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The Integral Cosine Addition and Sine Subtraction Laws

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Abstract. In the present paper we characterize the solutions of each of the integral functional equations

$$\int_G g(xyt)d\mu(t) = g(x)g(y) - f(x)f(y), \quad x, y \in G,$$

$$\int_G f(x\sigma(y)t)d\mu(t) = f(x)g(y) - f(y)g(x), \quad x, y \in G,$$

where G is a locally compact Hausdorff group, $\sigma : G \to G$ is a continuous homomorphism such that $\sigma \circ \sigma = I$, and μ is a regular, compactly supported, complex-valued Borel measure on G.

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1. Set Up, Notation and Terminology

Throughout the paper we work in the following framework and with the following notation and terminology. We use it without explicit mentioning. G is a topological group with neutral element e, $C(G)$ the algebra of continuous, complex valued functions on G, and $\sigma : G \to G$ is a continuous homomorphism such that $\sigma \circ \sigma = I$, where I denotes the identity map. The set of continuous homomorphisms $a : G \to (\mathbb{C}, +)$ will be called the additive maps and denoted by $\mathcal{A}(G)$. Those $a \in \mathcal{A}(G)$ for which $a \circ \sigma = -a$ will be denoted $\mathcal{A}^-(G)$.

A character χ of G is a homomorphism $\chi : G \to \mathbb{C}^*$, where $\mathbb{C}^*$ denotes the multiplicative group of non-zero complex numbers. So characters need not be unitary in the present paper. It is well known that the set of characters on G is a linearly independent subset of the vector space of all complex-valued functions on G (see [11, Corollary 3.20]).

If G is a locally compact Hausdorff group then $M_C(G)$ denotes the space of all regular, compactly supported, complex-valued Borel measures on G. For $\mu \in M_C(G)$, we use the notation

$$\mu(f) = \int_G f(t) d\mu(t),$$

for all $f \in C(G)$.

2. Introduction

The trigonometric addition and subtraction formulas have been studied in the context of functional equations by a number of mathematicians. The monographs by Aczél [1], by Kannappan [8], by Stetkær [11] and by Székelyhidi [15] have references and detailed discussions of the classic results.

Chung, Kannappan and Ng [3] solved the functional equation

$$f(xy) = f(x)g(y) + f(y)g(x) + h(x)h(y), \quad x, y \in G.$$

Poulsen and Stetkær [9] found the complete set of continuous solutions of each of the functional equations

$$g(x\sigma(y)) = g(x)g(y) - f(y)f(x), \quad x, y \in G,$$
$$f(x\sigma(y)) = f(x)g(y) \pm f(y)g(x), \quad x, y \in G,$$

in which G is an arbitrary topological group.

As a continuation of these investigations we find the continuous solutions $f, g : G \to \mathbb{C}$ of each of the following integral versions of the addition and subtraction formulas for sine and cosine:

$$\int_G g(xyt) d\mu(t) = g(x)g(y) - f(x)f(y), \quad x, y \in G, \tag{2.1}$$

$$\int_G f(x\sigma(y)t) d\mu(t) = f(x)g(y) - g(x)f(y), \quad x, y \in G, \tag{2.2}$$

where G is a locally compact Hausdorff group and $\mu \in M_C(G)$. To solve Eq. (2.1) we reduce it, for a fixed complex constant α, to the functional equation

$$g(xy) = g(x)g(y) - f(x)f(y) + \alpha f(xy), \quad x, y \in G,$$

where the solutions were given in [12, Theorem 3.1] and to find the solutions of Eq. (2.2), we reduce it to the functional equation

$$f(x\sigma(y)) = f(x)g(y) - f(y)g(y) + \alpha g(x\sigma(y)), \quad x, y \in G.$$

The solutions of which are given in Proposition 4.1.

Note that if $\sigma = I$ then Eq. (2.2) has only "trivial" solutions (see Remark 4.3) and that the functional equation

$$\int_G f(xyt) d\mu(t) = f(x)g(y) + g(x)f(y), \quad x, y \in G,$$

has been resolved in [21].

The papers [10,13,16] have been an inspiration in their treatment of similar functional equations on groups or semigroups. We refer also to [14, 17,20] for some contextual discussions and to [2,5–7,18,19] for other integral functional equations.

3. The Solutions of the Integral Cosine Addition Law

The following result concerns solutions of

$$\int_G g(xyt)d\mu(t) = g(x)g(y) - f(x)f(y), \quad x, y \in G. \tag{3.1}$$

Theorem 3.1. *Let G be a locally compact Hausdorff group. Assume that the pair $f, g \in C(G)$ is a solution of Eq. (3.1). Then we have the following possibilities:*

(1) *g is any function on $C(G)$ such that $\int_G g(xt)d\mu(t) = 0$ for all $x \in G$ and $f = \pm g$.*

(2) *There exist constants $\gamma \in \mathbb{C}$, $\beta \in \mathbb{C}^\star$ and a continuous character χ of G, with $\mu(\chi) = \frac{\beta^2-\gamma^2}{\beta}$, such that*

$$f = \gamma\chi \quad \text{and} \quad g = \beta\chi.$$

(3) *There exist constants $\gamma \in \mathbb{C}$, $q \in \mathbb{C}^\star$ and a continuous character χ of G, with $\mu(\chi) = -\gamma\frac{1\pm\sqrt{1+q^2}}{q}$, such that*

$$g = \gamma q\frac{\chi}{2}, \quad f = \gamma\left(1 \pm \sqrt{1+q^2}\right)\frac{\chi}{2}.$$

(4) *There exist constants $\gamma \in \mathbb{C}$, $q \in \mathbb{C}^\star$ and two different continuous characters χ_1 and χ_2 of G, with $\mu(\chi_1) = \frac{-\gamma(1\pm\sqrt{1+q^2})}{q}$ and $\mu(\chi_2) = \frac{-\gamma(-1\pm\sqrt{1+q^2})}{q}$, such that*

$$g = \gamma q\frac{\chi_1 - \chi_2}{2} \quad \text{and} \quad f = \gamma\left(\frac{\chi_1+\chi_2}{2} \pm \sqrt{1+q^2}\frac{\chi_1-\chi_2}{2}\right).$$

(5) *There exist constants $\gamma \in \mathbb{C}, \beta \in \mathbb{C}^\star$, $q \in \mathbb{C}\backslash\{\pm\frac{\gamma}{\beta}\}$ and two different continuous characters χ_1 and χ_2 of G, with $\mu(\chi_1) = \beta - \frac{\gamma^2+\gamma q}{1+2\delta}$ and $\mu(\chi_2) = \beta - \frac{\frac{\gamma^2}{\beta}-\gamma q}{1-2\delta}$, such that*

$$f = \gamma\frac{\chi_1+\chi_2}{2} + q\beta\frac{\chi_1-\chi_2}{2} \quad \text{and} \quad g = \beta\left(\frac{\chi_1+\chi_2}{2} + \delta\frac{\chi_1-\chi_2}{2}\right),$$

where $\delta := \pm\sqrt{1+q^2-\left(\frac{\gamma}{\beta}\right)^2}$.

