Results Math (2018) 73:117

1422-6383/18/030001-19
published online August 21, 2018
https://doi.org/10.1007/s00025-018-0879-5

Results in Mathematics

On Global Solvability and Uniform Stability of One Nonlinear Integral Equation

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Abstract. We consider nonlinear integral equations of a special type that appear in the inverse spectral theory of integral and integro-differential operators. We generalize the approach for solving equations of this type by introducing some abstract nonlinear equation and proving its global solvability. Moreover, we establish the uniform stability of such nonlinear equations.

Mathematics Subject Classification. 45G15, 45G10, 45M10, 44A35, 45D05.

Keywords. Nonlinear integral equation, convolution, global solvability, uniform stability.

1. Introduction

Consider the nonlinear integral equation

$$f(x)=\sum_{n=1}^{\infty}\Big(\psi_n(x)y^{*n}(x)+\int_0^x \Psi_n(x,t)y^{*n}(t)\,dt\Big), \quad 0<x<b, \qquad (1)$$

where $y(x)$ is an unknown function and

$$y^{*1}(x)=y(x), \quad y^{*(n+1)}(x)=y^{*n}*y(x)=\int_0^x y^{*n}(t)y(x-t)\,dt, \quad n\ge 1,$$

while $f(x)$ and $\psi_n(x)$, $\Psi_n(x,t)$, $n\ge 1$, are some known square-integrable functions. This equation appears in the inverse spectral theory of integral and integro-differential operators (see [1–3, 5, 10, 11] and the references therein). In particular, in [1] (see also [2]) it was proved that, under natural assumptions on the functional sequences $\{\psi_n(x)\}$, $\{\Psi_n(x,t)\}$ and the equality $\psi_1(x)=1$, Eq. (1) has a unique solution $y(x)\in L_2(0,b)$ for each left-hand side $f(x)\in$

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$L_2(0,b)$. In other words, Eq. (1) possesses a global unique solution. We note that in concrete applications to the inverse spectral theory the coefficient $\psi_1(x)$ usually equals to $b-x$, i.e., in addition to the nonlinearity, Eq. (1) is complicated also by the singularity at the end of the interval. In this situation one can guaranty, generally speaking, only the local square integrability of the solution, namely, $y(x) \in L_2(0,\beta)$ for each $\beta \in (0,b)$. However, in concrete cases (see, e.g., [1–3]) using additional properties of the functions $\psi_n(x)$ and $\Psi_n(x,t)$, inspired by the specifics of the problem, one can establish that the solution belongs to an appropriate weighted space, e.g., $(b-x)y(x) \in L_2(0,b)$.

Furthermore, the works [4,12,13] deal with more general nonlinear equations, which cannot be represented in the form (1), i.e. as a series with convolution powers of the unknown function. In this "non-convolutional" case the global solvability has been also established. Moreover, in [6,7,9] in connection with inverse spectral problems for integro-differential Dirac systems there arose a vectorial analog of Eq. (1).

The goal of the present paper is twofold. Firstly, we introduce an abstract equation generalizing Eq. (1) as well as all the cases having appeared in [1–7,9–13], including the vectorial and "non-convolutional" ones, and prove its global solvability and uniform stability with respect to the left-hand side. Secondly, applying the obtained results to the vectorial analog of Eq. (1), we establish its global solvability along with the uniform full stability (i.e. the uniform stability with respect to all components of the equation). We note that the stability of nonlinear equations of this type has not been previously studied even in simple situations.

Now let us clarify the key notions. For this purpose we consider the equation

$$f = \mathcal{P}y, \tag{2}$$

where $\mathcal{P} : \mathfrak{R}_1 \to \mathfrak{R}_2$ is some operator, $\mathfrak{R}_k$ is a metric space with the metric $\rho_k(\,\cdot\,,\,\cdot\,)$, $k = 1,2$. For definiteness, we assume that $\mathcal{P}$ is an injection, i.e. Eq. (2) has at most one solution. Equation (2) is called *locally* solvable if the image set $\mathcal{P}\mathfrak{R}_1$ is open in $\mathfrak{R}_2$. In particular, (2) is called solvable *globally* if $\mathcal{P}\mathfrak{R}_1 = \mathfrak{R}_2$, i.e. if $\mathcal{P}$ is a surjection. Assuming the local solvability of (2), we say that it is *stable*, if for any fixed $\tilde{y} \in \mathfrak{R}_1$ we have $\rho_1(y,\tilde{y}) \to 0$ as $\rho_2(f,\mathcal{P}\tilde{y}) \to 0$. A stronger and more useful version of stability involves an estimate of $\rho_1(y,\tilde{y})$ via $\rho_2(f,\mathcal{P}\tilde{y})$, e.g., when for any $\tilde{y} \in \mathfrak{R}_1$ there exist positive $\delta = \delta(\tilde{y},\mathcal{P})$ and $C = C(\tilde{y},\mathcal{P})$ such that for all $f \in \mathfrak{R}_2$, obeying $\rho_2(f,\tilde{f}) \le \delta$ with $\tilde{f} = \mathcal{P}\tilde{y}$, the following estimate holds:

$$\rho_1(y,\tilde{y}) \le C\rho_2(f,\tilde{f}), \tag{3}$$

where y is the solution of (2). Further, assuming the global solvability of (2), we call Eq. (2) *uniformly stable*, if for arbitrary fixed $R > 0$ and $g \in \mathfrak{R}_2$, and for $f,\ \tilde{f} \in \mathfrak{R}_2$ estimate (3) holds with $C = C(\mathcal{P},g,R)$ (i.e. depending only on $\mathcal{P}$, g and R) as soon as $\rho_2(f,g) \le R$ and $\rho_2(\tilde{f},g) \le R$. Here $\tilde{y}$ is the solution of the equation $\tilde{f} = \mathcal{P}\tilde{y}$, while y is the one of (2). The mentioned stabilities are ones with respect to the left-hand side of Eq. (2). *Full* stability applies, when the solution of (2) is stable also with respect to some perturbations of $\mathcal{P}$.

The paper is organized as follows. In the next section we introduce an abstract nonlinear equation, generalizing (1), and prove its global solvability (Theorem 1). In Sect. 3 we establish the uniform stability of this abstract equation with respect to its left-hand side (Theorem 2). In Sects. 4 and 5 we use the results obtained in Sects. 2 and 3 for proving the global solvability (Theorem 3) and the uniform full stability (Theorem 4) of the vectorial analog of Eq. (1).

2. The Abstract Nonlinear Equation

Fix $N \geq 1$ and consider the space $H_N(a,b) := (L_2(a,b))^N$ of vector-functions $y(x) = (y_\nu(x))_{\nu=\overline{1,N}}$, $a < x < b$, with the norm

$$\|y\|_{H_N(a,b)} := \max_{\nu=\overline{1,N}} \|y_\nu\|_{L_2(a,b)}. \tag{4}$$

Denote $\Delta_{a,b} := \{(x,t) : a < t < x < b\}$ and consider also the space $\mathcal{H}_N(a,b) := (L_2(\Delta_{a,b}))^{N^2}$ of matrix-functions $K(x,t) = (K_{\nu j}(x,t))_{\nu,j=\overline{1,N}}$, $a < t < x < b$, with the norm

$$\|K\|_{\mathcal{H}_N(a,b)} := \max_{\nu=\overline{1,N}} \sum_{j=1}^{N} \|K_{\nu j}\|_{L_2(\Delta_{a,b})}, \tag{5}$$

$$\|K_{\nu j}\|_{L_2(\Delta_{a,b})} = \sqrt{\int_a^b dx \int_a^x |K_{\nu j}(x,t)|^2\, dt}.$$

We also put $\|\cdot\|_a := \|\cdot\|_{L_2(0,a)}$, $\|\cdot\|_{\bar{a}} := \|\cdot\|_{H_N(0,a)}$ and $B_{a,r} := \{y \in H_N(0,a) : \|y\|_{\bar{a}} \leq r\}$.

In this section we establish the global solvability of the nonlinear equation

$$f(x) = y(x) + \mathcal{D}y(x), \quad 0 < x < b, \tag{6}$$

in the space $H_N(0,b)$, where $\mathcal{D}$ is a nonlinear operator of the special class introduced in the following definition.

