either surjective in R or injective in S. Then $f \in \mathcal{D}_n$ if and only if*

$$\sum_{j=0}^{n}(-1)^j\binom{n+1}{j}x^j f(x^{n+1-j})=0, \qquad x\in R. \tag{5}$$

Proof. First suppose Eq. (5) holds and define $\Phi : R^{n+1}\to S$ by

$$\Phi(x_1,\ldots,x_{n+1}) := \sum_{j=0}^{n}(-1)^j\sum_{\mathrm{card}(I)=j}\left(\prod_{i\in I}x_i\right) f\left(\prod_{r\in\{1,\ldots,n+1\}\setminus I}x_r\right)$$

for all $x_1,\ldots,x_{n+1}\in R$, where the second summation is taken over all subsets of cardinality j of the index set $\{1,\ldots,n+1\}$. Then Φ is $(n+1)$-additive and symmetric. Moreover

$$\Phi^*(x)=\Phi(x,\ldots,x)=\sum_{j=0}^{n}(-1)^j\binom{n+1}{j}x^j f(x^{n+1-j})=0$$

for all $x\in R$. Thus by Corollary 3 we get $\Phi=0$, hence $f\in\mathcal{D}_n$.

Conversely, if $f\in\mathcal{D}_n$ then $\Phi=0$, so $\Phi^*=0$ and therefore f satisfies (5). □

We illustrate with an example on a ring of finite characteristic.

Example 6. Let K be a finite field of order $q=p^r$ where $r\in\mathbb{N}$ and p is a prime number larger than $n+1$. Consider the polynomial ring $R=K[x]$, and let $S=R$. Then $\mathrm{char}(R)=\mathrm{char}(K)=p>n+1$. Therefore multiplication by $(n+1)!$ is bijective in R, hence $f:R\to R$ satisfies Eq. (5) if and only if $f\in\mathcal{D}_n$.

The next result generalizes Theorem 5 from [2], where it was stated for the case that R is an integral domain with $\mathrm{char}(R)=0$. It also extends Corollary 4 and Theorem 5 in [6], where it is stated for the case when R is a linear space over $\mathbb{Q}$ and $S=R$.

Theorem 7. *Let $n\in\mathbb{N}$, let S be a commutative ring with subring R, and suppose $f_i:R\to S$ is additive with $f_i(1)=0$ for each $i\in\{1,\ldots,n+1\}$. In addition suppose multiplication by $(n+1)!$ is bijective in S. Then the following are equivalent:*

$$\sum_{j=0}^{n}x^j f_{n+1-j}(x^{n+1-j})=0, \qquad x\in R. \tag{6}$$

$$\sum_{j=0}^{n}\frac{1}{\binom{n+1}{j}}\sum_{\mathrm{card}(I)=j}\left(\prod_{i\in I}x_i\right) f_{n+1-j}\left(\prod_{r\in\{1,\ldots,n+1\}\setminus I}x_r\right)=0, \qquad x_1,\ldots,x_{n+1}\in R. \tag{7}$$

$$f_{n+1-j}=(-1)^j\sum_{k=0}^{j}\binom{n+1-j+k}{k}d_{n-j+k} \qquad (0\le j\le n), \tag{8}$$

where $d_{n-j+k}\in\mathcal{D}_{n-j+k}$.

In short, $f_1,\ldots,f_{n+1}$ satisfy (6) or (7) if and only if each $f_i\in\mathcal{D}_n$.

Proof. We prove (6) ⇒ (7) ⇒ (8) ⇒ (6). Suppose $f_1, \ldots, f_{n+1}$ are additive functions with $f_i(1) = 0$ such that (6) holds. We define $\Phi : R^{n+1} \to S$ by

$$\Phi(x_1, \ldots, x_{n+1}) := \sum_{j=0}^{n} \frac{1}{\binom{n+1}{j}} \sum_{\mathrm{card}(I)=j} \left(\prod_{i \in I} x_i\right) f_{n+1-j}\left(\prod_{r \in \{1,\ldots,n+1\} \setminus I} x_r\right)$$

for all $x_1, \ldots, x_{n+1} \in R$, where the second summation is taken over all subsets of cardinality j of the index set $\{1, \ldots, n+1\}$. Then Φ is $(n+1)$-additive and symmetric. As in the proof of the previous theorem this leads immediately to $\Phi^* = 0$ and thence to $\Phi = 0$ which is (7).