(6) *There exist constants* $\gamma \in \mathbb{C}$, $\beta \in \mathbb{C}^{\star}\backslash\{\gamma\}$, *a function* $a \in \mathcal{A}(G)$, *and a continuous character* χ *of* G *with* $\mu(\chi) = \beta - \gamma$ *and* $\mu(a\chi) = \gamma - \frac{\gamma^2}{\beta}$, *such that*

$$f = \chi(\gamma + \beta a) \quad \textit{and} \quad g = \beta\chi(1 + a).$$

(7) *There exist constants* $\gamma \in \mathbb{C}$, $\beta \in \mathbb{C}^{\star}\backslash\{-\gamma\}$, *a function* $a \in \mathcal{A}(G)$, *and a continuous character* χ *of* G, *with* $\mu(\chi) = \beta + \gamma$ *and* $\mu(a\chi) = \gamma + \frac{\gamma^2}{\beta}$ *such that*

$$f = \chi(\gamma + \beta a) \quad \textit{and} \quad g = \beta\chi(1 - a).$$

Conversely, the formulas above for f *and* g *define solutions of* (3.1).

Proof. Let f, g be solutions of the Eq. (3.1). Letting $y = e$ in (3.1) we get that

$$\int_G g(xt)d\mu(t) = \beta g(x) - \gamma f(x), \quad x \in G, \tag{3.2}$$

where $\beta = g(e)$ and $\gamma = f(e)$. So, using (3.2), we can reformulate the form of Eq. (3.1) as

$$\beta g(xy) = g(x)g(y) - f(x)f(y) + \gamma f(xy), \quad x, y \in G. \tag{3.3}$$

Case 1 Suppose that $\beta = 0$ then (3.3) gives

$$-\gamma f(xy) = g(x)g(y) - f(x)f(y), \quad x, y \in G. \tag{3.4}$$

If $\gamma = 0$ then Eq. (3.4) becomes

$$g(x)g(y) = f(x)f(y), \quad x, y \in G,$$

thus g is any function and $f = \pm g$. On putting $f = \pm g$ in (3.1) we find that $\int_G g(xt)d\mu(t) = 0$, so we are in the case (1) of our statement.

If $\gamma \neq 0$ then from (3.4) we obtain the functional equation

$$F(xy) = F(x)F(y) - G(x)G(y), \quad x, y \in G,$$

where $F = \frac{1}{\gamma}f$ and $G = \frac{1}{\gamma}g$. The solutions of which were given in [12, Theorem 6.3]. We work our way through the 3 possibilities (a)–(c) presented by [12, Theorem 6.3] to see what the properties (3.2), that $g(e) = 0$ and $f(e) \neq 0$ entail.

(a) There exist constants $\gamma \in \mathbb{C}$, $q \in \mathbb{C}^{\star}$ and a continuous character χ of G such that

$$g = \gamma q \frac{\chi}{2} \quad \text{and} \quad f = \gamma(1 \pm \sqrt{1 + q^2})\frac{\chi}{2}.$$

If $q = 0$ then $g = 0$ and $f = 0$ so we are in (1) of our statement. If $q \neq 0$ then using (3.2) we infer that $\mu(\chi) = -\gamma\frac{1\pm\sqrt{1+q^2}}{q}$. So we are in (3).

(b) There exist constants $\gamma \in \mathbb{C}$, $q \in \mathbb{C}^{\star}$ and two different continuous characters χ_1 and χ_2 of G such that

$$g = \gamma q \frac{\chi_1 - \chi_2}{2} \quad \text{and} \quad f = \gamma\left(\frac{\chi_1 + \chi_2}{2} \pm \sqrt{1 + q^2}\frac{\chi_1 - \chi_2}{2}\right).$$

According to (3.2) and the linear independence of different characters we will have $\mu(\chi_1) = \frac{-\gamma(1\pm\sqrt{1+q^2})}{q}$ and $\mu(\chi_2) = \frac{-\gamma(-1\pm\sqrt{1+q^2})}{q}$. Then (4) holds.
(c) There exist a character χ of G, and a non zero additive function $a : G \to \mathbb{C}$ such that $g = \gamma\chi a$ and $f = \gamma\chi(1 \pm a)$. Using (3.2) we find that $\gamma = 0$, this case is excluded (because $\gamma \neq 0$).

Case 2 Suppose that $\beta \neq 0$ then the Eq. (3.3) becomes

$$G(xy) = G(x)G(y) - F(x)F(y) + \frac{\gamma}{\beta}F(xy), \quad x, y \in G,$$

where $G = \frac{1}{\beta}g$ and $F = \frac{1}{\beta}f$. Applying [12, Theorem 3.1] we infer that there are only the following cases:

(i) g is any function and $f = \pm g$, by using (3.1) we get $\int_G g(xt)d\mu(t) = 0$. So we are in the case (1) of our statement.

(ii) There exist constants $\beta \in \mathbb{C}^\star$, $q \in \mathbb{C}$ and a continuous character χ of G such that

$$f = \beta\left(q + \frac{\gamma}{\beta}\right)\frac{\chi}{2} \quad \text{and} \quad g = \beta\left(1 \pm \sqrt{1 + q^2 - \left(\frac{\gamma}{\beta}\right)^2}\right)\frac{\chi}{2}. \tag{3.5}$$

Since $f(e) = \gamma$, using (3.5), we get that $q = \frac{\gamma}{\beta}$ and so

$$f = \gamma\chi \quad \text{and} \quad g = \beta\chi.$$

A small computation based on (3.2) shows that $\mu(\chi) = \frac{\beta^2-\gamma^2}{\beta}$. So we are in the case (2) of our statement.
(iii) There exist a constant $q \in \mathbb{C}\backslash\{\pm\frac{\gamma}{\beta}\}$ and two different continuous characters χ_1 and χ_2 of G such that

$$f = \gamma\frac{\chi_1 + \chi_2}{2} + q\beta\frac{\chi_1 - \chi_2}{2} \quad \text{and}$$
$$g = \beta\left(\frac{\chi_1 + \chi_2}{2} \pm \sqrt{1 + q^2 - \left(\frac{\gamma}{\beta}\right)^2}\frac{\chi_1 - \chi_2}{2}\right).$$