Definition 1. We say that the operator $\mathcal{D} = (\mathcal{D}_\nu)_{\nu=\overline{1,N}} : H_N(0,b) \to H_N(0,b)$ belongs to the class $\mathcal{E}_{b,N}$, if the following four conditions are fulfilled:

(i) For each vector-function $y(x) \in H_N(0,b)$ and for each number $\gamma \in (0,b)$ the image vector-function $\mathcal{D}y(x)$ on the interval $(0,\gamma)$ does not depend on values of $y(x)$ on (γ,b);

(ii) For all $R > 0$ and $r > 0$ there exists $\delta \in (0,b]$ such that $\mathcal{D} : B_{\delta,R} \to B_{\delta,r}$;

(iii) For all $R > 0$ and $\alpha > 0$ there exists $\delta \in (0,b]$ such that

$$\|\mathcal{D}y - \mathcal{D}\tilde{y}\|_{\bar{\delta}} \leq \alpha \|y - \tilde{y}\|_{\bar{\delta}}$$

for all functions $y(x)$, $\tilde{y}(x) \in B_{\delta,R}$;

(iv) For all $\delta \in (0, b/2]$ and $y(x) \in H_N(0, 2\delta)$ the following representation holds:

$$\mathcal{D}_\nu y(x) = \mathcal{D}_\nu y_{(1)}(x) + \sum_{j=1}^N \int_\delta^x K_{\delta,\nu j}(x, t; y_{(1)}) y_{(2)j}(t)\, dt, \quad 0 < x < 2\delta, \quad \nu = \overline{1, N}, \tag{7}$$

where $K_\delta(x, t; y_{(1)}) := (K_{\delta,\nu j}(x, t; y_{(1)}))_{\nu,j=\overline{1,N}} \in \mathcal{H}_N(\delta, 2\delta)$, $y_{(j)}(x) = (y_{(j)\nu}(x))_{\nu=\overline{1,N}} \in H_N(0, 2\delta)$,

$$y_{(1)}(x) = \begin{cases} y(x), & x \in (0, \delta), \\ 0, & x \in (\delta, 2\delta), \end{cases} \qquad y_{(2)}(x) = \begin{cases} 0, & x \in (0, \delta), \\ y(x), & x \in (\delta, 2\delta), \end{cases} \tag{8}$$

and the matrix-function $K_\delta(x, t; y_{(1)})$ does not depend on $y_{(2)}(x)$.

Moreover, we say that $\mathcal{D} \in \mathcal{E}'_{b,N}$, if $\mathcal{D} \in \mathcal{E}_{b,N}$ and for all $\delta \in (0, b/2]$ the $\mathcal{H}_N(\delta, 2\delta)$-norm of the kernel $K_\delta(x, t; y_{(1)})$ determined in (iv) is a bounded (nonlinear) functional on $y_{(1)}$, i.e.

$$C_{\mathcal{D},\delta}(r) := \sup_{\|y\|_{\bar\delta} \le r} \|K_\delta(x, t; y)\|_{\mathcal{H}_N(\delta,2\delta)} < \infty, \quad r \ge 0. \tag{9}$$

Remark 1. In conditions (ii)–(iv) of Definition 1 one and the same symbol $\mathcal{D}$ denotes the natural extensions of the operator $\mathcal{D}$ to the spaces $H_N(0, \beta)$, $\beta \in (0, b)$, which, by virtue of condition (i), are determined uniquely.

Remark 2. It is easy to check that for each operator $\mathcal{D} \in \mathcal{E}_{b,N}$ in conditions (ii) and (iii) of Definition 1 one can take any sufficiently small $\delta > 0$. In other words, these conditions can be represented in the following equivalent form:

(ii′) For all R, $r > 0$ there exists $\delta \in (0, b]$ such that $\mathcal{D} : B_{\delta_1,R} \to B_{\delta_1,r}$ for all $\delta_1 \in (0, \delta]$.

(iii′) For all $R > 0$ and $\alpha > 0$ there exists $\delta \in (0, b]$ such that

$$\|\mathcal{D}y - \mathcal{D}\tilde{y}\|_{\overline{\delta_1}} \le \alpha \|y - \tilde{y}\|_{\overline{\delta_1}}$$

for each $\delta_1 \in (0, \delta]$ and for all functions $y(x), \tilde{y}(x) \in B_{\delta_1,R}$.

Indeed, let $y(x)$, $\tilde{y}(x) \in B_{\delta_1,R}$ for a certain $\delta_1 \in (0, \delta]$. Put

$$y_c(x) := \begin{cases} y(x), & x \in (0, \delta_1), \\ 0, & x \in (\delta_1, \delta), \end{cases} \qquad \tilde{y}_c(x) := \begin{cases} \tilde{y}(x), & x \in (0, \delta_1), \\ 0, & x \in (\delta_1, \delta). \end{cases}$$

Then, obviously, $y_c(x)$, $\tilde{y}_c(x) \in B_{\delta,R}$. According to (ii), we have $\mathcal{D}y_c(x) \in B_{\delta,r}$. On the other hand, by virtue of (i), we get

$$\|\mathcal{D}y\|_{\overline{\delta_1}} = \|\mathcal{D}y_c\|_{\overline{\delta_1}} \le \|\mathcal{D}y_c\|_{\bar\delta} \le r,$$

i.e. (ii′) is proven. Further, according to (iii), we have

$$\|\mathcal{D}y_c - \mathcal{D}\tilde{y}_c\|_{\bar\delta} \le \alpha \|y_c - \tilde{y}_c\|_{\bar\delta} = \alpha \|y - \tilde{y}\|_{\overline{\delta_1}}.$$

On the other hand, by virtue of (i) we get

$$\|\mathcal{D}y - \mathcal{D}\tilde{y}\|_{\overline{\delta_1}} = \|\mathcal{D}y_c - \mathcal{D}\tilde{y}_c\|_{\overline{\delta_1}} \le \|\mathcal{D}y_c - \mathcal{D}\tilde{y}_c\|_{\bar\delta} \le \alpha \|y - \tilde{y}\|_{\overline{\delta_1}}$$

and we arrive at (iii′).

The following theorem gives the global solvability of Eq. (6), when $\mathcal{D}$ is an operator of the class $\mathcal{E}_{b,N}$.

Theorem 1. *Let $f(x) \in H_N(0,b)$ and $\mathcal{D} \in \mathcal{E}_{b,N}$. Then Eq. (6) has a unique solution $y(x) \in H_N(0,b)$.*

Proof. By virtue of condition (i) in Definition 1, one can solve Eq. (6) by steps, i.e. first find its solution on $(0,\gamma)$ for some $\gamma \in (0,b)$ and then seek it on (γ,b).

Consider the operator $\Phi y(x) := f(x) - \mathcal{D}y(x)$. Choose $\delta > 0$ so that $\|f\|_{\bar{\delta}} \le 1$ and conditions (ii), (iii) in Definition 1 are met for $R = 2$, $r = 1$ and some $\alpha \in (0,1)$. Under these settings the operator Φ maps $B_{\delta,2}$ into $B_{\delta,2}$ and is a contracting mapping in $B_{\delta,2}$. Then, by virtue of the contracting mappings principle, Eq. (6) has a unique solution on the interval $(0,\delta)$ that belongs to the ball $B_{\delta,2}$.

Further, let for some $\delta \in (0,b/2]$ a solution of Eq. (6) has been already found on the interval $(0,\delta)$. Then we look for a solution on $(0,2\delta)$ in the form $y(x) = y_{(1)}(x) + y_{(2)}(x)$, where $y_{(1)}(x) = (y_{(1)\nu}(x))_{\nu=\overline{1,N}} = 0$ on $(\delta,2\delta)$ and $y_{(2)}(x) = (y_{(2)\nu}(x))_{\nu=\overline{1,N}} = 0$ on $(0,\delta)$. Thus, according to condition (iv) in Definition 1, Eq. (6) for $x \in (\delta,2\delta)$ takes the form

$$\xi_\nu(x) = y_{(2)\nu}(x) + \sum_{j=1}^{N} \int_\delta^x K_{\delta,\nu j}(x,t;y_{(1)}) y_{(2)j}(t)\,dt, \quad \delta < x < 2\delta, \quad \nu = \overline{1,N}, \tag{10}$$

where $(\xi_\nu(x))_{\nu=\overline{1,N}} = \Phi y_{(1)}(x) \in H_N(\delta,2\delta)$ and $(K_{\delta,\nu j}(x,t;y_{(1)}))_{\nu,j=\overline{1,N}} \in \mathcal{H}_N(\delta,2\delta)$ are known functions, which are independent of $y_{(2)}(x)$. Equation (10) has a unique solution $y_{(2)}(x) \in H_N(\delta,2\delta)$. Thus, we established that a solution of Eq. (6) on the interval $(0,\delta)$ can be uniquely extended to the interval $(\delta,2\delta)$. Continuing this process, in a finite number of steps we find a solution $y(x) \in H_N(0,b)$ of Eq. (6) on the entire interval $(0,b)$.

