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Inverse Source Problem for Heat Equation with Nonlocal Wentzell Boundary Condition

Mansur I. Ismailov

Abstract. The inverse source problem for one-dimensional heat equation is investigated with nonlocal Wentzell–Neumann boundary and integral overdetermination conditions. The generalized Fourier method is used to show the existence, uniqueness and stability of the classical solution under some regularity, consistency and orthogonality conditions on the data. The considered inverse problem gives an idea of how total energy might be externally controlled.

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1. Introduction and Problem Formulation

In theory of differential equations, the boundary conditions associated with a second order partial differential operator usually involve the function and its first derivative (including Dirichlet, Neumann and Robin conditions). In Markov process theory, following the work of Wentzell [1], it was recognized that it is natural to include boundary conditions involving the operator itself. In recent years, a number of authors have focused on such type of non-standard boundary conditions. This behavior is in contrast with the totally absorbing Dirichlet boundary condition, the totally reflecting Neumann boundary condition, or the partially absorbing, partially reflecting Robin boundary condition. One may think of the Wentzell boundary condition as describing a situation like the Robin boundary condition, with the additional feature that the boundary has the capacity for storing heat. For the classical results along these lines go back to [2], see also e.g. [3–6] and, more recently, [7] and [8]. We refer to

[9] and [10] and to the references contained therein for the motivations and the study of general Wentzell boundary conditions in the theory of parabolic partial differential equations.

This article considers the heat equation

$$u_t = u_{xx} + r(t)f(x,t), \quad (x,t) \in \Omega_T \tag{1.1}$$

with the initial condition

$$u(x,0) = \varphi(x), \quad x \in [0,1], \tag{1.2}$$

Dirichlet boundary condition

$$u(0,t) = 0, \quad t \in [0,T], \tag{1.3}$$

and nonlocal Wentzell–Neumann boundary condition

$$u_x(0,t) + \alpha u_{xx}(1,t) = 0, t \in [0,T] \tag{1.4}$$

for $\alpha > 0$, where $\Omega_T = \{(x,t) : 0 < x < 1,\ 0 < t \leq T\}$ with fixed $T > 0$. The functions f, φ are given functions in $\bar{\Omega}_T$ and $0 \leq x \leq 1$.

Note that (1.4) is a dynamic boundary condition involving motion on the boundary. Assuming that (1.1) is valid at $x = 1$, we plug (1.1) into (1.4), then the boundary condition becomes

$$u_x(0,t) + \alpha u_t(1,t) = -\alpha r(t)f(1,t).$$

The dynamic boundary conditions are not very common in the mathematical literature. Nevertheless, they appear in many mathematical models including heat transfer in a solid in contact with a moving fluid, thermoelectricity, diffusion phenomena, problems in fluid dynamics. Such conditions arise in the applications and have been studied in [11–13] and more recently in [14,15].

When the coefficient $r(t)$, $0 \leq t \leq T$ is also given, the problem of finding $u(x,t)$ from using Eq. (1.1), initial condition (1.2) and boundary conditions (1.3) and (1.4) is termed as the direct (or forward) problem.

When the function $r(t)$, $0 \leq t \leq T$ is unknown, the inverse problem is formulated as a problem of finding a pair of functions $\{r(t), u(x,t)\}$ which satisfy the Eq. (1.1), initial condition (1.2), boundary conditions (1.3) and (1.4) and overdetermination condition

$$\int_0^1 u(x,t)dx = E(t), 0 \leq t \leq T, \tag{1.5}$$

where $E(t)$ is a given function whilst the prescription of the total energy, or mass. This inverse problem for the model of microwave heating gives an idea of how total energy content might be externally controlled. However, the dielectric constant of the target material varies in space and time, resulting in spatially heterogeneous conversion of electromagnetic energy to heat. This can correspond to source term $r(t)f(x,t)$, where $r(t)$ is proportional to power of external energy source and $f(x,t)$ is local conversion rate of microwave energy. In ths way, the external energy is supplied to a target at a controlled level by the microwave generating equipment.