Putting $\delta = \pm\sqrt{\frac{1}{2}1 + q^2 - (\frac{\gamma}{\beta})^2}$. A small computation based on (3.2) shows that

$$\beta\left(\frac{1}{2} + \delta\right)\mu(\chi_1)\chi_1(x) + \beta\left(\frac{1}{2} - \delta\right)\mu(\chi_2)\chi_2(x)$$
$$= \left[\beta^2\left(\frac{1}{2} + \delta\right) - \gamma\beta\frac{\gamma + q}{2}\right]\chi_1(x) + \left[\beta^2\left(\frac{1}{2} - \delta\right) - \gamma\beta\frac{\gamma - q}{2}\right]\chi_2(x).$$

By the linear independence of different characters we infer that

$$\left(\frac{1}{2} + \delta\right)\mu(\chi_1) = \beta\left(\frac{1}{2} + \delta\right) - \frac{\gamma(\gamma + q)}{2},$$

and

$$\left(\frac{1}{2}-\delta\right)\mu(\chi_1)=\beta\left(\frac{1}{2}-\delta\right)-\frac{\gamma(\gamma+q)}{2}.$$

Since $q\in\mathbb{C}\backslash\{\pm\frac{\gamma}{\beta}\}$, then $\frac{1}{2}+\delta\neq 0\neq\frac{1}{2}-\delta$ and so we find that

$$\mu(\chi_1)=\beta-\frac{\gamma(\gamma+q)}{1+2\delta}\quad\text{ and }\quad\mu(\chi_2)=\beta-\frac{\gamma(\gamma-q)}{1-2\delta}.$$

So we are in the case (5) of our statement.

(iv) $\gamma\neq 0$, and there exist two different characters χ_1 and χ_2 of G such that $f=\gamma\chi_1$ and $g=\beta\chi_2$. Using (3.2) we get that

$$\beta\mu(\chi_2)\chi_2(x)=\beta^2\chi_2(x)-\gamma^2\chi_1(x)\quad\text{ for all }\ x\in G,$$

which shows that $\mu(\chi_2)=\beta$ and $\gamma=0$. So this case is excluded.

(v) There exist a character χ of G and $a\in\mathcal{A}(G)\backslash\{0\}$ such that

$$f=\beta\left(\frac{\gamma}{\beta}+a\right)\chi\quad\text{ and }\quad g=\beta(1+a)\chi.$$

Using (3.2) we obtain

$$(\beta-\gamma-\mu(\chi))a(x)=\frac{\gamma^2}{\beta}-\beta+\mu(a\chi)+\mu(\chi),$$

which implies that $\mu(\chi)=\beta-\gamma$ and $\mu(a\chi)=\gamma-\frac{\gamma^2}{\beta}$. So we are in the case (6) of our statement.

(vi) There exist a character χ of G and a function $a\in\mathcal{A}(G)\backslash\{0\}$ such that

$$f=\beta\left(\frac{\gamma}{\beta}+a\right)\chi\quad\text{ and }\quad g=\beta(1-a)\chi.$$

Using (3.2) we get that that $\mu(\chi)=\beta+\gamma$ and $\mu(a\chi)=\gamma+\frac{\gamma^2}{\beta}$. So we are in the case (7) of our statement.

Conversely, simple computations prove that the formulas above for f and g define solutions of (3.1). □

In the following corollary we solve the functional equation

$$g(xyz_0)=g(x)g(y)-f(x)f(y),\quad x,y\in G, \tag{3.6}$$

for a fixed complex constant z_0.

Corollary 3.2. *Let G be a topological group. Assume that the pair $f,g\in C(G)$ is a solution of Eq. (3.6). Then we have the following possibilities:*

(1) $g\equiv 0$ *and* $f\equiv 0$.

(2) *There exist constants $\gamma\in\mathbb{C}$, $\beta\in\mathbb{C}^\star\backslash\{\pm\gamma\}$ and a continuous character χ of G, with $\chi(z_0)=\frac{\beta^2-\gamma^2}{\beta}$, such that*

$$f=\gamma\chi\qquad\textit{and}\qquad g=\beta\chi.$$

(3) *There exist constants* γ, $q \in \mathbb{C}^\star$ *and a continuous character* χ *of* G *with* $\chi(z_0) = -\gamma\frac{1\pm\sqrt{1+q^2}}{q}$ *such that*

$$g = \gamma q\frac{\chi}{2} \quad \text{and} \quad f = \gamma\left(1 \pm \sqrt{1+q^2}\right)\frac{\chi}{2}\,.$$

(4) *There exist constants* γ,$q \in \mathbb{C}^\star$ *and two different continuous characters* χ_1 *and* χ_2 *of* G, *with* $\chi_1(z_0) = \frac{-\gamma(1\pm\sqrt{1+q^2})}{q}$ *and* $\chi_2(z_0) = \frac{-\gamma(-1\pm\sqrt{1+q^2})}{q}$ *such that*

$$g = \gamma q\frac{\chi_1 - \chi_2}{2} \quad \text{and} \quad f = \gamma\left(\frac{\chi_1+\chi_2}{2} \pm \sqrt{1+q^2}\frac{\chi_1-\chi_2}{2}\right).$$

(5) *There exist constants* $\gamma \in \mathbb{C}$, $\beta \in \mathbb{C}^\star$, $q \in \mathbb{C}\backslash\{\pm\frac{\gamma}{\beta}\}$ *and two different continuous characters* χ_1 *and* χ_2 *of* G, *with* $\chi_1(z_0) = \beta - \frac{\gamma^2+\gamma q}{1+2\delta}$ *and* $\chi_2(z_0) = \beta - \frac{\frac{\gamma^2}{\beta}-\gamma q}{1-2\delta}$, *such that*

$$f = \gamma\frac{\chi_1+\chi_2}{2} + q\beta\frac{\chi_1-\chi_2}{2} \quad \text{and} \quad g = \beta\left(\frac{\chi_1+\chi_2}{2} + \delta\frac{\chi_1-\chi_2}{2}\right).$$

where $\delta := \pm\sqrt{1+q^2-(\frac{\gamma}{\beta})^2}$.

(6) *There exist constants* $\gamma \in \mathbb{C}$, $\beta \in \mathbb{C}\backslash\{0,\gamma\}$, *a function* $a \in \mathcal{A}(G)$, *a continuous character* χ *of* G, *with* $\chi(z_0) = \beta - \gamma$ *and* $a(z_0) = \frac{\gamma}{\beta}$, *such that*

$$f = \chi(\gamma + \beta a) \quad \text{and} \quad g = \beta\chi(1+a).$$

(7) *There exist constants* $\gamma \in \mathbb{C}$, $\beta \in \mathbb{C}^\star\backslash\{-\gamma\}$, *a function* $a \in \mathcal{A}(G)$, *and a continuous character* χ *of* G, *with* $\chi(z_0) = \beta + \gamma$ *and* $a(z_0) = \frac{\gamma}{\beta}$, *such that*

$$f = \chi(\gamma + \beta a) \quad \text{and} \quad g = \beta\chi(1-a).$$

Conversely, the formulas above for f *and* g *define solutions of* (3.6).

