Results Math (2018) 73:34

1422-6383/18/010001-8
published online February 13, 2018
https://doi.org/10.1007/s00025-018-0800-2

Results in Mathematics

On an Inverse Spectral Problem for the Convolution Integro-Differential Operator of Fractional Order

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Abstract. An inverse spectral problem for the convolution integro-differential operator of fractional order $\alpha > 2$ is studied. We show that specification of one spectrum determines such operator uniquely independently of particular value of α. The convolution kernel can be recovered by solving a certain nonlinear equation.

Mathematics Subject Classification. 34A55, 45J05, 47G20, 34B09, 34B10
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Keywords. Integro-differential operator, nonlocal operator, inverse spectral problem.

1. Introduction

Let D^α be the Riemann–Liouville fractional differential operator and M be the convolution operator:

$$Mf(x) = (M * f)(x) = \int_0^x M(x-t)f(t)dt,$$

$M \in L_2(0,1)$. We consider the inverse problem of recovering the integro-differential fractional order operator:

$$L = D^\alpha + MD^{\alpha-1},\ \alpha > 2, \alpha \notin \mathbb{N} \tag{1.1}$$

from given spectrum of the boundary value problem:

$$Ly = \lambda y,\ D^{\alpha-k}y(0) = 0, k = \overline{2, [\alpha]+1},\ D^{\alpha-1}y(1) = 0, \tag{1.2}$$

where (and everywhere below) $[\alpha]$ denote an integer part of α.

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Inverse problems of spectral analysis consisting in recovering operators from their spectral characteristics often appear in mathematics, mechanics, physics and other branches of natural sciences and engineering. The greatest success in inverse problem theory has been achieved for Sturm–Liouville operators and afterwards for differential operators of an arbitrary (integer) order.

Inverse problems for integro-differential operators were found to be essentially more difficult. "Non-local" nature of such operators is insuperable obstacle for classical methods of inverse problem theory. Sporadic results obtained in this direction [3,5–7,9,10] do not form a comprehensive picture.

Most deep and nontrivial results were obtained in special (and important) case of convolution integro-differential operators. In [2] (see also [1]) it was shown that the Dirichlet spectrum of the operator:

$$Ly(x) = -y''(x) + \int_0^x M(x-t)y'(t)dt$$

determines uniquely the kernel $M(\cdot)$, which can be recovered by solving some nonlinear (uniquely solvable) equation. Moreover, similar result is true for convolution operators of *any* integer order, namely, specification of one spectrum determines uniquely the operator:

$$Ly(x) = y^{(n)}(x) + \int_0^x M(x-t)y^{(n-1)}(t)dt.$$

On can notice that these results are in contrast with the results of classical inverse problem theory for differential operators: it is well-known, for instance, that specification of one (say, Dirichlet) spectrum of the Sturm–Liouville operator does not determine the potential uniquely.

In this paper, we use some results of the author [4] to provide further development of the method presented in [2]. We show that main results of [2] can be, in general, extended to the fractional order case, namely, the fractional order operator (1.1) is determined uniquely by specification of (one) spectrum of the boundary value problem (1.2); the kernel $M(\cdot)$ can be recovered by solving some nonlinear (uniquely solvable) equation.

2. Construction of the Transformation Operator

Let $y(x,\lambda)$ be the solution of the following Cauchy problem:

$$Ly = \lambda y,\ D^{\alpha-k}y(0) = \delta_{k,1}, k = \overline{1, [\alpha]+1}, \tag{2.1}$$

where $\delta_{j,k}$ denote the Kroenecker delta.