Let us show that this solution $y(x)$ is unique. Indeed, let $\tilde{y}(x) = (\tilde{y}_\nu(x))_{\nu=\overline{1,N}} \in H_N(0,b)$ be another solution. Then, according to the first our choice of δ, for some $\delta_1 \in (0,\delta)$ the both solutions $y(x)$ and $\tilde{y}(x)$ on $(0,\delta_1)$ belong to the ball $B_{\delta_1,2}$ and, hence, by virtue of Remark 2 and the contracting mappings principle, $y(x) = \tilde{y}(x)$ a.e. on $(0,\delta_1)$. In view of uniqueness of the continuation of solution to (δ_1,b), they will coincide a.e. on $(0,b)$. □

Remark 3. One can see that Theorem 1 would hold also if in conditions (ii) and (iii) of Definition 1 one required only any fixed R and r such that $R > r > 0$ and any fixed $\alpha \in (0,1)$. However, requiring arbitrary R allows one to prove the uniform stability of equation (2) (for $\mathcal{D} \in \mathcal{E}'_{b,N}$ satisfying Condition $\mathcal{A}$, see Sect. 3). Moreover, the arbitrariness of r and α allows the class $\mathcal{E}_{b,N}$ to be closed with respect to the linear operations (see also [13]).

3. Uniform Stability of the Abstract Equation

In this section we prove the uniform stability of Eq. (6) with respect to its left-hand side, when the operator $\mathcal{D}$ belongs to $\mathcal{E}'_{b,N}$ and satisfies the following additional condition.

Condition $\mathcal{A}$. For each pair of functions $(y,\tilde{y}) \in (H_N(0,b))^2$ there exists a matrix-function $P(\,\cdot\,,\,\cdot\,;y,\tilde{y}) = (P_{\nu j}(\,\cdot\,,\,\cdot\,;y,\tilde{y}))_{\nu,j=\overline{1,N}} \in \mathcal{H}_N(0,b)$ such that the following relation holds:

$$\mathcal{D}_\nu y(x) - \mathcal{D}_\nu \tilde{y}(x) = \sum_{j=1}^{N} \int_0^x P_{\nu j}(x,t;y,\tilde{y})\hat{y}_j(t)\,dt, \quad 0<x<b, \quad \nu=\overline{1,N}, \tag{11}$$

where $\hat{y}_j(x) = y_j(x) - \tilde{y}_j(x)$, $j=\overline{1,N}$, and the $\mathcal{H}_N(0,b)$-norm of this function $P(x,t;y,\tilde{y})$ is a bounded (nonlinear) functional on the pair $(y,\tilde{y})$, i.e.

$$C_{P_\mathcal{D}}(r) := \sup_{\max\{\|y\|_{\bar{b}},\|\tilde{y}\|_{\bar{b}}\}\le r} \|P(x,t;y,\tilde{y})\|_{\mathcal{H}_N(0,b)} < \infty, \quad r \ge 0. \tag{12}$$

For the future needs, we provide the stability in the case, when the operator $\mathcal{D}$ is not fixed but belongs to some fixed set $\mathfrak{D}$. For this purpose we introduce the following two definitions.

Definition 2. We say that $\mathfrak{D}$ is an *uniform* subset of $\mathcal{E}'_{b,N}$, if $\mathfrak{D} \subset \mathcal{E}'_{b,N}$ and additionally the following three conditions are fulfilled:

- (ii$_+$) For all $R>0$ there exist $R_1>R$ and $\delta\in(0,b]$ such that $\mathcal{D}: B_{\delta,R_1} \to B_{\delta,R_1-R}$ for any $\mathcal{D}\in\mathfrak{D}$;
- (iii$_+$) For all $R>0$ there exist $\alpha\in(0,1)$ and $\delta\in(0,b]$ such that

$$\|\mathcal{D}y - \mathcal{D}\tilde{y}\|_{\bar{\delta}} \le \alpha \|y-\tilde{y}\|_{\bar{\delta}}$$

  for any operator $\mathcal{D}\in\mathfrak{D}$ and for all functions $y(x),\,\tilde{y}(x)\in B_{\delta,R}$;
- (iv$_+$) For $\mathcal{D}\in\mathfrak{D}$ the functions $C_{\mathcal{D},\delta}(r)$ determined in (9) are uniformly majorated by some nondecreasing function, i.e.

$$C_{\mathfrak{D},\delta}(r) := \sup_{\mathcal{D}\in\mathfrak{D}} C_{\mathcal{D},\delta}(r) < \infty, \quad r\ge 0, \quad \delta \in \Big(0,\frac{b}{2}\Big]. \tag{13}$$

Definition 3. Let $\mathfrak{D}$ be a set of operators $\mathcal{D}$, satisfying Condition $\mathcal{A}$. We say that the set $\mathfrak{D}$ *uniformly* satisfies Condition $\mathcal{A}$, if

$$C_{P_\mathfrak{D}}(r) := \sup_{\mathcal{D}\in\mathfrak{D}} C_{P_\mathcal{D}}(r) < \infty, \quad r \ge 0, \tag{14}$$

and the set $\{\mathcal{D}0 : \mathcal{D}\in\mathfrak{D}\}$ is bounded.

The following theorem gives the uniform stability of Eq. (6) with respect to the left-hand side, when the operator $\mathcal{D}$ belongs to a fixed uniform subset $\mathfrak{D}$ of $\mathcal{E}'_{b,N}$, uniformly satisfying Condition $\mathcal{A}$.

Theorem 2. *Fix* $R > 0$ *and let* $\|f\|_{\bar{b}}, \|\tilde{f}\|_{\bar{b}} \le R$. *Let also* $\mathfrak{D}$ *be an uniform subset of* $\mathcal{E}'_{b,N}$, *uniformly satisfying Condition* $\mathcal{A}$.

Then there exists a constant $C = C(R, \mathfrak{D})$ *such that for all* $\mathcal{D} \in \mathfrak{D}$ *the estimate*

$$\|\hat{y}\|_{\bar{b}} \le C\|\hat{f}\|_{\bar{b}} \tag{15}$$

holds, where $\hat{f}(x) = f(x) - \tilde{f}(x)$ *and* $\hat{y}(x) = y(x) - \tilde{y}(x)$, *while* $y(x)$ *is the solution of Eq.* (6) *and* $\tilde{y}(x)$ *is the one of the equation*

$$\tilde{f}(x) = \tilde{y}(x) + \mathcal{D}\tilde{y}(x), \quad 0 < x < b. \tag{16}$$

Before proceeding directly to the proof, let us provide the following three auxiliary assertions.

Lemma 1. *Let* $K(x,t) = (K_{\nu j}(x,t))_{\nu,j=\overline{1,N}} \in \mathcal{H}_N(a,b)$ *be the kernel of the Volterra linear integral operator* $K = (K_\nu)_{\nu=\overline{1,N}} : H_N(a,b) \to H_N(a,b)$ *acting by the formula*

$$K_\nu y(x) = \sum_{j=1}^N \int_a^x K_{\nu j}(x,t) y_j(t)\, dt, \quad a < x < b, \quad \nu = \overline{1,N}.$$

Then for the resolvent kernel $R(x,t) = (R_{\nu j}(x,t))_{\nu,j=\overline{1,N}}$, *i.e. for the kernel of the integral operator* $R = (R_\nu)_{\nu=\overline{1,N}} := (I+K)^{-1} - I$, *where* I *is the identity operator and*

$$R_\nu y(x) = \sum_{j=1}^N \int_a^x R_{\nu j}(x,t) y_j(t)\, dt, \quad a < x < b, \quad \nu = \overline{1,N},$$

the estimate

$$\|R\|_{\mathcal{H}_N(a,b)} \le F\Big(\|K\|_{\mathcal{H}_N(a,b)}\Big) \tag{17}$$

holds with some nondecreasing function $F(\,\cdot\,) : [0,\infty) \to [0,\infty)$ *independent of* $K(x,t)$.