Proof. As the proof of Theorem 3.1 with $\mu = \delta_{z_0}$ and the fact that $\chi(z_0) \neq 0$. □

4. The Solutions of the Integral Sine Subtraction Law with Involution

The purpose of this section is first to give an explicit description of the continuous solutions of the functional equation

$$f(x\sigma(y)) = f(x)g(y) - g(x)f(y) + \alpha g(x\sigma(y)), \quad x, y \in G, \tag{4.1}$$

where $f, g : G \to \mathbb{C}$ are unknown functions. And secondly to determine the continuous solutions $f, g : G \to \mathbb{C}$ of the functional equation (2.2), namely

$$\int_G f(x\sigma(y)t)d\mu(t) = f(x)g(y) - g(x)f(y), \quad x, y \in G, \tag{4.2}$$

where $\mu \in M_C(G)$, in terms of characters and additive maps of G.

If $\alpha = 0$, the solutions of (4.2) were given in [9, Theorem II.2], so in the following proposition we are only concerned with the case $\alpha \neq 0$.

Proposition 4.1. *Let G be a topological group and $\alpha \in \mathbb{C}^\star$. The solutions $f, g \in C(G)$ of the functional equation (4.1) are the following.*

(1) *f and g are linearly dependant and g any function.*

(2)

$$g = \frac{\chi + \chi\circ\sigma}{2} + c\frac{\chi - \chi\circ\sigma}{2},$$
$$f = \alpha\frac{\chi + \chi\circ\sigma}{2} + c_1\frac{\chi - \chi\circ\sigma}{2} \quad \text{and} \quad \chi \neq \chi\circ\sigma.$$

(3)

$$g = \chi(1 + ca), \quad f = \alpha\chi(1 + (c+1)a) \quad \text{and} \quad \chi = \chi\circ\sigma.$$

where χ is a continuous character of G, $a \in \mathcal{A}^-(G)$ and c, c_1 are complex constants.

Proof. We define the function $F(x) := \frac{1}{\alpha}f(x) - g(x)$. Then the Eq. (4.1) becomes

$$F(x\sigma(y)) = F(x)g(y) - F(y)g(x), \ x, y \in G.$$

In view of [9, Theorem II.2] we infer that there are only the following three cases (1)–(3) listed above.

Conversely, simple computations prove that the formulas above for f and g define solutions of (4.1). □

In the following theorem we solve the functional equation (4.2).

Theorem 4.2. *Let G be a locally compact Hausdorff group. Assume that the pair $f, g \in C(G)$ is a solution of Eq. (4.2). Then we have the following possibilities:*

(1) *g and f are linearely dependant and f is any function such that $\int_G f(\cdot t) d\mu(t) = 0$.*

(2) *There exist constants $\gamma, \beta \in \mathbb{C}^\star$ and a continuous character χ of G, with $\chi \neq \chi\circ\sigma$ and $\mu(\chi) = 0$, such that*

$$f = \gamma\chi \quad \text{and} \quad g = \beta\chi.$$

(3) *There exist constants* $c, c_1 \in \mathbb{C}$, $\beta \in \mathbb{C}^\star$, $\gamma \in \mathbb{C}\backslash\{\pm c_1\}$ *and a continuous character* χ *of* G, *with* $\chi \neq \chi\circ\sigma$ *and* $\mu(\chi) = \frac{\beta c_1 - \gamma\beta c}{\gamma + c_1}$ *and* $\mu(\chi\circ\sigma) = \frac{-\beta c_1 + \gamma\beta c}{\gamma - c_1}$, *such that*

$$f = \gamma\frac{\chi + \chi\circ\sigma}{2} + c_1\frac{\chi - \chi\circ\sigma}{2} \qquad \textit{and} \qquad g = \beta\left(\frac{\chi + \chi\circ\sigma}{2} + c\frac{\chi - \chi\circ\sigma}{2}\right).$$

(4) *There exist constants* $c, c_1 \in \mathbb{C}$, $\gamma \in \mathbb{C}\backslash\{0, \pm c\}$, *and a continuous character* χ *of* G, *with* $\chi \neq \chi\circ\sigma$, $\mu(\chi) = -\frac{\gamma c_1}{\gamma + c}$ *and* $\mu(\chi\circ\sigma) = \frac{\gamma c_1}{\gamma - c}$, *such that*

$$f = \gamma\left(\frac{\chi + \chi\circ\sigma}{2} + c\frac{\chi - \chi\circ\sigma}{2}\right) \qquad \textit{and} \qquad g = c_1\frac{\chi - \chi\circ\sigma}{2}.$$

(5) *There exist constants* $\gamma, c \in \mathbb{C}^\star$, *a function* $a \in \mathcal{A}^-(G)\backslash\{0\}$ *and a continuous character* χ *of* G, *with* $\chi = \chi\circ\sigma$ *and* $\mu(a\chi) = (\mu(\chi))^2$, *such that*

$$f = \gamma\chi(1 - \frac{1}{\mu(\chi)}a) \qquad \textit{and} \qquad g = \chi a.$$

(6) *There exist constants* $c \in \mathbb{C}$, *a function* $a \in \mathcal{A}^-(G)\backslash\{0\}$ *and a continuous character* χ *of* G, *with* $\chi = \chi\circ\sigma$ *and* $\mu(a\chi) = 0$, *such that*

$$f = \chi a \qquad \textit{and} \qquad g = \mu(\chi)\chi(1 + ca).$$

(7) *There exist constants* $\beta, \gamma \in \mathbb{C}^\star$, $c \in \mathbb{C}\backslash\{-1\}$, *a function* $a \in \mathcal{A}^-(G)\backslash\{0\}$ *and a continuous character* χ *of* G, *with* $\chi = \chi\circ\sigma$ *and* $\mu(\chi) = -(c+1)\mu(a\chi) = \frac{-\beta}{c+1}$, *such that*

$$f = \gamma\chi(1 + (c+1)a), \qquad g = \beta\chi(1 + ca)\,.$$

Conversely, the formulas above for f *and* g *define solutions of* (4.2).

Proof. Putting $y = e$ in (4.2) we get that

$$\int_G f(xt)d\mu(t) = \beta f(x) - \gamma g(x), \qquad x \in G, \tag{4.3}$$

where $\gamma := f(e)$ and $\beta := g(e)$. So, using (4.3), we can reformulate the form of Eq. (4.1) as

$$\beta f(x\sigma(y)) = f(x)g(y) - f(y)g(x) + \gamma g(x\sigma(y)), \qquad x, y \in G. \tag{4.4}$$

We break the job into two cases: $\beta = 0$ and $\beta \neq 0$.