It is clear that the spectrum of the problem (1.1)–(1.2) coincides with the set of roots of the following *characteristic function* $\Delta(\lambda) := D^{\alpha-1}y(1,\lambda)$. Define $\psi(x,\lambda) := D^{\alpha-1}y(x,\lambda)$. Then $\Delta(\lambda) = \psi(1,\lambda)$. Let $\tilde{y}(x,\lambda)$ be the solution of the "simplest" Cauchy problem:

$$D^{\alpha}y = \lambda y,\ D^{\alpha-k}y(0) = \delta_{k,1}, k = \overline{1, [\alpha]+1}. \tag{2.2}$$

Then for $\hat{y} := y - \tilde{y}$ one has:

$$D^{\alpha}\hat{y} - \lambda\hat{y} = -MD^{\alpha-1}y = M\psi.$$

Since $D^{\alpha-k}\hat{y}(0) = 0, k = \overline{1, [\alpha]+1}$ this yields:

$$\hat{y} = -J^{\alpha}(E - \lambda J^{\alpha})^{-1}M\psi,$$

where

$$Jf(x) := \int_0^x f(t)dt$$

and J^{α} denote the Riemann–Liouville fractional integration operator. Acting with the operator $D^{\alpha-1}$ we obtain finally the following integral equation with respect to $\psi(x,\lambda)$:

$$\psi = -\Phi_{\lambda}M\psi + \varphi, \tag{2.3}$$

where $\Phi_{\lambda} = (E - \lambda J^{\alpha})^{-1}J$, $\varphi = D^{\alpha-1}\tilde{y} = (E - \lambda J^{\alpha})^{-1}1$ or in more details:

$$\varphi(x,\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n x^{n\alpha}}{\Gamma(n\alpha+1)},\ \Phi_{\lambda}f(x) = \int_0^x \varphi(x-t,\lambda)f(t)dt.$$

We use the successive approximation method and obtain the solution of (2.3) in the form:

$$\psi(x,\lambda) = \sum_{n=0}^{\infty}(-1)^n\varphi_n(x,\lambda),\ \varphi_0 = \varphi,\ \varphi_{n+1} = \Phi_{\lambda}M\varphi_n.$$

Main technical tool for our further considerations is contained in the following lemma [4].

Lemma 2.1. *For $x, y > 0$, $\lambda \in \mathbb{C}$ the following relations hold:*

$$\varphi\left(x\omega^j,\lambda\right) = \varphi(x,\lambda),\ x > 0, j = \overline{-[\alpha/2],[\alpha/2]},\ \omega := \exp\left(i\frac{2\pi}{\alpha}\right),$$

$$\varphi(x,\lambda)\varphi(y,\lambda) = \frac{1}{\alpha}\sum_{j=-[\alpha/2]}^{[\alpha/2]} \varphi\left(x + \omega^j y,\lambda\right) + \int_0^x g(x-t,y)\varphi(t,\lambda)dt + \int_0^y g(y-t,x)\varphi(t,\lambda)dt,$$

where

$$g(x,y) = -\frac{\sin\alpha\pi}{\pi}\cdot\frac{x^{\alpha-1}y^{\alpha}}{x^{2\alpha} - 2x^{\alpha}y^{\alpha}\cos\alpha\pi + y^{2\alpha}}.$$

Using the assertion of Lemma 2.1 we obtain for $\varphi_1(x,\lambda), \varphi_2(x,\lambda), \dots$ successively the representations:

$$\varphi_n(x,\lambda) = \int_0^x K_n(x-t,t)\varphi(t,\lambda)dt, \tag{2.4}$$

$$K_n(\eta,\xi) = \theta_n(\xi)M^{*n}(\eta) + \sum_{\nu=0}^{n-1} \theta_\nu(\xi)\left(A_{n\nu} * M^{*n}\right)(\eta), \tag{2.5}$$

where (and below) we use the notations:

$$f^{*n} := f * f^{*(n-1)}, f^{*1} := f,\ \theta_n(\xi) := \frac{1}{n!}\left(\frac{\xi}{\alpha}\right)^n = 1^{*n}\left(\frac{\xi}{\alpha}\right).$$