In contrast to direct problems, the inverse parabolic problems with dynamic (and the related general Wentzell) boundary conditions are scarce [16–18] and needs additional consideration.

This paper studies the inverse source problem for the heat equation with a boundary condition including both nonlocal and dinamic character. Under some regularity, consistency and orthogonality conditions on the initial data and on the known part of source term, the well posedness of the classical solution are shown by using the generalized Fourier method. The mathematical motivation of considering such kind of inverse problems are the papers [19–22]. The first and second papers are devoted to direct and inverse initial boundary value problems (IBVPs) for heat equation with nonlocal boundary and integral overdetermination conditions. In third and fourth papers, the time dependent source term is determined under nonlocal boundary and overdetermination conditions of general form. In these papers the Green's function representation of the solution is obtained by the techniques of fundamental solution of heat equation. Some of the cases of nonlocal boundary conditions, when the fundamental solution of the heat equation does not work, we used the expansion in terms of eigenfunctions for the auxiliary spectral problem corresponding to the considered initial boundary value problem in [23,24]. Then this method is extended to some of IBVPs with Wentzell boundary conditions in [18].

There are some important recent works [25–29] on evolution equations subject to Wentzell/dynamic boundary conditions. These problems are generally analysed from the point of view of semigroup theory. We are pointed the evolution problems with Wentzell bounady conditions from the point of view of spectral theory. In contrast to a disadvantage that the spectral theory approach can not be extended to the multidimentional case, some of the classes of the problems with Wentzell bounady condition can not be solved by another approach, for example, semigroup approach.

The auxiliary spectral problem of the IBVP (1.1)–(1.4) is

$$\begin{cases} -y''(x) = \lambda y(x), 0 \leq x \leq 1, \\ y(0) = 0, y'(0) - \alpha\lambda y(1) = 0. \end{cases} \tag{1.6}$$

We consider this spectral problem with $\alpha \neq \frac{1}{x_i \sin x_i}$ where x_i are the roots of the equation $sinx + xcosx = 0$ on $(0, +\infty)$. It is known from [30,31] that this problem has at most infinitely many complex eigenvalues and their numbers depend on α and the system of eigenfunctions with one deleted is a Riesz basis in $L_2[0,1]$. For the case $\alpha = \frac{1}{x_0 \sin x_0}$ where x_0 is one of the positive roots of the equation $sinx + xcosx = 0$, the spectral problem has the eigenfunctions as well as one associated function correponding to eigenvalue $\lambda = x_0$. Unlike the previous case, the entire system of eigenfunctions and associated functions is a Riesz basis in $L_2[0,1]$.

The paper is organized as follows. In Sect. 2, we recall the eigenvalues and eigenfunctions of the auxiliary spectral problem and some of their properties. In Sect. 3, the well-posedness of the inverse problem (1.1)–(1.5) are proved.

2. Some Properties of the Auxiliary Spectral Problem

Consider the spectral problem (1.6) with $\alpha \neq \frac{1}{x_i \sin x_i}$ for all i, where the x_i are the roots of the equation $\sin x + x\cos x = 0$ on $(0, +\infty)$. In this case problem (1.6) has only the eigenfunctions

$$X_n(x) = \sqrt{2}\sin\left(\sqrt{\lambda_n}x\right), n = 0, 1, 2, \ldots$$

where the eigenvalues $\lambda_n, n = 0, 1, 2, \ldots$ satisfy the equation $\alpha\sqrt{\lambda}\sin\sqrt{\lambda} = 1$, $\operatorname{Re}\sqrt{\lambda} > 0$. This problem can have at most finite many complex eigenvalues, and their number depends on the parameter α.

Let $\lambda_n, n = 0, 1, 2, \ldots, n_\alpha$ are complex eigenvalues and $X_n(x), n = 0, 1, 2, \ldots, n_\alpha$ are corresponding eigenfunctions.

The asymptotic estimate for the eigenvalues

$$\sqrt{\lambda_n} = \pi n + \frac{(-1)^n}{\pi\alpha n} + O\left(\frac{1}{n^3}\right)$$

is valid for large n.