Case 1 Suppose that $\beta = 0$, then from (4.4) we get that

$$\gamma g(x\sigma(y)) = g(x)f(y) - g(y)f(x), \qquad x, y \in G.$$

First we suppose that $\gamma = 0$, then $f(x)g(y) = f(y)g(x)$ for all $x, y \in G$, thus f and g are proportional and f is any function such that $\int_G f(\cdot t)d\mu(t) = 0$. So we are in the case (1) of our statement.

If $\gamma \neq 0$ then Eq. (4.4) becomes

$$g(x\sigma(y)) = g(x)F(y) - g(y)F(x), \qquad x, y \in G, \tag{4.5}$$

where $F := \frac{1}{\gamma}f$. This functional equation was solved in [9, Theorem II.2] according to which there exist constants $c, c_1 \in \mathbb{C}^\star$, a function $a \in \mathcal{A}^-(G)$ and a continuous character χ of G such that:

(i) $g = 0$ and f any function such that $\int_G f(\cdot t)d\mu(t) = 0$. So we are in the case (1).

(ii) $g = c_1\frac{\chi - \chi\circ\sigma}{2}$, $f = \gamma[\frac{\chi+\chi\circ\sigma}{2} + c\frac{\chi-\chi\circ\sigma}{2}]$, $\chi \neq \chi\circ\sigma$. Using (4.3) we get

$$(\gamma + c)\mu(\chi) = -\gamma c_1 \qquad \text{and} \qquad (\gamma - c)\mu(\chi\circ\sigma) = \gamma c_1.$$

Suppose that $\gamma = \pm c$ then $c_1 = 0$ so $g = 0$ and we are also in the case (1). If $\gamma \neq \pm c$ then $\mu(\chi) = \frac{-\gamma c_1}{\gamma + c}$ and $\mu(\chi\circ\sigma) = \frac{\gamma c_1}{\gamma - c}$. So we are in (4).

(iii) $g = \chi a$, $f = \gamma\chi(1 + ca)$. Using (4.3) we get

$$1 + c\mu(\chi) = 0 \qquad \text{and} \qquad \mu(\chi) + c\mu(a\chi) = 0,$$

then $c \neq 0$ and so $\mu(\chi) = \frac{-1}{c}$ and $\mu(a\chi) = \frac{1}{c^2} = (\mu(\chi))^2$. So we are in (5).

Case 2 Suppose that $\beta \neq 0$. we can reformulate the form of Eq. (4.4) as

$$f(x\sigma(y)) = f(x)G(y) - f(y)G(x) + \gamma G(x\sigma(y)), \quad x, y \in G, \tag{4.6}$$

where $G := \frac{1}{\beta}g$, the solutions of which, in the subcase $\gamma = 0$, are given in [9, Theorem II.2]. When we analyse them we find, using the notation in [9, Theorem II.2] with m replaced here by χ, that:

(i) $f = 0$, g is any function. So we are in (1).

(ii) $f = c_1\frac{\chi-\chi\circ\sigma}{2}$, $g = \beta[\frac{\chi+\chi\circ\sigma}{2} + c\frac{\chi-\chi\circ\sigma}{2}]$, $\chi \neq \chi\circ\sigma$. A small computation based on (4.3) shows that

$$c_1\mu(\chi) = \beta c_1 \qquad \text{and} \qquad c_1\mu(\chi\circ\sigma) = \beta c_1.$$

If $c_1 = 0$ then $f = 0$. So we are also in the case (1) of our statement and if $c_1 \neq 0$ we get $\mu(\chi) = \mu(\chi\circ\sigma) = \beta$. So we are in the case (3) (because here $\gamma = 0$).

(iii) $f = \chi a$, $g = \beta\chi(1 + ca)$ and $\chi = \chi\circ\sigma$. A computation based on (4.3) shows that

$$\mu(\chi) = \beta \qquad \text{and} \qquad \mu(a\chi) = 0,$$

so we are in (6).

For **the subcase** $\gamma \neq 0$, according to Proposition (4.1) the solutions of (4.6). are the following, where χ is a continuous character, $a \in \mathcal{A}^-(G)\backslash\{0\}$ and c, c_1 are complex constants.

(a) $f = \gamma g$ and g any function, using (4.2) we get that $\int_G f(\cdot t)d\mu(t) = 0$, so we are in (1).

(*b*)

$$g=\beta\left(\frac{\chi+\chi\circ\sigma}{2}+c\frac{\chi-\chi\circ\sigma}{2}\right),\quad f=\gamma\frac{\chi+\chi\circ\sigma}{2}+c_1\frac{\chi-\chi\circ\sigma}{2}$$
$$\text{and}\qquad \chi\neq\chi\circ\sigma.$$

A small computation based on (4.3) shows that

$$\begin{aligned}&(\gamma+c_1)\mu(\chi)\chi(x)+(\gamma-c_1)\mu(\chi\circ\sigma)\chi\circ\sigma(x)\\&\quad=\left[\beta(\gamma+c_1)-\gamma\beta(1+c)\right]\chi(x)+\left[\beta(\gamma-c_1)-\gamma\beta(1-c)\right]\chi_2(x).\end{aligned}$$

By the linear independence of different characters we infer that

$$(\gamma+c_1)\mu(\chi)=\beta(\gamma+c_1)-\gamma\beta(1+c),$$

and

$$(\gamma-c_1)\mu(\chi\circ\sigma)=\beta(\gamma-c_1)-\gamma\beta(1-c).$$

If $\gamma=c_1$ then $c=1$ therefore $\mu(\chi)=0$. So we are in the case (2) of our statement.

If $\gamma=-c_1$ then $c=-1$ therefore $\mu(\chi\circ\sigma)=0$. So we are also in case (2) of our statement with χ replaced by $\chi\circ\sigma$.

If $c_1\neq\gamma\neq-c_1$ then $\mu(\chi)=\beta-\gamma\beta\frac{1+c}{\gamma+c_1}=\frac{\beta c_1-\gamma\beta c}{\gamma+c_1}$ and $\mu(\chi\circ\sigma)=\beta-\gamma\beta\frac{1-c}{\gamma-c_1}=\frac{-\beta c_1+\gamma\beta c}{\gamma-c_1}$. So we are in the case (3) of our statement.

