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Results in Mathematics

On Hankel Determinant $H_2(3)$ for Univalent Functions

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Abstract. In this paper we consider the Hankel determinant $H_2(3) = a_3a_5 - {a_4}^2$ defined for the coefficients of a function f which belongs to the class $\mathcal{S}$ of univalent functions or to its subclasses: S^* of starlike functions, $\mathcal{K}$ of convex functions and $\mathcal{R}$ of functions whose derivative has a positive real part. Bounds of $|H_2(3)|$ for these classes are found; the bound for $\mathcal{R}$ is sharp. Moreover, the sharp results for starlike functions and convex functions for which $a_2 = 0$ are obtained. It is also proved that $\max\{|H_2(3)| : f \in \mathcal{S}\}$ is greater than 1.

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1. Introduction

Let Δ be the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ be the family of all functions f analytic in Δ, normalized by the condition $f(0) = f'(0) - 1 = 0$. Hence, the functions in $\mathcal{A}$ are of the form

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots . \tag{1.1}$$

Pommerenke (see, [19,20]) defined the k-th Hankel determinant for a function f as

$$H_k(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+k-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+k} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+k-1} & a_{n+k} & \cdots & a_{n+2k-2} \end{vmatrix}, \tag{1.2}$$

where $n, k \in \mathbb{N}$.

In recent years many mathematicians have investigated Hankel determinants for various classes of functions contained in $\mathcal{A}$. These studies focus on

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the main subclasses of class $\mathcal{S}$ consisting of univalent functions (see, [1,3,8–10,14–16,21,22,24,25]). A few papers are devoted to some subclasses of $\mathcal{S}_\sigma$ of bi-univalent functions (see, [4,17]). In fact, the majority of papers discuss the determinants $H_2(2)$ and $H_3(1)$. The case $H_2(1) = a_3 - {a_2}^2$ is also very well known. It is the classical Fekete-Szegö functional, which has been considered since the 1930's and is still of great interest, especially in a modified version $a_3 - \mu {a_2}^2$.

From the explicit form of $H_3(1)$ we can see that it involves the second Hankel determinant $H_2(k)$, where $k = 1, 2, 3$. Indeed,

$$H_3(1) = a_3 H_2(2) + a_4 I + a_5 H_2(1), \tag{1.3}$$

or equivalently,

$$H_3(1) = H_2(3) + a_2 J + a_3 H_2(2), \tag{1.4}$$

where $I = a_2 a_3 - a_4$, $J = a_3 a_4 - a_2 a_5$. Surprisingly, the determinant

$$H_2(3) = a_3 a_5 - {a_4}^2$$

has not been discussed yet. In this paper we want to consider $H_2(3)$ for $\mathcal{S}$ and some its subclasses: S^* of starlike functions, $\mathcal{K}$ of convex functions and $\mathcal{R}$ of functions whose derivative has a positive real part. The main idea we put forward in this paper is to express the coefficients of a function f in a given class by the coefficients of a corresponding function p from the class $\mathcal{P}$ of functions with a positive real part. After that, we can use the known and some new estimates of coefficient functionals for $p \in \mathcal{P}$.

2. Auxiliary Lemmas

To obtain our results we need a few lemmas for functions p in $\mathcal{P}$ having the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots . \tag{2.1}$$

Lemma 1 [6]. *If $p \in \mathcal{P}$, then the sharp estimate $|p_n - \mu p_k p_{n-k}| \le 2$ holds for $n, k = 1, 2, \ldots,$ $n > k$, $\mu \in [0, 1]$.*

From this lemma we obtain the very well known bound for coefficients of $p \in \mathcal{P}$, i.e $|p_n| \le 2$, as well as the inequality $|p_n - p_k p_{n-k}| \le 2$ proved by Livingston [13].

