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Results in Mathematics

Convexity Properties of Some Entropies

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Dedicated to Professor Heiner Gonska on the occasion of his 70th birthday.

Abstract. We consider a family of probability distributions depending on a real parameter x, and study the logarithmic convexity of the sum of the squared probabilities. Applications concerning bounds and concavity properties of Rényi and Tsallis entropies are given. Finally, some extensions and an open problem are presented.

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1. Introduction

Let $c \in \mathbb{R}$. Set $I_c = \left[0, -\frac{1}{c}\right]$ if $c < 0$, and $I_c = [0, +\infty)$ if $c \geq 0$. For $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}_0$ the binomial coefficients are defined as usual by

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} \quad \text{if } k \in \mathbb{N}, \text{ and } \binom{\alpha}{0} := 1.$$

In particular, $\binom{m}{k} = 0$ if $m \in \mathbb{N}$ and $k > m$.

Let $n > 0$ be a real number, $k \in \mathbb{N}_0$ and $x \in I_c$. Define

$$p_{n,k}^{[c]}(x) := (-1)^k \binom{-\frac{n}{c}}{k} (cx)^k (1+cx)^{-\frac{n}{c}-k}, \qquad \text{if } c \neq 0,$$

$$p_{n,k}^{[0]}(x) := \lim_{c \to 0} p_{n,k}^{[c]}(x) = \frac{(nx)^k}{k!} e^{-nx}, \qquad \text{if } c = 0.$$

Then $\sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) = 1$. Throughout the paper we shall suppose that $n > c$ if $c \geq 0$, or $n = -cl$ with some $l \in \mathbb{N}$ if $c < 0$.

Let us remark that the cases $c = -1, c = 0, c = 1$ correspond, respectively, to the binomial, Poisson, and negative binomial distributions.

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Details concerning the positive linear operators defined in terms of the probability distribution $(p^{[c]}_{n,k}(x))_{k\geq 0}$ can be found in [4]; see also [7]. The degree of non-multiplicativity of these operators was investigated in [5]. In this context, the convexity of the index of coincidence

$$S_{n,c}(x) := \sum_{k=0}^{\infty} \left(p^{[c]}_{n,k}(x)\right)^2, \quad x \in I_c,$$

was studied, as well as its relations with Rényi entropy $R_{n,c}(x) := -\log S_{n,c}(x)$ and Tsallis entropy $T_{n,c}(x) := 1-S_{n,c}(x)$; for details see [3,7] and the references therein.

It was conjectured in [8] that $S_{n,c}$ is logarithmically convex on I_c. This conjecture was validated for $c \geq 0$ in [1]. The aim of this paper is to prove the conjecture for $c < 0$. This is done in Sect. 2; for a preliminary version, see [9]. Section 3 contains some applications, extensions, and an open problem.

This paper is an echo of the discussions between Heiner Gonska, Margareta Heilmann, Maria Rusu, Elena Stănilă, and the author, during the summer of 2012.

2. Main Results

Without loss of generality we may restrict to the case $c = -1$. Briefly, we shall prove the following

Theorem 1. *The function*

$$F_n(x) := S_{n,-1}(x) = \sum_{k=0}^{n} \left(\binom{n}{k} x^k (1-x)^{n-k}\right)^2,$$

is log-convex on $[0,1]$.

Proof. The proof is inspired by the method used in [6] in order to prove that F_n is convex on $[0,1]$.

First, we need the inequalities

$$0 \leq u_n(t) \leq \frac{2n^2}{\sqrt{4n^2(t^2-1) + (t-\sqrt{t^2-1})^2} + t - \sqrt{t^2-1}}, \quad t \geq 1, \tag{2.1}$$

where $u_n := P'_n/P_n$ and $P_n(t)$ are the classical Legendre polynomials on $[-1,1]$. The first inequality is well-known (see, e.g., [6, (1.2)]). We prove the second one by induction with respect to n.

It is easy to verify it for $n = 1$. Suppose that it is true for a certain $n \geq 1$. Then, according to [6], (3.2) and the subsequent remark,

$$\begin{aligned} u_{n+1}(t) &= (n+1)\frac{n+1+tu_n(t)}{(n+1)t+(t^2-1)u_n(t)} \\ &\leq (n+1)\frac{(n+1)\left[\sqrt{4n^2(t^2-1)+(t-\sqrt{t^2-1})^2}+t-\sqrt{t^2-1}\right]+2n^2t}{(n+1)t\left[\sqrt{4n^2(t^2-1)+(t-\sqrt{t^2-1})^2}+t-\sqrt{t^2-1}\right]+2n^2(t^2-1)} \\ &\leq \frac{2(n+1)^2}{\sqrt{4(n+1)^2(t^2-1)+(t-\sqrt{t^2-1})^2}+t-\sqrt{t^2-1}}, \end{aligned}$$

where the last inequality can be proved by a straightforward calculation using, e.g., the substitution $t = y/\sqrt{y^2-1}$, $y > 1$. Thus (2.1) is completely proved.