It is shown in [30] that the system of eigenfunctions $X_n(x)$, $n = 1, 2, ..$, that is, the system of eigenfunctions of problem (1.6) with one deleted, is a Riesz basis in $L_2[0, 1]$ and the system

$$Y_n(x) = \sqrt{2}\frac{\overline{\sqrt{\lambda_0}}\sin\overline{\sqrt{\lambda_0}}(1-x) - \overline{\sqrt{\lambda_n}}\sin\overline{\sqrt{\lambda_n}}(1-x)}{\overline{\overline{\sqrt{\lambda_n}}\cos\overline{\sqrt{\lambda_n}} + \sin\overline{\sqrt{\lambda_n}}}}, n = 1, 2, \ldots$$

is a biortogonal to the system $X_n(x)$, $n = 1, 2, \ldots$

Definition 1. The pair $\{r(t), u(x, t)\}$ from the class $C[0, T] \times (C^{2,1}(D_T) \cap C^{2,0}(\bar{D}_T))$ for which the conditions (1.1)–(1.4) are satisfied, is called a classical solution of the inverse problem (1.1)–(1.5).

The uniformly convergence of the Fourier series expansion in the system $X_n(x)$, $n = 1, 2, \ldots$ is important, since we are considering the classical solution of the inverse problem (1.1)–(1.5).

Lemma 1 (Theorem 1, [31]). *Suppose that a function $\varphi \in C[0, 1]$ has a uniformly convergent Fourier series expansion in the system $\sin(\pi nx)$, $n = 1, 2, \ldots$ on the interval $[0, 1]$. Then this function can be expanded in a Fourier series in the system $X_n(x)$, $n = 0, 1, 2, \ldots$ and this expansion is uniformly convergent on [0,1] if* $\left(\varphi, \sin\sqrt{\lambda_0}(1-x)\right) = 0$.

Because the temperature distribution $u(x,t)$ is real valued and some of the eigenvalues are complex, we choose the method of annihilation of complex terms of Fourier series expansions.

Notation 1. *The class of functions which also contains the conditions of the Lemma 1 will be denoted by*

$$\Phi_{n_\alpha} \equiv \left\{ \begin{array}{c} \varphi(x) \in C^3[0,1] : \varphi(0) = \varphi'(0) = \varphi''(0) = 0, \varphi(1) = \varphi''(1) = 0, \\ \int_0^1 \varphi(x) \sin(\sqrt{\lambda_n}(1-x))dx = 0, n = 0,1,2,\ldots,n_\alpha. \end{array} \right\}$$

3. Classical Solution of the Inverse Problem

Let $r(t)$, $t \in [0,T]$ be unknown function. The smoothness conditions $f(x,t) \in C\left(\overline{D}_T\right)$, $\varphi(x) \in C^2[0,1]$, $E(t) \in C[0,T]$ and the consistency conditions

$$\varphi(0) = 0, \varphi'(0) + \alpha\varphi''(1) = 0, E(0) = \int_0^1 \varphi(x)dx$$

are necessary for the existence of a classical solution of the problem (1.1)–(1.5).

Lemma 2. *If* $\varphi(x) \in C^3[0,1]$ *satisfies the conditions* $\varphi(1) = \varphi''(1) = 0$, $\varphi(0) = \varphi'(0) = \varphi''(0) = 0$, *then the inequality*

$$\sum_{n=1}^{\infty} |\lambda_n \varphi_n| \le c \left\|\varphi'''\right\|^2_{L_2(0,1)}, c = const > 0$$

holds, where $\varphi_n = (\varphi, Y_n)$.