Proof. is analogous to the scalar case $N = 1$ (see [14]). For convenience of the reader, we briefly provide the arguments for arbitrary N. Obviously, the resolvent kernel $R(x,t)$ satisfies the linear integral equation

$$R_{\nu j}(x,t) = -K_{\nu j}(x,t) - \sum_{s=1}^N \int_t^x K_{\nu s}(x,\tau) R_{sj}(\tau,t)\, d\tau,$$
$$a < t < x < b, \quad \nu, j = \overline{1,N}.$$

The method of successive approximations gives $R_{\nu j}(x,t) = \sum_{n=1}^\infty R_{\nu j}^{(n)}(x,t)$, $\nu, j = \overline{1,N}$, where

$$R_{\nu j}^{(1)}(x,t) = -K_{\nu j}(x,t), \quad R_{\nu j}^{(n+1)}(x,t) = -\sum_{s=1}^N \int_t^x K_{\nu s}(x,\tau) R_{sj}^{(n)}(\tau,t)\, d\tau,$$
$$a < t < x < b.$$

Having put

$$A_\nu(x) := \sqrt{\sum_{s=1}^N \int_a^x |K_{\nu s}(x,\tau)|^2\,d\tau}, \quad \nu = \overline{1,N},$$

$$B_j(t) := \sqrt{\sum_{s=1}^N \int_t^b |K_{sj}(\tau,t)|^2\,d\tau}, \quad j = \overline{1,N},$$

$$F_0(x,t) := 1, \quad F_{k+1}(x,t) := \sum_{\nu=1}^N \int_t^x A_\nu^2(\tau) F_k(\tau,t)\,d\tau, \quad k \ge 0,$$

by induction we get

$$F_k(x,t) \le \frac{F_1^k(x,t)}{k!}, \ \ k \ge 0, \quad |R_{\nu j}^{(n)}(x,t)|^2 \le A_\nu^2(x) B_j^2(t) F_{n-2}(x,t), \\ \nu, j = \overline{1,N}, \ \ n \ge 2.$$

Since $F_1(x,t) \le N\|K\|^2_{\mathcal{H}_N(a,b)}$ for $a \le t \le x \le b$ and

$$\int_a^b A_\nu^2(x)\,dx = \sum_{s=1}^N \|K_{\nu s}\|^2_{\mathcal{H}_1(a,b)} \le \|K\|^2_{\mathcal{H}_N(a,b)}, \quad \nu = \overline{1,N}$$

$$\int_a^b B_j^2(t)\,dt = \sum_{s=1}^N \|K_{sj}\|^2_{\mathcal{H}_1(a,b)}, \ \ j = \overline{1,N},$$

we arrive at

$$\|R_{\nu j}^{(n)}\|_{\mathcal{H}_1(a,b)} \le \frac{(\|K\|_{\mathcal{H}_N(a,b)}\sqrt{N})^{n-2}}{\sqrt{(n-2)!}} \|K\|_{\mathcal{H}_N(a,b)} \sum_{s=1}^N \|K_{sj}\|_{\mathcal{H}_1(a,b)}, \\ \nu, j = \overline{1,N}, \quad n \ge 2.$$

This yields

$$\|R_{\nu j}\|_{\mathcal{H}_1(a,b)} \le \|K_{\nu j}\|_{\mathcal{H}_1(a,b)} + \|K\|_{\mathcal{H}_N(a,b)} \sum_{s=1}^N \|K_{sj}\|_{\mathcal{H}_1(a,b)} \\ \times \sum_{n=0}^\infty \frac{(\|K\|_{\mathcal{H}_N(a,b)}\sqrt{N})^n}{\sqrt{n!}}, \quad \nu, j = \overline{1,N},$$

and, hence,

$$\|R\|_{\mathcal{H}_N(a,b)} \le \|K\|_{\mathcal{H}_N(a,b)} + \sum_{n=0}^\infty \frac{(\|K\|_{\mathcal{H}_N(a,b)}\sqrt{N})^{n+2}}{\sqrt{n!}},$$

which finishes the proof. □

According to the proof of Lemma 1, in (17) one can take

$$F(x) = x + \sum_{n=0}^{\infty} \frac{(x\sqrt{N})^{n+2}}{\sqrt{n!}}.$$

Lemma 2. *Any nonlinear operator $\mathcal{D}$, satisfying Condition $\mathcal{A}$, is bounded, i.e.*

$$C_{\mathcal{D}}(r) := \sup_{\|y\|_{\bar{b}} \le r} \|\mathcal{D}y\|_{\bar{b}} < \infty, \quad r \ge 0. \tag{18}$$

Moreover, if a set $\mathfrak{D}$ uniformly satisfies Condition $\mathcal{A}$, then

$$C_{\mathfrak{D}}(r) := \sup_{\mathcal{D} \in \mathfrak{D}} C_{\mathcal{D}}(r) < \infty, \quad r \ge 0. \tag{19}$$

Proof. By virtue of (11), we get

$$\mathcal{D}_\nu y(x) = \mathcal{D}_\nu 0(x) + \sum_{j=1}^{N} \int_0^x P_{\nu j}(x, t; y, 0) y_j(t)\, dt, \quad 0 < x < b, \quad \nu = \overline{1, N}.$$

Using the Cauchy–Bunyakovsky–Schwarz inequality along with (4), (5) and (12), we arrive at

$$\|\mathcal{D}y\|_{\bar{b}} \le \|\mathcal{D}0\|_{\bar{b}} + \|P(x,t;y,0)\|_{\mathcal{H}_N(0,b)} \|y\|_{\bar{b}} \le \|\mathcal{D}0\|_{\bar{b}} + \|y\|_{\bar{b}} C_{P_{\mathcal{D}}}(\|y\|_{\bar{b}}),$$

which implies (18) with $C_{\mathcal{D}}(r) \le \|\mathcal{D}0\|_{\bar{b}} + rC_{P_{\mathcal{D}}}(r)$. Further, by virtue of (14), the latter estimate gives (19), where $C_{\mathfrak{D}}(r) \le \sup_{\mathcal{D}\in\mathfrak{D}} \|\mathcal{D}0\|_{\bar{b}} + rC_{P_{\mathfrak{D}}}(r)$. □

Lemma 3. *Fix $R > 0$ and let $\|f\|_{\bar{b}} \le R$. Let also $\mathfrak{D}$ be an uniform subset of $\mathcal{E}'_{b,N}$, uniformly satisfying Condition $\mathcal{A}$.*

Then there exists a constant $r_0 = r_0(R, \mathfrak{D})$, depending only on R and $\mathfrak{D}$, such that the solution of Eq. (6) belongs to the ball B_{b,r_0} for all $\mathcal{D} \in \mathfrak{D}$.

Proof. According to conditions (ii$_+$) and (iii$_+$) in Definition 2, there exist $R_1 > R$, $\alpha \in (0,1)$ and $\delta \in (0, b]$ such that $\mathcal{D} : B_{\delta,R_1} \to B_{\delta,R_1-R}$ and

$$\|\mathcal{D}z - \mathcal{D}\tilde{z}\|_{\bar{\delta}} \le \alpha \|z - \tilde{z}\|_{\bar{\delta}}, \quad z, \tilde{z} \in B_{\delta,R_1},$$

for all $\mathcal{D} \in \mathfrak{D}$. As in Remark 2, it is easy to show that here one can take the minimal $\delta > 0$ from conditions (ii$_+$) and (iii$_+$). Thus, the contracting mappings principle yields $y(x) \in B_{\delta,R_1}$.

Further, supposing $y(x) \in B_{\delta,r_1}$ for some $\delta \in (0, b/2]$, let us prove that $y(x) \in B_{2\delta,r_2}$ for some $r_2 > 0$ independent of $f(x)$ and $\mathcal{D}$. Indeed, solving Eq. (10) with respect to $y_{(2)}(x) = (y_{(2)\nu}(x))_{\nu=\overline{1,N}}$ on $(\delta, 2\delta)$ we get

$$y_{(2)\nu}(x) = \xi_\nu(x) + \sum_{j=1}^{N} \int_\delta^x R_{\delta,\nu j}(x, t; y_{(1)}) \xi_j(t)\, dt, \quad \delta < x < 2\delta, \quad \nu = \overline{1, N},$$

where $R_\delta(x,t;y_{(1)}) := (R_{\delta,\nu j}(x,t;y_{(1)}))_{\nu,j=\overline{1,N}} \in \mathcal{H}_N(\delta,2\delta)$ is the resolvent kernel for the kernel $K_\delta(x,t;y_{(1)}) = (K_{\delta,\nu j}(x,t;y_{(1)}))_{\nu,j=\overline{1,N}}$ determined by (7). Thus, we have

$$\|y_{(2)}\|_{H_N(\delta,2\delta)} \le \Big(1 + \|R_\delta(x,t;y_{(1)})\|_{\mathcal{H}_N(\delta,2\delta)}\Big)\|\xi\|_{H_N(\delta,2\delta)}, \tag{20}$$

where $\xi(x) := (\xi_\nu(x))_{\nu=\overline{1,N}} = f(x) - \mathcal{D}y_{(1)}(x)$. By virtue of Lemmas 1, 2 and (9), (13), we get

$$\|R_\delta(x,t;y_{(1)})\|_{\mathcal{H}_N(\delta,2\delta)} \le F\Big(\|K_\delta(x,t;y_{(1)})\|_{\mathcal{H}_N(\delta,2\delta)}\Big) \le F(C_{\mathcal{D},\delta}(r_1))$$
$$\le F(C_{\mathfrak{D},\delta}(r_1)),$$
$$\|\mathcal{D}y_{(1)}\|_{H_N(\delta,2\delta)} \le \|\mathcal{D}y_{(1)}\|_{\overline{2\delta}} \le C_{\mathcal{D}}(r_1) \le C_{\mathfrak{D}}(r_1),$$

which along with (20) give

$$\|y_{(2)}\|_{H_N(\delta,2\delta)} \le \Big(1 + F(C_{\mathfrak{D},\delta}(r_1))\Big)(R + C_{\mathfrak{D}}(r_1)) =: r_2 - r_1.$$