(*c*)

$$g=\beta\chi(1+ca),\ f=\gamma\chi(1+(c+1)a)\qquad\text{and}\qquad\chi=\chi\circ\sigma.$$

A small computation based on (4.3) shows that

$$\begin{aligned}&\left[(c+1)\mu(\chi)+\beta(c+1)-\beta c\right]a(x)\\&\quad=-\mu(\chi)-(c+1)\mu(a\chi),\end{aligned}$$

then

$$(c+1)\mu(\chi)=-\beta(c+1)+\beta c,\tag{4.7}$$

and

$$\mu(\chi)=-(c+1)\mu(a\chi),$$

and so $c\neq-1$ because otherwise (4.7) implies $\beta c=-\beta=0$. Therefore

$$\mu(\chi)=-\beta+\frac{\beta c}{c+1}=-(c+1)\mu(a\chi).$$

So we are in (7).

Conversely, simple computations prove that the formulas above for f define solutions of (4.2). $\square$

Remark 4.3. Let G be a locally compact Hausdorff group. If we analyse the different assertions of Theorem 4.2 in the case where $\sigma = I$, we find that the continuous solutions of the following functional equation

$$\int_G f(xyt)d\mu(t) = f(x)g(y) - f(y)g(x), \quad x, y \in G,$$

are the pair (f, g) such that g and f are linearely dependant and f is any function on $C(G)$ satisfying $\int_G f(\cdot t)d\mu(t) = 0$. Indeed, in the cases (2)–(4) of Theorem 4.2 we have $\chi \neq \chi \circ \sigma$ so they do not occur here, and in the cases (5)–(7) we have $a = 0$ and so $f = 0$ or $g = 0$.

In the following corollary we solve the functional equation

$$f(xyz_0) = f(x)g(y) - f(y)g(x), \quad x, y \in G, \tag{4.8}$$

for a fixed complex constant z_0.

Corollary 4.4. *Let G be a topological group. Assume that the pair $f, g \in C(G)$ is a solution of Eq. (4.8). Then we have the following possibilities:*

(1) $f \equiv 0$ *and g is arbitrary in $C(\mathbb{R})$.*

(2) *There exist constants $c, c_1 \in \mathbb{C}$, $\beta \in \mathbb{C}^\star$, $\gamma \in \mathbb{C}\backslash\{\pm c_1\}$ and a continuous character χ of G, with $\chi \neq \chi \circ \sigma$ and $\chi(z_0) = \frac{\beta c_1 - \gamma\beta c}{\gamma + c_1}$ and $\chi \circ \sigma(z_0) = \frac{-\beta c_1 + \gamma\beta c}{\gamma - c_1}$, such that*

$$f = \gamma\frac{\chi + \chi\circ\sigma}{2} + c_1\frac{\chi - \chi\circ\sigma}{2} \qquad \text{and} \qquad g = \beta\left[\frac{\chi + \chi\circ\sigma}{2} + c\frac{\chi - \chi\circ\sigma}{2}\right].$$

(3) *There exist constants $c \in \mathbb{C}, c_1 \in \mathbb{C}^\star$, $\gamma \in \mathbb{C}\backslash\{0, \pm c\}$, and a continuous character χ of G, with $\chi \neq \chi \circ \sigma$, $\chi(z_0) = -\frac{\gamma c_1}{\gamma + c}$ and $\chi \circ \sigma(z_0) = \frac{\gamma c_1}{\gamma - c}$, such that*

$$f = \gamma\left[\frac{\chi + \chi\circ\sigma}{2} + c\frac{\chi - \chi\circ\sigma}{2}\right] \qquad \text{and} \qquad g = \frac{c_1}{2}(\chi - \chi\circ\sigma).$$

(4) *There exist constants $\gamma, c \in \mathbb{C}^\star$, $a \in \mathcal{A}^-(G)\backslash\{0\}$ and a continuous character χ of G, with $\chi = \chi \circ \sigma$ and $a(z_0) = \chi(z_0)$, such that*

$$f = \gamma\chi\left(1 - \frac{1}{\chi(z_0)}a\right) \qquad \text{and} \qquad g = \chi a.$$

(5) *There exist a constant $c \in \mathbb{C}$, $a \in \mathcal{A}^-(G)\backslash\{0\}$ and a continuous character χ of G, with $\chi = \chi \circ \sigma$ and $a(z_0) = 0$, such that*

$$f = \chi a \quad \text{and} \quad g = \chi(z_0)\chi(1 + ca).$$

(6) *There exist constants $\beta, \gamma \in \mathbb{C}^\star$, $c \in \mathbb{C}\backslash\{-1\}$, $a \in \mathcal{A}^-(G)\backslash\{0\}$ and a continuous character χ of G, with $\chi = \chi\circ\sigma$, $a(z_0) = \frac{-1}{c+1}$ and $\chi(z_0) = \frac{-\beta}{c+1}$ such that*

$$f = \gamma\chi(1 + (c+1)a) \quad \text{and} \quad g = \beta\chi(1 + ca).$$

Conversely, the formulas above for f and g define solutions of (4.1).

Proof. As the proof of Theorem 4.2 with $\mu = \delta_{z_0}$ and the fact that $\chi(z_0) \neq 0$. □

5. Examples

Example 5.1. Let $G = (\mathbb{R}, +)$, $\sigma(x) = -x$ for all $x \in \mathbb{R}$, $z_0 \in \mathbb{R}\backslash\{0\}$ be a fixed element, and let $\mu = \delta_{z_0}$.

We indicate here the corresponding continuous solutions of Eq. (4.2) [i.e. (4.8)] by the help of Corollary 4.4.

The continuous characters on $\mathbb{R}$ are known to be $\chi(x) = e^{\lambda x}$, $x \in \mathbb{R}$, where λ ranges over $\mathbb{C}$ (see for instance [11, Example 3.7(a)]). The condition $\chi \circ \sigma(z_0) = \chi(z_0)$, i.e. $\chi(2z_0) = 1$, of Theorem 4.2(3) i.e (of course in the case $\gamma = 0$) becomes $e^{2\lambda z_0} = 1$, which reduces to $\lambda = i\frac{n\pi}{z_0}$, where $n \in \mathbb{Z}$. The relevant characters are thus

$$\chi_n(x) := e^{i\frac{n\pi}{2z_0}x}, \quad x \in \mathbb{R}, \quad \text{and} \quad n \in \mathbb{Z}.$$

The continuous additive functions on $\mathbb{R}$ are the functions of the form $a(x) = \alpha x$, $x \in \mathbb{R}$, where the constant α ranges over $\mathbb{C}$ (see for instance [11, Corollary 2.4]). In the point (6) of Corollary 4.4 we have $a(z_0) = 0$ which reduces to $\alpha = 0$ i.e. $a = 0$ and $f = 0$. So this case does not occur here.