Coefficients $A_{n\nu}(\eta)$ in (2.5) can be calculated recursively as follows:

$$A_{10} = -\frac{2}{\alpha}\left[\frac{\alpha}{2}\right] + g_{10} + g_{20}, \tag{2.6}$$

$$A_{n+1,n} = A_{n,n-1} + \frac{1}{\alpha}\sum_{0<|j|<\alpha/2} \frac{\omega^j}{1-\omega^j} + g_{20}, \tag{2.7}$$

$$\begin{aligned} A_{n+1,\nu} = A_{n,\nu-1} &+ \frac{1}{\alpha}\sum_{0<|j|<\alpha/2} \frac{\omega^j}{1-\omega^j}\left(\theta_{n-\nu,j} + \sum_{\mu=\nu}^{n-1}\theta_{\mu-\nu,j} * A_{n\mu}\right) \\ &+ g_{2,n-\nu} + \sum_{\mu=\nu}^{n-1} g_{2,\mu-\nu} * A_{n\mu},\ \nu = \overline{1,n-1}, \end{aligned} \tag{2.8}$$

$$\begin{aligned} A_{n+1,0} = -\frac{1}{\alpha}&\sum_{0<|j|<\alpha/2}\left(\theta_{n,j} + \sum_{\mu=0}^{n-1}\theta_{\mu,j} * A_{n\mu}\right) \\ &+ g_{2n} + g_{1n} + \sum_{\mu=0}^{n-1}(g_{2\mu} + g_{1\mu}) * A_{n\mu}, \end{aligned} \tag{2.9}$$

where

$$g_{1\nu}(\eta) := \int_0^\eta g(\eta-t,t)\theta_\nu(t)dt,\ g_{2\nu}(\eta) := \int_0^\eta g(t,\eta-t)\theta_\nu(t)dt, \tag{2.10}$$

$$\theta_{nj}(\xi) := \theta_n\left(\frac{\xi}{1-\omega^j}\right). \tag{2.11}$$

Observing that

$$g_{1\nu}(\eta) = \gamma_{1\nu}\theta_\nu(\eta),\ g_{2\nu}(\eta) = \gamma_{2\nu}\theta_\nu(\eta),$$

where the constants $\gamma_{1\nu}, \gamma_{2\nu}$ are bounded as $\nu \to \infty$ and using (2.6)–(2.11) one can deduce the estimate:

$$\max_{\eta\in(0,1)} |A_{n\nu}(\eta)| \le C^n. \tag{2.12}$$

From (2.5) and (2.12) it follows that the series $\sum_{n=1}^{\infty}(-1)^n K_n(\eta,\xi)$ converges absolutely and uniformly on the simplex $\{\eta,\xi \geq 0, \eta+\xi \leq 1\}$ and we obtain the following result.

Theorem 2.1. *The function $\psi(x,\lambda)$ admits the integral representation:*

$$\psi(x,\lambda) = \varphi(x,\lambda) + \int_0^x K(x-t,t)\varphi(t,\lambda)dt,$$
$$K(\eta,\xi) = \sum_{n=1}^{\infty}(-1)^n \left(\theta_n(\xi)M^{*n}(\eta) + \sum_{\nu=0}^{n-1}\theta_\nu(\xi)\left(A_{n\nu} * M^{*n}\right)(\eta)\right).$$

3. Inverse Problem

Let $\tilde{\Delta}(\lambda) = \varphi(1,\lambda)$ be a characteristic function of the boundary value problem (1.2) with "simplest" operator $\tilde{L} = D^\alpha$. Since $\varphi(t,\lambda) = E_{1/\alpha}\left(\lambda t^{1/\alpha};1\right)$ one can deduce the properties of $\varphi(t,\lambda)$, $\tilde{\Delta}(\lambda)$ and the corresponding spectrum from the well-known properties of Mittag–Leffler functions $E_p(z;\mu)$ and their roots [8].

Lemma 3.1. *The characteristic function $\tilde{\Delta}(\lambda)$ has infinite sequence $\{\tilde{\lambda}_k\}_{k=1}^{\infty}$ of roots. All roots are simple and real. For $k \to \infty$ the following asymptotics hold:*

$$\tilde{\lambda}_k = \tilde{\rho}_k^\alpha,\ \tilde{\rho}_k = \exp\left(i\frac{\pi}{\alpha}\right)\frac{\pi}{\sin(\pi/\alpha)}\left(k-\frac{1}{2}\right) + O(\exp(-\gamma k)),$$

where $\gamma > 0$.