Hence, $\|y\|_{\overline{2\delta}} \le \|y_{(1)}\|_{\bar\delta} + \|y_{(2)}\|_{H_N(\delta,2\delta)} \le r_2$ with r_2 depending only on R, r_1 and $\mathfrak{D}$.

Repeating the previous step finitely many times, we establish $\|y\|_{\bar b} \le r_0$ with the constant r_0 independent of concrete $f(x) \in B_{b,R}$ and $\mathcal{D} \in \mathfrak{D}$. □

Now we are in position to give the proof of Theorem 2.

Proof of Theorem 2. Subtracting Eq. (16) from (6) and using Condition $\mathcal{A}$, we get

$$\hat f_\nu(x) = \hat y_\nu(x) + \mathcal{D}_\nu y(x) - \mathcal{D}_\nu \tilde y(x) = \hat y_\nu(x)$$
$$+ \sum_{j=1}^N \int_0^x P_{\nu j}(x,t;y,\tilde y)\hat y_j(t)\,dt, \quad 0 < x < b, \quad \nu = \overline{1,N},$$

which along with Lemma 1 give

$$\|\hat y\|_{\bar b} \le \Big(1 + F\Big(\|P(x,t;y,\tilde y)\|_{\mathcal{H}_N(0,b)}\Big)\Big)\|\hat f\|_{\bar b}.$$

Using (12), (14) and Lemma 3, we arrive at (15) with the constant $C = (1 + F(C_{P_{\mathfrak{D}}}(r_0)))$ depending only on R and $\mathfrak{D}$. □

Obviously, any finite subset $\mathfrak{D} \subset \mathcal{E}'_{b,N}$ is automatically uniform. Moreover, if all operators in a finite set $\mathfrak{D}$ satisfy Condition $\mathcal{A}$, then $\mathfrak{D}$ automatically satisfies Condition $\mathcal{A}$ uniformly. Thus, the following corollary from Theorem 2 holds.

Corollary 1. *Fix $R > 0$ and let $\|f\|_{\bar b}, \|\tilde f\|_{\bar b} \le R$. Let also the operator $\mathcal{D}$ belong to $\mathcal{E}'_{b,N}$ and satisfy Condition $\mathcal{A}$. Then there exists $C = C(R,\mathcal{D}) > 0$ such that estimate (15) holds.*

4. Global Solvability of the Convolutional Equation in the Vectorial Case

In this section we apply the results of Sect. 2 to the vectorial analog of Eq. (1), having appeared in the inverse spectral theory for integro-differential Dirac systems (see [6,7,9]):

$$f_\nu(x) = \sum_{n=1}^{\infty} \sum_{j=1}^{m_{n,N}} \Big(\psi_{\nu nj}(x) Q_{nj}[y](x) + \int_0^x \Psi_{\nu nj}(x,t) Q_{nj}[y](t)\,dt\Big),$$
$$0 < x < b, \nu = \overline{1,N}, \tag{21}$$

where $m_{n,N}$ is the number of all possible convolutional monomials $Q_{nj}[y]$ of the form

$$Q_{nj}[y](x) = y_{i_1} * y_{i_2} * \cdots * y_{i_n}(x), \quad 1 \le i_1 \le i_2 \le \cdots \le i_n \le N. \tag{22}$$

In particular, $m_{1,N} = N$ and we agree that

$$Q_{1j}[y](x) = y_j(x), \quad \psi_{\nu 1j}(x) = \delta_{\nu,j}, \quad j = \overline{1,N}, \tag{23}$$

where $\delta_{\nu,j}$ is the Kronecker delta. Assume that $\psi_{\nu nj}(x) \in L_2(0,b)$ and $\Psi_{\nu nj}(x,t) \in \mathcal{H}_1(0,b)$ for all ν, n, j and

$$\|\psi_{\nu nj}\|_b \le A^n, \quad \|\Psi_{\nu nj}\|_{\mathcal{H}_1(0,b)} \le A^n, \quad \nu = \overline{1,N}, \quad j = \overline{1,m_{n,N}}, \quad n \ge 2, \tag{24}$$

for some fixed $A > 0$ independent of ν, n and j.

The following theorem gives the global solvability of Eq. (21) in $H_N(0,b)$.

Theorem 3. *For each vector-function $f(x) = (f_\nu(x))_{\nu=\overline{1,N}} \in H_N(0,b)$ Eq. (21) has a unique solution $y(x) = (y_\nu(x))_{\nu=\overline{1,N}} \in H_N(0,b)$.*

Rewrite Eq. (21) in the form

$$f_\nu(x) = y_\nu(x) + \mathcal{D}_\nu y(x), \quad 0 < x < b, \quad \nu = \overline{1,N},$$

where

$$\mathcal{D}_\nu y(x) = \sum_{n=1}^{\infty} \sum_{j=1}^{m_{n,N}} \Big((1-\delta_{n,1})\psi_{\nu nj}(x) Q_{nj}[y](x) + \int_0^x \Psi_{\nu nj}(x,t) Q_{nj}[y](t)\,dt\Big),$$
$$\nu = \overline{1,N}. \tag{25}$$

Thus, Theorem 3 is a direct corollary of Theorem 1 and the following assertion.

Lemma 4. *The operator $\mathcal{D} = (\mathcal{D}_\nu)_{\nu=\overline{1,N}}$ determined by (25) belongs to the class $\mathcal{E}_{b,N}$.*

Proof. Since, according to (22), for any $\gamma \in (0,b)$ all monomials $Q_{nj}[y](x)$ on $[0,\gamma]$ do not depend on the values of the function $y(x)$ on the interval (γ,b), condition (i) in Definition 1 is obvious. In order to check conditions (ii) and (iii), we prove the following proposition. □

Proposition 1. *For $n \ge 1$, $j = \overline{1, m_{n,N}}$ and $a > 0$ the following estimates hold:*

$$\|Q_{nj}[y]\|_a \le \frac{a^{\frac{n-1}{2}}}{\sqrt{(n-1)!}}\|y\|_{\bar a}^n, \tag{26}$$

$$\|Q_{nj}[y] - Q_{nj}[\tilde y]\|_a \le n\frac{a^{\frac{n-1}{2}}}{\sqrt{(n-1)!}}(\max\{\|y\|_{\bar a}, \|\tilde y\|_{\bar a}\})^{n-1}\|\hat y\|_{\bar a}. \tag{27}$$

Proof. Let $z_\nu(x) \in L_2(0,a)$, $\nu \ge 1$. Then

$$\|z_1 * \dots * z_n\|_a \le \frac{a^{\frac{n-1}{2}}}{\sqrt{(n-1)!}}\prod_{\nu=1}^n \|z_\nu\|_a, \quad n \ge 1. \tag{28}$$

Indeed, for $n = 1$ estimate (28) is obvious. Using Cauchy–Bunyakovsky–Schwarz inequality and assuming by induction that (28) holds for some $n = n_1 \ge 1$, we get

$$|z_1 * \dots * z_{n_1+1}(x)| \le \|z_1 * \dots * z_{n_1}\|_x \|z_{n_1+1}\|_x \le \frac{x^{\frac{n_1-1}{2}}}{\sqrt{(n_1-1)!}}\prod_{\nu=1}^{n_1+1}\|z_\nu\|_a,$$
$$0 \le x \le a.$$

Taking the $L_2(0,a)$-norm of the both sides in this inequality, we arrive at (28) for $n = n_1 + 1$. Estimate (28) along with (4), (22) and (23) give (26).