Lemma 2 [2]. *If $p \in \mathcal{P}$, then the sharp estimate $|p_{n+m} - p_n| \le 2\sqrt{2 - Re p_m}$ holds for $n, m = 1, 2, \ldots$.*

Lemma 3 [5]. *The power series for p given in (2.1) converges in Δ to a function in $\mathcal{P}$ if and only if the Toeplitz determinants*

$$T_n = \begin{vmatrix} 2 & p_1 & p_2 & \dots & p_n \\ p_{-1} & 2 & p_1 & \dots & p_{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ p_{-n} & p_{-n+1} & p_{-n+2} & \dots & 2 \end{vmatrix}, \tag{2.2}$$

where $p_{-n} = \overline{p}_n$, $n = 1, 2, \dots$ *are all non-negative.*

Directly from this lemma one can obtain sets of variability of the initial coefficients of functions in $\mathcal{P}$. In what follows, we need a set Ω of variability of the pair $(|p_1|, |p_2|)$, where p_1, p_2 are the first two coefficients of $p \in \mathcal{P}$. Let $\Omega = \{(x, y) : \max\{0, x^2 - 2\} \leq y \leq 2\}$.

Lemma 4. *If* $p \in \mathcal{P}$, *then* $(|p_1|, |p_2|) \in \Omega$.

Proof. According to Lemma 3, if $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is in $\mathcal{P}$, then in order to determine Ω it is enough to consider the first two Toeplitz determinants. For this reason

$$\begin{vmatrix} 2 & p_1 \\ \overline{p}_1 & 2 \end{vmatrix} \geq 0$$

and

$$\begin{vmatrix} 2 & p_1 & p_2 \\ \overline{p}_1 & 2 & p_1 \\ \overline{p}_2 & \overline{p}_1 & 2 \end{vmatrix} \geq 0.$$

Hence,

$$|p_1| \leq 2 \quad \text{and} \quad 4 + Re p_2 \overline{p}_1{}^2 - |p_2|^2 - 2|p_1|^2 \geq 0.$$

Denoting $x = |p_1|$, $y = |p_2|$, $x, y \geq 0$, the above inequalities lead to

$$x \leq 2 \quad \text{and} \quad 4 + x^2 y - y^2 - 2x^2 \geq 0.$$

Consequently, $0 \leq y \leq 2$ for $x \in [0, \sqrt{2}]$ and $x^2 - 2 \leq y \leq 2$ for $x \in [\sqrt{2}, 2]$. □

What plays a crucial role in estimating expressions involving coefficients of $p \in \mathcal{P}$ is the result obtained by Libera and Złotkiewicz.

Lemma 5 [12]. *Let* $p_1 \in [0, 2]$. *A function* p *belongs to* $\mathcal{P}$ *if and only if*

$$2p_2 = p_1{}^2 + x(4 - p_1{}^2)$$

and

$$4p_3 = p_1{}^3 + 2p_1(4 - p_1{}^2)x - p_1(4 - p_1{}^2)x^2 + 2(4 - p_1{}^2)(1 - |x|^2)z$$

for some x *and* z *such that* $|x| \leq 1$, $|z| \leq 1$.

Let f be given by (1.1) and let

$$f_\varphi(z) = e^{-i\varphi} f(z e^{i\varphi}), \varphi \in \mathbb{R}. \tag{2.3}$$

Lemma 6. *If* A *is one of the classes:* $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{K}$, $\mathcal{R}$ *and* $\Phi(f) = |a_3 a_5 - a_4{}^2|$ *is a functional defined for* $f \in A$ *given by* (1.1), *then* $\Phi(f) = \Phi(f_\varphi)$ *for every* $\varphi \in \mathbb{R}$.

This lemma makes it possible to assume that when estimating $\Phi(f)$, one selected coefficient of f is a non-negative real number.