Lemma 3.2. *The following representation holds:*

$$\varphi(t,\lambda) = \frac{1}{\alpha}\sum_{j=-[\alpha/2]}^{[\alpha/2]}\exp(\rho\omega^j t) + \int_0^\infty g(t,x)\exp(-\rho x)dx,\ \lambda = \rho^\alpha, Re\rho > 0,$$

where $g(\cdot,\cdot)$ is the same is in Lemma 2.1.

Now we consider the characteristic function $\Delta(\lambda)$ of general boundary value problem (1.2). From Theorem 2.1 it follows the representation:

$$\Delta(\lambda) = \tilde{\Delta}(\lambda) + \int_0^1 w(1-t)\varphi(t,\lambda)dt, \tag{3.1}$$

where $w(t) = K(t,1-t) \in L_2(0,1)$. Using Lemmas 3.1, 3.2, representation (3.1) and repeating standard arguments based on the Rouche theorem we conclude that the characteristic function $\Delta(\lambda)$ has infinitely many roots and these roots (being counted with their multiplicities) $\{\lambda_k\}_{k=1}^{\infty}$ admits the asymptotics:

$$\lambda_k = \rho_k^\alpha,\ \rho_k = \tilde{\rho}_k + o(1). \tag{3.2}$$

Moreover, one can obtain more detailed result by substituting

$$\lambda_k = \left(z_k \exp\left(i\frac{\pi}{\alpha}\right)\right)^{\alpha}$$

to the equation $\Delta(\lambda_k) = 0$ and using the asymptotics (as $k \to \infty$):

$$z_k = \frac{\pi}{s}\left(k - \frac{1}{2}\right) + o(1),\ \Delta(\lambda_k) = \frac{2}{\alpha}\exp(cz_k)\cos(sz_k) + O\left(\exp(cz_k)k^{-1/2}\right),$$
$$c := \cos\left(\frac{\pi}{\alpha}\right),\ s := \sin\left(\frac{\pi}{\alpha}\right)$$

that follow from (3.1), (3.2) and Lemmas 3.1, 3.2. On this way we arrive at the following result.

Theorem 3.1. *Boundary value problem (1.2) has infinite sequence* $\{\lambda_k\}_{k=1}^{\infty}$ *of eigenvalues. The eigenvalues have the form:*

$$\lambda_k = \rho_k^{\alpha},\ \rho_k = \tilde{\rho}_k + \hat{\rho}_k,\ k^{-1}\hat{\rho}_k \in l_1.$$

Now we can give precise formulation of the inverse problem we consider.

Inverse Problem 1 Given $\{\lambda_k\}_{k=1}^{\infty}$, where the eigenvalues λ_k are counted with their algebraic multiplicities. Find $M(\eta)$, $\eta \in (0, 1)$.

The first step in our solution of the inverse problem consists in recovering the characteristic function $\Delta(\lambda)$ from given spectrum $\{\lambda_k\}_{k=1}^{\infty}$. The Hadamard theorem implies:

$$\Delta(\lambda) = C\lambda^r \prod_{k=r+1}^{\infty}\left(1 - \frac{\lambda}{\lambda_k}\right). \tag{3.3}$$

Since $\tilde{\Delta}(0) = \varphi(1, 0) = 1$ the Hadamard theorem for $\tilde{\Delta}(\lambda)$ reads as follows:

$$\tilde{\Delta}(\lambda) = \prod_{k=1}^{\infty}\left(1 - \frac{\lambda}{\tilde{\lambda}_k}\right). \tag{3.4}$$