Further, estimate (27) for $n = 1$ is obvious. Assume that it holds for some $n = n_1 \ge 1$. Let us prove it for $n = n_1 + 1$. It is clear that there exist $s \in \{1, \dots, m_{n_1,N}\}$ and $\nu \in \{1, \dots, N\}$ such that

$$(Q_{n_1+1,j}[y] - Q_{n_1+1,j}[\tilde y])(x) = (Q_{n_1 s}[y] * y_\nu - Q_{n_1 s}[\tilde y] * \tilde y_\nu)(x)$$

and, hence,

$$|(Q_{n_1+1,j}[y] - Q_{n_1+1,j}[\tilde y])(x)| \le \|Q_{n_1 s}[y]\|_x \|\hat y_\nu\|_x + \|Q_{n_1 s}[y] - Q_{n_1 s}[\tilde y]\|_x \|\tilde y_\nu\|_x.$$

By virtue of (26) and (27) for $n = n_1$, we get

$$|(Q_{n_1+1,j}[y] - Q_{n_1+1,j}[\tilde y])(x)| \le \frac{x^{\frac{n_1-1}{2}}}{\sqrt{(n_1-1)!}}\Big(\|y\|_{\bar x}^{n_1}\|\hat y_\nu\|_x$$
$$+ n_1(\max\{\|y\|_{\bar x}, \|\tilde y\|_{\bar x}\})^{n_1-1}\|\hat y\|_{\bar x}\|\tilde y_\nu\|_x\Big),$$

which gives

$$|(Q_{n_1+1,j}[y] - Q_{n_1+1,j}[\tilde y])(x)| \le (n_1+1)\frac{x^{\frac{n_1-1}{2}}}{\sqrt{(n_1-1)!}}(\max\{\|y\|_{\bar a}, \|\tilde y\|_{\bar a}\})^{n_1}\|\hat y\|_{\bar a},$$
$$0 \le x \le a.$$

Taking the $L_2(0,a)$-norm of the both sides, we arrive at (27) for $n = n_1 + 1$. □

Let us return to the proof of Lemma 4. According to (23)–(25), we have

$$\|\mathcal{D}_\nu y\|_\delta \le \sum_{j=1}^{N} \|\Psi_{\nu 1j}\|_{\mathcal{H}_1(0,\delta)} \|y_j\|_\delta + 2\sum_{n=2}^{\infty}\sum_{j=1}^{m_{n,N}} A^n \|Q_{nj}[y]\|_\delta, \quad \nu = \overline{1,N}.$$

By virtue of (26) and the inequality $m_{n,N} \le N^n$, we arrive at

$$\|\mathcal{D}y\|_{\bar\delta} \le C_\delta \|y\|_{\bar\delta} + 2AN\|y\|_{\bar\delta} \sum_{n=1}^{\infty} \frac{(AN\|y\|_{\bar\delta}\sqrt{\delta})^n}{\sqrt{n!}}, \quad C_\delta = \|\Psi\|_{\mathcal{H}_N(0,\delta)}, \quad (29)$$

where $\Psi(x,t) = (\Psi_{\nu 1j}(x,t))_{\nu,j=\overline{1,N}}$. Thus, condition (ii) in Definition 1 is fulfilled. Indeed, it is sufficient to choose $\delta > 0$ so that

$$C_\delta R + 2ANR\sum_{n=1}^{\infty} \frac{(ANR\sqrt{\delta})^n}{\sqrt{n!}} \le r.$$

Further, according to (23)–(25), we have

$$\|\mathcal{D}_\nu y - \mathcal{D}_\nu \tilde y\|_\delta \le \sum_{j=1}^{N} \|\Psi_{\nu 1j}\|_{\mathcal{H}_1(0,\delta)} \|\hat y_j\|_\delta$$
$$+2\sum_{n=2}^{\infty}\sum_{j=1}^{m_{n,N}} A^n \|Q_{nj}[y] - Q_{nj}[\tilde y]\|_\delta, \nu = \overline{1,N}.$$

Substituting estimate (27) into this inequality, we arrive at

$$\|\mathcal{D}y - \mathcal{D}\tilde y\|_{\bar\delta} \le \Big(C_\delta + 2AN\sum_{n=1}^{\infty}(n+1)\frac{(AN\max\{\|y\|_{\bar\delta}, \|\tilde y\|_{\bar\delta}\}\sqrt{\delta})^n}{\sqrt{n!}}\Big)\|\hat y\|_{\bar\delta}. \quad (30)$$

Thus, condition (iii) in Definition 1 is fulfilled, where it is sufficient to choose $\delta > 0$ so that

$$C_\delta + 2AN\sum_{n=1}^{\infty}(n+1)\frac{(ANR\sqrt{\delta})^n}{\sqrt{n!}} \le \alpha.$$

In order to check condition (iv) we prove the following proposition.

Proposition 2. *For all $n \ge 2$ and $j = \overline{1, m_{n,N}}$ the following representation holds:*

$$Q_{nj}[y](x) = Q_{nj}[y_{(1)}](x) + \sum_{p=1}^{N} y_{(2)p} * \sum_{s=1}^{m_{n-1,N}} \beta_{njs}^{(p)} Q_{n-1,s}[y_{(1)}](x), \quad 0 \le x \le 2\delta, \quad (31)$$

where $y_{(j)}(x) = (y_{(j)p}(x))_{p=\overline{1,N}}$, $j = 1, 2$, are determined in (8), all $\beta_{njs}^{(p)} \ge 0$ and

$$\sum_{p=1}^{N}\sum_{s=1}^{m_{n-1,N}} \beta_{njs}^{(p)} = n. \quad (32)$$

Proof. By virtue of (22) and $y_p(x) = y_{(1)p}(x) + y_{(2)p}(x)$, $0 < x < 2\delta$, $p = \overline{1,N}$, we get

$$Q_{nj}[y] = \sum_{\substack{(j_1,j_2,\ldots,j_n)\in\{1,2\}^n \\ 1\le i_1\le i_2\le\cdots\le i_n\le N.}} y_{(j_1)i_1} * y_{(j_2)i_2} * \cdots y_{(j_n)i_n}, \tag{33}$$

Since $y_{(2)j} * y_{(2)p}(x) = 0$ on $[0, 2\delta]$ for all $j, p = \overline{1,N}$, formula (33) takes the form

$$Q_{nj}[y] = y_{(1)i_1} * y_{(1)i_2} * \cdots * y_{(1)i_n} + \sum_{k=1}^{n} y_{(1)i_1} * \cdots * y_{(1)i_{k-1}} * y_{(2)i_k} * y_{(1)i_{k+1}} * \cdots * y_{(1)i_n}$$

Combining terms with equal i_k's, we arrive at (31). □

Return to the proof of Lemma 4. Substituting (31) for $\delta \in (0, b/2]$ into (25), we get (7) with $K_{\delta,\nu j}(x, t; y_{(1)})$, $\delta < t < x < 2\delta$, $\nu, j = \overline{1,N}$, determined by the formula

$$K_{\delta,\nu j}(x,t;y_{(1)}) = \Psi_{\nu 1j}(x,t) + \sum_{n=2}^{\infty}\sum_{k=1}^{m_{n,N}}\sum_{s=1}^{m_{n-1,N}} \beta_{nks}^{(j)} \Big(\psi_{\nu nk}(x) Q_{n-1,s}[y_{(1)}](x-t) + \int_t^x \Psi_{\nu nk}(x,\tau) Q_{n-1,s}[y_{(1)}](\tau - t)\, d\tau\Big).$$

By virtue of (24), (26), we get

$$\|K_{\delta,\nu j}(x,t;y_{(1)})\|_{\mathcal{H}_1(\delta,2\delta)} \le \|\Psi_{\nu 1j}\|_{\mathcal{H}_1(\delta,2\delta)} + (1+\sqrt{\delta}) \sum_{n=2}^{\infty} \frac{A^n \delta^{\frac{n-2}{2}} \|y_{(1)}\|_{\bar{\delta}}^{n-1}}{\sqrt{(n-2)!}} \sum_{k=1}^{m_{n,N}}\sum_{s=1}^{m_{n-1,N}} \beta_{nks}^{(j)},$$

which along with (5), (32) and the inequality $m_{n,N} \le N^n$ imply

$$\|K_\delta(x,t;y_{(1)})\|_{\mathcal{H}_N(\delta,2\delta)} \le \|\Psi\|_{\mathcal{H}_N(\delta,2\delta)} + (1+\sqrt{\delta})(AN)^2 \|y_{(1)}\|_{\bar{\delta}} \sum_{n=0}^{\infty} (n+2) \frac{(AN\|y_{(1)}\|_{\bar{\delta}}\sqrt{\delta})^n}{\sqrt{n!}}, \tag{34}$$

i.e. $K_\delta(x, t; y_{(1)}) \in \mathcal{H}_N(\delta, 2\delta)$ and, hence, condition (iv) in Definition 1 is proven. □