Remark 1. A similar observation can be made for functions of the class $\mathcal{P}$. Namely, if $p \in \mathcal{P}$ is given by (2.1), $p_\varphi(z) = p(ze^{i\varphi})$ and $\Phi(p)$ is one of the expressions $|p_n - \mu p_k p_{n-k}|$ or $|p_{n-1}p_{n+1} - {p_n}^2|$, $n, k = 1, 2, \ldots$, $n > k$, $\mu \in \mathbb{R}$, then $\Phi(p) = \Phi(p_\varphi)$ for every $\varphi \in \mathbb{R}$.

From Lemma 5 we immediately obtain two results which are generalizations of Lemma 1 in the case $n = 2$, $k = 1$ and in the case $\mu = 1$, $n = 3$, $k = 1$.

Lemma 7. *If $p \in \mathcal{P}$, then for $\mu \in \mathbb{R}$ the following sharp estimate holds*

$$|p_2 - \mu {p_1}^2| \leq \begin{cases} 2 - \mu|p_1|^2, & \mu \leq 1/2, \\ 2 - (1-\mu)|p_1|^2, & \mu \geq 1/2. \end{cases}$$

Lemma 8. *If $p \in \mathcal{P}$, then $|p_3 - p_1 p_2| \leq \frac{1}{4}\left(8 - 2|p_1|^2 + |p_1|^3\right)$.*

The proofs of both lemmas are not difficult to obtain, so they are omitted.

Remark 2. Considering $p(z^n)$, we can obtain related versions of these lemmas writing p_{kn} instead of p_k, $k = 1, 2, \ldots$. For example,

$$|p_4 - \mu {p_2}^2| \leq \begin{cases} 2 - \mu|p_2|^2, & \mu \leq 1/2, \\ 2 - (1-\mu)|p_2|^2, & \mu \geq 1/2 \end{cases} \tag{2.4}$$

or

$$|p_6 - p_2 p_4| \leq \frac{1}{4}\left(8 - 2|p_2|^2 + |p_2|^3\right). \tag{2.5}$$

Now, we can prove one more auxiliary lemma.

Lemma 9. *If $p \in \mathcal{P}$, then $|p_2 p_4 - {p_3}^2| \leq 4$. The equality holds only for functions*

$$p(z) = \frac{1+z^3}{1-z^3}, \tag{2.6}$$

$$p(z) = \frac{1+z^2}{1-z^2} \tag{2.7}$$

and their rotations.

Proof. Since

$$|p_2 p_4 - {p_3}^2| \leq |p_2 p_4 - p_6| + |p_6 - {p_3}^2|, \tag{2.8}$$

we immediately obtain the declared bound of $|p_2 p_4 - {p_3}^2|$.

The equality

$$|p_2 p_4 - {p_3}^2| = 4 \tag{2.9}$$

holds if both expresions $|p_2 p_4 - p_6|$ and $|p_6 - {p_3}^2|$ are equal to 2. From (2.5) we can conclude that $|p_2 p_4 - p_6| = 2$ only if $p_2 = 0$ or $|p_2| = 2$. We shall discuss these two cases.

I. Let $|p_2| = 2$. According to Lemma 6, we can assume that $p_2 = 2$. From Lemma 2 for $m = 2$ we know that all odd coefficients of p are equal and $p_{2k} = 2$, $k = 1, 2, \ldots$. In this case, Lemma 3 for $n = 2$ leads to

$$\begin{vmatrix} 2 & p_1 & 2 \\ \overline{p}_1 & 2 & p_1 \\ 2 & \overline{p}_1 & 2 \end{vmatrix} \geq 0 \tag{2.10}$$

which is equivalent to $Imp_1 = 0$. Consequently, $p_1 = p_3 = \cdots = p_{2k+1} = \cdots = a$, $a \in \mathbb{R}$ for all $k = 1, 2, \ldots$. For this reason, (2.9) can be written as $|4 - a^2| = 4$, so $a = 0$. This means that (2.9) holds for the functions p of the form (2.7) and their rotations.