The equation $\Phi = 0$ [that is (7)] is solved in Theorem 5 of [6] under the assumption that S is a linear space over $\mathbb{Q}$, however an examination of the proof shows that it only requires the ability to divide uniquely by $(n+1)!$ in S. The solution is exactly of the form (8).

The implication (8) ⇒ (6) is a straightforward computation. □

From this we get the following corollary, which encompasses Theorem 4.8 of [1] together with Theorem 4 and Corollary 3 of [6]. The second half of this result is proved in [1,6] for the case $R = S =$ an integral domain of characteristic 0. The new thing here is the case of integral domains of sufficiently large finite characteristic.

Corollary 8. *Let $n \in \mathbb{N}$, let S be a commutative ring with subring R, and suppose multiplication by $(n+1)!$ is bijective in S. Suppose also that $f : R \to S$ is additive with $f(1) = 0$. If there exist constants $a_1, \ldots, a_{n+1} \in S$ such that f satisfies either*

$$\sum_{j=0}^{n} a_{n+1-j} x^j f(x^{n+1-j}) = 0, \qquad x \in R, \tag{9}$$

or

$$\sum_{j=0}^{n} \frac{a_{n+1-j}}{\binom{n+1}{j}} \sum_{\mathrm{card}(I)=j} \left(\prod_{i \in I} x_i\right) f\left(\prod_{r \in \{1,\ldots,n+1\} \setminus I} x_r\right) = 0, \qquad x_1, \ldots, x_{n+1} \in R, \tag{10}$$

then $a_i f \in \mathcal{D}_n$ for each $i \in \{1, \ldots, n+1\}$, with

$$\sum_{k=1}^{n+1} k a_k f = 0. \tag{11}$$

Now suppose in addition that R is an integral domain and S is the field of fractions of R. Then the following are equivalent:

(i) $f \in \mathcal{D}_n$.
(ii) f *satisfies* (9) *with at least one of the constants* $a_1, \ldots, a_{n+1}$ *nonzero.*
(iii) f *is a solution of* (10) *with at least one of the constants* $a_1, \ldots, a_{n+1}$ *nonzero.*

Moreover if $f \neq 0$ then we also have

$$\sum_{k=1}^{n+1} k a_k = 0.$$

Proof. Suppose there exist constants $a_1, \dots, a_{n+1} \in S$ such that either (9) or (10) holds. Defining $f_i := a_i f$ for $1 \leq i \leq n+1$, we get respectively Eqs. (6) or (7). So by Theorem 7 we have

$$a_{n+1-j} f = (-1)^j \sum_{k=0}^{j} \binom{n+1-j+k}{k} d_{n-j+k} \tag{12}$$

for $0 \leq j \leq n$, where $d_i \in \mathcal{D}_i$ for each $i \in \{0, \dots, n\}$. In particular, $a_i f \in \mathcal{D}_n$ for each i. The appropriate weighted sum of Eq. (12) for $0 \leq j \leq n$ yields (11) in a straightforward calculation.

For the second part, suppose also that R is an integral domain, S is the field of fractions of R, and at least one of the constants $a_1, \dots, a_{n+1}$ is nonzero. We get immediately (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) from the first part. For the reverse implications we put $a_{n+1-j} = (-1)^j$ for each j. Theorem 5 shows that (i) $\Rightarrow$ (ii), and (i) $\Rightarrow$ (iii) is verified by a simple inductive argument using the definition of $\mathcal{D}_n$ as noted in the Introduction. □

Also we have the following generalization of Theorem 7 in [3]. It demonstrates how the degrees of the derivations in the solution are reduced when an equation has "gaps," *i.e.* missing terms. Specifically, if there are only m nonzero terms in Eq. (13) below, then each f_i is a derivation of order at most $m-1$.