Proof. Because $\varphi(0) = \varphi'(0) = \varphi''(0) = 0$, $\varphi(1) = \varphi''(1) = 0$, the equality

$$\varphi_n = -\frac{\sqrt{2}}{\lambda_n\sqrt{\lambda_n}} \frac{\sqrt{\lambda_n}}{\sqrt{\lambda_n}\cos\sqrt{\lambda_n} + \sin\sqrt{\lambda_n}} \int_0^1 \varphi'''(1-s)\cos\left(\sqrt{\lambda_n}s\right)ds$$

holds by three times integrating by parts. The functions $\sqrt{2}\cos\left(\sqrt{\lambda_n}x\right)$, $n = 1,2,\ldots$ are eigenfunctions of the differential operator generated by differential expression $l(y) = -y''$ and regular boundary conditions

$$y'(0) = 0, \quad y(0) + \alpha y(1) = 0.$$

The eigenfunctions and associated functions of this operator form a Riesz basis in the space $L_2(0,1)$ [25], which implies the Bessel type inequality

$$\sum_{n=1}^{\infty} \left| \left(h, \sqrt{2}\cos\left(\sqrt{\lambda_n}x\right)\right)\right|^2 \le c\left\|h\right\|^2_{L_2(0,1)}$$

for each $h(x) \in L_2(0,1)$.

From the earlier discussion, by using the Schwarz and Bessel inequalities, we obtain

$$\sum_{n=1}^{\infty}|\lambda_n\varphi_n| \leq \sum_{n=1}^{\infty}\left|\mu_n\int_0^1 \varphi'''(1-s)\sqrt{2}\cos\left(\sqrt{\lambda_n}s\right)ds\right|$$
$$\leq \left[\sum_{n=1}^{\infty}|\mu_n|^2\right]^{\frac{1}{2}}\left[\sum_{n=1}^{\infty}\left|\int_0^1 \varphi'''(1-s)\sqrt{2}\cos\left(\sqrt{\lambda_n}s\right)ds\right|^2\right]^{\frac{1}{2}}$$
$$\leq c_1\left\|\varphi'''\right\|_{L_2(0,1)} \leq c_1\left\|\varphi\right\|_{C^3[0,1]},$$

for some constant c, where $\mu_n = \frac{1}{\sqrt{\lambda_n}\cos\sqrt{\lambda_n}+\sin\sqrt{\lambda_n}}$ and $\sum_{n=1}^{\infty}|\mu_n|^2 < +\infty$, since $\mu_n \sim \frac{1}{\pi n}$, $n \to +\infty$. □

The main result is presented as follows.

Theorem 1 (Existence and uniqueness). *Let the following conditions be satisfied:*

- $E(t) \in C^1[0,T]$;
- $\varphi(x) \in \Phi_{n_\alpha}$ *and* $E(0) = \int\limits_0^1 \varphi(x)dx$;
- $f(x,t) \in C\left(\overline{D}_T\right)$ *and* $f(x,t) \in \Phi_{n_0}$, $\int\limits_0^1 f(x,t)dx \neq 0$, $\forall t \in [0,T]$;

Then the inverse source problem has a unique classical solution $\{r(t), u(x,t)\} \in C[0,T] \times (C^{2,1}(D_T) \cap C^{2,0}(\bar{D}_T))$. *Moreover,* $u(x,t) \in C^{2,1}(\bar{D}_T)$.

Proof. To construct the formal solution of the problem (1.1)–(1.4) for arbitrary $r(t) \in C[0,T]$, we will use the generalized Fourier method. In accordance with this method, the solution $u(x,t)$ is sought in a Fourier series in terms of the eigenfunctions $X_n(x) = \sqrt{2}\sin\left(\sqrt{\lambda_n}x\right)$, $n = 0,1,2,\ldots$ of the auxiliary spectral problem (1.6).

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)X_n(x), u_n(t) = (u(x,t), Y_n(x)).$$

For the functions $u_n(t)$, $n = 0,1,2,\ldots$ we obtain the Cauchy problem

$$u_n'(t) + \lambda_n u_n(t) = r(t)f_n(t),$$
$$u_n(0) = \varphi_n,$$

where $\varphi_n = (\varphi, Y_n)$, $f_n(t) = (f(x,t), Y_n)$. The solutions of these Cauchy problems are

$$u_n(t) = \varphi_n e^{-\lambda_n t} + \int_0^t r(\tau)f_n(\tau)e^{-\lambda_n(t-\tau)}d\tau$$