II. Let $p_2 = 0$. Condition (2.9) is satisfied if $|p_3| = 2$. In this case, given Lemma 6, we assume that $p_3 = 2$. From Lemma 2 for $m = 3$ we obtain that $p_1 = p_4 = \cdots = p_{3k+1} = \cdots$, $p_2 = p_5 = \cdots = p_{3k+2} = \cdots$ and $p_{3k} = 2$ for all $k = 1, 2, \ldots$. Now we apply Lemma 3 with $n = 2$ and $n = 3$. We have

$$\begin{vmatrix} 2 & p_1 & 0 \\ \overline{p}_1 & 2 & p_1 \\ 0 & \overline{p}_1 & 2 \end{vmatrix} \geq 0 \tag{2.11}$$

and

$$\begin{vmatrix} 2 & p_1 & 0 & 2 \\ \overline{p}_1 & 2 & p_1 & 0 \\ 0 & \overline{p}_1 & 2 & p_1 \\ 2 & 0 & \overline{p}_1 & 2 \end{vmatrix} \geq 0, \tag{2.12}$$

which are equivalent to

$$|p_1| \leq \sqrt{2} \quad \text{and} \quad |p_1|^4 - 4Rep_1{}^3 - 8|p_1|^2 \geq 0. \tag{2.13}$$

Writing $p_1 = re^{i\theta}$, we obtain

$$r \leq \sqrt{2} \quad \text{and} \quad r^2(r^2 - 4r\cos(3\theta) - 8) \geq 0, \tag{2.14}$$

which is satisfied only if $r = 0$. This means that p_1 is equal to 0 and, consequently, the extremal functions are rotations of that given by (2.6). □

Finally, applying (2.5) in (2.8), we obtain an improvement of the inequality from Lemma 9.

Lemma 10. *If* $p \in \mathcal{P}$, *then* $|p_2p_4 - p_3{}^2| \leq 4 - \frac{1}{2}|p_2|^2 + \frac{1}{4}|p_2|^3$.

3. Determinant $H_2(3)$ for $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{R}$

Let f, g, h be univalent. Then

$$f \in \mathcal{S}^* \Leftrightarrow \frac{zf'(z)}{f(z)} \in P, \tag{3.1}$$

$$g \in \mathcal{K} \Leftrightarrow 1 + \frac{zg''(z)}{g'(z)} \in P, \tag{3.2}$$

$$h \in \mathcal{R} \Leftrightarrow h'(z) \in P. \tag{3.3}$$

From now on, we assume that $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, $g(z) = z + b_2z^2 + b_3z^3 + \cdots$, $h(z) = z + c_2z^2 + c_3z^3 + \cdots$ and $p(z) = 1 + p_1z + p_2z^2 + \cdots$ are in $\mathcal{S}^*$, $\mathcal{K}$, $\mathcal{R}$ and P, respectively.

From (3.1) we obtain

$$(n-1)a_n = \sum_{j=1}^{n-1} a_j p_{n-j}. \tag{3.4}$$

Hence,

$$\begin{aligned} a_2 &= p_1, \\ a_3 &= \tfrac{1}{2}(p_2 + {p_1}^2), \\ a_4 &= \tfrac{1}{3}\left(p_3 + \tfrac{3}{2}p_1p_2 + \tfrac{1}{2}{p_1}^3\right), \\ a_5 &= \tfrac{1}{4}\left(p_4 + \tfrac{4}{3}p_1p_3 + \tfrac{1}{2}{p_2}^2 + {p_1}^2p_2 + \tfrac{1}{6}{p_1}^4\right). \end{aligned} \tag{3.5}$$