In conclusion, by help of Corollary 4.4 we find that the continuous solutions $f, g : \mathbb{R} \to \mathbb{C}$ of the functional equation (4.2), which is here

$$f(x - y + z_0) = f(x)g(x) - f(y)g(x), \quad x, y \in \mathbb{R},$$

are

(1) $f = 0$ and g is arbitrary in $C(\mathbb{R})$.

(2)

$$f(x) = \gamma \cos \lambda x + c_1 \sin \lambda x \text{ and } g = \beta[\cos \lambda x + c \sin \lambda x], \quad x \in \mathbb{R},$$

for some constants $c_1 \in \mathbb{C}^\star$, $c \in \mathbb{C}$, $\lambda, \beta \in \mathbb{C}^\star$, and $\gamma \in \mathbb{C}\backslash\{\pm c_1\}$ such that

$$e^{i\lambda z_0} = \frac{\beta c_1 - \gamma\beta c}{\gamma + c_1} = \frac{\gamma - c_1}{-\beta c_1 + \gamma\beta c}.$$

In particular, if $\gamma = 0$ we obtain

$$f(x) = c_1 \sin \frac{n\pi}{z_0} x \quad \text{and} \quad g(x) = (-1)^n \left[\cos \frac{n\pi}{z_0} x + c \sin \frac{n\pi}{z_0} x\right], \quad x \in \mathbb{R},$$

for some constants $c_1 \in \mathbb{C}^\star$, $c \in \mathbb{C}$ and $n \in \mathbb{Z}$.

(3)

$$f(x) = \gamma[\cos \lambda x + c \sin \lambda x], \quad x \in \mathbb{R} \quad \text{and} \quad g(x) = \frac{c_1}{2} \sin \lambda x,$$

for some constants $c, c_1 \in \mathbb{C}$, $\lambda \in \mathbb{C}^\star$, and $\gamma \in \mathbb{C}\backslash\{0, \pm c\}$ such that

$$e^{i\lambda z_0} = \frac{-\gamma c_1}{\gamma + c} = \frac{\gamma - c}{\gamma c_1}.$$

(4)

$$f(x)=\gamma\left(1-\frac{x}{z_0}\right) \qquad \text{and} \qquad g(x)=\frac{x}{z_0}, \quad x\in\mathbb{R},$$

for some $\gamma\in\mathbb{C}^\star$.

(5)

$$f(x)=\gamma\left(1-\frac{x}{z_0}\right) \qquad \text{and} \qquad g(x)=\beta\left(1-\frac{cx}{(c+1)z_0}\right), \quad x\in\mathbb{R},$$

for some $\gamma,\beta\in\mathbb{C}^\star$ and $c\in\mathbb{C}\backslash\{-1\}$ such that $\beta=c+1$.

Example 5.2. Let z_0 be a non-zero real number, we seek the solutions $f,g\in C(\mathbb{R})$ of the functional equation (3.1) which is here

$$g(x+y+z_0)=g(x)g(y)-f(x)f(y), \qquad x,y\in\mathbb{R}.$$

According to Corollary 3.2, we have only the following cases:

(1) $g=0$ and $f=0$.

(2)

$$f(x)=\gamma e^{\lambda x} \quad \text{and} \quad g(x)=\beta e^{\lambda x},\ x\in\mathbb{R},$$

for some constants $\beta,\lambda\in\mathbb{C}^\star$, $\gamma\in\mathbb{C}$ such that $e^{\lambda z_0}=\frac{\beta^2-\gamma^2}{\beta}$.

(3)

$$g(x)=\gamma q\frac{e^{\lambda x}}{2} \text{ and : } f(x)=\gamma(1\pm\sqrt{1+q^2})\frac{e^{\lambda x}}{2} \quad \text{for all } x\in\mathbb{R}.$$

for some constants $q\in\mathbb{C}^\star$ and $\lambda\in\mathbb{C}$ such that $e^{\lambda z_0}=-\gamma\frac{1\pm\sqrt{1+q^2}}{q}$.

(4)

$$g(x)=\gamma q\frac{e^{\lambda_1 x}-e^{\lambda_2 x}}{2},$$

and

$$f(x)=\gamma\left(\frac{e^{\lambda_1 x}+e^{\lambda_2 x}}{2}\pm\sqrt{1+q^2}\frac{e^{\lambda_1 x}-e^{\lambda_2 x}}{2}\right), \quad x\in\mathbb{R},$$

for some constants $q\in\mathbb{C}^\star$ and λ_1, $\lambda_2\in\mathbb{C}$ such that

$$\lambda_1\neq\lambda_2,\ e^{\lambda_1 z_0}=\frac{-\gamma(1\pm\sqrt{1+q^2})}{q} \quad \text{and} \quad e^{\lambda_2 z_0}=\frac{-\gamma(-1\pm\sqrt{1+q^2})}{q}.$$

(5)

$$f(x)=\gamma\frac{e^{\lambda_1 x}+e^{\lambda_2 x}}{2}+q\beta\frac{e^{\lambda_1 x}-e^{\lambda_2 x}}{2},$$

and

$$g(x)=\beta\left(\frac{e^{\lambda_1 x}+e^{\lambda_2 x}}{2}+\delta\frac{e^{\lambda_1 x}-e^{\lambda_2 x}}{2}\right) \qquad \text{for all } x\in\mathbb{R},$$

for some constants $\gamma \in \mathbb{C}$, $\beta \in \mathbb{C}^\star$, $q \in \mathbb{C}\backslash\{\pm\frac{\gamma}{\beta}\}$ and two different complex numbers λ_1, λ_2 such that $e^{\lambda_1 z_0} = \beta - \frac{\gamma^2+\gamma q}{1+2\delta}$ and $e^{\lambda_2 z_0} = \beta - \frac{\frac{\gamma^2}{\beta}-\gamma q}{1-2\delta}$, where $\delta := \pm\sqrt{1+q^2-(\frac{\gamma}{\beta})^2}$.

(6)

$$f(x) = e^{\lambda x}\left(\gamma + \frac{\gamma x}{z_0}\right) \quad \text{and} \quad g(x) = e^{\lambda x}\left(\beta + \frac{\gamma x}{z_0}\right), \quad x \in \mathbb{R},$$

for some constants $\lambda, \gamma \in \mathbb{C}$ and $\beta \in \mathbb{C}^\star\backslash\{\gamma\}$ such that $e^{\lambda z_0} = \beta - \gamma$.

(7)

$$f(x) = e^{\lambda x}\left(\gamma + \frac{\gamma x}{z_0}\right) \quad \text{and} \quad g(x) = e^{\lambda x}\left(\beta - \frac{\gamma x}{z_0}\right), \quad x \in \mathbb{R},$$

for some constants $\lambda, \gamma \in \mathbb{C}$ and $\beta \in \mathbb{C}^\star\backslash\{-\gamma\}$ such that $e^{\lambda z_0} = \beta + \gamma$.