Now let $x \in \left(0, \frac{1}{2}\right)$ and $t = \frac{2x^2-2x+1}{1-2x} > 1$. Then, according to [6, (2.3) and (2.4)],

$$\frac{F_n'(x)}{F_n(x)} = \frac{2\sqrt{t^2-1}}{t-\sqrt{t^2-1}}\left(u_n(t) - \frac{n}{\sqrt{t^2-1}}\right). \tag{2.2}$$

Denote $X := x(1-x)$, $X' = 1-2x$, and let $z_1 < z_2$ be the roots of the equation

$$XX'z^2 + [1+4(n-1)X]\,z + 2nX' = 0.$$

Then $z_1 = \frac{-r-p}{q}$, $z_2 = \frac{r-p}{q}$, where $p = 1+4(n-1)X$, $q = 2XX'$, $r = \sqrt{16n^2X^2+(1-4X)^2}$. It is easy to express p, q, r as functions of t, and to verify that (2.1) implies

$$\left|\frac{2\sqrt{t^2-1}}{t-\sqrt{t^2-1}}\left(u_n(t) - \frac{n}{\sqrt{t^2-1}}\right) + \frac{p}{q}\right| \leq \frac{r}{q}.$$

Combined with (2.2), this yields $z_1 \leq \frac{F_n'(x)}{F_n(x)} \leq z_2$, hence

$$XX'\frac{(F_n'(x))^2}{F_n(x)} + [1+4(n-1)X]\,F_n'(x) + 2nX'F_n(x) \leq 0. \tag{2.3}$$

On the other hand, it was proved in [7, (33)], [8, (4.13)] that F_n is a solution of the differential (Heun) equation

$$XX'F_n''(x) + [1+4(n-1)X]\,F_n'(x) + 2nX'F_n(x) = 0. \tag{2.4}$$

From (2.3) and (2.4) we get $(F_n'(x))^2 \leq F_n''(x)F_n(x)$, and so F_n is log-convex on $\left[0, \frac{1}{2}\right]$. To conclude the proof, it suffices to remark that $F_n(1-x) = F_n(x)$ for all $x \in [0,1]$. □

3. Applications and Extensions

(a) The Rényi and Tsallis entropies of order 2 corresponding to the binomial distribution are defined, respectively, by $R_n(x) = -\log F_n(x)$ and $T_n(x) = 1 - F_n(x)$, $x \in [0,1]$.
So we have the following

Corollary 1. *R_n is concave and T_n is log-concave on* $[0,1]$.

(b) The logarithmic convexity of $S_{n,c}$ was used in [3] to establish bounds for this function and for the entropies $R_{n,c}, T_{n,c}$.

(c) Consider the Bernstein basis on the unit square $Q = [0,1]^2$ (see, e.g., [2, (6.3.101)]):

$$\binom{n}{i}\binom{n}{j}x^i y^j (1-x)^{n-i}(1-y)^{n-j},\ (x,y) \in Q,\ 0 \le i,j \le n.$$

The sum of squares of these functions is $F_n(x)F_n(y)$; consequently, this sum is logarithmically convex.

(d) Let $T := \{(x,y) \in \mathbb{R}^2 : x \ge 0, y \ge 0, x+y \le 1\}$. Then, for $(x,y) \in T$,

$$A_n(x,y) := \sum_{i+j \le n} \left(\frac{n!}{i!j!(n-i-j)!}\right)^2 x^{2i} y^{2j} (1-x-y)^{2(n-i-j)}$$

is the sum of squares of the functions from the Bernstein basis on the unit triangle T; see, e.g., [2, (5.2.89)].
For $0 \le y < 1$ we have

$$A_n(x,y) = \sum_{j=0}^{n} \binom{n}{j}^2 y^{2j}(1-y)^{2(n-j)} \sum_{i=0}^{n-j} \binom{n-j}{i}^2 \left(\frac{x}{1-y}\right)^{2i} \left(1-\frac{x}{1-y}\right)^{2(n-j-i)}$$
$$= \sum_{j=0}^{n} \binom{n}{j}^2 y^{2j}(1-y)^{2(n-j)} F_{n-j}\left(\frac{x}{1-y}\right),\ 0 \le x \le 1-y.$$

Geometrically, this means that A_n restricted to each segment parallel to Ox is logarithmically convex. Analogously, A_n is logarithmically convex when restricted to each segment parallel to one of the other two sides of T. With the terminology of [2, Section 6.3.1], we conclude that A_n is *axially-logarithmically-convex* on the triangle T. It is easy to see that A_1 is not logarithmically convex on T.

We end this paper with the following

Problem 1. Is A_n convex on T?

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