Therefore, if $f \in \mathcal{S}^*$, then $H_2(3) = F(p_1, p_2, p_3, p_4)$, where

$$\begin{aligned} F(p_1,p_2,p_3,p_4) = \frac{1}{144}&\left(-{p_1}^6 - 3{p_1}^4p_2 + 8{p_1}^3p_3 - 9{p_1}^2{p_2}^2\right. \\ &\left.+18{p_1}^2p_4 - 24p_1p_2p_3 + 9{p_2}^3 + 18p_2p_4 - 16{p_3}^2\right). \end{aligned} \tag{3.6}$$

According to the Alexander relation, $nb_n = a_n$. Substituting it into the definition of $H_2(3)$ for a convex function and applying formulae (3.5), we obtain $H_2(3) = G(p_1, p_2, p_3, p_4)$, where

$$\begin{aligned} G(p_1,p_2,p_3,p_4) = \frac{1}{2880}&\left(-{p_1}^6 - 2{p_1}^4p_2 + 12{p_1}^3p_3 - 9{p_1}^2{p_2}^2\right. \\ &\left.+24{p_1}^2p_4 - 28p_1p_2p_3 + 12{p_2}^3 + 24p_2p_4 - 20{p_3}^2\right). \end{aligned} \tag{3.7}$$

Finally, if $f \in \mathcal{R}$, then $nc_n = p_{n-1}$. Directly from (1.2) it follows that $H_2(3) = H(p_1, p_2, p_3, p_4)$, where

$$H(p_1,p_2,p_3,p_4) = \frac{1}{15}p_2p_4 - \frac{1}{16}{p_3}^2. \tag{3.8}$$

Taking into account the explicit formulae of $H_2(3)$, it is a difficult problem to find the exact bounds of these expressions. For this reason, we start with a particular case.

Theorem 1. *Let f be given by (1.1) with an additional assumption that $a_2 = 0$.*

1. *If* $f \in \mathcal{S}^*$, *then* $|H_2(3)| \leq 1$.
2. *If* $f \in \mathcal{K}$, *then* $|H_2(3)| \leq \frac{1}{15}$.

Both results are sharp.

Proof. Assume that for f given by (1.1) there is $a_2 = 0$. From (3.5) it follows that $p_1 = 0$.

If $f \in \mathcal{S}^*$, then

$$|H_2(3)| = \frac{1}{144}\left|9{p_2}^3 + 2p_2p_4 + 16(p_2p_4 - {p_3}^2)\right|. \tag{3.9}$$

For $f \in \mathcal{K}$, we can write

$$|H_2(3)| = \frac{1}{720}\left|3{p_2}^3 + p_2p_4 + 5(p_2p_4 - {p_3}^2)\right|. \tag{3.10}$$

It is enough to apply Lemmas 1 and 9 to obtain the bounds 1 and 1/15 for $\mathcal{S}^*$ and $\mathcal{K}$, respectively.

The expression (3.9) is equal to 1 if and only if $|p_2| = 2$, $|p_4| = 2$ and $|p_2p_4 - {p_3}^2| = 4$. It is possible only for rotations of (2.7). This means that the extremal starlike functions are $f(z) = \frac{z}{1-z^2}$ and its rotations.

The same argument as above is also valid for the equality in the estimation of $|H_2(3)|$ for convex functions. This argument leads to the conclusion that only for rotations of $f(z) = \frac{1}{2}\log\frac{1+z}{1-z}$ there is $|H_2(3)| = 1/15$. □

Observe that the extremal functions in Theorem 1 are odd. This results in the following corollary. The symbol $A^{(2)}$ stands for a subclass of A consisting of all odd functions.

Corollary 1. *Let* f *be given by* (1.1).

1. *If* $f \in \mathcal{S}^{*(2)}$, *then* $|H_2(3)| \leq 1$.
2. *If* $f \in \mathcal{K}^{(2)}$, *then* $|H_2(3)| \leq \frac{1}{15}$.

Both results are sharp.

It is worth saying that the result from Theorem 1 for starlike functions generalizes the result of Jakubowski, who proved in [7] that the inequality $|a_3a_5| \leq 1$ holds for all odd starlike functions with real coefficients given by (1.1).