Under the condition that $\varphi(x) \in \Phi_{n_\alpha}$ and $f(x,t) \in \Phi_{n_0}$, $\forall t \in [0,T]$, in particular, $\int_0^1 \varphi(x)\sin(\sqrt{\lambda_n}(1-x))dx = 0$, $\int_0^1 f(x,t)\sin(\sqrt{\lambda_n}(1-x))dx = 0$, $n =$

$0, 1, 2, \ldots, n_\alpha$, the Fourier coefficients $\varphi_n = 0,\ f_n(t) = 0,\ n = 0, 1, 2, \ldots, n_\alpha$ and

$$\varphi_n = \sqrt{2}\frac{\sqrt{\lambda_n}}{\sqrt{\lambda_n}\cos\sqrt{\lambda_n} + \sin\sqrt{\lambda_n}}\int_0^1 \varphi(x)\sin\left(\sqrt{\lambda_n}(x-1)\right)dx, n > n_\alpha$$

are real numbers and

$$f_n(\tau) = \sqrt{2}\frac{\sqrt{\lambda_n}}{\sqrt{\lambda_n}\cos\sqrt{\lambda_n} + \sin\sqrt{\lambda_n}}\int_0^1 f(x,t)\sin\left(\sqrt{\lambda_n}(x-1)\right)dx, n > n_\alpha$$

are real valued functions.

The formal solution of the mixed problem (1.1)–(1.4) is the series

$$u(x,t) = \sum_{n=n_\alpha+1}^{\infty}\left[\varphi_n e^{-\lambda_n t} + \int_0^t r(\tau) f_n(\tau) e^{-\lambda_n(t-\tau)}d\tau\right] X_n(x). \tag{3.1}$$

Under the smoothness assumptions $\varphi(x) \in C^3[0,1],\ f(x,t) \in C\left(\overline{D}_T\right),$ $f(x,t) \in C^3[0,1]$ for $\forall t \in [0,T]$ and $\varphi(1) = \varphi''(1) = 0,\ \varphi(0) = \varphi'(0) = \varphi''(0) = 0,\ f(1,t) = f_{xx}(1,t) = 0,\ f(0,t) = f_x(0,t) = f_{xx}(0,t) = 0$ the series $\sum_{n=1}^{\infty}|\lambda_n\varphi_n|$ and $\sum_{n=1}^{\infty}|\lambda_n f_n(\tau)|$ are convergent by Lemma 2. The series (3.1) and its x-partial derivative converge uniformly in D_T since their majorizing sums are absolutely convergent. Therefore, their sums $u(x,t)$ and $u_x(x,t)$ are continuous in D_T. In addition, the t-partial derivative and the xx-second order partial derivative series are uniformly convergent for $t > \varepsilon$ (ε is an arbitrary positive number). Thus, $u(x,t) \in C^{2,1}(D_T) \cap C^{1,0}(\bar{D}_T)$ and satisfies the conditions (1.1)–(1.4). In addition, $u_t(x,t)$ is continuous in $\bar{D}_T$ because the majorizing sum of t-partial derivative series is absolutely convergent.

The formulas (3.1) and (1.5) yields a following Volterra integral equation with respect to $r(t)$:

$$\int_0^t K(t,\tau) r(\tau) d\tau + F(t) = E(t), \tag{3.2}$$

where

$$F(t) = \sum_{n=n_\alpha+1}^{\infty}\left[\frac{\sqrt{2}}{\sqrt{\lambda_n}}(1 - \cos\sqrt{\lambda_n})\varphi_n\right] e^{-\lambda_n t},$$

$$K(t,\tau) = \sum_{n=n_\alpha+1}^{\infty}\left[\frac{\sqrt{2}}{\sqrt{\lambda_n}}(1 - \cos\sqrt{\lambda_n}) f_n(\tau)\right] e^{-\lambda_n(t-\tau)}.$$