5. Uniform Full Stability of the Convolutional Equation

In this section, using Theorem 2, we establish the uniform stability of Eq. (21) with respect to its left-hand side $f(x)$ as well as to the functional sequences $\{\psi_{\nu nj}(x)\}$ and $\{\Psi_{\nu nj}(x,t)\}$ in the right-hand one. For this purpose, along with Eq. (21) we consider the equation

$$\tilde{f}_\nu(x) = \sum_{n=1}^{\infty} \sum_{j=1}^{m_{n,N}} \Big(\tilde{\psi}_{\nu nj}(x) Q_{nj}[\tilde{y}](x) + \int_0^x \tilde{\Psi}_{\nu nj}(x,t) Q_{nj}[\tilde{y}](t)\, dt \Big), \quad 0 < x < b, \quad \nu = \overline{1, N}, \tag{35}$$

where $\tilde{f}(x) := (\tilde{f}_\nu(x))_{\nu=\overline{1,N}} \in H_N(0,b)$, $\tilde{\psi}_{\nu nj}(x) \in L_2(0,b)$ and $\tilde{\Psi}_{\nu nj}(x,t) \in \mathcal{H}_1(0,b)$ for all ν, n, j and

$$\|\tilde{\psi}_{\nu nj}\|_b \le A^n, \quad \|\tilde{\Psi}_{\nu nj}\|_{\mathcal{H}_1(0,b)} \le A^n, \quad \nu = \overline{1,N}, \quad j = \overline{1, m_{n,N}}, \quad n \ge 2. \tag{36}$$

Moreover, let $\tilde{\psi}_{\nu 1j}(x) = \delta_{\nu,j}$ for $\nu, j = \overline{1,N}$ and denote $\hat{\psi}_{\nu nj}(x) := \psi_{\nu nj}(x) - \tilde{\psi}_{\nu nj}(x)$ and $\hat{\Psi}_{\nu nj}(x,t) := \Psi_{\nu nj}(x,t) - \tilde{\Psi}_{\nu nj}(x,t)$. The following theorem holds.

Theorem 4. *Fix A, $R > 0$, $\theta \in [0,1)$ and a function $G(x,t) \in \mathcal{H}_N(0,b)$. Let $\|f\|_{\bar{b}}$, $\|\tilde{f}\|_{\bar{b}} \le R$ and, besides (24) and (36), the following estimates hold:*

$$\|G - \Psi\|_{\mathcal{H}_N(0,b)} \le \theta, \quad \|G - \tilde{\Psi}\|_{\mathcal{H}_N(0,b)} \le \theta, \tag{37}$$

where $\Psi(x,t) = (\Psi_{\nu 1j}(x,t))_{\nu,j=\overline{1,N}}$ and $\tilde{\Psi}(x,t) = (\tilde{\Psi}_{\nu 1j}(x,t))_{\nu,j=\overline{1,N}}$.

Then there exists $C > 0$, depending only on A, R, θ and $G(x,t)$, such that

$$\|\hat{y}\|_{\bar{b}} \le C\|\hat{f}\|_{\bar{b}} + C \max_{\nu,j=\overline{1,N}} \|\hat{\Psi}_{\nu 1j}\|_{\mathcal{H}_1(0,b)} + \sum_{n=2}^{\infty} \frac{C^n}{\sqrt{n!}} \max_{\substack{\nu=\overline{1,N} \\ j=\overline{1,m_{n,N}}}} \Big(\|\hat{\psi}_{\nu nj}\|_b + \|\hat{\Psi}_{\nu nj}\|_{\mathcal{H}_1(0,b)} \Big).$$

Consider the set $\mathfrak{D}_{A,G}^{\theta}$ of operators $\mathcal{D} = (\mathcal{D}_\nu)_{\nu=\overline{1,N}}$ determined by (25) and obeying conditions (24) and (37). Before proceeding directly to the proof of Theorem 4, let us provide the following two auxiliary assertions.

Lemma 5. *For all $A > 0$, $\theta \in [0,1)$ and any function $G(x,t) \in \mathcal{H}_N(0,b)$ the set $\mathfrak{D}_{A,G}^{\theta}$ is an uniform subset of $\mathcal{E}'_{b,N}$.*

Proof. Put

$$\varepsilon := \frac{1-\theta}{2}, \quad R_1 := \frac{2R}{\varepsilon}, \quad \alpha := 1 - \varepsilon + 2AN \sum_{n=1}^{\infty} (n+1) \frac{(ANR_1\sqrt{\delta})^n}{\sqrt{n!}}.$$

Thus, according to (29), (30), (37) and the estimate $C_\delta = \|\Psi\|_{\mathcal{H}_N(0,\delta)} \le \|G\|_{\mathcal{H}_N(0,\delta)} + \theta$, for conditions (ii$_+$) and (iii$_+$) in Definition 2 to hold, it is sufficient to choose $\delta \in (0,b]$ so that

$$\|G\|_{\mathcal{H}_N(0,\delta)} \le \varepsilon, \quad 2AN \sum_{n=1}^{\infty} (n+1) \frac{(ANR_1\sqrt{\delta})^n}{\sqrt{n!}} < \varepsilon.$$

Condition (iv$_+$) in Definition 2 is fulfilled by virtue of (34). □

Lemma 6. *The set $\mathfrak{D}_{A,G}^{\theta}$ uniformly satisfies Condition $\mathcal{A}$.*

Proof. Consider an alternative representation for monomials $Q_{nj}[y]$. Namely, there exist $\nu_s = \nu_s(n,j) \ge 0$, $s = \overline{1,N}$, such that $\nu_1 + \ldots + \nu_N = n$ and (22) takes the form

$$Q_{nj}[y] = y_1^{*\nu_1} * y_2^{*\nu_2} * \ldots * y_N^{*\nu_N}.$$

We agree that here and below $f^{*0} * g = g * f^{*0} = g$ for any f and g. It is easily seen that

$$Q_{nj}[y] - Q_{nj}[\tilde{y}] = \sum_{s=1}^{N} \tilde{y}_1^{*\nu_1} * \ldots * \tilde{y}_{s-1}^{*\nu_{s-1}} * (y_s^{*\nu_s} - \tilde{y}_s^{*\nu_s}) * y_{s+1}^{*\nu_{s+1}} * \ldots * y_N^{*\nu_N},$$

where we have

$$y_s^{*\nu_s} - \tilde{y}_s^{*\nu_s} = \begin{cases} 0, & \nu_s = 0, \\ \hat{y}_s, & \nu_s = 1, \\ q_{s,\nu_s} * \hat{y}_s, & \nu_s > 1, \end{cases} \qquad q_{s,\nu_s} = \sum_{p=0}^{\nu_s - 1} y_s^{*p} * \tilde{y}_s^{*(\nu_s - 1 - p)}.$$

Thus, for $n \ge 2$ we arrive at the representation

$$Q_{nj}[y] - Q_{nj}[\tilde{y}] = \sum_{s=1}^{N} h_s^{nj} * \hat{y}_s, \quad h_s^{nj} = \begin{cases} 0, & \nu_s = 0, \\ \displaystyle\sum_{p=1}^{\nu_s} Q_{n-1,\theta_p}[y,\tilde{y}], & \nu_s \ge 1, \end{cases} \tag{38}$$

where $\theta_p = \theta_p(n,j) \in \{1, \ldots, m_{n-1,2N}\}$ and $Q_{nj}[y,\tilde{y}]$, $n \ge 1$, $j = \overline{1, m_{n,2N}}$, are all possible monomials of the form

$$Q_{nj}[y,\tilde{y}] = y_1^{*\alpha_1} * \cdots * y_N^{*\alpha_N} * \tilde{y}_1^{*\tilde{\alpha}_1} * \cdots * \tilde{y}_N^{*\tilde{\alpha}_N}, \quad \alpha_s \ge 0, \quad \tilde{\alpha}_s \ge 0,$$
$$\sum_{s=1}^{N} (\alpha_s + \tilde{\alpha}_s) = n.$$