Now we can prove two general theorems.

Theorem 2. *Let* f *be given by* (1.1).

1. *If* $f \in \mathcal{S}^*$, *then* $|H_2(3)| \leq 1.573\ldots$.
2. *If* $f \in \mathcal{K}$, *then* $|H_2(3)| \leq 0.096\ldots$.

Theorem 3. *If* $f \in \mathcal{R}$, *then* $|H_2(3)| \leq \frac{4}{15}$. *The equality holds only for* $f(z) = \log\frac{1+z}{1-z} - z = z + \frac{2}{3}z^3 + \frac{2}{5}z^5 + \cdots$ *and its rotations.*

Proof of Theorem 2. Let $f \in \mathcal{S}^*$. From (3.6) it follows that

$$F(p_1, p_2, p_3, p_4) = \frac{1}{144}\left[16(p_4 - p_1p_3)\left(p_2 - \tfrac{1}{2}{p_1}^2\right) - 20(p_2 - {p_1}^2)\left(p_4 - \tfrac{7}{10}{p_2}^2\right)\right.$$
$$+16(p_2p_4 - {p_3}^2) - 5(p_2 + \tfrac{1}{5}{p_1}^2)(p_2 - {p_1}^2)^2$$
$$\left. + 6{p_1}^2\left(p_4 - \tfrac{2}{3}{p_2}^2\right) + 6p_2(p_4 - p_1p_3) - 2p_1p_2p_3\right], \tag{3.11}$$

Similarly, if $f \in \mathcal{K}$, then from (3.7),

$$G(p_1, p_2, p_3, p_4) = \frac{1}{2880}\left[16(p_4 - p_1p_3)\left(p_2 - \tfrac{1}{2}{p_1}^2\right) + 8p_2(p_4 - p_1p_3)\right.$$
$$-20(p_2 - {p_1}^2)\left(p_4 - \tfrac{9}{20}{p_2}^2\right) + 20(p_2p_4 - {p_3}^2) + 3{p_2}^3$$
$$+12{p_1}^2\left(p_4 - \tfrac{1}{3}{p_2}^2\right) + 2{p_1}^4\left(p_2 - \tfrac{1}{2}{p_1}^2\right)$$
$$\left.-4p_1(p_2 - {p_1}^2)(p_3 - p_1p_2)\right]. \tag{3.12}$$

In both cases, it is enough to apply the triangle inequality and Lemmas 1 and 7. Denoting $p = |p_1|$ and $q = |p_2|$, we have

$$|F(p_1, p_2, p_3, p_4)|$$
$$\le \frac{1}{144}\left[32\left(2 - \tfrac{1}{2}p^2\right) + 40\left(2 - \tfrac{3}{10}q^2\right) + 16\left(4 - \tfrac{1}{2}q^2 + \tfrac{1}{4}q^3\right)\right.$$
$$\left. +4(5q + p^2) + 6p^2\left(2 - \tfrac{1}{3}q^2\right) + 12q + 4pq\right]$$
$$= \frac{1}{144}\left[210 + 32q - 20q^2 + 4q^3 - 2(1 - pq)^2\right] \le \frac{1}{144}\left(210 + 32q - 20q^2 + 4q^3\right),$$

which achieves its greatest value in $[0, 2]$ for $q = 4/3$; hence, the result follows. Similarly,

$$|G(p_1, p_2, p_3, p_4)|$$
$$\le \frac{1}{2880}\left[32\left(2 - \tfrac{1}{2}p^2\right) + 16q + 40\left(2 - \tfrac{9}{20}q^2\right)\right.$$
$$\left.+20\left(4 - \tfrac{1}{2}q^2 + \tfrac{1}{4}q^3\right) + 3q^3 + 12p^2\left(2 - \tfrac{1}{3}q^2\right) + 2p^4\left(2 - \tfrac{1}{2}p^2\right) + 2p(8 - 2p^2 + p^3)\right]$$
$$= \frac{1}{2880}\left[224 + 16p + 8p^2 - 4p^3 + 6p^4 - p^6 + 16q - 28q^2 + 8q^3 - 4p^2q^2\right].$$