By using Lemma 2, the term $F(t)$ and the kernel $K(t,\tau)$ are continuously differentiable functions in $[0,T]$ and $[0,T] \times [0,T]$, respectively. It is easy to show that

$$K(t,t) = \sum_{n=n_\alpha+1}^{\infty}\frac{\sqrt{2}}{\sqrt{\lambda_n}}(1 - \cos\sqrt{\lambda_n}) f_n(t) = \int_0^1 f(x,t)dx,$$

since

$$\int_0^1 X_n(x)dx = \frac{\sqrt{2}}{\sqrt{\lambda_n}}(1-\cos\sqrt{\lambda_n}) \text{ and } f(x,t) = \sum_{n=1}^{\infty} f_n(t)X_n(x).$$

Further, under the consistency assumption $\int_0^1 \varphi(x)dx = E(0)$, the Eq. (3.2) yields the following Volterra integral equation of the second kind:

$$K(t,t)R(t) + \int_0^t K_t(t,\tau)R(\tau)d\tau + F'(t) = E'(t).$$

Note that the function $K(t,t)$ is never equal to zero because of the assumption $\int_0^1 f(x,t)dx \neq 0$, $\forall t \in [0,T]$. In addition, the functions $F'(t)$, $E'(t)$ and the kernel $K_t(t,\tau)$ are continuous functions in $[0,T]$ and $[0,T]\times[0,T]$, respectively. We therefore obtain a unique function $r(t)$, continuous on $[0,T]$, which, together with the solution of the problem (1.1)–(1.4) given by the Fourier series (3.1), form the unique solution of the inverse problem. □

The following result on continuously dependence on the data of the solution of the inverse problem holds.

Theorem 2 (Continuous dependence upon the data). *Let $\digamma$ be the class of triples in the form of $\{f, \varphi, E\}$ which satisfy the assumptions of Theorem 1 and*

$$\|f\|_{C^{3,0}(\overline{D}_T)} \leq M_0, \|\varphi\|_{C^3[0,1]} \leq M_1, \|E\|_{C^1[0,T]} \leq M_2, 0 < M_3 \leq \left|\int_0^1 f(x,t)dx\right|,$$

for some positive constants M_i, $i = 0, 1, 2, 3$.

Then the solution pair (u,r) of the inverse problem (1.1–1.5) depends continuously upon the data in $\digamma$.

The proof of this Theorem is omitted since it is similar to Theorem 2 in [18].

4. Conclusion

We investigate the inverse source problem for the heat equation with a nonlocal Wentzell–Neumann boundary condition and integral overdetermination condition. Under some regularity, consistency and orthogonality conditions on initial data and known part of source term, the well-posedness of the classical solution are shown by using the generalized Fourier method. Theorems 1 and 2 establish that the inverse problem under investigation given by Eqs. (1.1)–(1.5) is well-posed in appropriate spaces of regular functions. However, in practice the input data, especially the measured one, such as the energy (1.5), is non-smooth and hence, the solution of the inverse problem becomes unstable under unregularised inversion. The discretisation of the inverse problem using the different methods and the discussion of the regularization of the numerical solution is for the future investigations. This inverse problem enables us to arrive at some conclusions on an externally controlled problem. The external energy is supplied to a target at a controlled level by a microwave generating equipment. However, the dielectric constant of the target material varies in space and time, resulting in spatially heterogeneous conversion of electromagnetic energy to heat. This can correspond to a source term $r(t)f(x,t)$, where $r(t)$ is proportional to the power of external energy source and $f(x,t)$ is the local conversion rate of microwave energy. It is needed to notice that this spatial variation of absorbing material does not greatly affect the thermal diffusivity, which is due to another material at higher concentration. It is also needed to say that the temperature is not so high that the temperature dependence of the dielectric constant is important, as in thermal runaway studies [32]. If $u(x,t)$ denotes the concentration of absorbed energy, then its integral over all volume of material determining the time dependenent absorbed energy. The above mentioned inverse problem for such a model gives an idea of how total energy content might be externally controlled when the boundary of material is supported by the nonlocal Wentzell–Neumann condition.

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Mansur I. Ismailov
Department of Mathematics
Gebze Technical University
41400 Gebze Kocaeli
Turkey
e-mail: `mismailov@gtu.edu.tr`

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