In order to find the constant C in (3.3) we notice that by virtue of (3.1) and Lemmas 3.1, 3.2 we have:

$$\lim_{\lambda\to+\infty}\frac{\Delta(\lambda)}{\tilde{\Delta}(\lambda)} = 1.$$

On the other hand, from Theorem 3.1 it follows that the infinite product $\prod_{k=r+1}^{\infty}(\tilde{\lambda}_k/\lambda_k)$ converges and

$$\lim_{\lambda\to+\infty}\prod_{k=r+1}^{\infty}\frac{\lambda - \lambda_k}{\lambda - \tilde{\lambda}_k} = 1.$$

This yields:

$$C = \prod_{k=1}^{r}\left(\frac{-1}{\tilde{\lambda}_k}\right)\prod_{k=r+1}^{\infty}\left(\frac{\lambda_k}{\tilde{\lambda}_k}\right)$$

and finally we obtain the representation:

$$\Delta(\lambda) = \prod_{k=1}^{\infty} \frac{\lambda_k - \lambda}{\tilde{\lambda}_k}. \tag{3.5}$$

Our next observation is that the function $w(t) := K(t, 1-t)$ is determined uniquely via (3.1) once the characteristic function $\Delta(\lambda)$ has been calculated since the system $\{\varphi(t,\lambda)\}_{\lambda\in\mathbb{C}}$ is complete in $L_2(0,1)$.

Further, Theorem 2.1 yields:

$$\begin{aligned} w(t) = K(t,1-t) &= -\theta_1(1-t)M(t) - (A_{10} * M)(t) \\ &\quad + \sum_{n=2}^{\infty} (-1)^n \left(\theta_n(1-t) M^{*n}(t) + \sum_{\nu=0}^{n-1} \theta_\nu(1-t) \left(A_{n\nu} * M^{*n}\right)(t) \right). \end{aligned}$$

We rewrite this in the following form:

$$\begin{aligned} M(t) = &-\frac{\alpha}{1-t} w(t) - A_{10} \int_0^t \frac{\alpha}{1-t} M(\tau) d\tau \\ &+ \sum_{n=2}^{\infty} (-1)^n \left\{ \frac{\alpha\theta_n(1-t)}{1-t} M^{*n}(t) \right. \\ &\left. + \int_0^t \left(\sum_{\nu=0}^{n-1} \frac{\alpha\theta_\nu(1-t)}{1-t} A_{n\nu}(t-\tau) \right) M^{*n}(\tau) d\tau \right\} \end{aligned} \tag{3.6}$$

and consider (3.6) as a nonlinear equation w.r.t. desired function $M(\cdot)$.

Theorem 3.2. *For any fixed $T \in (0,1)$ equation (3.6) is uniquely solvable in $L_2(0,T)$.*

Theorem 3.2 follows immediately from estimates (2.12) and the general lemma below that can be obtained by repeating the arguments from the proof of Theorem 4 in [1].

Lemma 3.3. *Consider the integral equation:*

$$y(t) = f(t) + \sum_{n=1}^{\infty} \left(f_n(t) y^{*n}(t) + \int_0^t F_n(t,\tau) y^{*n}(\tau) d\tau \right),$$

where

$$\|f_n\|_{L_2(0,T)} \le C^n, \ \ \|F_n\|_{L_2((0,T)\times(0,T))} \le C^n.$$

Suppose that $f_1(t) \equiv 0$. Then for any function $f \in L_2(0,T)$ the equation has a unique solution in $L_2(0,T)$.

Thus, we arrive at the following main result.

Theorem 3.2. *Specification of the spectrum $\{\lambda_k\}_{k=1}^{\infty}$ of the boundary value problem (1.2) determines uniquely operator (1.1). The kernel $M(\cdot)$ can be recovered by making the following steps:*

1) Calculate $\Delta(\cdot)$ *via* (3.5).
2) Find $w(\cdot)$ *from* (3.1).
3) Find $M(\cdot)$ *by solving main Eq.* (3.6).

Acknowledgements

This work was supported by the Russian Science Foundation (Project No. 17-11-01193).

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Received: December 13, 2017.
Accepted: January 27, 2018.