Denoting $h(p, q) = 224 + 16p + 8p^2 - 4p^3 + 6p^4 - p^6 + 16q - 28q^2 + 8q^3$, we can see that $h(p, q)$ is an increasing function of the variable $p \in [0, 2]$. Due to Lemma 4,

$$|G(p_1, p_2, p_3, p_4)| \le \frac{1}{2880}h(p, q) \le \frac{1}{2880}h(\sqrt{2 + q}, q),\ \ q \in [0, 2].$$

Observe that it is equivalent to discuss the function $h(p, p^2 - 2)$, $p \in [\sqrt{2}, 2]$ instead of $h(\sqrt{2 + q}, q)$, $q \in [0, 2]$. Considering the derivative of $h(p, p^2 - 2)$, we find that its greatest value in $[\sqrt{2}, 2]$ is achieved for $p_0 = 1.666\ldots$, which is the only positive solution of the equation $4 + 118p + 56p^2 - 42p^3 - 21p^4 = 0$. Consequently, $\max\{h(p, q), (p, q) \in \Omega\} = h(p_0, {p_0}^2 - 2) = 278.503\ldots$. This completes the proof. □

Proof of Theorem 3. Observe that for $f \in \mathcal{R}$ we can write

$$H(p_1, p_2, p_3, p_4) = \tfrac{1}{16}\left(p_2 p_4 - {p_3}^2\right) + \tfrac{1}{240} p_2 p_4. \tag{3.13}$$

The desired estimate follows directly from the triangle inequality and Lemmas 1 and 9.

The equality

$$|H_2(3)| = \tfrac{4}{15} \tag{3.14}$$

holds only if $|p_2 p_4 - {p_3}^2| = 4$ and $|p_2| = |p_4| = 2$. By Lemma 9, it is possible only for $p(z) = \frac{1+z^2}{1-z^2}$. As a consequence, we obtain the extremal functions $f \in \mathcal{R}$ for which (3.14) holds. □

4. Determinant $H_2(3)$ for $\mathcal{S}$

The problem of finding a sharp bound of $|H_2(3)|$ for all univalent functions seems to be extremely difficult. The same can also be said about a sharp bound of $|H_2(2)|$ for $f \in \mathcal{S}$.

Let us recall that in [18] the following conjecture was posed:

$$|H_2(n)| = |a_n a_{n+2} - {a_{n+1}}^2| \le 1 \tag{4.1}$$

for all $f \in \mathcal{S}$ and $n = 2, 3, \ldots$.

According to the result of Schaeffer and Spencer [23], this conjecture is false for $n \ge 4$ (see also [11]). In [21] it was shown that for univalent functions $|H_2(2)|$ can exceed 1. Namely, for the function f_ε, $\varepsilon \in (0,1)$ which is a composition of the function $s(z)$ satisfying

$$\frac{s}{(1+s)^2} = \frac{4\varepsilon}{(1+\varepsilon)^2} \cdot \frac{z}{(1+z)^2}, \tag{4.2}$$

and the function

$$w(s) = \frac{(1+\varepsilon)^2}{4} \cdot \frac{s(1-\varepsilon s)}{\varepsilon - s}, \tag{4.3}$$

we have

$$\begin{aligned} f_\varepsilon(z) = z &+ \frac{2(1-\varepsilon)(1+3\varepsilon)}{(1+\varepsilon)^2} z^2 + \frac{(1-\varepsilon)(3+15\varepsilon+33\varepsilon^2-19\varepsilon^3)}{(1+\varepsilon)^4} z^3 \\ &+ \frac{4(1-\varepsilon)(1+7\varepsilon+18\varepsilon^2+54\varepsilon^3-59\varepsilon^4+11\varepsilon^5)}{(1+\varepsilon)^6} z^4 \\ &+ \frac{(1-\varepsilon)(5+45\varepsilon+185\varepsilon^2+145\varepsilon^3+1855\varepsilon^4-2537\varepsilon^5+899\varepsilon^6-85\varepsilon^7)}{(1+\varepsilon)^8} z^5 + \cdots. \end{aligned} \tag{4.4}$$