Theorem 9. *Let $n, m \in \mathbb{N}$ with $m \leq n$, let S be a commutative ring with subring R, and suppose multiplication by $(n+1)!$ is bijective in S. Suppose also that $f_i : R \to S$ is additive and $f_i(1) = 0$ for $i \in I$, where $I \subset \{1, \dots, n+1\}$ with* $\operatorname{card}(I) = m$. *If*

$$\sum_{i \in I} x^i f_i(x^{n+1-i}) = 0, \quad x \in R, \tag{13}$$

then $f_i \in \mathcal{D}_{m-1}$ for each $i \in I$.

Proof. We follow the path of the proof of Theorem 7 in [3], making only two changes: (i) we replace each reference to Proposition 4 in [3] by a reference to Theorem 7 above, and (ii) we take $c_i = 0$ for every i in the proof of Theorem 7 since we are now assuming $f_i(1) = 0$. □

4. Further Remarks

Theorem 7, Corollary 8, and Theorem 9 also have their counterparts when it is not assumed that the unknown additive functions vanish at the unity of R. We state one of these counterparts here.

Remark 10. In the setup of Theorem 7, assume only that each $f_i : R \to S$ is an additive solution of (6) or (7), but not that $f_i(1) = 0$. Then we define $\tilde{f}_i : R \to S$ by

$$\tilde{f}_i(x) := f_i(x) - x f_i(1), \quad x \in R, \quad 1 \le i \le n+1,$$

so each $\tilde{f}_i$ is additive and vanishes at 1. By Theorem 7 we find that $\tilde{f}_i$ has the form (8), so f_i has the form

$$f_{n+1-j}(x) = (-1)^j \sum_{k=0}^{j} \binom{n+1-j+k}{k} d_{n-j+k}(x) + c_{n-j+k} \cdot x, \quad (0 \le j \le n)$$

where $d_{n-j+k} \in \mathcal{D}_{n-j+k}$ and $c_{n-j+k} := f_{n-j+k}(1)$. Furthermore the constants c_i must satisfy

$$\sum_{i=1}^{n+1} c_i = 0.$$

The counterparts of Corollary 8 and Theorem 9 are analogous.

For equations with missing powers (i.e. (13) where $I \neq \{1, \ldots, n+1\}$) Gselmann et al. [6] introduced a multivariate method in which they first expand the equation so that it has as many free variables as the degree of homogeneity, and then substitute the value 1 as many times as the number of missing powers. They state that the multivariate version provides a "more effective and subtle way" [than the univariate version] to determine the structure of the unknown functions. We claim that this is not always the case, and the relative efficiency of one method as compared to another method depends on the functional equation. In particular, the univariate method (specifically Theorem 9) seems to be at least as efficient and perhaps simpler for sparse equations (i.e. equations where $\operatorname{card}(I)$ is very small). We illustrate with three examples.

Example 11. Let S be a commutative ring with subring R, and let multiplication by 6 be bijective in S. Suppose $f, g : R \to S$ are additive functions such that $f(1) = g(1) = 0$ and

$$f(x^3) + x^2 g(x) = 0, \qquad x \in R. \tag{14}$$

By Theorem 9 we get immediately that $f, g \in \mathcal{D}_1$, so Eq. (14) reduces to

$$x^2(3f(x) + g(x)) = 0.$$

Applying Theorem 9 once more we have $3f + g = 0$ (since $3f + g \in \mathcal{D}_0$) and the solution is

$$f \in \mathcal{D}_1, \quad g = -3f.$$

For comparison, the solution by the multivariate method is given in Example 5 of [6].

Example 12. Let S be a commutative ring with subring R, and suppose multiplication by 17! is bijective in S. Suppose $f, g : R \to S$ are additive functions such that $f(1) = g(1) = 0$ and

$$f(x^{17}) + x^4 g(x^{13}) = 0, \quad x \in R. \tag{15}$$

By Theorem 9 we get immediately that $f, g \in \mathcal{D}_1$, so Eq. (15) reduces to

$$x^{16}(17f(x) + 13g(x)) = 0.$$

Applying Theorem 9 once more we have $17f + 13g = 0$ and the solution is

$$f \in \mathcal{D}_1, \quad g = -\frac{17}{13}f.$$

For comparison we give also the solution by the multivariate method. Since the equation is of degree 17 we introduce 17 free variables and expand (15) to the multivariate equation