Example 5.3. For an application of our results on a non-abelian group, consider the 3-dimensional Heisenberg group $G = H_3$ described in [11, Example A.17(a)], and take as the involutive automorphism

$$\sigma\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -b & -c+ab \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for} \quad a, b, c \in \mathbb{R}.$$

Let $Z_0 = \begin{pmatrix} 1 & a_0 & c_0 \\ 0 & 1 & b_0 \\ 0 & 0 & 1 \end{pmatrix}$ be a fixed element of H_3 such that $a_0 + b_0 \neq 0$, and let $\mu = \delta_{Z_0}$. We write down the continuous solutions of Eq. (4.2).

The continuous characters on H_3 are parametrized by $(u, v) \in \mathbb{C}^2$ as follows (see, e.g., [4, Example 5.2]).

$$\chi_{u,v}\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = e^{ua+vb} \quad \text{for} \quad a, b, c \in \mathbb{R}.$$

We compute that $\chi_{u,v} \circ \sigma = \chi_{-v,-u}$, so $\chi_{u,v} \circ \sigma = \chi_{u,v}$, if and only if $v = -u$, and in that case

$$\chi_{u,-u}\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = e^{u(a-b)} \quad \text{for} \quad a, b, c \in \mathbb{R}.$$

In view of [4, Example 5.2], the continuous additive functions A on H_3, satisfying $A \circ \sigma = -A$, are parametrized by $\lambda \in \mathbb{C}$ as follows

$$A_\lambda\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \lambda(a+b) \qquad \text{for} \quad a, b, c \in \mathbb{R}.$$

Now we are in the position to describe the solutions $f, g \in C(H_3)$ of the functional equation (4.2), which is here

$$f(X\sigma(Y)Z_0) = f(X)g(Y) - f(Y)g(X), \qquad X, Y \in H_3.$$

By Corollary 4.4, there exist constants $\alpha, \beta \in \mathbb{C}$ such that

(1) $f = 0$ and g is arbitrary in $C(H_3)$.
(2)

$$f\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \gamma \frac{e^{ua+vb} + e^{-va-ub}}{2} + c_1 \frac{e^{ua+vb} - e^{-va-ub}}{2}, \qquad a, b, c \in \mathbb{R},$$

$$g\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \beta \left[\frac{e^{ua+vb} + e^{-va-ub}}{2} + c \frac{e^{ua+vb} - e^{-va-ub}}{2} \right], \qquad a, b, c \in \mathbb{R},$$

for some constants $c_1 \in \mathbb{C}^\star$, $c \in \mathbb{C}$, $u, v, \beta \in \mathbb{C}^\star$, and $\gamma \in \mathbb{C}\backslash\{\pm c_1\}$ such that $v \neq -u$, $e^{ua_0+vb_0} = \frac{\beta c_1 - \gamma\beta c}{\gamma + c_1}$ and
$e^{-va_0-b_0} = \frac{-\beta c_1 + \gamma\beta c}{\gamma - c_1}$.

(3)

$$f\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \gamma \left[\frac{e^{ua+vb} + e^{-va-ub}}{2} + c \frac{e^{ua+vb} - e^{-va-ub}}{2} \right], \qquad a, b, c \in \mathbb{R},$$

$$g\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \frac{c_1}{2} \left[\frac{e^{ua+vb} - e^{-va-ub}}{2} \right], \qquad a, b, c \in \mathbb{R},$$

for some constants $c, c_1 \in \mathbb{C}$, $u, v \in \mathbb{C}$, and $\gamma \in \mathbb{C}\backslash\{0, \pm c\}$ such that $v \neq -u$, $e^{ua_0+vb_0} = \frac{-\gamma c_1}{\gamma + c}$ and $e^{-va_0-b_0} = \frac{\gamma c_1}{\gamma - c}$.

(4)

$$f\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \gamma e^{u(a-b)} \left[1 - \frac{a+b}{a_0+b_0} \right], \qquad a, b, c \in \mathbb{R},$$

$$g\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \frac{a+b}{a_0+b_0} e^{u(a+a_0-b-b_0)}, \qquad a, b, c \in \mathbb{R},$$

for some $u \in \mathbb{C}$ and $\gamma \in \mathbb{C}^\star$.

(5)

$$f\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix} = \gamma e^{u(a-b)}\left[1-\frac{a+b}{a_0+b_0}\right], \qquad a,b,c\in\mathbb{R},$$

$$g\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix} = \beta e^{u(a-b)}(1+c\frac{e^{u(a_0-b_0)}}{a_0+b_0}(a+b))$$

$$= \beta e^{u(a-b)} + c\frac{a+b}{a_0+b_0}e^{u(a+a_0-b-b_0)}, \qquad a,b,c\in\mathbb{R},$$

for some $u\in\mathbb{C}$, $\gamma,\beta\in\mathbb{C}^\star$ and $c\in\mathbb{C}\backslash\{-1\}$.

Example 5.4. Consider the $(ax+b)$-group $G := \left\{\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix} \mid a>0, b\in\mathbb{R}\right\}$ and let σ be the involutive automorphism defined by

$$\sigma\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix} = \begin{pmatrix}a & -b\\ 0 & 1\end{pmatrix} \qquad \text{for } a>0 \quad \text{and} \quad b\in\mathbb{R}.$$

Let Z_0 be a fixed element on G and let $\mu=\delta_{Z_0}$. We indicate here the corresponding continuous solutions of Eq. (4.2).

The continuous characters on G are parametrized by $\lambda\in\mathbb{C}$ as follows (see, e.g., [11, Example 3.13]).

$$\chi_\lambda\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix} = a^\lambda \qquad \text{for } a>0 \quad \text{and} \quad b\in\mathbb{R}.$$

So the condition $\chi_\lambda\circ\sigma=\chi_\lambda$ is always satisfied.

In view of [11, Example 2.10], the continuous additive functions on G are parametrized by $\lambda\in\mathbb{C}$ as follows

$$a_\lambda\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix} = \lambda\ln a \qquad \text{for } a>0 \quad \text{and} \quad b\in\mathbb{R}.$$

Then $a_\lambda\circ\sigma=a_\lambda$ for all $\lambda\in\mathbb{C}$ and so $\mathcal{A}^-(G)=\{0\}$.

In conclusion, by help of Corollary 4.4, we find that the continuous solutions $f,g\in C(G)$ of the functional equation (4.2), which is here

$$f(X\sigma(Y)Z_0) = f(X)g(Y)-f(Y)g(X), \qquad X,Y\in G,$$

are: $f=0$ and g is arbitrary in $C(G)$.

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