The function f_ε maps Δ univalently onto the set

$$\mathbb{C} \setminus \left((-\infty, -d_\varepsilon] \cup \{d_\varepsilon e^{i\theta}, \theta_\varepsilon \le |\theta| \le \pi\}\right), \tag{4.5}$$

where

$$d_\varepsilon = \frac{(1+\varepsilon)^2}{4} \quad \text{and} \quad \theta_\varepsilon = 2\arccos\varepsilon. \tag{4.6}$$

Admitting the cases $\varepsilon = 0$ and $\varepsilon = 1$, we obtain a univalent passage between the identity function ($f_1 = id$) and the Koebe function ($f_0 = k$, $k(z) = \frac{z}{(1-z)^2}$).

For the function f_ε we have $H_2(2) = 1.175\ldots$ (see, Theorem 2 in [21]) and

$$H_2(3) = -K(\varepsilon),$$

where

$$K(\varepsilon) = \frac{(1-\varepsilon)^4}{(1+\varepsilon)^{12}}(1 + 16\varepsilon - 4\varepsilon^2 + 1136\varepsilon^3 \\ +4678\varepsilon^4 + 4976\varepsilon^5 - 2692\varepsilon^6 - 240\varepsilon^7 + 321\varepsilon^8),\ \varepsilon \in [0,1]. \tag{4.7}$$

It is easily seen that $K(\varepsilon) \geq 0$ for $\varepsilon \in [0,1]$. Moreover,

$$K'(\varepsilon) = -\frac{256(1-\varepsilon)\varepsilon}{(1+\varepsilon)^{13}}\left(1 - 14\varepsilon - 2\varepsilon^2 + 173\varepsilon^3 + 301\varepsilon^4 - 220\varepsilon^5 - 4\varepsilon^6 + 21\varepsilon^7\right).$$

For ε_0, which is a greater solution of $1 - 14\varepsilon - 2\varepsilon^2 + 173\varepsilon^3 + 301\varepsilon^4 - 220\varepsilon^5 - 4\varepsilon^6 + 21\varepsilon^7 = 0$ in $(0,1)$, i.e., for $\varepsilon_0 = 0.205\ldots$, we can write

$$\max\{K(\varepsilon) : \varepsilon \in [0,1]\} = K(\varepsilon_0) = 1.012\ldots. \tag{4.8}$$

We have proven

Theorem 4. *If f is given by* (1.1), *then*

$$\max\{|H_2(3)| : f \in \mathcal{S}\} \geq 1.012\ldots. \tag{4.9}$$

Remark 3. Theorem 4 with previously obtained results lead to the conclusion that the conjecture in (4.1) is false for all $n \geq 2$. From (4.5) it follows that f_ε, $\varepsilon \in (0,1)$ is neither a starlike nor a close-to-convex function.

On the other hand, $|H_2(n)| = 1$ for the Koebe function $k(z) = \frac{z}{(1-z)^2}$ (as well as for its rotations). We also know that for all $f \in \mathcal{S}^*$ we have $|H_2(1)| \leq 1$ and $|H_2(2)| \leq 1$. Although we do not know the sharp bound of $|H_2(3)|$, taking into account Theorem 1, point 1, and Theorem 2, point 1, it is likely that (4.1) may hold just for functions in the class $\mathcal{S}^*$.

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