$$f(x_1 \cdots x_{17}) + \sum_{\operatorname{card}(J)=4} \frac{1}{\binom{17}{4}} \left(\prod_{j\in J} x_j\right) g\left(\prod_{k\in\{1,\dots,17\}\setminus J} x_k\right) = 0,$$

for all $x_1, \dots, x_{17} \in R$, where the summation is taken over all four-element subsets of the index set $\{1, \dots, 17\}$. Since there are 15 missing powers in (15), we set 15 of the variables equal to 1 and set the remaining variables equal to each other. Say $x_1 = x_2 = x$ and $x_3 = \cdots = x_{17} = 1$. The result is

$$f(x^2) + \frac{1}{\binom{17}{4}}\left[2\binom{15}{3}xg(x) + \binom{15}{4}g(x^2)\right] = 0,$$

which simplifies to

$$\left[f(x^2) + \frac{39}{68}g(x^2)\right] + \frac{13}{34}xg(x) = 0.$$

According to Theorem 7 the solution of this equation is

$$f + \frac{39}{68}g = d_1 \in \mathcal{D}_1, \qquad \frac{13}{34}g = -2d_1,$$

from which it follows that

$$f = 4d_1, \qquad g = -\frac{68}{13}d_1.$$

This solution is clearly equivalent to the one found by the univariate method.

Example 13. Let S be a commutative ring with subring R, and suppose multiplication by 5! is bijective in S. Suppose $f, g, h : R \to S$ are additive functions such that $f(1) = g(1) = h(1) = 0$ and

$$f(x^5) + xg(x^4) + x^4 h(x) = 0, \qquad x \in R. \tag{16}$$

By Theorem 9 we get immediately that $f, g, h \in \mathcal{D}_2$, so let us put $f = d_2 \in \mathcal{D}_2$. We also use Corollary 6 from [3] which shows how to express $d(x^n)$ for any $n > 2$ in terms of $d(x^2)$ and $d(x)$ when $d \in \mathcal{D}_2$. The results are

$$f(x^5) = d_2(x^5) = 10x^3 d_2(x^2) - 15x^4 d_2(x),$$
$$g(x^4) = 6x^2 g(x^2) - 8x^3 g(x).$$

Substituting these into (16) and rearranging we get

$$x^3(10d_2 + 6g)(x^2) + x^4(h - 8g - 15d_2)(x) = 0.$$

Applying Theorem 9 to this equation we find that

$$10d_2 + 6g = d_1 \in \mathcal{D}_1,$$
$$h - 8g - 15d_2 = -2d_1.$$

Simple calculations lead to the solution

$$f = d_2,$$
$$g = -\frac{5}{3}d_2 + \frac{1}{6}d_1,$$
$$h = \frac{5}{3}d_2 - \frac{2}{3}d_1.$$

For comparison, the solution by the multivariate method is given in Example 6 of [6]. The two solutions are in agreement via the correspondence $(D_1, D_2) = (-\frac{3}{2}d_1, \frac{5}{3}d_2 + \frac{1}{3}d_1)$.

We close with a small curiosity.

Remark 14. In [5] Gselmann observed that most characterizations of derivations require a pair of functional equations. She proved an alienation-type theorem and from that produced a single functional equation that characterizes derivations on fields. Here we present a single functional equation characterizing derivations on (not necessarily commutative) rings.

Let S be a ring with subring R, and suppose $f : R \to S$ is any function satisfying the functional equation

$$f(xy + z) = xf(y) + f(x)y + f(z), \quad x, y, z \in R. \tag{17}$$

Putting $x = 1$ here we have

$$f(y + z) = f(y) + f(1)y + f(z), \quad y, z \in R.$$

Since addition is commutative in any ring this means

$$f(y) + f(1)y + f(z) = f(z) + f(1)z + f(y), \quad y, z \in R,$$

therefore $f(1)y = f(1)z$. Taking $y = 1$ and $z = 0$ we get that $f(1) = 0$, thus f is additive. Now Eq. (17) reduces to the Leibniz (or product) rule

$$f(xy) = xf(y) + f(x)y, \quad x, y \in R,$$

hence $f \in \mathcal{D}_1$.

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