Thus, inequality (26) yields the estimate

$$\|h_s^{nj}\|_b \le \nu_s(n,j) \frac{b^{\frac{n-2}{2}}}{\sqrt{(n-2)!}} (\max\{\|y\|_{\bar{b}}, \|\tilde{y}\|_{\bar{b}}\})^{n-1}. \tag{39}$$

Further, let $\mathcal{D} = (\mathcal{D}_\nu)_{\nu=\overline{1,N}} \in \mathfrak{D}_{A,G}^{\theta}$. Then, according to (25), we get

$$\begin{aligned} &\mathcal{D}_\nu y(x) - \mathcal{D}_\nu \tilde{y}(x) \\ &\quad = \sum_{j=1}^{N} \int_0^x \Psi_{\nu 1 j}(x,t) \hat{y}_j(t)\, dt + \sum_{n=2}^{\infty} \sum_{j=1}^{m_{n,N}} \Big(\psi_{\nu n j}(x) (Q_{nj}[y] - Q_{nj}[\tilde{y}])(x) \\ &\qquad + \int_0^x \Psi_{\nu n j}(x,t) (Q_{nj}[y] - Q_{nj}[\tilde{y}])(t)\, dt \Big), \quad 0 < x < b, \quad \nu = \overline{1,N}. \end{aligned}$$

Using (38), we arrive at

$$\mathcal{D}_\nu y(x) - \mathcal{D}_\nu \tilde{y}(x) = \sum_{s=1}^{N} \int_0^x P_{\nu s}(x,t;y,\tilde{y}) \hat{y}_s(t)\, dt, \quad 0 < x < b, \quad \nu = \overline{1,N},$$

where

$$P_{\nu s}(x,t;y,\tilde{y}) = \Psi_{\nu 1s}(x,t) + \sum_{n=2}^{\infty}\sum_{j=1}^{m_{n,N}} \Big(\psi_{\nu nj}(x)h_s^{nj}(x-t) + \int_t^x \Psi_{\nu nj}(x,\tau)h_s^{nj}(\tau-t)\,d\tau\Big).$$

Estimates (24) and (39) yield

$$\|P_{\nu s}(x,t;y,\tilde{y})\|_{\mathcal{H}_1(0,b)} \le \|\Psi_{\nu 1s}\|_{\mathcal{H}_1(0,b)} + (1+\sqrt{b})\sum_{n=2}^{\infty} A^n \sum_{j=1}^{m_{n,N}} \|h_s^{nj}\|_b$$

$$\le \|\Psi_{\nu 1s}\|_{\mathcal{H}_1(0,b)} + (1+\sqrt{b})\sum_{n=2}^{\infty} \frac{A^n b^{\frac{n-2}{2}}(\max\{\|y\|_{\bar{b}}, \|\tilde{y}\|_{\bar{b}}\})^{n-1}}{\sqrt{(n-2)!}} \sum_{j=1}^{m_{n,N}} \nu_s(n,j).$$

Since $\nu_1(n,j) + \cdots + \nu_N(n,j) = n$, for the kernel $P(x,t;y,\tilde{y}) = (P_{\nu s}(x,t;y,\tilde{y}))_{\nu,s=\overline{1,N}}$ inequality (12) holds and the function $C_{P_{\mathcal{D}}}(r)$ satisfies the estimate

$$C_{P_{\mathcal{D}}}(r) \le \|G\|_{\mathcal{H}_N(0,b)} + \theta + (1+\sqrt{b})(AN)^2 r \sum_{n=0}^{\infty}(n+2)\frac{(ANr\sqrt{b})^n}{\sqrt{n!}},$$

where the right-hand side is independent of concrete $\mathcal{D} \in \mathfrak{D}_{A,G}^{\theta}$. It remains to note that $\mathcal{D}0 = 0$. □

Now we are in position to give the proof of Theorem 4.

Proof of Theorem 4. Consider the operators $\mathcal{D} = (\mathcal{D}_\nu)_{\nu=\overline{1,N}}$, $\tilde{\mathcal{D}} = (\tilde{\mathcal{D}}_\nu)_{\nu=\overline{1,N}} \in \mathfrak{D}_{A,G}^{\theta}$, where $\mathcal{D}_\nu$ are determined in (25), while $\tilde{\mathcal{D}}_\nu$ are determined by the formula

$$\tilde{\mathcal{D}}_\nu z(x) = \sum_{n=1}^{\infty}\sum_{j=1}^{m_{n,N}} \Big((1-\delta_{n,1})\tilde{\psi}_{\nu nj}(x)Q_{nj}[z](x) + \int_0^x \tilde{\Psi}_{\nu nj}(x,t)Q_{nj}[z](t)\,dt\Big), \quad \nu = \overline{1,N}. \tag{40}$$

According to Lemmas 3, 5 and 6, there exists $r_0 = r_0(A,R,G,\theta) > 0$, depending only on A, R, θ and the function $G(x,t)$, such that $y(x)$, $\tilde{y}(x) \in B_{b,r_0}$.

Further, by virtue of (40), Eq. (35) can be represented in the form

$$\xi(x) = \tilde{y}(x) + \mathcal{D}\tilde{y}(x), \quad 0 < x < b,$$

where $\xi(x) = (\xi_\nu(x))_{\nu=\overline{1,N}} = \tilde{f}(x) + \mathcal{D}\tilde{y}(x) - \tilde{\mathcal{D}}\tilde{y}(x)$. According to Lemma 2, we have the estimate $\|\xi\|_{\bar{b}} \le R + 2C_{\mathfrak{D}_{A,G}^{\theta}}(r_0)$. Using Theorem 2, we arrive at the estimate

$$\|\hat{y}\|_{\bar{b}} \le C_1\|f - \xi\|_{\bar{b}}, \tag{41}$$

where C_1 depends only on A, R, G and θ. On the other hand, according to (25) and (40), for $0 < x < b$ and $\nu = \overline{1,N}$ we have

$$\xi_\nu(x) = \tilde{f}_\nu(x) + \sum_{j=1}^{N} \int_0^x \hat{\Psi}_{\nu 1 j}(x,t)\tilde{y}_j(t)\,dt + \sum_{n=2}^{\infty} \sum_{j=1}^{m_{n,N}} \Big(\hat{\psi}_{\nu n j}(x) Q_{nj}[\tilde{y}](x) + \int_0^x \hat{\Psi}_{\nu n j}(x,t) Q_{nj}[\tilde{y}](t)\,dt \Big).$$

By virtue of (26) along with the estimate $\|\tilde{y}\|_{\bar{b}} \le r_0$, this yields

$$\|f - \xi\|_{\bar{b}} \le \|\hat{f}\|_{\bar{b}} + N r_0 \max_{\nu,j=\overline{1,N}} \|\hat{\Psi}_{\nu 1 j}\|_{\mathcal{H}_1(0,b)} + \sum_{n=2}^{\infty} \frac{C_2^n}{\sqrt{n!}} \max_{\substack{\nu=\overline{1,N} \\ j=\overline{1,m_{n,N}}}} \Big(\|\hat{\psi}_{\nu n j}\|_b + \|\hat{\Psi}_{\nu n j}\|_{\mathcal{H}_1(0,b)} \Big),$$

where C_2 depends only on $r_0 = r_0(A, R, G, \theta)$, which along with (41) finishes the proof. □

6. Summary

Let us briefly summarize the key points of the paper. In Sect. 2 we established the global solvability of Eq. (6), when the operator $\mathcal{D}$ belongs to the class $\mathcal{E}_{b,N}$ (Theorem 1). In Sect. 3 we proved the uniform stability of Eq. (6) with respect to the left-hand side, when $\mathcal{D} \in \mathfrak{D}$, while $\mathfrak{D}$ is an uniform subset of $\mathcal{E}'_{b,N}$ and uniformly satisfies Condition $\mathcal{A}$ (Theorem 2). In particular, this implies the uniform stability, when the operator $\mathcal{D} \in \mathcal{E}'_{b,N}$ is fixed and satisfies Condition $\mathcal{A}$ (Corollary 1). In Sect. 4, as a corollary of Theorem 1, we proved the global solvability of the vectorial analog (21) of Eq. (1) (Theorem 3). In Sect. 5, using Theorem 2, we established the uniform full stability of Eq. (21) (Theorem 4).

Acknowledgements

This work was supported in part by the Ministry of Education and Science of Russian Federation (Grant 1.1660.2017/4.6) and by the Russian Foundation for Basic Research (Grants 16-01-00015 and 17-51-53180).

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Received: July 23, 2018.
Accepted: August 11, 2018.